Fundamental, yet imprecise?

Michael E. Miller†

Many physical theories characterize their observables with unlimited precision. Non-fundamental theories do so needlessly: they are more precise than they need to be to capture the matters of fact about their observables. A natural expectation is that a truly fundamental theory would require unlimited precision in order to exhaustively capture all of the fundamental physical matters of fact. I argue against this expectation and I show that there could be a fundamental theory with limited precision.

1. Superfluous precision. Suppose you have been tasked with measuring the value of my height at some particular instant in time. You might proceed by asking me to stand up straight against a wall at that instant, making a mark just above the top of my head with a very fine tipped pen, and measuring the distance between the mark and the floor with a meter stick. Were you to do this you would likely find that the mark on the wall falls between two millimeter markings on your meterstick, say the seventh and eighth millimeter markings between the 95th and 96th centimeter markings.[1] Having already measured one full length of the meter stick, you would come to the conclusion that you have measured me to be 1.957m $\pm$ 0.001m tall.

The precision of this measurement can obviously be improved. If only one additional decimal place of precision is required you could simply obtain a rule with finer markings. With the aid of an electron microscope you could determine the value with nine or ten decimal places of precision.[2] In fact, on first inspection it seems that the only limit to the precision with which my height can be accurately determined is the resolution provided by currently available technology. In order for the *only* limit on the precision to come from such pragmatic factors, there must be a physical fact of the matter not just about the tenth decimal place of my height but also about the nth decimal place for any arbitrary n. If there is some level precision beyond which there is no longer such a physical matter of fact, that marks a principled, not merely pragmatic, limit on the precision with which my height can be measured.

†Department of Philosophy, University of Toronto
Email: mike.miller@utoronto.ca

[1] The argument presented here generalizes straighforwardly to the case where the mark overlaps with one of the millimeter markings.

[2] This would of course require indicating the position of the top of my head more precisely than is possible with a fine tipped pen.

It turns out that there is such a principled limit. To see this note that by a measurement of my height we mean a measurement of the distance between the floor and the highest point on my head. Consider the determination of the position of the highest point on my head.[3] Suppose we can all agree which cells are part of me and which electrons are parts of those cells and which are not.[4] Determining which of those electrons happens to be the furthest from the floor at a given instant requires exactly determining each of their positions at that instant. But this is precisely the sort of thing that quantum mechanics indicates that there will not be a physical matter of fact about.[5] At a certain level of resolution there is simply no longer a physical matter of fact for the measurement to track. A real number provides more precision than is required to say everything that there is to say about my height.

This conclusion tells against the expectation that there are only pragmatic limitations to the precision with which quantities such as height can be measured. The failure of this expectation invites the following question: where does the expectation that one should in principle be able to make meaningful statements about my height with unlimited precision come from? I suspect that it is the practice of physics that has led us to this expectation. Consider, for example, the elementary physics problem of determining the vertical displacement as a function of time $y(t)$, of a block of mass, m, suspended from a spring with spring constant, k. The dynamics of classical mechanics holds that this displacement is governed by the equation,

$$m\frac{d^2y(t)}{dt^2} = -ky(t), \tag{1}$$

which is solved by,

$$y(t) = A\cos(\omega t + \phi) \;\text{ for }\; \omega = \sqrt{\frac{k}{m}}. \tag{2}$$

[3]Issues similar to those that follow obviously also affect the determination of the position of the floor, but this is of no consequence for my argument.

[4]The issue that I am introducing here is completely distinct from the problem of the many introduced in (Unger 1980).

[5]On most interpretations of quantum mechanics this is obvious from any one of a number of different observations, such as that position measurements do not yield exact position eigenstates. One might think that Bohmian mechanics presents a counterexample as it assigns definite positions to each of the the electrons in the complicated superposition that constitutes the top of my head. However, the measurement of any one of the electrons instantaneously influences the positions of all of the others. As a result, any attempt to determine whether or not a given electron is the highest will affect the positions of all of the others, and thus there is not a unique, identifiable highest point, independent of how one goes about the measurement process.

Note that once m and k are fixed, and A and ϕ are determined by the initial conditions of the system at t_0, this solution assigns a real number to the value of the displacement for all times t. Thus, assuming that m, k, A, and ϕ are fixed with real number precision, the solution makes statements with maximal precision in the following sense:

> ($\text{MP}_{\mathbb{R}}$) A statement about a quantity Q is maximally precise iff it ascribes a real number $d \in \mathbb{R}$ to that quantity.

