Lüders conditionalization: Conditional probability, transition probability, and updating in quantum probability theory

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Guerra Bobo (2013) has questioned whether Lüders conditionalization for quantum probabilities supplies a notion of conditional probability worthy of the name. I agree in large part with Guerra Bobo's critique; indeed, I show how her critique can be sharpened. But while valuable in itself, the main virtue of the critique derives from the fact that it prompts questions about the nature of quantum probabilities that engage some of the more important and contentious issues in the foundations of QM. I show how understanding the role of Lüders conditionalization in quantum probability theory helps to illuminate these issues.

1 Introduction

A natural question to raise about quantum probability is: In QM what plays the role of the concept of conditional probability familiar from classical probability? The widely accepted answer is that the rightful occupant of this role is Lüders conditionalization. Guerra Bobo (2013) has argued that the common wisdom is wrong and, more strongly, that there simply is no satisfactory extension to quantum probability of the classical concept of conditional probability. I agree in large part with Guerra Bobo's critique; indeed, I show how her critique can be sharpened. But while valuable in itself, the main virtue of the critique derives from the fact that prompts questions about the nature of quantum probabilities that engage some of the more important and contentious issues in the foundations of QM.

To begin, what does Lüders conditionalization supply, if not a conditional probability in a sense anywhere close to classical conditionalization? My answer is that Lüders conditionalization supplies a rule for updating quantum probabilities on the outcomes of measurements. Furthermore, this updating rule is well-motivated on both the objectivist and personalist interpretations of quantum probability; indeed, it brings the two interpretations into harmonious agreement as regards quantum state preparation. This account, not surprisingly, engages the notorious measurement problem.

The discussion is organized as follows. Section 2 gives a brief review of the formalism of quantum probability theory, construed as the study of probability measures on the projection lattice of a von Neumann algebra. The relation between quantum probability measures and quantum states is discussed, and it is seen how this relation can be used to underwrite both personalist and objectivist interpretations of quantum probabilities. Section 3 reviews the argument that Lüders conditionalization is the unique quantum analog of classical conditionalization satisfying the condition that it reduces to classical conditionalization in the commutative case. Section 4 underscores and sharpens Guerra Bobo's contention that, despite serving as the closest quantum counterpart of classical conditionalization, Lüders conditionalization lacks core features of its classical counterpart. Section 5 reviews arguments for classical conditionalization as supplying a rule for updating personal probabilities, and it is shown how that the arguments can be carried over to the quantum context to motivate using Lüders conditionalization to update personal probabilities on a quantum projection lattice. Section 6 gives an example of how using Lüders updating reveals one of the most characteristic features of QM—interference effects. Section 7 discusses the updating of quantum probabilities construed as objective probabilities induced by observer-independent states. Section 8 indicates how Lüders updating serves to meld the personalist and objectivist take on quantum probabilities in state preparation, and also why rational credence over quantum events must track objective probabilities. But the story that generates this seeming harmony depends on accepting the Lüders/von Neumann projection postulate, a generalized version of state vector reduction (aka collapse of wave packet). Conclusions are presented in Section 9.

2 Quantum probability theory, quantum states, and the nature of quantum probability

2.1 Quantum probability

Quantum probability theory may be construed as the study of quantum probability measures on the projection lattice $\mathcal{P}(\mathfrak{N})$ of a von Neumann algebra $\mathfrak{N}$ acting on a Hilbert space $\mathcal{H}$ (see Hamhalter 2003). A projection E is an "observable," i.e. a self-adjoint operator, and it is idempotent, i.e., $E^2 = E$. $\mathcal{P}(\mathfrak{N})$ is equipped with a natural partial order, viz. for $E, F \in \mathcal{P}(\mathfrak{N})$, $F \leq F$ iff $range(E) \subseteq range(F)$. $\mathcal{P}(\mathfrak{N})$ is closed under meet $\wedge$ (the least upper bound) and join $\vee$ (greatest lower bound). Elements $E_1, E_2 \in \mathcal{P}(\mathfrak{N})$ are mutually orthogonal iff $E_1 E_2 = E_2 E_1 = O$ (the null projection). When E_1 and E_2 are mutually orthogonal $E_1 \vee E_2 = E_1 + E_2$. Complementation (or negation) is understood as orthocomplementation. The elements of $\mathcal{P}(\mathfrak{N})$ are variously referred to as events, propositions, or Yes-No questions. It is assumed that for any $E \in \mathcal{P}(\mathfrak{N})$ it is possible in principle to to design an experiment that will answer the Yes-No question posed by E. The theory does not come with a manual specifying how to build an apparatus to carry out such an experiment; this is part of the practical art of applying QM to the world.

A quantum probability measure on $\mathcal{P}(\mathfrak{N})$ is a map $\Pr : \mathcal{P}(\mathfrak{N}) \to [0, 1]$ satisfying

(i) $\Pr(I) = 1$ (I the identity projection)

(ii) $\Pr(E_1 \vee E_2) = \Pr(E_1 + E_2) = \Pr(E_1) + \Pr(E_2)$ whenever $E_1, E_2 \in \mathcal{P}(\mathfrak{N})$ are mutually orthogonal.

