The Later Wittgenstein’s Guide to Contradictions

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## **Abstract**

This paper portrays the later Wittgenstein’s conception of contradictions and his therapeutic approach to them. I will focus on and give relevance to the *Lectures on the Foundations of Mathematics* (LFM 1976), plus the *Remarks on the Foundations of Mathematics* (RFM 2001). First, I will explain why Wittgenstein’s attitude towards contradictions is rooted in: (a) a rejection of the debate about realism and anti-realism in mathematics; and (b) Wittgenstein’s endorsement of logical pluralism. Then, I will explain Wittgenstein’s therapeutic approach towards contradictions, and why it means that a contradiction is not a problem for logic and mathematics. Rather, contradictions are problematic when we do not know what to infer from them. Once a meaning is established through a new rule of inference, the contradiction becomes a usable expression like many others in our inferential apparatus. Thus, the apparent problem is dissolved. Finally, I will take three examples of dissolved contradictions from Wittgenstein to clarify further his notion. I will conclude considering why his position on contradictions led him to clash with Alan Turing, and whether the latter was convinced by the Wittgensteinian proposal.

**Keywords:**Wittgenstein, Logic, Contradiction, Mathematics, Alan Turing, Rules, Paraconsistent Logic

# Introduction

I aim to reconstruct the later Wittgenstein’s notion of contradiction and clarify his approach to dissolving them. My explanation will investigate three interwoven points in Wittgenstein: (1) how his notion of contradiction follows from his rejection of the realism\anti-realism debate; (2) his “therapy” to dissolve contradictions; and (3) why (1)-(2) led Wittgenstein and Alan Turing to disagree.[[1]](#footnote-1) In particular, I will focus on the *Lectures on the Foundations of Mathematics* (LFM 1976) and the *Remarks on the Foundations of Mathematics* (RFM 2001) as the main sources.

I will reconstruct through the following steps. Firstly, I will argue that Wittgenstein’s consideration of the realism\anti-realism debate in mathematics and his philosophy of logic influenced his treatment of contradictions (§2).[[2]](#footnote-2) Subsequently, I will describe Wittgenstein’s therapeutic approach to contradictions and the reasons for his belief that they are not harmful (§3). I will then examine some concrete examples that Wittgenstein discussed (§4). Finally, I will explicate why Wittgenstein and Turing disagreed on contradictions (§5). I conclude summarising the key points of my reconstruction (§6).

# Dispelling a Conceptual Confusion: Mathematics and Logic

It would be hard to make sense of Wittgenstein’s notion of contradictions without also considering his views on mathematics and logic.[[3]](#footnote-3) To clarify why Wittgenstein in his later period arrived at a certain position, I must show how his philosophy of mathematics and logic influenced his later attitude to contradictions (i.e. 1929 and onwards). In particular, I will show how his therapeutic approach brought him to two conclusions: firstly, the rejection of the metaphysical confusion about realism and anti-realism in mathematics; secondly, the assumption of a pluralistic attitude in logic against monism.[[4]](#footnote-4)

His later conception of contradiction arises from the attempt to dispel the wrong setup of the realism\anti-realism debate in mathematics. For Wittgenstein, mathematics is a cluster of rules that govern certain language-games. As he specifies in the RFM: “mathematics is a motley of techniques of proof” (RFM, III-46 p. 176); and “Mathematics forms a network of norms” (RFM, VII-67 p. 431). Rather than discovering facts in the world, mathematics is a grammar for different activities; e.g. counting, calculating the perimeter of a geometrical figure, etc. On many occasions in the LFM and RFM, he specifies clearly that mathematics is not an enterprise of discovering facts like natural sciences; for instance:

‘Let us return to the discovery. 'The point was whether we should say that part of the discovery was a mathematical discovery. It was not a discovery that 125 ÷ 5 = 25; for this result is merely part of the use of the symbols. —This has to do with what I said, that 'mathematical discoveries' are better called inventions’ (LFM, VIII p. 82).

‘The mathematician is an inventor, not a discoverer’ (RFM, I-168 p. 99).

Wittgenstein tries to warn us that in mathematics we do not speak of discoveries in the same sense of natural sciences. For Wittgenstein, “discovery” in mathematics means rather developing new propositions and ways of doing things rather than ascertain new facts in the world.[[5]](#footnote-5) Severin Schroeder (2015) and Juliet Floyd (2005) have argued in favour of this reading: the later Wittgenstein regards mathematics as different language-games that express the same information in relation to multiple anthropological, social and psychological circumstances.

Thus, Wittgenstein considers mathematics as a grammar used in order to fix certain techniques for a variety of purposes. In a family of cases, mathematics describes empirical experience (cf. LFM, IV p. 43); i.e. using mathematics to make calculations in reality. Nevertheless, the possibility of practical calculations through mathematical tools could lead to a conceptual mistake about the nature of mathematics — as Hardy’s case exemplifies:

‘Consider Professor Hardy's article (“Mathematical Proof”) and his remark that “to mathematical propositions there corresponds — in some sense, however sophisticated — a reality”. […] Taken literally, this seems to mean nothing at all— what *reality*? I don't know what this means. —But it is obvious what Hardy compares mathematical propositions with: namely physics” (LFM, XXV p. 239).