In this way, our theoretical representation of the displacement leads us to expect that there will be a physical fact about $y(t)$ with $\text{MP}_{\mathbb{R}}$ precision. But considerations of the sort we just made in the case of my height show that there is no physical matter of fact about the position of the edge of the block with unlimited precision. Just as we found in the height case, the theory is more precise than it needs to be to exhaust the physical matters of fact about the displacement of the block.

This phenomenon is very general and consultation with an elementary physics text will allow for examples to be elaborated ad nauseum. A moment's reflection will show that the same is true of a thermodynamic account of the temperature of the room in which you are reading this paper, or a classical electrodynamic account of the electric field at a point between two charged plates, or the time it takes a mass to fall from the top of a building. In each case, it is possible to convince one's self that the $\text{MP}_{\mathbb{R}}$ statements provided by the theory are more precise than they need to be to get the facts about the world right. Moreover, in each case the argument for the claim proceeds by appeal to a more a fundamental theory.[6]

This collection of observations leads naturally to the thought that if we had a truly fundamental theory, it would make maximally precise statements in the sense of $\text{MP}_{\mathbb{R}}$ about the fundamental physical quantities. Moreover, the maximal precision of those statements would be necessary in order to say everything that there is to say about the fundamental quantities. After all, if there is a more precise truthful statement to make about the quantities, then the theory fails to be complete, and hence fails to be fundamental. Call quantities about which $\text{MP}_{\mathbb{R}}$ statements are required to say everything that there is to say $\text{MP}_{\mathbb{R}}$-quantities. For $\text{MP}_{\mathbb{R}}$-quantities, there are physical matters of fact about the 50th decimal place and the 10^{500}th decimal place, and more generally, d_N for arbitrarily large values of N:

$$d_0.d_1d_2d_3 \ldots d_{10^{50}} \ldots d_{10^{500}} \ldots d_N \ldots .$$

[6] In the temperature case, consider statistical mechanics. In the electric field case, consider quantum field theory. For the time case, the relevant considerations are similar to those given for the height case.

The full precision of $MP_{\mathbb{R}}$ statements are required to get the facts about such quantities right.

The remainder of this paper provides an argument for the claim that it is mistaken to assume that a fundamental theory must necessarily make $MP_{\mathbb{R}}$ statements, as we do not have good grounds to assume that the fundamental quantities of our world are $MP_{\mathbb{R}}$-quantities. In section two I show that quantum field theory, our best effort at a fundamental theory, fails to provide statements about the purported fundamental physical quantities which satisfy $MP_{\mathbb{R}}$. I then show that there are alternative criteria of maximal precision, and we default to assuming a fundamental theory will make $MP_{\mathbb{R}}$ statements because it is simply difficult to imagine worlds at which this fails to be the relevant standard of precision. In section three I provide an account of worlds in which $MP_{\mathbb{R}}$ statements are more precise than they need to be to capture all of the fundamental physical facts, that is, an account of world with fundamental quantities that are not $MP_{\mathbb{R}}$-quantities. The fourth section contains concluding remarks.

2. Standards of precision. In the last section I argued that if we want to find a domain where the full precision of $MP_{\mathbb{R}}$ statements aren't superfluous, then we should look to the domain described by a fundamental theory. The Standard Model of particle physics is our current best effort at a theory of the fundamental constituents of matter and the interactions between them.[7] This is a natural place to look for a regime whose complete description requires $MP_{\mathbb{R}}$ statements.

The first interesting thing to note is that the Standard Model *does not* make $MP_{\mathbb{R}}$ statements about its observables, and the reasons for its failure to do so are diverse. There are a number of sources of finiteness to the precision of the statements of quantum field theory which stem from the structure of the theory itself. Issues with the short distance structure of the theory make it best understood as an effective field theory which results in a limit to the precision of the predictions of the theory.[8] The infrared divergences of the theory are best treated by including the energy resolution of the detector into the theoretical expression for the observables, which produces a limit on the precision of the prediction.[9] Finally, the use of perturbation theory to extract predictions from the theory also results in limits on their precision.[10]

In practice, these theoretical limits on the precision of the predictions of

[7] I will proceed with this intuitive understanding of fundamentality. Whether or not the arguments below depend on one's particular account of fundamentality is not a question I will pursue in this paper, though it merits further investigation.

[8] For discussion see (Fraser 2018; Williams 2019).

[9] A helpful discussion of this issue is given in (Duncan 2012).