The requirement (ii) of finite additivity can be strengthened to complete additivity

(ii*) For any family $\{E_a\} \in \mathcal{P}(\mathfrak{N})$ of mutually orthogonal projections, $\Pr(\vee_a E_a) = \sum_a \Pr(E_a)$.[1]

For sake of simplicity I concentrate on the case of ordinary nonrelativistic QM where $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$, the von Neumann algebra of all bounded operators acting on $\mathcal{H}$. For most applications it suffices to use a separable $\mathcal{H}$, in which case any family of mutually orthogonal projections is countable and, thus, complete additivity (ii*) reduces to countable additivity. And, of course, when $\mathcal{H}$ is finite dimensional (ii*) reduces to finite additivity (ii).

[1] When the sum $\sum_a \Pr(E_a)$ is over a transfinite collection it is to understood as the the supremum of sums over finite subcollections. The sums over ever large subcollections form a bounded non-decreasing sequence of real numbers which always has a least upper bound.

2.2 Quantum states

A quantum state ω is a complex valued, normed, positive linear functional on $\mathfrak{B}(\mathcal{H})$. A state is pure iff it cannot be written as a convex linear combination of other distinct states; impure states are often referred to as mixed states. A normal state is a state that admits a density operator representation or, equivalently, is countably additive on any family of mutually orthogonal projections.[2] On $\mathfrak{B}(\mathcal{H})$ the vector states coincide with the normal pure states. That ω is a vector state means that there is a $|\psi\rangle \in \mathcal{H}$ such that $\omega(A) = \langle\psi|A|\psi\rangle$ for all $A \in \mathfrak{B}(\mathcal{H})$.

Any state ω on $\mathfrak{B}(\mathcal{H})$, pure or impure, normal or non-normal, induces a quantum probability measure on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$, viz. $\Pr^{\omega}(E) = \omega(E)$, $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$, satisfies the axioms of quantum probability listed above. A normal (respectively, non-normal) state induces a countably additive (respectively, merely finitely additive) probability on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$. Gleason's theorem provides an almost general converse:

> *Gleason*: Let Pr be a quantum probability on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ where $\dim(\mathcal{H}) > 2$. Then Pr has a unique extension to a state on $\mathfrak{B}(\mathcal{H})$ which is normal (respectively, non-normal) if Pr is countably additive (respectively, merely finitely additive).[3]

2.3 The nature of quantum probabilities

The two-way traffic between quantum probability measures on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ and states on $\mathfrak{B}(\mathcal{H})$ enables a variety of viewpoints on the nature of quantum probabilities. An objectivist interpretation would claim that quantum states codify objective, observer-independent features of quantum systems and, thus, the probabilities that states induce on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ are objective

[2]See Kadison and Ringrose (1991), Vol. 2, Theorem 7.1.12. The density operator representation allows expectation values to be calculated via the trace prescription, viz. $\omega(A) = Tr(\varrho_\omega A)$, $A \in \mathfrak{B}(\mathcal{H})$, where ϱ_ω is the density operator corresponding to the normal state ω. Philosophical discussions of Lüders conditionalization tend to make heavy use of calculations using the trace formalism, which has the drawback of obscuring the connection to the associated probabilities on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$.

[3]The original version of Gleason's theorem did not deal with the case of merely finitely additive probabilities. A detailed treatment of Gleason's theorem and its generalizations can be found in Hamhalter (2003).

chances. Such a viewpoint can exist side-by-side with a personalist interpretation of probability a measure on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ as representing the degrees of belief of a physicist about quantum events. But then there arises the question about the relation between rational credence and objective chance. Quantum Bayesians (QBians as they style themselves) think that this question is meaningless since they insist that all probabilities in QM are to be given a personalist interpretation.[4] Enabled by Gleason's theorem QBians take quantum states to be merely representational devices that can be used to bookkeep the credence functions of Bayesian agents. Lüders conditionalization will prove to be crucial to understanding and evaluating there various viewpoints on the nature of quantum probabilities.

3 Classical and quantum conditionalization

An elegant presentation of the argument that for the probabilities on the projection lattice $\mathcal{P}(\mathfrak{N})$ of a general von Neumann algebra $\mathfrak{N}$ Lüders conditionalization supplies the correct quantum analog of classical conditionalization is provided by Cassinelli and Zanghi (1983). Here I restrict attention to the case of $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$.[5]

The argument starts by noting the following feature of the standard definition of classical conditional probability:

> *Prop 1.* Let (Ω, Σ, pr) be a classical probability space, and let $F \in \Sigma$ be such that $pr(F) \neq 0$. Then there is a unique functional $pr(\bullet/F)$ on Σ such that (a) $pr(\bullet/F)$ is a probability measure on Σ, and (b) for all $E \in \Sigma$ such that $E \subseteq F$, $pr(E/F) = pr(E)/pr(F)$.[6]

And, no surprise, this unique functional $pr(\bullet/F)$ is just the familiar classical conditional probability functional $pr(\bullet \cap F)/pr(F)$.

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[4] See von Baeyer (2016) for an accessible overview of this program. For a critical reaction, see Timpson (2008) and Earman (2018a).

[5] The special case of $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ was treated in Bub (1977).

[6] Ω is the sample space; Σ (the measurable sets) consists of a set of subsets of Ω; and $\Pr : \Sigma \to [0, 1]$ satisfies (i′) $\Pr(\Omega) = 1$ and (ii′) $\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2)$ for any $E_1, E_2 \in \Sigma$ such that $E_1 \cap E_2 = \emptyset$, the classical counterparts of the requirements for a quantum probability.

