There is no invariant, four-dimensional stuff

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May 28, 2019

I've heard a few philosophers say that Einstein's special theory of relativity (STR) calls for some kind of four-dimensional ontology. They say that four-dimensional stuff is invariant in some sense that three-dimensional stuff is not. For example, Balashov claims that "an object viewed as a 4d being is relativistically invariant in a sense in which its 3d parts are not" (1999, p 659).[1] Similarly, Hofweber and Lange argue against Kit Fine's fragmentalist interpretation of STR on the basis that "the spacetime interval, as a frame-invariant fact, is the reality, whereas the facts related by the coordinate transformations are frame-dependent facts and hence are appearances of that reality" (2017, p 876).

In this note, I show that these philosophers are mistaken about what is invariant in STR. In particular, there is no invariant four-dimensional stuff in STR, neither objects nor facts. To this end, we first need a couple definitions.

1. A *4d object* is represented by a four-dimensional subset of Minkowski spacetime. For simplicity, we will assume that a 4d object is represented (in some coordinate system) by a region of the form $\mathbb{R} \times I$, where I is the unit cube. The result proven below is easily generalizable to other regions that might represent 4d objects.

2. An object O is *relativistically invariant* just in case O is invariant under Lorentz transformations.

We now show that no objects are relativistically invariant.

Theorem. *For any region O of Minkowski spacetime representing a 4d object, there is a Lorentz transformation L such that $L(O) \neq O$.*

[1] Balashov's claim was contested by Davidson (2013), who argues that 4d objects themselves fail to be relativistically invariant. However, Balashov (2014) and Calosi (2015) argue that Davidson's conclusion and the reasoning behind it are in error. We will show here that Davidson's conclusion, if not his reasoning, is most certainly correct.

Proof. For simplicity, I will look at the case of two-dimensional Minkowski spacetime, where $O = \mathbb{R} \times I$, and $I = [0, 1]$ is the unit interval. The reasoning is unchanged for the full four-dimensional case. Now choose some particular time $0 \in \mathbb{R}$, and consider the 3d part $\{0\} \times I$ of $\mathbb{R} \times I$. Fix two points $p, q \in I$ such that $p \neq q$. For simplicity, we can take $p = \langle 0, 0\rangle$ and $q = \langle 0, 1\rangle$. The action of a Lorentz boost L_v centered at p results in the x coordinate of a point being transformed to $x' = (1 - \frac{v^2}{c^2})^{-1/2} x$. In particular, for the point $q = \langle 0, 1\rangle$, we have $x' = (1 - \frac{v^2}{c^2})^{-1/2}$. Hence, for large enough v, $L_v(p)$ lies outside the region O. That is, O is not invariant under Lorentz transformations. $\square$

Thus, there is nothing in Minkowski spacetime that looks like a physical object — whether three or four dimensional — and that is relativistically invariant. We now turn our attention to facts. Hofweber and Lang claim that the spacetime interval is an invariant fact. But it's not. Let's write $\eta(p, q)$ for the spacetime distance between two points p, q of Minkowski spacetime. Of course, the Minkowski metric η itself is invariant under Lorentz transformations. But η itself is not a fact, nor is it used to represent a fact.[2] What's more, to say that η is Lorentz invariant means that $\eta(p, q) = \eta(Lp, Lq)$ for any Lorentz transformation L. But the points p and q themselves are *not* invariant under Lorentz transformations, which means that it doesn't make sense to say that the spacetime distance between *these* points p and q is invariant. The correct thing to say is that the distance between p and q is the same as the distance between the points Lp and Lq, which are related to the former by a Lorentz transformation. Even then, we needn't say that the distance between p and q is the same, in some ontological sense, as the distance between Lp and Lq. After all, distances aren't things.

In conclusion, arguments from invariance don't support a four-dimensionalist ontology for STR.

References

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[2] The logical type of η is to map pairs of vectors to numbers. An equation, such as $\eta(p, q) = r$ can represent a fact, but the tensor η is not itself used to represent facts.

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