[10] See (Fraser, forthcoming) and (Blinded for Review) for further discussion.

the theory are often dominated by more pragmatic limitations. The pragmatic limitations include errors from numerical integration and lattice simulations. They also include the need to input measured values for physical constants in the expressions for theoretical predictions. Measured values come with measurement uncertainties which translate to limits on the precision of the theoretical predictions. Recall, for example, the spring example from Section 1. The mass, m, and the spring constant, k, can only be measured with some finite level of precision. When the measurement error is included in Equation (2), the solution no longer provides $\mathrm{MP}_{\mathbb{R}}$ statements about $y(t)$. In practice the error from measured values typically dominate the limit on the precision of the theoretical value of Standard Model observables as well.

A second interesting thing to note is that even if the Standard Model did make $\mathrm{MP}_{\mathbb{R}}$ statements, the full precision of those statements would not be required to get the measured values completely correct.[11] This is the case because in practice, measurements will always have limited precision. The measured values of some representative fundamental quantities and their associated uncertainties are given in Table 1. As a result, it follows that there is nothing about the way we use our theories, including our best effort at a fundamental theory, to match onto the world that requires maximal precision in the sense of $\mathrm{MP}_{\mathbb{R}}$. Our best precision tests of the Standard Model involve agreement between theory and experiment to on the order of 10 decimal places. This, to put it mildly, falls rather short of the precision of $\mathrm{MP}_{\mathbb{R}}$ statements.[12] We can account for the best precision tests of our best efforts at a fundamental theory with statements having the precision of rational numbers.

A standard of precision more coarse than $\mathrm{MP}_{\mathbb{R}}$ might properly be thought of as maximal for comparisons between theory and experiment. We can define a class of more coarse standards as follows:

> ($\mathrm{MP}^n_{\mathbb{Q}}$) A statement about a quantity Q is maximally precise iff it ascribes a rational number $d \in \mathbb{Q}$, which requires n digits to specify as a decimal[13], to that quantity.

This is an infinite class of standards, one for each value of n. For large n, $\mathrm{MP}^n_{\mathbb{Q}}$ closely approximates $\mathrm{MP}_{\mathbb{R}}$, but for small n $\mathrm{MP}^n_{\mathbb{Q}}$ is significantly more coarse

[11] A similar observation is made in (Teller 2018).

[12] Purcell once quipped that "There's not enough carbon in the universe to print out the value of one classical variable" where by classical variable he means a variable with $\mathrm{MP}_{\mathbb{R}}$ precision. Though I will not give a detailed defense here, considerations of this sort suggest that statements with $\mathrm{MP}_{\mathbb{R}}$ precision could not ever possibly be required to exhaust the precision of the best measured values.

[13] If d is a repeating decimal, n will be the number of digits that repeat. If d terminates with repeating zeros, n will be the number of digits occuring before the repeating zeros.

Quantity	Value	Unit
Gravitational constant (G)	$6.674\,08(31) \times 10^{-11}$	$m^3kg^{-1}s^{-2}$
Planck Constant (h)	$6.626\,070\,040(81) \times 10^{-34}$	Js
Elementary Charge (e)	$1.602\,176\,6208(98) \times 10^{-19}$	C
Fine-structure Constant (α)	$7.297\,352\,5664(17) \times 10^{-3}$	–
Weak Mixing Angle ($\sin^2 \theta_W$)	$0.2223(21)$	–
Electron Mass (m_e)	$9.109\,383\,56(11) \times 10^{-31}$	kg
Magnetic-Moment Anomaly (a_e)	$1.159\,652\,180\,91(26) \times 10^{-3}$	–
Proton Mass (m_p)	$1.672\,621\,898(21) \times 10^{-27}$	kg

Table 1: CODATA recommended values of some fundamental constants after (Mohr, Newell, and Taylor 2016).

than $\text{MP}_{\mathbb{R}}$. Once it is realized that there is a weaker standard of precision that one might count as maximal for some purposes, one might wonder if there is a standard of precision stronger than $\text{MP}_{\mathbb{R}}$. Such a standard can be formulated as follows:

> ($\text{MP}_{*\mathbb{R}}$) A statement about a quantity Q is maximally precise iff it ascribes a hyperreal[14] number $d \in {}^*\mathbb{R}$ to that quantity.

Up to this point, I have referred to $\text{MP}_{\mathbb{R}}$ statements as having unlimited precision. It should now be apparent that their precision is in fact very limited if the relevant standard of precision is $\text{MP}_{*\mathbb{R}}$. If one wants to demand that a theory is maximally precise, or that a theory will need to be maximally precise if it is to exhaustively characterize a particular physical domain, then one needs to specify the relevant standard of maximal precision. Similarly, the claim that a given statement is imprecise is relative to a given standard of precision. $\text{MP}^n_{\mathbb{Q}}$ statements are imprecise with respect to $\text{MP}_{\mathbb{R}}$, and $\text{MP}_{\mathbb{R}}$ statements are imprecise with respect to $\text{MP}_{*\mathbb{R}}$. Adjudicating whether or not there could be a fundamental theory that is imprecise requires specifying which standard it is imprecise with respect to.