It seems beyond dispute that a quantum analog of classical conditionalization should agree with the latter in the case of commuting projections. Using '//' to indicate quantum conditionalization, the requirement would be

> (*) If Pr is a quantum probability on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ and $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ is such that $\Pr(F) \neq 0$ then quantum conditionalization $\Pr(\bullet//F)$ on F is a quantum probability on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$, and $\Pr(E//F)$ agrees with classical conditionalization $\Pr(E/F) := \dfrac{\Pr(EF)}{\Pr(F)}$ for all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ such that $EF = FE$, i.e. $\Pr(E//F) = \Pr(E/F)$.

Note that $E \leq F$ then $EF = FE = E$. So by (*) when $\Pr(F) \neq 0$

$$\Pr(E//F) = \Pr(E/F) := \frac{\Pr(EF)}{\Pr(F)} = \frac{\Pr(E)}{\Pr(F)}.$$

Then use the following quantum counterpart of Prop. 1:

> *Prop. 2.* Let Pr be a countably additive quantum probability measure on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ for separable $\mathcal{H}$ with $\dim(\mathcal{H}) > 2$, and let $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ be such that $\Pr(F) \neq 0$. Then there is a unique functional $\Pr(\bullet//F)$ on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ such that (a) $\Pr(\bullet//F)$ is a quantum probability, and (b) for all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ such that $E \leq F$, $\Pr(E//F) = \Pr(E)/\Pr(F)$ (Cassinelli and Zanghi 1983).

Conclude that for $\dim(\mathcal{H}) > 2$ and Pr countably additive there is a unique quantum conditionalization rule satisfying (*).

What is this unique rule, known as Lüders conditionalization? For $\dim(\mathcal{H}) > 2$ and Pr countably additive Gleason's theorem applies and there is a unique extension of Pr to a normal state ω on $\mathfrak{B}(\mathcal{H})$. It is easy to verify that if ω is a normal state and $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ is such that $\omega(F) \neq 0$ then $\omega_F(E) := \omega(FEF)/\omega(F)$, $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$, defines a normal state ω_F. Hence, $\Pr^{\omega_F}(\bullet) := \omega_F(\bullet)$ defines a countably additive quantum probability. The unique functional in question $\Pr(\bullet//F)$ is given by $\Pr^{\omega_F}(\bullet)$, i.e. $\Pr(\bullet//F) = \dfrac{\omega(F \bullet F)}{\omega(F)} = \dfrac{\omega(F \bullet F)}{\Pr(F)}$ (see Fig. 1). Note that when E and F do not commute the numerator $\omega(FEF)$ on the rhs cannot be written as $\Pr(FEF)$ since $FEF \notin \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ and, hence, its probability is not defined. This is what necessitates the detour through the space of states in going from $\Pr(\bullet)$ to $\Pr(\bullet//F)$.

If $\mathcal{H}$ is infinite dimensional and Pr is merely finitely additive it extends uniquely to a non-normal state so that Lüders conditionalization is defined. But the Cassinelli and Zanghi (1983) proof of uniqueness of Lüders conditionalization uses countable additivity; I conjecture that uniqueness fails for Pr merely finite additive, but do not have an example demonstrating non-uniqueness. When $\dim(\mathcal{H}) = 2$ countable additivity reduces to finite additivity and all states are normal; but there there are quantum probability measures on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ that do not extend to any state on $\mathfrak{B}(\mathcal{H})$. Whether or not there is a plausible conditionalization rule for such measures is an interesting issue that will not be tackled here.

4 Quantum conditional probability (?)

"*Abandon hope all ye who enter here seeking conditional probability*"[7]

Granted that Lüders conditionalization provides the correct quantum analog of classical conditionalization, it does not follow that this analog supplies what deserves to be called a quantum conditional probability. Guerro Bobo's (2013) central complaint is that, except for the case of commuting projections, $\Pr(E//F)$ does not express a quantum analog of the core idea of classical conditional probability; namely, the idea that the probability of E conditional on F is the probability of what is "common" to E and F, normalized by dividing by the probability of F . This seems correct since $\Pr(E//F)$ is equal to $\dfrac{\omega(FEF)}{\Pr(F)}$, and the numerator is not the probability of what is common to E and F since it is not the probability of any quantum proposition when E and F do not commute.[8]

This criticism can be sharpened. Trying to read a Lüders conditional probability statement '$\Pr(E//F) = q$' (or any other candidate for quantum conditional probability) classically as 'Given that F is true, the probability that E is also true equals q' is a non-starter. Indeed, trying to read an

[7]Supposedly Dante's reaction to reading a text on QM.

[8]The meet $E \wedge F$ is in $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ even when E and F do not commute. However, as Guerro Bobo notes, $\widetilde{\Pr}(\bullet) := \Pr(\bullet \wedge F)/\Pr(F)$ does not define a quantum probability on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$.

unconditional quantum probability statement 'Pr$(E) = p$' classically as 'The probability that E is true equals p' is non-starter. The reason is simply that such reading presuppose that the elements of $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ have simultaneous truth values. But under the natural constraints on a truth value assignment this presupposition is impossible to satisfy except when $\dim(\mathcal{H}) = 2$.

Applied to $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ the quantum analog of a classical truth value assignment is a map $V : \mathcal{P}(\mathfrak{B}(\mathcal{H})) \to \{True, False\}$ satisfying (at a minimum)

(α) $V(I) = True$

(β) For any mutually orthogonal E_1 and E_2, if $V(E_1) = True$ then $V(E_2) = False$

(γ) For any mutually orthogonal E_1 and E_2, $V(E_1 \vee E_2) = True$ if either $V(E_1) = True$ or $V(E_2) = True$, and $V(E_1 \vee E_2) = False$ if both $V(E_1) = False$ and $V(E_2) = False$.