Having mathematical results in the empirical domain mislead us into thinking of mathematics as consisting of hard facts inside a Platonic realm and to think of mathematical rules as something that work in the same way as physical laws. Hence, I have no control because mathematics is independent of my consideration. Likewise, if mathematics has its own reality and it is fixed independently from us, then the laws of non-contradiction and the excluded middle cannot be ignored; i.e. mind-independent facts demand which laws I have to adopt. Wittgenstein in the RFM points an objection against the assumptions of Hardy’s picture:

‘“To be practical, mathematics must tell us facts.”—But do these facts have to be the mathematical facts? — But why should not mathematics, instead of “teaching us facts”, create the form of what we call facts? […] *There are no* causal connexions in a calculation, only the connexions of the pattern’ (RFM, VII-18 pp. 381-382).

This conceptual error, for Wittgenstein, produces a fallout for our understanding of contradictions. If I consider mathematics as a discovery, then I am led to a certain conception of contradictions, that there are fatal flaws inside a system under which we have limited power of decision. From this perspective, the system is rigid and there is nothing that any mathematician, logician or philosopher can do to alter it. But for Wittgenstein, if mathematics has a grammatical role (rather than working like a set of natural laws), then we can make sense of contradictions by revising our grammar.[[6]](#footnote-6) The error of Hardy is to frame the problem as if mathematical rules and numbers are respectively laws and objects of which we must inquire their natures and ask what makes an arithmetic rule “correct”, in which sense do numbers exist, etc. Instead, we recognise that the terms “number” and “rule” assume different meanings according to their context of use (cf. LFM, I pp. 15, 18). It does not make sense to ask the same questions that we would ask about physical laws in the context of mathematics. Wittgenstein quoted Hardy in order to make aware us of this conceptual pitfall.

The approach that Wittgenstein takes to mathematics is reflected in his consideration of logic. He criticises the picture that there is only one single correct model. Wittgenstein diagnoses the kind of conceptual confusion causing this: the fact that we agree on a series of certain propositions on logic leads us to find an explanation of the phenomenon; and the explanation is that we have an agreement because it is a mind-independent reality (cf. LFM, XVIII pp.172-173). But this conception deceives us into believing that rules are immutable and detached from our practice. It leads us to forget their grammatical role and convinces us that logic is something that cannot be changed, prescribing normatively which kind of propositions can be accepted as true. And, models radically different from classical logic are wrong. Wittgenstein challenges this account with these two particular examples from the RFM:

‘Let us imagine having been taught Frege’s calculus, contradiction and all. But the contradiction is not presented as a disease. […] Now we are set the task of changing this calculus, of which the contradiction is an entirely respectable part, into another one, in which this contradiction is not to exist, as the new calculus is wanted for purposes which make a contradiction undesirable. —What sort of problem is this?’ (RFM, III-80 pp. 209-210).

‘I formed a system of rules of calculation which were modelled on those of another calculus. I took the latter as a model. But it exceeded its limits. This was even an advantage; but now the new calculus became unusable in certain parts (at least in the former purpose). I therefore seek to alter it: that is, to replace it by one that is *to some extent* different. […] Is there such a thing —it might also be asked—as *the right* logical calculus, only without contradictions?’ (RFM, II-85 p. 217).

From Wittgenstein’s perspective, there is no one single correct logic: there are many calculi according to different circumstances in which we find ourselves employing a certain genre of inferences. If in a certain scenario I need to avoid contradictions, I will take axioms and rules that forbid it. But the same model is useless when I have to handle contradictions. If we understand that the same calculus is extremely limited to express certain propositions, we need to abandon the idea of one single logic as normatively correct; therefore, we must develop a model for each necessity and problem we encounter. Hence, for Wittgenstein consistency is not a problem at all, but just a requirement for one kind of logic; in the LFM he concludes: ‘If that's logic, it doesn't contain any contradictions worth talking about. […] But you might say: This is only one logic, and in others you may have as many contradictions as you like.’ (LFM, XXII p. 213).

My discussion up to this point has shown that Wittgenstein tried to reject some conceptual confusions determining a wrong mindset towards contradictions. To summarise Wittgenstein’s target:

**Classic Realism\Anti-Realism Opposition (CRA):** *T* is either parasitic on mind-independent facts or *T* exists thanks to conventional warrant conditions.

CRA sees contradictions in mathematics or logic as contrasting either mind-independent laws or a series of stipulated conventions. This false dichotomy misleads us to see contradictions as a problem in accord or against certain facts rather than an issue related to grammar. The ambition to dispel this point in philosophy of logic and mathematics shaped the discussion of contradictions in the RFM and the LFM.

