Jesús Mosterín

HOW SET THEORY IMPINGES ON LOGIC

The set-theoretical universe

Reality often cannot be grasped and understood in its unfathomable richness and mind-blowing complexity. Think only of the trivial case of the shape of the Earth. Every time the wind blows, a bird flies, a tree drops a leave, every time it rains, a car moves or we get a haircut, the form of the Earth changes. No available or conceivable geometry can describe the ever changing form of the surface of our planet. Sometimes the best we can do is to apply the method of theoretical science: to pick up a mathematical structure from the set-theoretical universe, a structure that has some formal similarities with some features of the real world situation we are interested in, and to use that structure as a model of that parcel of the world. In the case of the Earth, the structure can be an Euclidean sphere, or a sphere flattened at the poles, or an ellipsoid, but of course these structures do not represent the car and the hair, and so are realistic only up to a point.

The largest part of scientific activity results in data, in contributions to history (in a broad sense). Only exceptionally does scientific activity result in abstract schemata, in formulas, in theories. In history there is truth and falsity, but we are not sure whether it makes sense to apply these same categories to an abstract theory. We pick up a mathematical structure and construct a theory. We still have to determine its scope of application or validity, the range of its realizations. If it is consistent, it will have at least mathematical realizations, and that we can know *a priori*. But the range of its real world applications is a matter for empirical research to ascertain.

Mathematical (or theoretical) science departs very drastically from usual ways of representing and understanding. In order to realize how utterly different its method of representation is from ordinary language, it suffices to ponder how remote the set-theoretical universe (the reservoir of mathematical structures) is from anything in ordinary language or everyday experience.

Mathematicians create the set-theoretical universe from almost nothing, from just the empty set, by means of successive (and never ending) iterations of the operations of the power set and the union. The ordinal numbers include the natural numbers and extend the possibility of carrying out iterations into the transfinite. The usual ordinal numbers β such that (for some ordinal α) $\beta = \alpha+1$ are the successor ordinals. The limit ordinals are the ordinals that are not successor ordinals. Von Neumann proved a general recursion theorem, that allows us to define ordinal functions by transfinite recursion over all the ordinals. So he was able to give a precise definition of the cumulative hierarchy of sets by means of the ordinal function V_α:

$V_0 = \varnothing$

$V_{\alpha+1} = \wp(V_\alpha)$

$V_\lambda = \bigcup_{\beta<\lambda} V_\beta$

Assuming the axiom of foundation (i.e. that all sets are in the cumulative hierarchy), von Neumann defined the set-theoretical universe V as the union of all the V_β (for any ordinal β): $V = \bigcup_{\beta\in\Omega} V_\beta$. (Of course, the universe V and the class Ω of all ordinals are proper classe).This definition has become canonical. The set-theoretical universe is usually figured as an inverted cone, whose (inverted) vertex is the empty set. This inverted cone proceeds upward by iterations of the operations of power set and union, indexed by the ordinals. To each ordinal α corresponds a new slice of sets (the sets with rank α). The union of all those slices is the set-theoretical universe, V.

All the mathematical structures used in science for modeling the real world (the natural and real numbers, the Euclidean and non-Euclidean spaces, the probability spaces, the vector spaces, the Hilbert spaces, the differential manifolds, the tensor fields, ...), all of them appear somewhere in this inverted cone. The rank of a structure is the least ordinal β such that the structure is in the slice V_β of the set-theoretical universe.

Concerning the inverted cone of the set-theoretical universe, we can ask (1) how wide it is, and (2) how tall it is. Different axioms and hypothesis determine the width and the depth (or height) of the cone. The Continuum Hypothesis and Gödel's axiom of constructibility (V = L) concern the width of the cone. If we accept the Generalized Continuum Hypothesis (GCH), the cone is narrower. If we reject it, it becomes wider. The axiom of constructibility makes for an especially narrow cone. On the other hand, large cardinal axioms concern the height of the cone. The axiom of infinity, the existence of $\aleph_1$, of inaccessible cardinals, of Mahlo cardinals, of weakly compact cardinals, of measurable cardinals, of Woodin cardinals, and so on, make for deeper and deeper (or higher and higher) cones. Set theory is deep and full of open questions. One could be tempted to think that logic is a harmless pursuit, independent of set-theoretical assumptions, but it is not.

