There is no invariant, four-dimensional stuff

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I've heard a few philosophers say that Einstein's special theory of relativity (STR) favors a four-dimensional ontology. They say that in STR, four-dimensional stuff is invariant in some sense that three-dimensional stuff is not. For example, Balashov claims that "an object viewed as a 4d being is relativistically invariant in a sense in which its 3d parts are not" (1999, p 659).[1] Similarly, Sattig says that "there is a permanent shape standing behind the different three-dimensional shapes of the object, namely, an invariant four-dimensional shape, rendering the various three-dimensional shapes different perspectival representations of the single invariant shape" (2015, p 220). Finally, Hofweber and Lange argue against Kit Fine's fragmentalist interpretation of STR on the basis that "the spacetime interval, as a frame-invariant fact, is the reality, whereas the facts related by the coordinate transformations are frame-dependent facts and hence are appearances of that reality" (2017, p 876).

In this note, I show that these philosophers are operating with a false picture of invariance and frame-dependence in STR. First I show the precise sense in which there are no invariant four-dimensional objects. Then I explain why it's misleading to say that facts about the spacetime interval are frame-invariant.

I begin with a couple of definitions. First, a *4d object* is represented by a four-dimensional subset of Minkowski spacetime. For simplicity, I will assume that a 4d object is represented (in some coordinate system) by a region of the form $\mathbb{R} \times I$, where I is the unit cube. The result proven below is easily generalizable to other regions that might represent 4d objects. Second, an object O is *relativistically invariant* just in case O is invariant under Lorentz transformations.

[1] Balashov's claim was contested by Davidson (2013), who argues that 4d objects themselves fail to be relativistically invariant. However, Balashov (2014) and Calosi (2015) argue that Davidson's conclusion and the reasoning behind it are in error. I show here that Davidson's conclusion is correct.

The following result rules out the existence of invariant four dimensional objects.

Proposition. *For any region O of Minkowski spacetime representing a 4d object, there is a Lorentz transformation L such that $L(O) \neq O$.*

Proof. The idea behind the proof is simple: fix $p \in O$, and consider Lorentz boosts centered at p. If $q \in O$ and q is spacelike separated from p, then the orbit of q under these Lorentz boosts is an infinite hyperboloid that cannot be contained within O. Thus, there is a Lorentz boost L such that $L(q) \notin O$.

For more precision, consider the case of two-dimensional Minkowski spacetime, where $I = [0, 1]$ is the unit interval, and let $p = \langle 0, 0 \rangle \in O$. The action of a Lorentz boost L_v centered at p transforms a point $\langle t, x \rangle$ to a point $\langle t', x' \rangle$, where $x' = (1 - \frac{v^2}{c^2})^{-1/2} x$. In particular, L_v transforms $\langle 0, 1 \rangle \in O$ to a point with x-coordinate $x' = (1 - \frac{v^2}{c^2})^{-1/2}$. Clearly, this x-coordinate grows unboundedly large as v approaches c. Therefore $L(O) \neq O$. $\square$

Thus, there is nothing in Minkowski spacetime that could be called a physical object — whether three or four dimensional — and that is relativistically invariant.

We now turn our attention to facts. Hofweber and Lange claim that the spacetime interval is an invariant fact. But depending on how we disambiguate "spacetime interval," either it's not a fact, or it's not invariant. Let's write $\eta(p, q)$ for the spacetime distance between two events p, q. Of course, the Minkowski metric η itself is invariant under Lorentz transformations. But η is not a fact, nor is it used (without inputs) to represent a fact.[2] What's more, to say that η is Lorentz invariant means that $\eta(p, q) = \eta(Lp, Lq)$ for any Lorentz transformation L. But events like p and q are not relativistically invariant; i.e. usually $Lq \neq q$. Hence "the spacetime interval between events p and q" is not invariant. The correct thing to say is that the distance between events (say p and q) is the same as the distance between Lorentz-related events (say Lp and Lq). But a distance by itself is neither a thing nor a fact.

Let's try to be charitable. What Hofweber and Lange probably mean is that the spacetime distance between events is something that all observers can agree upon. But what does it mean to say that all observers can agree

[2] As a contravariant tensor, η is a function from pairs of vectors to numbers. Physicists do not take functions (without inputs) as representing facts. For example, physicists don't think of the dot product between two vectors as a fact. In contrast, physicists do use equations such as $\eta(p, q) = r$ to represent facts. But that equation is *not* invariant under Lorentz transformations.

on this fact? We have already seen that it does *not* mean that the fact is invariant. What's more, a fact doesn't have to be invariant for all observers to agree upon it. For example, all observers can agree upon the spatial distance between two events, relative to a spacelike hypersurface Σ that contains both events. Perhaps, though, the former fact is somehow intrinsic to Minkowski spacetime, i.e. it's a relation that the events bear to each other merely in virtue of their being events in Minkowski spacetime. (One might be tempted to suggest that it's a "frame-independent fact".) But here is another relation that these events bear to each other merely in virtue of their being events in Minkowski spacetime: they are contained in the spacelike hypersurface Σ, and they have a certain spatial distance within this hypersurface.

In conclusion, it's simply not true that invariance features of Minkowski spacetime favor a four-dimensional ontology over a three-dimensional ontology.

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