What is an elementary particle in the first-quantized Standard Model?

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Abstract

The purpose of this paper is to elucidate the concept of an elementary particle in the first-quantized Standard Model. The emphasis is upon the mathematical structures involved, rather than numerical computations. After the general concepts and philosophical outlook are introduced in the opening section, Section 2 addresses the question of what a *free* elementary particle is, paying particular attention to the relationships between the configuration representation and momentum representation. Section 3 deals with gauge fields, Section 4 deals with interactions between particles and gauge fields, Section 5 deals with composite systems, and Section 6 deals with the representation of Baryons, Mesons, and Hadrons. Section 7 addresses the interpretational question of whether an elementary particle has only one intrinsic state, and Section 8 attempts to elucidate what an *interacting* elementary particle is in the Standard Model.[1]

1 Introduction

In the Standard Model of elementary particle physics, the two basic types of thing which are represented to exist are matter fields and gauge force fields. A gauge force field mediates the interactions between the matter fields.

This paper will deal primarily with first-quantized, or 'semi-classical' particle physics, in which it is possible to represent interacting fields in a tractable mathematical manner. The first-quantized approach is empirically adequate in the sense that it enables one to accurately represent many of the structural features of the physical world. Second-quantization, quantum field theory proper, is required to generate quantitatively accurate predictions, but quantum field theory proper is incapable of directly representing interacting fields.

In the first-quantized theory, a matter field can be represented by a cross-section of a vector bundle, and a gauge force field can be represented by a connection upon a principal fibre bundle.

[1]For a comprehensive account of all these subjects, see McCabe, [1].

The first-quantized theory is rather curious in that the matter particles have undergone the first step of quantization, apparently turning from point-like objects into field-like objects, whilst, at first sight, the gauge fields have undergone no quantization at all. On both counts, this appearance may be deceptive. One of the outputs from the first quantized theory is a state space for each type of elementary particle, which becomes the so-called 'one-particle subspace' of the second-quantized theory. The vector bundle cross-sections which represent a matter particle in the first-quantized theory, are vectors from the one-particle subspace of the second-quantized theory. The connections which represent a gauge field can be shown, under a type of symmetry breaking called a 'choice of gauge', to correspond to cross-sections of a direct sum of vector bundles. The cross-sections of the individual direct summands are vectors from the one-particle subspaces of particles called 'interaction carriers', or 'gauge bosons'. Hence, neither the matter fields nor the gauge fields of the first-quantized theory can be treated as classical fields. Given these complexities, the terms 'particle' and 'field' will be used interchangeably throughout the text, without the intention of conveying any interpretational connotations.

A particle is an elementary particle in a theory if it is not represented to be composed of other particles. All particles, including elementary particles, are divided into fermions and bosons according to the value they possess of a property called 'intrinsic spin'. If a particle possesses a non-integral value of intrinsic spin, it is referred to as a fermion, whilst if it possesses an integral value, it is referred to as a boson. The elementary matter fields are fermions and the interaction carriers of the gauge force fields are bosons. The elementary fermions represented in the Standard Model number six leptons and six quarks. The six leptons consist of the electron and electron-neutrino (e, ν_e), the muon and muon-neutrino (μ, ν_μ), and the tauon and tauon-neutrino (τ, ν_τ). The six quarks consist of the up-quark and down-quark (u, d), the charm-quark and strange-quark (c, s), and the top-quark and bottom-quark (t, b). The six leptons have six anti-leptons, $(e^+, \overline{\nu}_e)$, $(\mu^+, \overline{\nu}_\mu)$, $(\tau^+, \overline{\nu}_\tau)$, and the six quarks have six anti-quarks $(\overline{u}, \overline{d})$, $(\overline{c}, \overline{s})$, $(\overline{t}, \overline{b})$. These fermions are partitioned into three generations. The first generation, (e, ν_e, u, d), and its anti-particles, is responsible for most of the macroscopic phenomena we observe. Triples of up and down quarks bind together with the strong force to form protons and neutrons. Residual strong forces between these hadrons bind them together to form atomic nuclei. The electromagnetic forces between nuclei and electrons leads to the formation of atoms and molecules. (Manin, [2], p3).

The general interpretational doctrine adopted in this paper can be referred to as 'structuralism', in the sense advocated by Patrick Suppes [3], Joseph Sneed [4], Frederick Suppe [5], and others. This doctrine asserts that, in mathematical physics at least, the physical domain of a theory is conceived to be an instance of a mathematical structure or collection of mathematical structures. The natural extension of this principle proposes that an entire physical universe is an instance of a mathematical structure or collection of mathematical structures. In particular, each type of particle is considered to be an instance of some species of mathematical structure. Whilst the definition of structuralism is most often expressed in terms of the set-theoretical, Bourbaki notion of a species of mathematical structure, one could reformulate the definition in terms of mathematical Category theory. One could assert that our physical universe is an object in a mathematical Category, or a collection of such objects. In particular, one could assert that each type of particle is an object in a mathematical Category.