One might argue that there is something natural about the choice of real number values for physical quantities. After all, our best theories typically employ the resources of calculus and so $\text{MP}_{\mathbb{R}}$ seems like a natural standard of precision. But the assumption that a theory will need to make $\text{MP}_{\mathbb{R}}$ state-

[14]The hyperreal numbers contain the real numbers as a subset and the order relation on the real numbers is a subset of the order relation on the real numbers, establishing a clear sense in which $\text{MP}_{*\mathbb{R}}$ is strictly sharper than $\text{MP}_{\mathbb{R}}$. A clear introduction to the hyperreal numbers can be found in (Keisler 2012).

ments in order to exhaustively characterize the fundamental physical level makes the mistake of imputing structural features of our theoretical representation to the world itself. There are of course circumstances in which we come to have grounds for thinking that our theoretical representation tracks the structure of the world itself. In this section I have argued that the standard of precision that comes along with our best theories does not fall into this category. We do not have good grounds to expect that the precision of our mathematical representation of fundamental quantities tracks the fineness-of-grain of those quantities themselves. $\text{MP}_{\mathbb{Q}}^{100}$ are sharper than what is required to account for the best measurements of Standard Model observables, but $\text{MP}_{\mathbb{Q}}^{100}$ is exceedingly coarse compared to $\text{MP}_{\mathbb{R}}$.

3. Fundamental, yet imprecise. In the last section I argued that we do not use maximally precise statements, understood as $\text{MP}_{\mathbb{R}}$ statements, when we confront the domain of fundamental particle physics. I expect that many will remain unpersuaded to drop $\text{MP}_{\mathbb{R}}$ as the standard of precision we expect to be required of a truly fundamental theory. While the argument of the last section provided epistemic grounds for abandoning this standard, it is hard to imagine worlds where metaphysically there fail to be facts of the matter about the fundamental quantities with $\text{MP}_{\mathbb{R}}$ precision. In this section I will provide an account of such worlds. That is, I will make the case that there are possible worlds where the fundamental quantities are not $\text{MP}_{\mathbb{R}}$ quantities.

Consider a fundamental quantity with decimal expansion,

$$d_0.d_1d_2d_3\ldots d_{10^{50}}\ldots d_{10^{500}}\ldots d_N\ldots$$

An $\text{MP}_{\mathbb{R}}$ statement is required to say everything there is to say about this quantity if there is a physical matter of fact about the Nth decimal place d_N for N arbitrarily large. One way to deny this is to suppose that there exists a d_W in the expansion,

$$d_0.d_1d_2d_3\ldots d_W\ldots d_N\ldots,$$

such that, i) there is a physical matter of fact about d_i for $0 \leq i \leq W$, but ii) there is no physical matter of fact about d_i for any $i > W$. $\text{MP}_{\mathbb{R}}$ statements are more precise than they need to be to say everything there is to say about quantities that satisfy i) and ii), and so quantities satisfying i) and ii) are not $\text{MP}_{\mathbb{R}}$ quantities.[15]

To understand this type of non-$\text{MR}_{\mathbb{R}}$ quantity, we need to find a way to

[15]It is worth emphasizing that this is just one way to arrive at a notion of non-$\text{MP}_{\mathbb{R}}$ quantities. There may well be others worth exploring.

make sense of the indeterminacy of the digits after d_W. The account of metaphysical indeterminacy due to Wilson, which holds that worldly indeterminacy occurs when an entity has a determinable property but no determinate of that determinable, is well-suited to this task.[16] Wilson's determinable-based account of metaphysical indeterminacy holds that:

> What it is for a [state of affairs] to be [metaphysically indeterminate] in a given respect R at a time t is for the [state of affairs] to constituitively involve an object (more generally, entity) O such that (i) O has a determinable property P at t, and (ii) for some level L of determination of P, O does not have a unique level-L determinate of P at t. (Wilson 2013, p. 366)

This view can be used to understand the nature of the non-$\mathrm{MP}_{\mathbb{R}}$ quantities as follows. Say that a fundamental physical quantity has a unique level-L determinate of the quantity if there is a physical matter of fact about the value of d_L, the Lth decimal in the expansion for the quantity. In the expression,

$$d_0.d_1d_2d_3 \ldots d_W \ldots d_N \ldots,$$

there are unique level-L determinates of d_L for all $L \leq W$, and there are not unique level-L determinates for all $L > W$.