However, one can argue that Wittgenstein’s rejection of platonism in mathematics and logic leads to classifying his approach in a minimal sense as “anti-realist” rather than therapeutic. I intend anti-realism here as the family of positions that explain the nature and the applicability of mathematics appealing to either fictional mental entities or conventions (cf. Balauger, 2009, pp. 44–45). But anti-realism would merely be a different way for Wittgenstein to repeat the same mistake: the anti-realist substitutes abstract entities and mind-independent laws with fictional entities and conventions, continuing then to see the problem as factual and not as grammatical. It is the same error with a different flavour: i.e. the realist develops theories in mathematics to explain facts regarding mind-independent entities, while the anti-realist does the same with facts concerning fictional entities and conventions. This is a mistake that J.H. McDowell (1989) diagnosed in his criticism to M. Dummett about platonism and anti-realism: both platonistic realists and anti-realist philosophers of mathematics treats mathematics as a series of empirical objects to investigate, they just disagree on their nature (cf. McDowell, 1989, pp. 185–186, 189). They do not realise that philosophical problems in logic and mathematics must be dissolved through emendation of grammar, not through proposing a theory about the nature of something.[[7]](#footnote-7) Regarding the rejection of CRA, B. Clark (2017) makes a good point: Wittgenstein did not want to substitute a realistic theory with an anti-realist theory in mathematics and logic; instead, he saw the whole problem as the result of proposing theories explaining mathematics (cf. Clark, 2017, p. 132). Hence, the ascription of anti-realism to Wittgenstein is a misunderstanding of his attempt to dissolve philosophical problems through grammatical analysis without appealing to a theory. This means a “therapeutic approach”: a grammatical analysis aimed to dissolve a pseudo-problem (e.g. a dichotomy) that is taken seriously by a philosopher who understands in the wrong way a specific phenomenon; also, a therapy must be designed to stop that same philosopher from worrying about the problem (cf. Jacquette, 2014; Plant, 2004). So, the rejection by Wittgenstein of platonism in philosophy of mathematics does not entail that Wittgenstein would have accepted an anti-realist theory in its place. For this reason, we can attribute anti-platonism to Wittgenstein without being an anti-realist, describing at the same time his approach as therapeutic.

# A Meaning for Contradictions: Exposition of Wittgenstein’s Later Notion

A brief historical clarification is required before discussing Wittgenstein’s notion. Initially, in the conversations with the Vienna Circle (WWK 1979), Wittgenstein built on a Fregean thesis: he agreed with Frege that the core of a formal language is the rules embedded inside it. He quoted a passage directly from Part II §106 of Frege’s *Grundsetze der Arithmetik*. There, Frege compares mathematics to the game of chess: like chess, the meaning of a particular scheme is not given by the pieces themselves, but by the rules involved (cf. Frege, 1960, p. 202). I can understand a certain scheme of the chessboard if and only if I know the rules that led the game to that configuration. Indeed, Wittgenstein identified the problem in this point made by Frege: if the understanding of a game is related to the rules and not the pieces, then the contradiction must occur due to the former. In particular, the annotation of Sunday 28th December 1930 at Schlick’s house makes this explicit:

‘The idea of contradiction—and this is something I hold fast to—is that of a logical contradiction, and this can occur only in the *true-false game*, that is, when we make mistake. This means that a contradiction can occur only among the *rules of a game*’ (WWK, p. 124).

But Wittgenstein adds more to the Fregean account: he not only gives a diagnosis, but he proposes the kernel of the approach that he will develop later. He claims:

‘I can, for example, have a rule of a game that says: A white piece has to move by jumping over a black one. If a black piece, then, is at the edge of the board, the rule fails. Thus it may be the case that I do not know what to do. The rule tells me nothing further. What would I do in such a case? Nothing than removing the contradictions—I must take a decision, i.e. *introduce another rule*’ (WWK, pp. 124-125).

At this stage, Wittgenstein is convinced that the problem lies in the rules of inference, and not in the propositions themselves. The solution consists in introducing a new rule to settle the contrast between two or more rules provoking the contradiction. So, he considers contradictions as not related to the propositions themselves. The footnote of the *Insertion* dated back to December 1931 confirms this thesis:

‘“And no contradiction must ever follow from a rule.” This we again fail to understand. Let us take the Euclidean axioms. The axioms are rules, that is, propositions of grammar. The rules according to which the game is played within geometry are the rules of logic. Where then could a contradiction be looked for?’ (WWK, p. 197).

Indeed, if the propositions involved in a logical calculus are not combined in a certain way through rules, then they do not alone yield contradictions. Wittgenstein explains the issue with an example: only when I write down the formula $q . ∼p≡q$ is there the contradictory claim that *q* and not *p* is equal to *q* (cf. WWK, p. 201). Hence, the problem relies in how the conjunction interacts with the operator of identity “$≡$”: before that, *q* and *p* are innocuous. Nevertheless, in the Remarks (cf. RFM, III-12 p. 120, III-18 p. 122, III-80 p. 208) and in the Lectures (cf. LFM, XVIII pp. 175-176, 179, XX p. 193, XXI pp. 200-201, 206), Wittgenstein enlarges the WWK’s conception also to cases of material inferences. His notion began to include contradictions due to conflicting extra-logical relationships expressed by their semantic content. What he analyses are anthropological cases where there are issues of incompatible claims, not examples of contradictions yielded by two contrasting rules.[[8]](#footnote-8) This is the main difference between WWK and the position of RFM and LFM; although, as I will show, the suggestion of to how to dissolve them does not change.[[9]](#footnote-9)

As explained in §2, the tolerant attitude in Wittgenstein stems from his rejection of the realism\anti-realism dichotomy in philosophy of mathematics and logic. Wittgenstein rejects every kind of monism rooted in CRA which sustains that consistency is a mandatory requirement and contradictions are banned. The crux of Wittgenstein’s discussion is to see contradictions as non-threatening. He admits that his goal towards the issue is therapeutic: ‘My aim is to alter the *attitude* to contradictions and to consistency proofs’ (RFM, III-82 p. 213). The expression “therapeutic” is not casual: Paul Horwich (2013) has argued that Wittgenstein’s later philosophy is entirely focused on dissolving problems rather than find genuine solutions; i.e. he takes a certain problem *P*, and he proposes a set of examples [*E1*…*En*] of why *P* is a pseudo-problem, showing that *P* is the fruit of conceptual confusion. The later Wittgenstein regards contradictions as an anxiety to debunk. They are not catastrophic and they do not give rise to any issue. Basically, the problem occurs because we do not know what to do or think when we meet a contradiction.