Second-order logic and set theory

At first sight, second-order logic looks like the most natural framework for formulating such mathematical theories as natural number arithmetic, Euclidean geometry, mathematical analysis, and even ZFC set theory, all of which are categorical in second-order language. Only in second-order logic are all these theories complete, and only in second-order logic can their corresponding mathematical structures be characterized uniquely, up to isomorphism. Unfortunately it is impossible to mine all these alleged riches from second order logic. There cannot be any complete calculus for deducing all the theorems of a second-order theory from its axioms. And second-order logic itself is indeterminate. The set of its valid formulas cannot be known. If it was known, it would settle all the many open questions of set theory. As a matter of fact, we need to settle all the open questions of set theory before second-order logic can be made determinate.

A sentence of pure second-order logic is a closed formula of second-order logic that does not contain any symbol besides logical constants and quantified variables. It does not contain any specifically mathematical or set-theoretical symbol. Corresponding to each open question in set theory there is a sentence (a closed formula) of pure second-order logic, such that the formula is logically valid if and only if the set-theoretical question has an affirmative answer.

The axiom of choice

No one really doubts the axioms of ZF. The first doubts were expressed in relation to the axiom of choice (AC), that, in one of its versions, says that there is a universal choice function, i.e., a function that assigns to each non-empty set in the universe one of its members. It is equivalent to many different mathematical statements, like the well-ordering theorem. The well-ordering theorem was conjectured by Cantor in 1883. It was a necessary keystone of the Cantorian construction of set theory. Only if every set can be well ordered can we be sure that every set has an ordinal as its order type and an aleph as its cardinality. Cantor tried to prove it, but failed. It was first proved by Zermelo in 1904, but only under the assumption of the axiom of choice, to which it is equivalent.

The following sentence of pure second-order logic (interpreted on any universe) says that the universe can be well-ordered.

$$\exists W\,[(\forall xyu\,(Wxy \wedge Wyu \Rightarrow Wxu) \wedge \neg Wxx \wedge \forall xy(Wxy \vee Wyx \vee x{=}y)$$
$$\wedge\ \forall Z(\exists xZx \Rightarrow \exists u(Zu \wedge \forall x(Zx \Rightarrow Wux \vee u{=}x)))]$$

This formula is satisfied by a structure iff the universe of the structure can be well-ordered (i.e., is well-ordered by some relation).

The Continuum Hypothesis

The other conjecture Cantor unsuccessfully tried to prove was the continuum hypothesis. With the benefit of hindsight, we now know that it could not be proved, as it is independent of the rest of axioms of set theory.
In order to be able to formulate a formula of pure second-order logic equivalent to the continuum hypothesis in a compact way, we have to introduce some abbreviations or definitions. Notice that these abbreviatory devices are fully dispensable. Any formula formulated with the so defined symbols is just an abbreviation of another and longer formula of pure second-order logic. Here are some definitions:

Y is injectable into Z (smaller than or equal to Z): $Y \preceq Z$

$$Y \preceq Z \Leftrightarrow_{\mathrm{df}} \exists W\,[\forall xyz\,((Wxy \wedge Wxz \Rightarrow y{=}z) \wedge (Wyx \wedge Wzx \Rightarrow y = z))$$
$$\wedge\ \forall u\,(Yu \Rightarrow \exists x\,(Wux \wedge Zx))]$$

Y is bijectable onto Z (equinumerous with Z): $Y \sim Z$

$Y \sim Z \Leftrightarrow_{\mathrm{df}} Y \preceq Z \wedge Z \preceq Y$

Y is smaller than Z: $Y \prec Z$

$Y \prec Z \Leftrightarrow_{\mathrm{df}} Y \preceq Z \wedge \neg\, Y \sim Z$

Z is infinite: Inf(Z) [Dedekind]

$\mathrm{Inf}(Z) \Leftrightarrow_{\mathrm{df}} \exists Y\, [\forall x\, (Yx \Rightarrow Zx) \wedge \exists x\, (Zx \wedge \neg Yx) \wedge Z \sim Y]$