Now consider a world at which a d_W exists for each of the fundamental physical quantities. At such a world, $\mathrm{MP}_{\mathbb{R}}$ statements are more precise than they need to be to say everything there is to say about the fundamental physical quantities. Thus, if there are such d_W, a complete fundamental theory with be imprecise with respect to $\mathrm{MP}_{\mathbb{R}}$. In this sense, there could be a complete fundamental theory that is imprecise. This shows that the concept of fundamentality is not inextricably linked with what many regard as the natural standard of precision.

This conclusion leads naturally to the question of whether or not there are in fact such d_W for the quantities we regard as fundamental. Consider, for example, the mass of the electron, a quantity figuring in the Standard Model. Perhaps surprisingly, all of the available direct empirical evidence is compatible with the mass of the electron having a finite d_W. To see this note that there are empirical consequences to the existence of such a finite d_W. Suppose that there is a finite d_W for the electron mass and suppose we

[16] There is another account of metaphysical indeterminacy which is based on the resources of semantic supervaluationism which might also be employed for my purposes (Barnes 2010). Whether or not this can be done successfully is not a question that I will pursue here.

repeatedly measure the mass of a single electron.[17] If we accurately measure with precision less than d_W we will always obtain the same result since there is a physical matter of fact about each of the digits. Now suppose that we accurately measure with precision of N decimal places for $N > W$. In this case, each measurement will agree on all of the decimal places up to and including d_W. But there are no physical matters of fact about d_N for $N > W$ and thus each measurement can yield something different for these decimal places. We should expect the digits after d_W to be randomly distributed, perhaps with some bias due to the particularities of the experimental arrangement we choose to employ.

For a fixed W and a given quantity, the hypothesis that d_W exists is thus falsifiable. To falsify the hypothesis one simply needs to measure the quantity repeatedly. If each measurement yields the same result for all d_N with $N < W$, this falsifies the hypothesis that there exists a d_W for this value of W. Currently available direct measurements of the electron mass falsify the existence of a d_W for $W = 30$, but those same measurements are compatible with the existence of a d_W for $W = 50$. The qualification about the direct nature of the measurements is required because when the quantities figure in dynamical theories, the dynamics may cause the indeterminacy to affect the determinacy of other quantities in the theory. In my view, the arguments here motivate a systematic study of the the empirical constraints on d_W. A better understanding of these constraints is essential as they determine the extent to which we understand what it would mean for there to be fundamental non-$\mathrm{MP}_{\mathbb{R}}$ quantities.[18]

4. Conclusion. We are accustomed to thinking that some aspects of a theory structurally correspond to aspects of the world itself. We are similarly accustomed to thinking that some aspects of our theory are surplus and do not represent structural aspects of reality. Our theoretical representations of

[17] We have independent grounds for thinking that all electrons are identical, so the experiment could just as well be carried out on a collection of electrons.

[18] It is worth noting that a view related to the one that I have argued for in this section has recently been advocated in (Gisin 2018). For example he claims that,

> The view I am suggesting is that the first bits in the expression of [a quantity] x are 'really real' . . . , while the very far away bits are totally random. As time passes they get shifted to the left, one position at each time-step. Hence, step by step they acquire some definite value. As time passes they have a changing disposition (or propensity) to hold their eventual value. (Gisin 2018, p. 5)

Gisin is concerned primarily with the determinacy of initial data and how it evolves over time. How this is connected to the view I have advocated here is an interesting question, but one which I will not pursue further in this paper.

the world come along with a natural notion of maximal precision. In most theories it is the one captured by $MP_{\mathbb{R}}$. I have argued here that it is a mistake to think that this particular notion of maximal precision corresponds directly with the fineness-of-grain of the world itself and so we should not associate a particular notion of precision with our conception of what it is for a theory to be fundamental.

The arguments presented here have several consequences worth emphasizing. There has been recent discussion of metaphysical indeterminacy in the case of quantum mechanics.[19] The arguments presented here open the possibility that this quantum mechanical indeterminacy is really a subspecies of a more general form of metaphysical indeterminacy of physical quantities. Second, while limitations on theoretical precision with respect to $MP_{\mathbb{R}}$ have been thought to be deficiencies of theories, the arguments presented here show that such limits do not necessarily undermine a theory's claim to fundamentality.

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[19] See, for example, (Skow 2010; Bokulich 2014; Calosi and Wilson 2018).

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