Suppose such a V exists. Define $\Pr : \mathcal{P}(\mathfrak{B}(\mathcal{H})) \to \{1, 0\}$ by: $\Pr(E) = 1$ if $V(E) = True$ and $\Pr(E) = 0$ if $V(E) = False$. Verify that Pr is a dispersion free quantum probability on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$. But if $\dim(\mathcal{H}) > 2$ no dispersion free probability exists (see Hamhalter 2003, p. 90) and, hence, no truth evaluation exists for $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$. The contrast here with classical observables is dramatic. If $\mathfrak{N}$ is an abelian von Neumann algebra then the projection lattice $\mathcal{P}(\mathfrak{N})$ admits dispersion free probability measures galore; indeed, every pure state on $\mathfrak{N}$ induces a dispersion free probability measure on $\mathcal{P}(\mathfrak{N})$.

The alternative to reading $\Pr(E) = p$ as 'The probability that E is true equals p' is 'The probability that a Yes-No measurement of E will give a Yes answer equals p' where there is no presumption that prior to measurement E has a definite truth value. Similarly, $\Pr(E//F) = q$ is not to be read as 'Given that F is true, the probability that E is also true equals q' but rather as 'Given that a Yes-No measurement of F has given a Yes answer, the probability that a Yes-No measurement of E will give a Yes answer equals q' where again there is no presumption about the truth values of E and F prior to their measurements. These alternative readings will be adopted here.

Disanalogies between classical conditionalization and Lüders conditionalization could be paraded. For instance, classical conditional probability exhibits symmetry of relevance/irrelevance relations, i.e. if $pr(E) \neq 0 \neq$

$pr(F)$ then $pr(E/F) = pr(F)$ iff $pr(F/E) = pr(F)$. However, for quantum probability when $EF \neq FE$, $\Pr(E) \neq 0 \neq \Pr(F)$ does not entail that $\Pr(E//F) = \Pr(F)$ iff $\Pr(F//E) = \Pr(F)$. Other disanalogies will emerge below. But perhaps enough has been said to underscore Guerra Bobo's point that, except in the commutative case when it reduces to classical conditionalization, Lüders conditionalization diverges in numerous and striking ways from its classical counterpart.

Such divergences are quite typical in theory change. To use a well-worn example, a key feature of mass in Newtonian mechanics is that it is a scalar invariant, whose value is independent of the state of motion. By contrast, in special relativistic mechanics mass depends on velocity. This divergence is often acknowledged by putting qualifiers on the term 'mass' by speaking of 'Newtonian mass' and 'relativistic mass'. In cases where theory change produces a pronounced divergence between the old and new concepts one may wonder whether there has been a change is reference as well as meaning, and whether it may be best to coin a completely different term for the new concept. I will not enter into these troubled waters here because, whatever the decision as regards Lüders conditionalization, there are other more pressing matters.

Classical conditionalization is widely touted as supplying a rule for updating classical probabilities. Does Lüders conditionalization, despite its shortcomings as supplying a quantum concept of conditional probability, play an analogous role for updating quantum probabilities? Answering this question is, I will argue, crucial to understanding how quantum probabilities work.

5 Updating: the personalist perspective

5.1 Classical updating of personal probabilities

On the personalist interpretation of classical probability, on which probability is construed as rational degrees of belief, an agent can ask: 'If my initial probability function is pr, what should my new probability function pr_F be upon learning that F is true, and how is pr_F related to pr?' The widely accepted answer is updating by classical conditionalization on the item learned (aka Bayes updating): $pr_F(\bullet) := pr(\bullet \cap F)/pr(F)$, assuming $pr(F) \neq 0$.[9] If

[9] How to handle the zero-probability cases is a matter of dispute. We will not enter the fray here since it would take us too far afield.

it is agreed that a desirable feature of classical updating is that when $E \subseteq F$, the F-updated probability $pr_F(E)$ should equal $pr(E)/pr(F)$, provided that $pr(F) \neq 0$, then Prop. 1 justifies equating classical updating with Bayes updating.

For those who will not join the agreement the diachronic Dutch book argument due to David Lewis and Paul Teller (see Teller 1976) may be more persuasive. A bookie employs a two-stage suite of bets using the agent's personal probability pr as the fair betting quotient.[10]

> *Stage 1.* The bookie sells the agent an unconditional bet on F with betting quotient $pr(F)$ and stakes S_F.[11] At the same time the bookie buys from the agent a bet on E conditional on F using the betting quotient $pr(E/F) = pr(E \cap F)/pr(F)$ and stakes $S_{E/F}$. ('Conditional on F' means that if F is found to be false, the bet is called off.) The truth value of F is then ascertained and the result is announced to both the agent and the bookie. If F is found to be false the bookie collects on the unconditional bet, calls off the conditional bet and closes shop. If F is found to be true the bookie proceeds to the next stage.
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> *Stage 2.* The bookie sells the agent an unconditional bet on E with stakes S_E and betting quotient given by the agent's probability $pr_F(E)$ of E updated on the knowledge that F is true. The truth of E is ascertained and the remaining bets are settled.

If the agent's updated $pr_F(E)$ is not equal to the conditional probability $pr(E/F)$ then the stakes of the three bets can be chosen so that, come what may, the agent loses money; conversely, if $pr_F(E) = pr(E/F)$ then the agent is protected from diachronic Dutch book.