What does it mean for Wittgenstein to dissolve the issue of contradictions? Let me consider these two analogies made by Alan Turing in the LFM (and the Wittgenstein’s response):

1. ‘*Turing:* The sort of case which I had in mind was the case where you have a logical system, a system of calculations, which you use in order to build bridges. You give this system to your clerks and they build a bridge with it and the bridge falls down. You then find a contradiction in the system’ (LFM, XXII p. 212).
2. ‘*Turing:* Could one take as an analogy a person having blocks of wood having two squares on them, like dominoes. If I say to you “White-green”, you then have to paint one of the squares on the domino which I give you white and the other green. If the point of this procedure is to be able to distinguish the two squares, you will probably hesitate when I say “White-white”. —Your suggestion comes to saying that when I say ‘White-white’ you are to paint one of the squares white and the other grey. *Wittgenstein:* Yes, exactly. And where does the cheating come in? What is the wrong continuation I have suggested? Why is this continuation in your analogy a *wrong* continuation? Might it not be the ordinary jargon among painters?’ (LFM, XIX p. 186)

For Wittgenstein, the collapse of the bridge in (a) is not caused by the contradiction: the issue is that the clerks could not handle the contradiction, it is not the inconsistency in itself. Wittgenstein describes the failure in the following way: ‘You give him a rule for multiplying; and when he gets to a certain point he can go in either of two ways, one of which leads him all wrong’ (LFM, XXII p. 218). So, the bridge falls down because the clerks have chosen an application of the contradiction that led to that result. In disagreement with Turing, Wittgenstein argues that there is no problem with the system employed in the calculation. Contradictions like that in (a) cannot harm anyone: it is just a string of symbols within the system until the clerks decide what to do. As Wittgenstein observes:

‘*Wittgenstein*: No, that is NOT what I mean at all.—The trouble described is something you get into if you apply the calculation in a way that leads to something breaking. This you can do with any calculation, contradiction or no contradiction’ (LFM, XXII p. 219).

‘And now about contradictions. Whether we're to say they have a meaning I don't know-but it's clear they don't have a use. The point is: Don't think of a contradiction as a 'wrong proposition' ("Surely this isn't so" etc.). But this doesn't mean that a contradiction can't be pernicious, if it actually misleads us’ (LFM, XXIII p. 223).

The clerks employ a system with a contradiction without understanding how to use it; this is the cause of the bridge falling down, not the contradiction itself. (b) is a case of inconsistency in a language-game: as Wittgenstein points out, painters have approved a certain use for a contradictory expression, like many other sentences in our language. So, when the contradiction occurs, it results in a new rule of inference. Therefore, there is no purpose in discussing whether the contradiction causes or not problems: it has been solved, and the grammar we apply to those circumstances does not provoke any further troubles. Mostly, a contradiction is the lack of a definite practice which tells us what to do; as Wittgenstein underlines: ‘This simply means that given a certain training, if I give you a contradiction (which I need not notice myself) you don't know what to do’ (LFM, XXII p. 213). The same for Wittgenstein happens in logic and mathematics.

So, contradictions are worrisome because no one has decided what to do with them: but they do not damage or compromise anything. A contradiction remains there, while the rest of our rules and axioms continue to function. For example, in the RFM, Wittgenstein tries to argue that contradictions are not harmful, but it is our consideration of them that creates a problem:

‘Can we say: “Contradiction is harmless if it be sealed off”? But what prevents us from sealing it off? That we do not know our way about in the calculus. Then *that* is the harm. And this is what one means when one says: the contradiction indicates that there is something wrong about our calculus. It is merely the (local) *symptom* of a sickness of the whole body. But the body is only sick if we do not know our way about’ (RFM, III-80 p. 209).

Additionally, Wittgenstein affirms: ‘We have seen that if we didn't recognize a contradiction, or if we allowed a contradiction but, for example, did not draw any further conclusions from it, we could not then say we must come into conflict with any facts’ (LFM, XXIII p. 230). They are meaningless parts without any role in our apparatus of inferences. However, if mathematics and logic have a grammatical role and they do not act like facts, then reframing is enough to dissolve them: for Wittgenstein, the solution is to assign an agreed employment for each contradiction — like the White-white dominos’ example. In other words, the invention of one rule is required to fix a meaning for each contradiction in our inferences. An issue arises only if we infer from the contradiction something that could lead us to a breakdown. But this would be an error caused by the choosing a rule of inference for the contradiction that is not convenient for us.

To clarify further with an example, consider a board game with its set of rules: suppose that the participants play the game until they find an unexpected contradiction between the rules. At that point, the game does not stop: the players would simply establish a course of action to resolve the difficulty. Thus, the board game is still playable, even if there was a contradiction. If it appears again, then the players will not recognize it as contradictory. Nothing prohibits the extension of this analogy to the case of contradictions in mathematics and logic. What Wittgenstein has in mind is this:

“But you can’t allow a contradiction to stand! —Why not? We do sometimes use this form in our talk, of course not often—but one could imagine a technique of language in which it was a regular instrument” (RFM, VII-11 p. 370).