One frequently finds in the literature the assertion that an elementary particle 'is' an irreducible, unitary representation of the local space-time symmetry group G, (e.g. Sternberg, [6], p149; Streater, [7], p144). As such, this is an expression of structuralism. In this paper, a *free* elementary particle is considered to be an irreducible, unitary representation of the universal cover of the *restricted* Poincare group. To be precise, in first-quantized quantum theory, the *state space* of a free elementary particle is represented to be a Hilbert space equipped with an irreducible, unitary representation of the universal cover of the *restricted* Poincare group. The structure of the state space of a particle does not itself represent the type of thing which the particle is. In first-quantized quantum theory, a particle is the type of thing which is represented to be a cross-section of such-and-such a bundle over space-time, satisfying such-and-such conditions. However, first-quantized quantum theory does not provide the final word on what type of thing a particle is. One cannot answer a question such as, 'What type of thing is an electron?', without a final, definitive theory, but one can answer a question such as, 'What type of thing is an electron represented to be in first-quantized quantum theory?'. The answer is that an electron is represented to be a cross-section of such-and-such a bundle over space-time, satisfying such-and-such conditions. When a final, definitive theory is obtained, one will be able to remove the phrase 'represented to be' from such an answer, and one will be able to state that an electron *is* a such-and-such mathematical object.

2 What is a free elementary particle?

Free matter fields ('free particles') are matter fields which are idealized to be free from interaction with force fields. In particular, this section is concerned with the representation of free elementary particles, i.e. particles which are not represented to be composed of other particles.

To specify the free elementary particles which can exist in a universe. i.e. the free elementary 'particle ontology' of a universe, one specifies the *projective*, infinite-dimensional, irreducible unitary representations of the 'local' symmetry group of space-time.

The large-scale structure of a universe is represented by a pseudo-Riemannian manifold $(\mathcal{M}, g)$. The dimension n of the manifold $\mathcal{M}$, and the signature (p, q) of the metric g, determine the largest possible local symmetry group of the space-time. The automorphism group of a tangent vector space $T_x\mathcal{M}$, equipped with the inner product $\langle \ , \ \rangle = g_x(\ , \ )$, defines the largest possible local symmetry group of such a space-time, the semi-direct product $O(p,q) \circledS \mathbb{R}^{p,q}$. If there is no reason to restrict to a subgroup of this, then one specifies the possible free elementary particles in such a universe by specifying the *projective*, infinite-dimensional, irreducible unitary representations of $O(p,q) \circledS \mathbb{R}^{p,q}$.

In the case of our universe, the dimension $n = 4$, and the signature $(p,q) = (3,1)$, indicating three spatial dimensions and one time dimension. An n-dimensional pseudo-Riemannian manifold such as this, with a signature of $(n-1,1)$, is said to be a Lorentzian manifold. Each tangent vector space of a 4-dimensional Lorentzian manifold is isomorphic to Minkowski space-time, hence the automorphism group of such a tangent vector space is the Poincare group, $O(3,1) \circledS \mathbb{R}^{3,1}$, the largest possible symmetry group of Minkowski space-time. In the case of our universe the actual local space-time symmetry group is a subgroup of the Poincare group, called the *restricted* Poincare group, $SO_0(3,1) \circledS \mathbb{R}^{3,1}$. The *projective*, infinite-dimensional, irreducible unitary representations of the *restricted* Poincare group correspond to the infinite-dimensional, irreducible unitary representations of its universal covering group, $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$. Hence, one specifies the free elementary particle ontology of our universe by specifying the infinite-dimensional, irreducible unitary representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$.

It is assumed, or reasoned, that the free particle ontology of a universe equals the interacting particle ontology. In other words, although a realistic representation of particles involves representing their interaction with force fields, it is assumed, or reasoned, that the set of particle types which exists in a universe can be determined from the free particle ontology.