There are various qualms about the efficacy of the diachronic Dutch book argument, some of which I share.[12] But for present purposes I ignore them to concentrate on the issue of whether or not an analogous argument can be mounted in favor of using Lüders conditionalization as the rule for updating personal probabilities over the lattice of projections.

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[10] For an overview of Dutch book arguments, see Vineberg (2016).

[11] This bet is a contract whereby the agent agrees to pay the bookie $pr(F) \cdot S_A$ in order to collect S_A from the bookie if F is true and nothing if F is false.

[12] In particular, it has been claimed that diachronic Dutch book provides merely pragmatic grounds for adopting Bayes updating and does not show that an agent who departs from Bayes updating will necessarily have irrational credences (see Christensen 1991).

5.2 Quantum updating of personal probabilities

A physicist whose degrees of belief about quantum events are represented by a probability measure Pr on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ can ask herself the appropriate quantum counterpart of the updating question her classical counterpart asked; namely, 'If my initial probability function is Pr, what should my new probability function $\Pr_F$ be if I learn that a Yes-No measurement of $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ has given a Yes answer, and how is $\Pr_F$ related to Pr? If Lüders conditionalization is to supply the quantum updating rule then when Gleason's theorem applies and Pr is countably additive the answer is that the agent's updated $\Pr_F(\bullet)$ should be $\dfrac{\omega(F \bullet F)}{\Pr(F)}$ where ω is the unique normal state corresponding to Pr. I remark in passing that Lüders updating should cause the QBians to blush: they maintain that quantum states are merely representational devices for keeping track of the personal probabilities of QBian agents, but Lüders updating requires an indispensable use of quantum states.

The case for Lüders updating of personal quantum probabilities seems just as good—or just as inadequate—as the case for Bayes updating of personal classical probabilities. In parallel with classical probability, Prop. 2 would provide a justification for Lüders updating if it is agreed that whenever $E \leq F$, $E, F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$, an agent's F-updated $\Pr_F(E)$ should equal $\Pr(E)/\Pr(F)$, provided that $\Pr(F) \neq 0$.

To persuade those who resist joining this agreement, a quantum version the diachronic Dutch book argument can be trotted out, using a protocol for settling bets that is tailored to the proposed reading of quantum probabilities as being about the outcomes of Yes-No measurements of elements of $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$. The argument would fail for non-commuting $E, F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ if it required simultaneous measurability of E and F to settle the bets; but the two-stage construction avoids this pitfall. In Stage 1 the bookie sells the agent an unconditional bet on F with betting quotient $\Pr(F)$. At the same time the bookie buys from the agent a bet on E conditional on F using the betting quotient $\Pr(E//F)$. A Yes-No measurement of F is performed. If a No answer is obtained the bookie collects on the unconditional bet, calls off the conditional bet, and closes shop. If a Yes answer is obtained a second round of betting ensues. In Stage 2 the bookie sells the agent an unconditional bet on E using as the betting quotient the agent's probability $\Pr_F(E)$ of E updated on the knowledge of the Yes outcome of the measurement of F. A Yes-No measurement of E is then performed. If a Yes answer (respectively, a No answer) is obtained the agent (respectively, the bookie) collects on the Stage 2 unconditional bet on E but loses (respectively, wins) the Stage 1 conditional bet on E. As in the classical case, the stakes of the three bets can be chosen so that the agent is a net loser come what may just in case $\mathrm{Pr}_F(E) \neq \mathrm{Pr}(E//F)$.

The failure of Lüders conditionalization to deliver a conditional probability that exhibits core features of its classical counterpart is irrelevant to the above construction. Of course, the same qualms that were raised against the classical version of the diachronic Dutch book argument can also be raised against the proposed quantum version. In what follows I will assume, despite the qualms, that rational agents ought to use Lüders updating of their personal probabilities for $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$. The next task is to understand the consequences of following this updating rule.

5.3 Consequences of Lüders updating

An agent who Lüders updates will quickly discover that her updated credences differ markedly from those of her classical counterpart who Bayes updates. Here is one example. Suppose that a classical Bayesian agent with initial credence function pr learns that F is true, where $F = F_1 \cup F_2$ and $F_1 \cap F_2 = \emptyset$. What is her new credence function if she Bayes updates on F? The answer is supplied by a theorem of classical probability:

(†) If $F = F_1 \cup F_2$ and $F_1 \cap F_2 = \emptyset$ then

$$pr(E/F) = pr(E/F_1)\frac{pr(F_1)}{pr(F)} + pr(E/F_2)\frac{pr(F_2)}{pr(F)}.$$

A quantum Bayesian agent can ask herself a similar question. If her initial credence function is Pr and she learns that a Yes-No measurement of $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$, where $F = F_1 \vee F_2$ with F_1 and F_2 mutually orthogonal, has yielded a Yes answer, what is her new credence function if she Lüders updates on F? Supposing that Pr is a countably additive quantum probability measure and that Gleason's theorem applies, Pr extends uniquely to a normal state ω. For such a Pr the quantum version of (†) is

(‡) If $F = F_1 \vee F_2$ with F_1 and F_2 mutually orthogonal,

$$\Pr(E//F) = \Pr(E//F_1)\frac{\Pr(F_1)}{\Pr(F)} + \Pr(E//F_2)\frac{\Pr(F_2)}{\Pr(F)}$$

$$+\frac{\omega(F_1EF_2)}{\Pr(F)} + \frac{\omega(F_2EF_1)}{\Pr(F)}.$$

The extra terms are indicative of interference effects, one of the key features that separates quantum mechanics from classical mechanics. The reader is invited to apply (‡) to the famous two slit experiment.