Now, I will briefly explicate how the previous metaphor works for Wittgenstein at the level of formal machinery. Namely, from contradictory formulas like $p∧¬p$ or $\left(p\leftrightarrow q\right)∧(¬p∧q)$ I could infer everything and the contrary of everything, in virtue of the principle of explosion. Nevertheless, for Wittgenstein it would be legitimate to introduce a new rule of inference to govern the meaning of these inconsistent expressions. From this point of view, Wittgenstein’s ideas at formal level are not so distant from those of Graham Priest (2006) and contemporary paraconsistent logics: alternative rules of inference case by casereplace the *ex contradictione quodlibet* principle, e.g. $\left(p\leftrightarrow ¬p\right)∴p∧¬p$ as proposed by Priest.[[10]](#footnote-10) So, the key is to modify a certain model of calculus introducing rules that allow certain inferences from contradictory propositions. Wittgenstein provides some examples at the formal level comparing classic logic with possible alternatives in the LFM XIX. For example, he proposes $⊢p.\~p$ or $⊢p∨\~p $ as proper inferences to admit in a non-standard logic (cf. LFM XIX p. 189). Note this interesting comparison: Wittgenstein’s proposal resembles the conceptual strategy of designation for well-formed formulas in paraconsistent models. Designation is a family of strategies for paraconsistency: it consists into developing a counter-model *C\** of a model *C* (where consistency is a requirement in *C*) in which an argument *A2* in *C\** retains all the premises of *A1* in *C* but has different conclusions (cf. Ripley, 2015, pp. 773–775). In particular, the designation is a strategy invented to find counter-examples of cases of explosion. It seems that Wittgenstein anticipated the subsequent research on logic and inconsistencies in the 1950s and onwards (cf. da Costa et al., 1995a, 1995b).[[11]](#footnote-11)

In brief: after introducing the semantics which clarify what is permitted by a specific contradiction, the formula turns out to be a useful part of our system. Once the contradiction is identified and dissolved, there won’t be any further issues.

# Examples of Dissolved Contradictions

Now I will examine three examples of Wittgenstein that explain the mindset that one should adopt towards contradictions in logic and mathematics.

Neither in the LFM nor in the RFM, had Wittgenstein proposed a full formal model of an alternative kind of logic where contradictions are acceptable. Instead, he offered analogies in natural language that he extends to logic and mathematics. Both his later thesis on what logic is and his later philosophical method can explain this gap. S.S. Grève (2018) has observed that the later Wittgenstein broadens the task of logic: from the analysis and construction of formal languages to the activity to find what is essential in a language-game (cf. Grève, 2018, pp. 176–177); i.e. logic examines also the rules of language-games in ordinary language, in addition to expressions and models in formal logic. Hiroshi Ohtani (2018a) argues that Wittgenstein in his later period takes as method of analysis that of “pictures”: the method of pictures consists in taking a philosophical “stereotype” blinding us and demystifying its non-sensical nature through examples from ordinary language (cf. Ohtani, 2018a, pp. 2050–2053). These two remarks by Gréve and Ohtani clarify why Wittgenstein has employed his stance on contradictions through examples from ordinary language. This also makes clear why he has imagined but not sketched a full formal model of logic where consistency is not one of the features and contradictions are true. Moreover, this explains why Wittgenstein broadens his notion of contradiction to material inferences too, focusing on these sorts of examples in natural language.

Take the language-game of giving orders outlined in the Lectures (cf. LFM, XVIII pp. 174-175). The conjunction of orders is contradictory and I am undecided about what to do. For example, I can ask ‘Get out of the room and do not get out of the room’ or “Bring me the book and do not bring me the book”. The difficulty here is that I do not grasp the use of that statement in practice: no one has taught me how to cope with this sort of requests. But if I can give these expressions a meaning, then I would know what to do; e.g. maybe ‘Get out of the room and do not get out of the room’ will be interpreted as ‘Stay on the doorstep’ within the language-game. So, the problem here regards the fact that there is no action connected to the contradictory claim (cf. LFM, XVIII p. 179). Nonetheless, this does not imply that a course of action cannot be determined. This, again, means introducing a rule that prescribes what to infer.

Wittgenstein presents another case (cf. LFM, XX p.193): consider the use of “all” in a sentence like “All the men in this room are over twenty-five but he is not”. The use of two terms that affirm one generality (all) and the other exception (but) give rise to the contradiction. However, one can imagine a tribe — Wittgenstein suggests — where the expression “all but one” in these cases is not contradictory and its members do not regard it as such. And he observes:

‘This way of learning “contradicts” this way of using; or: It “contradicts” the meaning of "all" not to let *fa* follow from *(x).fx*. But it is here a matter of a peculiar picture being always connected with one use rather than with another use’ (LFM, XX p.193).

We consider in our language the utterance “All the men in this room are over twenty-five but he is not” as contradictory because our consideration is bound to a certain picture; a certain false assumption that leads us to consider something as falsely problematic (cf. Ohtani, 2018b, p. 117). In this case, the picture is the following: the conviction that what “ALL” expresses and what “BUT” means are incompatible in the same sentence; and therefore, the use of ALL is irreconcilable with the use of BUT. This picture deceives us into reckoning it as a contradiction without the possibility of seeing possible uses of it. But once I change the picture connected, the difficulty is erased; e.g. I can consider “all but one” as a case where a universal quantification accepts exceptions. And one could extend this kind of quantification from natural languages to formal languages, creating in this way a new rule to deal with this kind of contradictions.