Y is the power set of X: $\wp(X,Y)$

$\wp(X,Y) \Leftrightarrow_{\mathrm{df}} \exists W\, [\forall u\, (Xu \Leftrightarrow \exists z\ Wuz) \wedge \forall u\, (Yu \Leftrightarrow \exists z\ Wzu) \wedge$
$\wedge\, \forall Z\, (\forall u\, (Zu \Rightarrow Xu) \Rightarrow \exists v \forall x\, (Zx \Leftrightarrow Wxv)) \wedge \forall uv\, (\forall x\, (Wxu \Leftrightarrow Wxv) \Rightarrow u{=}v)]$

[Here the first formulas mean Dom(W) = X and Range(W) = Y. W mimics the membership relation between elements of X and subsets of X (elements of Y)].

The Generalized Continuum Hypothesis (GCH) says that, for any infinite set A, there are no intermediate cardinalities between $|A|$ and $|\wp(A)|$. It is equivalent to the validity of the pure second-order logic formula abbreviated below.

$\mathrm{GCH} \Leftrightarrow_{\mathrm{df}} \forall XYZ\, (\mathrm{Inf}(X) \wedge \wp(X,Y) \wedge Z \prec Y \Rightarrow Z \preceq X)$

In ZFC set theory we can neither prove nor disprove GCH, because it is independent of the rest of the axioms of standard set theory, as proved by Kurt Gödel and Paul Cohen. So, GCH is a logical truth iff the generalized continuum hypothesis is a set-theoretical truth, but we do not know whether it is (or we want it to be) true, and so neither we do know whether the corresponding formula is a logical truth.

Inaccessible cardinals

A cardinal κ is inaccessible iff
(1) κ is an uncountable cardinal (i.e., $\kappa > \aleph_0$)
(2) κ is a strong limit cardinal (i.e., the power set (or power of 2) construction on inferiors does not lead us to κ: for any cardinal λ: $\lambda < \kappa$ implies $2^\lambda < \kappa$)
(3) κ is regular (i.e., κ is not the supremum of a set of fewer than κ smaller ordinals).

Regularity of cardinals can also be defined in terms of cofinality. The cofinality of α, cf(α), is the minimum ordinal β such that there is a function f: $\alpha \to \beta$ with range(f) cofinal in α. The cofinality of a limit ordinal λ, cf(λ), is the smallest cardinal κ such that λ is the supremum of κ smaller cardinals. A cardinal κ is regular iff cf(κ) = κ.

If κ is an inaccessible cardinal, then κ is a fixed point of the aleph function: $\aleph_\kappa = \kappa$. If κ is an inaccessible cardinal, then all of ZFC axioms are true in V_κ (i.e., V_κ is a model of ZFC). A consequence of this last fact (via Gödel's 2nd incompleteness theorem) is that the existence of inaccessible cardinals is unprovable in ZFC. The relative consistency is also unprovable.

In ZFC we can neither prove nor disprove the existence of inaccessible cardinals. So the assertion that there are inaccessible cardinals adds new strength to the theory, is an axiom that adds new depth (or height) to the set-theoretical universe.

There are inaccessible cardinals iff there are sets of inaccessible cardinality. The predicate Inacc(Z) says that the set Z is of inaccessible cardinality. It can be defined in pure second order logic (remember that the symbols Inf and $\wp$ are just abbreviations):

$$
\begin{aligned}
&\mathrm{Inacc}(Z) \Leftrightarrow_{\mathrm{df}}\\
&\mathrm{Inf}(Z) \wedge \exists Y\,(Y < Z \wedge \mathrm{Inf}(Y)) \wedge \forall X\,(X < Z \Rightarrow \exists Y\,(\wp(X,Y) \wedge Y < Z)\\
&\wedge \forall XYW\,[X < Z \wedge \forall u\,(Xu \Leftrightarrow \exists x\, Wux) \wedge \forall u\,(Yu \Leftrightarrow \exists x\, Wxu)\\
&\wedge \forall y\,(Xy \Rightarrow \forall V\,(\forall x\,(Vx \Leftrightarrow Wyx) \Rightarrow V < Z)) \Rightarrow Y < Z]
\end{aligned}
$$

This formula says that Z is infinite and uncountable, that the power set construction on inferiors does not lead us to Z (strong limit), and that the range of any function that applies inferiors to the elements of inferiors does not lead us to Z (what is equivalent to regularity). So, the formula defines a set of inaccessible cardinality.