The reader can easily verify that the Lüders version of Bayes' theorem contains and extra factor:

$$(\natural)\quad \Pr(E//F) = \frac{\Pr(E)\Pr(F//E)}{\Pr(F)}\left[\frac{\omega(FEF)}{\omega(EFE)}\right]$$

where again ω is the normal state that extends Pr.

6 Quantum updating: the objectivist interpretation

On the objectivist reading of quantum probabilities, quantum states codify objective (= observer independent) features of quantum systems and, thus, the probabilities they induce on the projection lattice are objective. On this reading updating of probabilities can only concern a change in the state-induced probability due to a change of state. The change of state to be considered here follows the Lüders/von Neumann projection postulate:

> *L/vN Postulate*: If the pre-measurement state of a system is ω and a measurement of $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ returns a Yes answer, then the post-measurement state is $\omega_F(A) := \dfrac{\omega(FAF)}{\omega(F)}$, $A \in \mathfrak{B}(\mathcal{H})$, provided that $\omega(F) \neq 0$.

It is worth noting in passing that von Neumann's preferred version of the projection postulate differs from the version stated here in the case where the F's are the spectral projections of observables with degenerate spectra. Preliminary experimental evidence seems to favor the Lüders version stated here (see Hegerfeldt and Mayato 2012 and Kumar, Shukla, and Mahesh 2016).

Another way to arrive at the L/vN postulate is to reverse engineer Lüders updating of personal probabilities. The engineering is conduced under the assumption that is baked into the standard practice of ordinary QM; namely, that only normal states are physically realizable.[13] Let $\Pr^{\omega}$ be the probability measure on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ induced by the normal state ω on $\mathfrak{B}(\mathcal{H})$. Suppose that a Yes-No measurement of $F \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ returns a Yes answer. What post-measurement state ω_F induces a probability measure $\Pr^{\omega_F}(\bullet)$ equal to the Lüders updating $\Pr^{\omega}(\bullet//F)$ of $\Pr^{\omega}(\bullet)$? The answer, of course, is that ω_F is given by the L/vN postulate.

If the reader is beginning to suspect that quantum probability theory conspires to mesh personal and objective probabilities her suspicion will be confirmed below.

7 State preparation and the melding of the personalist and objectivist perspectives

One of the arguments for the objectivist stance on quantum probabilities is that (some) quantum states can be prepared and that the probabilities calculated from the prepared states are borne out by the frequency counts in repeated trials on systems all prepared in the same state. The formal account of state preparation of a normal pure state uses the L/vN projection postulate and the existence of a filter for such a state.

> *Def.* A projection $F_\varphi \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ is a filter for a normal state φ on $\mathfrak{B}(\mathcal{H})$ iff for any normal state ω (pure or impure) such that $\omega(E_\varphi) \neq 0$,
>
> $$\omega_{F_\varphi}(A) := \frac{\omega(F_\varphi A F_\varphi)}{\omega(F_\varphi)} = \varphi(A) \text{ for all } A \in \mathfrak{B}(\mathcal{H}).$$

The normal pure states on $\mathfrak{B}(\mathcal{H})$ are identical with the vector states, and as the reader can easily verify a filter F_ψ for a vector state ψ consists of

[13] Standard textbooks on QM assume that probabilities and expectation values are to be calculated via the Born rule, which is equivalent to the exclusive use of normal states. For a discussion of the status of this assumption, see Ruetsche (2011).

the projection onto the ray spanned by a vector $|\psi\rangle \in \mathcal{H}$ representative of ψ.[14] To prepare the state ψ make Yes-No measurements of F_ψ until a Yes answer is obtained. The pre-measurement state ω may be any normal state, pure or impure, provided only that $\omega(F_\psi) \neq 0$. When a Yes answer is obtained apply the L/vN postulate to conclude that the post-measuerment state is $\omega_{F_\psi}(\bullet) := \dfrac{\omega(F_\psi \bullet F_\psi)}{\omega(F_\psi)}$. By the filter property of F_ψ, $\omega_{F_\psi}(\bullet) = \psi(\bullet)$ regardless of the initial state of the system.

Now it should be apparent how Lüders updating of personal probabilities on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ meshes with this account of state preparation, at least in cases where Gleason's theorem applies and the agent is using a countably additive credence function. Suppose that a Bayesian agent using Lüders rule to update her personal probability function Pr over $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ learns that a Yes-No measurement of the filter F_ψ for a normal pure state ψ has yielded a Yes answer. She updates her probability to $\Pr(\bullet//F_\varphi)$, provided that $\Pr(F_\varphi) \neq 0$. Gleason's theorem ensures that Pr extends to a normal state ω. Her updated probability is $\Pr(\bullet//F_\varphi) = \dfrac{\omega(F_\psi \bullet F_\psi)}{\omega(F_\psi)}$, and by the filter property, $\Pr(\bullet//F_\psi) = \psi(\bullet)$.

Along the same lines let F_{ψ_1} and F_{ψ_2} be filters for the vector states ψ_1 and ψ_2 respectively. If Pr is any countably additive probability measure—representing, if you like, the credences of a Bayesian agent—then Gleason's theorem implies that $\Pr(F_{\psi_2}//F_{\psi_1}) = |\langle\psi_2|\psi_1\rangle|^2$, the rhs being the standard expression for the transition probability from ψ_1 to ψ_2.[15] More generally, from the objectivist perspective $\Pr(E//F)$ can be interpreted as the transition probability from a state in which F holds to a state in which E holds.