Similar is the case of the conflicting reports in the army (cf. LFM, XXI p. 201): there are two contrasting reports made by two soldiers about the number of enemies on the battlefield; one says 30.000 whilst the other affirms 40.000. The commander cannot believe that the enemies are 30.000 and 40.000 simultaneously. However, he could handle these contradictory statements by picking up a certain technique: for instance, he could consider it as a probabilistic estimate, or he could draw a blurred picture of the enemy’s forces etc. This means determining a new course of action to deal with the two conflicting claims. For Wittgenstein, the same reasoning is applicable to formal systems:

“*Lewy:* One might say that an entirely new meaning has been given to the contradiction.

*Wittgenstein:* Yes, one might say that. —And notice that contradictions are actually often used in this way. For instance, we say, "Well it is fine and it's not fine", meaning that the weather is mediocre. And one might even introduce this use into mathematics” (LFM, XVIII, pp. 174-175).

# Contradictions and Practical Applications: The Reply to Alan Turing

Now, I will discuss the consequences of the therapeutic approach to contradictions in the debate with Alan Turing. This is what arises from their conversations: Turing was still looking at contradictions through CRA, ignoring the therapeutic intention of Wittgenstein. This is the central point of their dispute: Wittgenstein tried to persuade him to abandon the realism\anti-realism dichotomy in order to understand his therapeutic aim; but he failed. Nonetheless, their discussion helps us to focus on the kind of errors regarding contradictions that Wittgenstein was trying to warn of.

The main objection raised by Turing against the Wittgensteinian account is the following:

‘*Wittgenstein:* There seems to me to be an enormous mistake there. For your calculus gives certain results, and you want the bridge not to break down. I'd say things can go wrong in only two ways: either the bridge breaks down or you have made a mistake in your calculation—for example, you multiplied wrongly. But you seem to think that there may be a third thing wrong: the calculus is wrong.

*Turing:* No. What I object to is the bridge falling down. […]

*Turing:* If one takes Frege's symbolism and gives someone the technique of multiplying in it, then by using a Russell paradox he could get a wrong multiplication’ (LFM XXII, p. 218).

Turing notices a conflict between the Wittgensteinian account and the common practices of applied mathematics. He defends implicitly this thesis: if there is an applied calculation, it is in virtue of some isomorphism between the system employed and the reality (cf. LFM, XII p. 118, XIV pp. 138-139, XV p. 150). Consequently, if there is an isomorphism as a condition of possibility for the effectiveness of a calculation, then every contradiction in mathematics or logic will have an effect on my calculations in practice (cf. LFM, XXII p. 211). Hence, contradictions cannot be amended arbitrarily with the introduction of a rule because they must respect this isomorphism with reality.

Wittgenstein’s response to Turing’s criticism consists in pointing out this confusion: contradictions are a grammatical illusion, that is why they do not clash with facts. Turing does not understand that the relationship between application and contradiction does not concern facts. Wittgenstein wanted to distinguish between mathematical propositions (that are grammatical) and empirical propositions expressed through mathematics; he has declared clearly this point in the LFM XII: ‘I said that the whole trend of these discussions was to show the difference between mathematical propositions and experiential propositions which look exactly like them’ (LFM, XII p. 113). In replying to Turing, Wittgenstein invites him to see the difference:

‘This correspondence may be called a correspondence of grammar. If you say, "This reality corresponds to this word”—this is a sentence of grammar; you are giving a grammatical explanation. Whereas if you say, "A reality corresponds to ‘There are six people in this room’”—this is not a sentence of grammar at all; you are affirming a proposition. This is an essential difference.’ (LFM, XXVI p. 248).

Likewise, in the RFM, Wittgenstein insists that mathematics is a problem concerning grammar:

‘What does mathematics need a foundation for? It no more needs one, I believe, than propositions about physical objects—or about sense impressions, need an *analysis*. What mathematical propositions do stand in need of is clarification of their grammar, just as do those other propositions’ (RFM, VII-16 p. 378).

Turing’s concerns are worrying if and only if we see contradictions through the realist/anti-realist opposition. Instead, Wittgenstein was trying to explain to him that contradictions do not clash with any fact because they are grammatical confusions. Wittgenstein identified this kind of factual speaking about logic and mathematics as non-sensical: ‘Arithmetic as the natural history (mineralogy) of numbers. But *who* talks likes this about it? Our whole thinking is penetrated with this idea’ (RFM, IV-11 p. 229). Wittgenstein argues that mathematics concerns the rules regulating our calculations and the arrangement of these rules is previous to any factual circumstance. So, contradictions create problems only when we have not decided a use for them in the grammar of the system employed, not because they conflict with a state of affairs. In RFM, he stressed again that the problem is at the grammatical level:

‘The steps which are not brought in question are logical inferences. But the reason why they are not brought in question is not that they “certainly correspond to the truth”— or something of the sort,—no, it is just this that is called “thinking”, “speaking, “inferring”, “arguing”. There is not any question at all here of some correspondence between what is said and reality; rather is logic antecedent to any such correspondence; in the same sense, that is, as that in which the establishment of a method of measurement is *antecedent* to the correctness or incorrectness of a statement of length’ (RFM, I-156 p. 96).