$\exists X$ Inacc(X) is satisfiable iff there is an inaccessible cardinal.

$\neg\forall X$ Inacc(X) is a logical truth iff there are no inaccessible cardinals.

Of course, still stronger axioms have been proposed, asserting the existence of larger and larger cardinals (like Mahlo cardinals, weakly compact cardinals, measurable cardinals, Woodin cardinals or supercompact cardinals). Each of these axioms is independent of the previous ones and implies them. But (with some ingenuity and much space) we could reformulate all these axioms as sentences of pure second-order logic.

First-order logic and set theory

Everyone agrees that second-order logic is set-theory in disguise, but first-order logic is often supposed not to be contaminated by any set-theoretical decisions. Is it so?

The set of logically valid sentences of first-order logic is the set of all sentences that are satisfied (or true) in all structures (or in all interpretations on all domains). Depending on which and how many sets there are, there will be more or less logically valid sentences. The more sets there are, the less formulas will be satisfied in all of them, i.e. the less formulas are logically valid.

First, a trivial point. Usually we do not admit in our semantics (or model theory) structures with an empty universe. Because of that, formulas like the following are logically valid:

$\exists x\, (Px \vee \neg\, Px)$ $\qquad \exists x\; x{=}x$ $\qquad \forall \mathrm{x}\; Zx \Rightarrow \exists x\; Zx$

If we admit structures with the empty set as universe, then these formulas become invalid, obviously.

Some first-order formulas (called infinite schemata by Quine) are only satisfied by infinite structures, like the following:

$\forall \mathrm{x} \exists \mathrm{y}\ \mathrm{Rxy} \wedge \forall \mathrm{x}\ \neg \mathrm{Rxx} \wedge \forall \mathrm{xyz}\ (\mathrm{Rxy} \wedge \mathrm{Ryz} \Rightarrow \mathrm{Rxz})$

$\forall x \exists y \forall z\ (Rxy \wedge \neg Rxx \wedge (Ryz \Rightarrow Rxz))$ [the same, in prenex normal form]

$\forall xy\ (f(x) = f(y) \Rightarrow x = y) \wedge \exists y \forall x\ f(x) \neq y$

The negations of these formulas, like

$\neg \forall x \exists y \forall z\ (Rxy \wedge \neg Rxx \wedge (Ryz \Rightarrow Rxz))$

are logically true if all sets are finite, but are not valid if there are infinite sets.

Many invalid (assuming infinite sets) first-order formulas become valid formulas, if we restrict our set-theoretical background to finite sets. The formula

$$\forall xyz\ (Rxy \wedge Ryz \Rightarrow Rxz) \wedge \forall x\ \neg Rxx \wedge \forall xy\ (Rxy \vee Ryx \vee x = y)$$
$$\Rightarrow \exists y \forall x\ (Rxy \vee x = y)$$

says that if R is a lineal order, then R has a maximum. If there are infinite sets, then some lineal orders have a maximum and some other lineal orders do not. In this case, this formula is true in certain structures and is false in other structures, so that it is not logically true. On the contrary, if there are no infinite sets, if all sets are finite, then every lineal order has a maximum, and the formula is logically true.

The first-order formula

$$\forall xyz\ (Rxy \wedge Ryz \Rightarrow Rxz) \wedge \forall x\ \neg Rxx \wedge \forall xy\ (Rxy \vee Ryx \vee x = y)$$
$$\Rightarrow \exists xy\ (Rxy \wedge x \neq y) \wedge \neg \exists z\ (Rxz \wedge Rzy)$$

says that if R is a lineal order, then R is not dense. Of course, no finite ordering is dense. If there are only finite sets, this formula is logically true. If there are infinite sets, this formula is not logically valid, as many infinite orderings (like the rationals or the reals) are dense.