These meshing results can be spun in two different ways. On the objectivist perspective the normal pure state ψ induces objective chances on the elements of $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$, and what the meshing result shows is that when an agent learns that what the objective chances are by learning that a Yes-No

[14] It takes a little more work to show that impure normal states do not have filters. This is one reason for thinking that only pure states induce objective probabilities.

[15] If Pr is countably additive and Gleason's theorem applies, Pr extends uniquely to a normal state ω and if $\Pr(F_{\psi_1}) \neq 0$ then $\omega(F_{\psi_1}) \neq 0$. We know that From the results of Section 3, $\Pr(\bullet//F_{\psi_1}) = \dfrac{\omega(F_{\psi_1} \bullet F_{\psi_1})}{\omega(F_{\psi_1})}$. Then using the fact that F_{ψ_1} is a filter for ψ_1 we have that $\dfrac{\omega(F_{\psi_1} \bullet F_{\psi_1})}{\omega(F_{\psi_1})} = \psi_1(\bullet)$. And using the fact that ψ_1 and ψ_2 are vector states we have $\psi_1(E_{\psi_2}) = \langle\psi_1|E_{\psi_2}|\psi_1\rangle = |\langle\psi_2|\psi_1\rangle|^2$.

measurement of F_ψ has given a Yes answer—and thus that the state ψ has been prepared—her updated degrees of belief are identical with the objective chances assigned by ψ. This is so as a theorem of quantum probability and, contrary to what the philosophical literature on Lewis Principal Principle assumes, no new principle of rationality is needed to enforce the alignment of rational credence and chance.[16]

The QBians, who reject the very notion of objective chance, will of course adapt a different spin. For them what the meshing results show is that all QBian agents who use countably additive credence functions and who assign a non-zero initial probability to F_ψ will experience a merger of opinion when Lüders updating on F_ψ: their posterior probabilities will all be the same because they all assign the same value $\psi(E)$ to all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$. They can speak with the vulgar in agreeing that the state ψ has been prepared, but for them this simply means that the same bookkeeping mark ψ represents all their credence functions.[17]

8 Conclusion

I endorsed Guerro Bobo (2013) claim that Lüders conditionalization fails to provide a notion of quantum conditional probability that bears anything more than a very distant kinship to classical Bayes conditionalization. The no-go result reviewed in Section 4 helps to strengthen the case for this claim. At the same time, however, I argued that Lüders conditionalization does provide a rule for updating quantum probabilities and that understanding the role that this updating rule plays in QM is vital to understanding different stances on the nature of quantum probabilities and how these stances factor into some of the key foundations problems in QM.

As became increasingly evident, one of the these problems is the notorious measurement problem which remains the source of dissension among physicists and philosophers of physics. In particular, the story told above about the role of Lüders conditionalization in harmonizing the updating of rational

[16] The original formulation of the Principal Principle is to found in Lewis (1980). The ensuing philosophical literature on the correct formulation and justification of this principle is voluminous. Curiously, this literature is incapable of treating quantum probabilities since it is couched in the language of classical probability theory. For more on the Principal Principle in QM, see Earman (2018b).

[17] This story doesn't work for $\dim(\mathcal{H}) = 2$ since in this case there are probability measures on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ that do not extend to any state on $\mathfrak{B}(\mathcal{H})$.

credence with the updating of objective state-induced probabilities depends on the Lüders version of the von Neumann projection postulate, a generalized version of state vector reduction (aka collapse of the wave packet). Does this mean that the story I told has to be tossed out the window by non-collapse interpretations of QM? No—the story remains but has to be reinterpreted. The reason is that, as judged by the statistics of outcomes of experiments, the L/vN postulate gives the correct connection between the pre- and post-measurement probabilities. Non-collapse interpretations must acknowledge that as far as the phenomena are concerned it is as-if the state of the system has undergone a change given by the L/vN postulate, and they must then tell a story about how this as-if comes about. I leave it to the reader to judge how plausibility of these as-if stories.[18]

Proponents of the literal construal of the L/vN postulate must either treat the notion of measurement as an unexplained primitive notion, or else must explain measurement results as arising from a dynamical interaction between an object system and a measuring instrument. The former is an unattractive dodge. The latter must confront the problem that numerous no-go results indicate that measurement outcomes cannot be explained by Schrödinger dynamics, strongly suggesting that new physics is required to implement a literal construal of the L/vN postulate.[19]

QBism offers a tertium quid. All probabilities in QM are to be given a personalist reading, and quantum states serve only as representational devices to bookkeep the credences of QBian agents. When an agent learns the outcome of a measurement she updates her credence function via Lüders conditionalization. The quantum state changes, perforce, in accordance with the L/vN postulate; but this change does not require an explanation by old or by new physics since it is just a change in bookkeeping entries tracking the agent's credences. Left unexplained is what counts as a measurement and how QBian agents acquire information about measurements. If the response is that such matters are not part of what QBianism seeks to explain, one can

[18] There are many non-collapse theories of quantum measurement. The three most widely discussed are the modal interpretations (see Lombardi and Dieks 2014), the many world interpretation (see Vaidman 2015), and Bohmian mechanics (see Goldstein 2013). In the 1980s there was considerable debate about whether the quantum logic interpretation or the Copenhagen interpretation gave a better explanation of the projection postulate; see Friedman and Putnam (1978), Bub (1982), and Stairs (1983). The new actor on the scene is QBism which accepts measurement collapse but views it as innocuous.