So, if contradictions are not a matter of fact but are a grammatical issue, we can dissolve them changing the grammar governing the calculation. This is enough to adjust our practice in order to avoid unpleasant results with contradictions. Wittgenstein does not deny that the grammar of mathematics and logic do not have an impact on practical applications.[[12]](#footnote-12) Wittgenstein defends rather the view that the bridge’s collapse is not a direct consequence of the existence of a contradiction: first, the problem concerns the confusion between factual reality and grammar caused by CRA; second, the problem occurs when we choose a new rule of inference for the contradiction that causes the incident. About this, he affirms in the RFM: ‘Does a misunderstanding about the possible application constitute an objection to the calculation as a part of mathematics? And apart from misunderstanding,—what about mere lack of clarity?’ (RFM, V-5 p. 261). So, it is the fault of the technique employed and not of the contradiction alone. The collapse of the bridge concerns which technique we have picked up for that goal. Indeed, a grammar regulates a particular technique, which in turn is constituted by a network of rules; thence, it concerns which rule we have chosen in our grammar to deal with that contradiction. As Wittgenstein points out to Turing:

‘If you look at it this way, the whole idea of mathematics as the physics of the mathematical entities breaks down. For which road you build is not determined by the physics of mathematical entities but by totally different considerations. The mathematical proposition says: The road goes there. Why we should build a certain road isn't because mathematics says that the road goes there—because the road isn't built until mathematics says it goes there. What determines it is partly practical considerations and partly analogies in the present system of mathematics’ (LFM, XIV pp. 138-139).

But still, Turing continued — influenced by CRA — to deem consistency as the condition to have successful applications. Wittgenstein proposed a dissolution for the whole problem of contradictions, but Turing did not grasp his therapeutic perspective. For this reason, Turing was not convinced. Indeed, Wittgenstein clearly had difficulties explaining this to Turing (and the other participants of the lectures):

‘One of the greatest difficulties I find in explaining what I mean is this: You are inclined to put our difference in one way, as a difference of *opinion*. But I am not trying to persuade you to change your opinion. I am only trying to recommend a certain sort of investigation. If there is an opinion involved, my only opinion is that this sort of investigation is immensely important, and very much *against the grain* of some of you’ (LFM, XI p. 103).

This is why certain interpretations of Wittgenstein do not render justice to his therapeutic method. Namely, P. Maddy (2000) in her reading of Wittgenstein claimed that the true mathematics is the applied one and pure mathematics is just a game of signs detached from the practice, and therefore pointless and misleading (cf. Maddy, 2000, p. 169). I deem this reading implausible. Wittgenstein never claimed that pure mathematics was less important or meaningless. Although critical about set theory (cf. WVC, p. 102; RFM, II-22 p. 132, II-23 p. 132, V-5 p. 260; LFM, I pp. 16-17, PI 2009, Part II §372 p. xiv), he never generalized this claim to pure mathematics. Indeed, Wittgenstein considers pure mathematics as a game among many others, and he affirms: ‘[…] when mathematics is divested of all content, it would remain that certain signs can be constructed from others according to certain rules’ (RFM, III-38 p. 169). This means that when detached from practical applications mathematics becomes a meaningful game only thanks to its rules. Again, Maddy poses the issue through the same CRA that misled Turing: there is no opposition between “true” versus “wrong” mathematics; instead, there are multiple mathematics and logics for different activities (including those where contradictions are employable). In addition, inside the literature, there is a strong tradition since M. Dummett (1996) which reads Wittgenstein as an anti-realist; for instance, S. Schroeder (2018) , S. M. Pinto (2015), M. Marion (2017, 2009), P. Garavaso (1991) and J. Bouveresse (1992). However, Wittgenstein aims to cure our intellect of the whole dichotomy between realism and anti-realism. An anti-realist ascribes a position that is incoherent with the Wittgensteinian programme of therapy from false dilemmas and dichotomies; i.e. antirealism poses the problem in mathematics and logic as genuine and not as an issue to be dissolved through conceptual clarification. Wittgenstein wanted to dissolve non-sensical dichotomies, not to take side in debates that he would have considered merely as ‘houses of cards’ (PI, §118 p. 54). As H. Putnam (2001) and P. Frascolla (2014) noticed, Wittgenstein clearly had no sympathy for realism, but this does not make him an anti-realist as well. This is exactly the kind of debate that Wittgenstein attempted to deflate. In the Wittgensteinian philosophy, the point is not the nature of mathematics or logic: instead, it regards both how certain pictures keep us captivated and how grammar defines our techniques. As I have shown, the case of contradictions is a good example of this sort of confusion.

To conclude, CRA led Turing to consider Wittgenstein’s proposal not as therapeutic, but as an anti-realist response to his criticism. In fact, Wittgenstein did not propose to solve contradictions, but rather to change our consideration of them. In turn, Turing considered as unsatisfying the Wittgensteinian attempt: Turing was expecting a solution to a problem set up accordingly to the CRA while Wittgenstein rejected this whole framework to the discussion. However, their debate does help us to comprehend which false picture misleads us to see contradictions as a problem.

# Conclusions

In the paper, I have outlined the later Wittgenstein’s therapeutic approach towards contradictions. Through this discussion, three main points have arisen.