The first-order formula

$$\forall xyz\ (Rxy \wedge Ryz \Rightarrow Rxz) \wedge \forall x\ \neg\ Rxx \wedge \forall xy\ (Rxy \vee Ryx \vee x = y)$$
$$\Rightarrow [\forall xy\ (Sxy \Rightarrow Rxy) \wedge \exists xy\ Sxy \Rightarrow \exists z \forall xy\ (Sxy \vee Syx \Rightarrow Szx \vee z = x)]$$

says that if R is a lineal order and S is a non-empty suborder of R, then S has a minimum. Its validity would imply that any lineal ordering is a well-ordering.

Indeed, every finite lineal ordering is a well ordering, and so the formula is valid if all sets are finite. But many infinite lineal orderings (like the integers or the rational or the reals) are not well-orderings. So, the formula is not valid, if there are infinite sets.

In contrast with second order validity (which is not well defined), first order validity is well determined, both if we accept infinite sets or if we keep to finite ones. But in both different cases we get different notions of validity, different sets of valid sentences.

If we reject infinite sets, and admit only finite sets, we get many more valid first-order formulas. As there are less sets, and so less structures, there are more formulas satisfied by the fewer structures. All the formulas usually considered valid continue to be valid, but many new ones become valid, like the formulas previously considered.

We have seen that second-order logic depends massively on set theory, but also first-order logic has some dependency. In the polemic about logicism in the first decades of the 20th century, the status of the axiom of infinity played a crucial role. The polemic closed with the agreement that the axiom of infinity belonged to set theory and had nothing to do with logic. But, as we have just seen, it has much to do with logic, even with first-order logic. Whether we accept it or not in the metatheory as background of our definitions, we get quite different (extensionally different) first-order logics.

The standard notion of first-order logic is the one that accepts infinite sets in the set-theoretic background. This logic is semantically complete, as Gödel first proved in 1930. That means that the set of the valid sentences of first-order logic is recursively enumerable, or, in other words, that it can be generated by the successive application of the rules of a deductive calculus. This is also equivalent to say that the calculus allows as to deduce all the consequences of a given set of premisses. All the proofs (Gödel's, Henkin's, ...) of the semantic completeness of first-order logic proceed by the construction of certain infinite sets (sets of terms, in Henkin's case), and lose all acceptability if we reject infinite sets. Furthermore, if we reject infinite sets, we reduce so drastically the amount of available structures, that the number of valid formulas increases considerably. The set of valid formulas becomes much larger and complex, so much so that it ceases to be recursively enumerable, as proved by Trakhtenbrot in 1950. So, first-order logic is only semantically complete in so far as we countenance infinite sets in the metatheoretic background.

The other way around, if we replace standard second-order logic (in which the second order variables vary over the full power set of the universe of the structure) by Henkin second-order logic (in which variables can vary on any particular subset of the power set of the universe), then there are many more structures available than in the standard case, and so there are fewer formulas satisfied in all of them, i.e., there are fewer valid formulas. The set of valid formulas becomes so much smaller and less complex, that it even becomes recursively enumerable.

REFERENCES

Cohen, Paul. (1963-64) The Independence of the Continuum Hypothesis, I and II. *Proceedings of the National Academy of Sciences USA*, **50**: 1143-1148 and **51**: 105-110.

Gödel, Kurt. (1940) *The Consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis with the Axioms of Set Theory*. Princeton University Press.

Henkin, Leon. (1950) Completeness in the theory of types. *Journal of Symbolic Logic*, **15**: 81-91.

Jané, Ignacio. (1993) A Critical Appraisal of Second-Order Logic. *History and Philosophy of Logic*, **14**: 67-86.

Kanamori, Akihiro. (1996) The Mathematical Development of set Theory from Cantor to Gödel. *The Bulletin of symbolic Logic*, **2**: 1-71.

Mosterín, Jesús. (1987) La frontera entre lógica y teoría de conjuntos. In Acero and Calvo (eds.), *Symposium Quine*, Universidad de Granada Press, pp. 209-227.

Trakhtenbrot, B. A. (1950) The impossibility of an algorithm for the decision problem in finite classes [in Russian]. *Dokl. Akad. Nauk SSSR*, 70: 569-572.

Von Neumann, John. (1928). Über die Definition durch transfinite Induktion und verwandte Fragen der allgemeinen Mengenlehre. *Mathematische Annalen*, **99**: 373-391.