[19] For an overview of some proposals for the physics of collapse, see Ghirardi (2018).

be forgiven for being disappointed by the limited ambitions of the enterprise.

And so it goes. Whatever the final resolution (if we ever get one) it is a safe bet that, in some guise or other, Lüders conditionalization will be part of the story.

$$
\begin{array}{ccc}
\text{Probabilities} & & \text{States} \\
\text{on } \mathcal{P}(\mathfrak{B}(\mathcal{H})) & & \text{on } \mathfrak{B}(\mathcal{H}) \\
& & \\
\Pr(\bullet) & \longrightarrow & \omega(\bullet) \\
& & \\
F \ \downarrow & & \downarrow \ F \\
& & \\
\Pr(\bullet//F) := \Pr^{\omega_F}(\bullet) & \longleftarrow & \omega_{_F}(\bullet) := \dfrac{\omega(F \bullet F)}{\omega(F)}
\end{array}
$$

Fig. 1 To Lüders conditionalize is to go from the upper left probability to the lower left probability; this is done circuitously by traversing the diagram clockwise, moving up to the state space, changing the state, and then moving back down to the probability space.

References

[1] Bub, J. 1977. "Von Neumann's Projection Postulate as a Possible Conditonalization Rule in Quantum Mechanics," *Journal of Philosophical Logic* 6: 381-390.

[2] Bub, J. 1982. "Quantum Logic, Conditional Probability, and Interference," *Philosophy of Science* 49: 402-421.

[3] Cassinelli, G. and Zanghi, N. 1983. "Conditional Probabilities in Quantum Mechanics," *Nuovo Cimento B* 73: 237-245.

[4] Christensen, D. 1991. "Clever Bookies and Coherent Beliefs," *Philosophical Review* 100: 229-247.

[5] Earman, J. 2018a. "Quantum Bayesianism Assessed," http://philsci-archive.pitt.edu/id/eprint/14824

[6] Earman, J. 2018b. "The Relation between Credence and Chance: Lewis' 'Principal Principle' Is a Theorem of Quantum Probability,"http://philsci-archive.pitt.edu/id/eprint/14822.

[7] Friedman, M. and Putnam, H. 1978. "Quantum Logic, Conditional Probability, and Interference," *Dialectica* 32: 305-315.

[8] Ghirardi, G. 2018. "Collapse Theories," *Stanford Encyclopedia of Philosophy* (Fall 2018 Edition), Edward N. Zalta (ed.), https://plato.stanford.edu/archives/fall2018/entries/qm-collapse/.

[9] Goldstein, S. 2013. "Bohmian Mechanics," *The Stanford Encyclopedia of Philosophy* (Spring 2013 Edition), Edward N. Zalta (ed.), http://plato.stanford.edu/archives/spr2013/entries/qm-bohm/.

[10] Guerra Bobo, I. 2013. "On Quantum Conditional Probability," *Theoria: Revista de Teoría, Historia y Fundamentos de la Ciencia* 28, 115-137.

[11] Hamhalter, J. 2003. *Quantum Measure Theory.* Dordrecht: Kluwer Academic.

[12] Hegerfeldt, G. C. and Mayato, R. 2012. "Discriminating between von Neumann and Lüders reduction rule," *Physical Review A* 85: 032116.

[13] Kadison, R. V. and Ringrose, J. R. 1991. *Fundamentals of the Theory of Operator Algebras*, 2 Vols. Providence, RI; American Mathematical Society.

[14] Kumar, C. S., Shukla, A. and Mahesh, T. S. 2016. "Discriminating between Lüders and von Neumann measuring devices: An NMR investigation," *Physics Letters A* 380: 3612-2616.

[15] Lewis, D. 1980. "A Subjectivist Guide to Objective Chance." In R. Carnap and R. Jeffrey (eds.), *Studies in Inductive Logic and Probability*, pp. 263-293. Berkeley: University of California Press.

[16] Lombardi, O. and Dieks, D. 2014. "Modal Interpretations of Quantum Mechanics," *The Stanford Encyclopedia of Philosophy* (Spring 2014 Edition), Edward N. Zalta (ed.), http://plato.stanford.edu/archives/spr2014/entries/qm-modal/.

[17] Stairs, A. 1982. "Quantum Logic and the Lüders Rule," *Philosophy of Science* 49: 422-436.

[18] Ruetsche, L. 2011. "Why Be Normal?" *Studies in History and Philosophy of Modern Physics* 42: 107-115.

[19] Teller, Paul, 1976, "Conditionalization, Observation, and Change of Preference", in W. Harper and C. A. Hooker (eds.), *Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science*. Dordrecht: D. Reidel.

[20] Timpson, C. G. 2008. "Quantum Bayesianism: A Study," *Studies in History and Philosophy of Modern Physics* 39: 579-609.

[21] Vaidman, L. 2015. "Many-Worlds Interpretation of Quantum Mechanics," *The Stanford Encyclopedia of Philosophy* (Spring 2015 Edition), Edward N. Zalta (ed.), http://plato.stanford.edu/archives/spr2015/entries/qm-manyworlds/.

[22] Vineberg, S. 2016. "Dutch Book Arguments," *Stanford Encyclopedia of Philosophy* https://plato.stanford.edu/entries/dutch-book/.

[23] von Baeyer, H. C. 2016. *QBism: The Future of Quantum Physics*. Cambridge. MA: Harvard University Press.