Firstly, Wittgenstein’s dissolution of the issue. For the later Wittgenstein, contradictions arise for two reasons: either because a conflict exists between two or more rules or because there are two contrasting claims in a material inference. However, for Wittgenstein these are not harmful: he positions himself against the prejudice that contradictions entail a breakdown in a formal or natural language. They are just parts of a grammar that miss a clear use defined by a rule. So, the Wittgensteinian dissolution to a formal contradiction is to give it a meaning; i.e. instead of leaving it alone, I rectify its use with a rule. So, the contradiction becomes a usable part of our inferences and techniques. Thus, the difficulty is dissolved as it was simply the product of a picture which sees contradictions as fatal flaws.

Secondly, I agree with Diego Marconi (1984) when he defined Wittgenstein as a forerunner of the paraconsistent logics: indeed, for Wittgenstein consistency is not a primary concern for the usability of logic or mathematics. When a semantics in terms of rules is established, contradictions have a role in our inferences.

Thirdly, the debate with Turing shows which picture constrained us towards the orthodox conception of contradictions: if we examine the issue through the CRA, then we see contradictions as conflicting with either a mind-independent law or a human stipulated convention. But for Wittgenstein this is not the point: CRA hinders us from seeing contradictions as a source of grammatical confusion. As Wittgenstein argues, contradictions are loopholes in our grammar which miss a rule to fill them. They do not clash with facts or conventions constituting mathematics and logic. Turing, in turn, insisted in seeing the issue as a problem to solve through a realist or anti-realist approach. For this reason, Turing was not convinced regarding the case of applied mathematics. But the debate between Wittgenstein and Turing helped us to focus on which kind of mindset gives rise to this pseudo-problem. A language-game is still healthy even if some contradictions occur: the conceptual illness is for us to think that there is a problem.

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1. My goal is purely historical. Hence, I will not take a stance on the role of Wittgenstein in contemporary logic or the philosophy of mathematics. For further reading on the contribution of Wittgenstein’s views in the contemporary debate, see S. Schroeder (2017), H. Putnam and J. Conant (1997) and M. Plebani (2011). [↑](#footnote-ref-1)
2. The relationship between logic and mathematics in Wittgenstein deserves a separate debate. It is not my aim to deeply explore the relationship in this article, though I need to face some aspects of it. [↑](#footnote-ref-2)
3. Note that Wittgenstein talks indifferently of contradiction in both mathematics and logic in RFM and LFM. He does not seem interested in drawing a distinction between the two cases. At the end of section §2, I will offer an explanation for this. [↑](#footnote-ref-3)
4. Monism in philosophy of logic claims that there is only one correct model of logic. [↑](#footnote-ref-4)
5. These remarks led Hao Wang (1962) to coin the term “anthropologism” to label Wittgenstein’s philosophy of mathematics (cf. Wang, 1962, pp. 39–41). According to Wang, for Wittgenstein, mathematics does not identify facts but introduces new techniques which enable us to undertake new activities. [↑](#footnote-ref-5)
6. I will supply some examples of this in the following sections §3 and §4. [↑](#footnote-ref-6)
7. This appears with clarity when in LFM XXVIII (p. 270) Wittgenstein talks about the problems in Russell’s logic as an issue concerning clarification of grammar. Similarly, in LFM XXVII referring to the same problem, Wittgenstein claims that an analysis: ‘[…] also cleared the grammar enormously and made certain misunderstandings impossible’ (p. 262). [↑](#footnote-ref-7)
8. See the example of Turing and Rhees trying to execute the contradictory instructions to be contemporarily at Trumpington at 3 o’clock and at Grantchester at 3:30 (cf. LFM, XII p. 212). In this case, the problem relies on the meaning of the two propositions, not in the operator conjoining them. [↑](#footnote-ref-8)
9. Different readers of Wittgenstein have underestimated the role that material inferences play in his later conception: for instance, M. Mathieu and M. Okada (2013), M. Potter (2012), Z.A. Sokuler (1988) and I. Dilman (1970) have depicted the problem of contradictions in Wittgenstein’s work as merely an issue regarding the rules; they overlook the aspect of material inferences. [↑](#footnote-ref-9)
10. A disclaimer here is required: Wittgenstein accepts contradictions as meaningful only in mathematics and logic; inversely, Graham Priest accepts contradictions as true also in reality, see Priest (2014, 2006, 1995; Smiley and Priest, 1993). Wittgenstein does not commit to contradictions in reality because he considers non-sensical the whole metaphysical question involved. Moreover, Graham Priest rejects in logic the principle of explosion for metaphysical reasons, whereas Wittgenstein rejects it because it is connected with a misleading conception of logic and mathematics. Furthermore, Wittgenstein is a logical pluralist whilst Graham Priest is a logical monist (cf. Read, 2006). So, my comparison between them is limited only at the level of formal semantics. [↑](#footnote-ref-10)
11. During the same period, besides Wittgenstein, the only logician proposing a similar approach to deal with contradictions was Stanisław Jaśkowski, precisely between 1910 and 1948 (cf. Perzanowski, 2004). [↑](#footnote-ref-11)
12. Repeatedly, he claims that the mathematics is not just signs but language-games with different purposes including practical calculations (cf. RFM, V-2 p. 257, V-4 pp. 258-259; LFM, III pp. 34-48, XV p. 150). Moreover, R. Dawson (2014) argued that for Wittgenstein it is legitimate in certain circumstances to talk of mathematical grammar as practical instructions to apply (cf. Dawson, 2014, p. 4144). [↑](#footnote-ref-12)