It is also assumed, or reasoned, that representations of the *local* symmetry group of space-time are an adequate means of determining the free elementary particle ontology. One could reason that elementary particles exist at small length scales, and the strong equivalence principle of General Relativity holds that Minkowski space-time, and its symmetries, are valid on small length scales. i.e. the strong equivalence principle holds that the global symmetry group of Minkowski space-time is the local symmetry group of a general space-time. One can choose a neighbourhood U about any point in a general space-time, which is sufficiently small that the gravitational field within the neighbourhood is uniform to some agreed degree of approximation, (Torretti, [8], p136). Such neighbourhoods provide the domains of 'local Lorentz charts'. A chart in a 4-dimensional manifold provides a diffeomorphic map $\phi : U \rightarrow \mathbb{R}^4$. If $\mathbb{R}^4$ is equipped with the Minkowski metric, a local Lorentz chart provides a map which is almost isometric, to some agreed degree of approximation, (Torretti, [8], p147). One can treat each elementary particle as 'living in' the domain of a local Lorentz chart within a general space-time $(\mathcal{M}, g)$. To simplify the representational task, the fibre bundles used in the Standard Model are usually assumed to be fibre bundles over Minkowski space-time. This is done with the understanding that the base space of such bundles represents the domain of an arbitrary local Lorentz chart, rather than the whole of space-time. Hence, the elementary particles which exist in a general Lorentzian space-time still transform under the global symmetry group of Minkowski space-time, namely the Poincare group, or a subgroup thereof.

A fully realistic representation of each individual elementary particle would begin with a Lorentzian manifold $(\mathcal{M}, g)$ which represents the entire universe, and would then identify a small local Lorentz chart which the particle 'lives in'. The particle would then be represented by the cross-sections and connections of vector bundles over this small local Lorentz chart. In terms of practical physics, this would be an act of representational *largesse*, but in terms of ontological considerations, it is important to bear in mind.

Where the gravitational field is very strong (i.e. where the space-time curvature is very large), it is no longer valid to assume that the gravitational field is uniform on the length scales at which elementary particles exist. Note that because gravity is geometrized in General Relativity, it is consistent to speak of *free* elementary particles in a gravitational field. Where the gravitational field is very strong, it is not valid to assume that free elementary particles transform under the global symmetry group of Minkowski space-time. When the gravitational field is very strong, elementary particles are represented by fibre bundles over general, curved space-times. Again, this is done with the understanding that the base space of such bundles represents a small region of space-time, rather than the whole universe. These considerations weaken the assumption that the representations of the Poincare group are an adequate means of determining the free elementary particle ontology in a universe. However, one might still be able to reason that the identity of elementary particles remains unchanged by a strong gravitational field, hence one can identify the free elementary particle ontology by studying the ontology under less extreme conditions.

The largest possible local symmetry group of our space-time, the Poincare group $O(3,1) \circledS \mathbb{R}^{3,1}$, is the group of diffeomorphic isometries of Minkowski space-time $\mathcal{M}$, a semi-direct product of the Lorentz group $O(3,1)$ with the translation group $\mathbb{R}^{3,1}$. The Lorentz group is the group of linear isometries of Minkowski space-time.

The Poincare group is a disconnected group which possesses four components. One component contains the isometry which reverses the direction of time, another component contains the isometry which performs a spatial reflection (it reverses parity), another component contains the isometry which reverses the direction of time and performs a spatial reflection, whilst the identity component $SO_0(3,1) \circledS \mathbb{R}^{3,1}$ preserves both the direction of time and spatial parity. The identity component $SO_0(3,1) \circledS \mathbb{R}^{3,1}$ is variously referred to as the *restricted* Poincare group, or the 'proper orthochronous' Poincare group, and is often denoted as $\mathscr{P}_+^\uparrow$ in the Physics literature. Similarly, the identity component of the Lorentz group, $SO_0(3,1)$, is variously referred to as the *restricted* Lorentz group, or the 'proper orthochronous' Lorentz group, and is often denoted as $\mathscr{L}_+^\uparrow$ in Physics literature.

The physical evidence indicates that the local symmetry group of our space-time is actually the restricted Poincare group. The spin group of Minkowski space-time, $Spin(3,1) = SL(2,\mathbb{C})$, provides the universal cover of the restricted Lorentz group $SO_0(3,1)$, hence $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ provides the universal cover of the restricted Poincare group. $Spin(3,1) = SL(2,\mathbb{C})$ is actually a double cover of $SO_0(3,1)$, hence $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ is a double cover of the restricted Poincare group.

One can present the infinite-dimensional, irreducible unitary representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ in either the momentum representation (the Wigner representation), or the configuration representation. In the momentum representation, each irreducible unitary representation of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ is provided by a Hilbert space of square-integrable cross-sections of a vector bundle over a mass hyperboloid or light cone in Minkowski (energy-)momentum space, $T^*_x\mathcal{M}$. In the configuration representation, each irreducible unitary representation is constructed from a space of mass-m solutions, of either positive or negative energy, to a linear differential equation over Minkowski space-time $\mathcal{M}$. The Hilbert space of a unitary irreducible representation in the configuration representation is provided by the completion of a space of mass-m, positive or negative energy solutions, which can be Fourier-transformed into square-integrable objects in Minkowski (energy-)momentum space.

The irreducible unitary representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ are parameterized by mass m and spin s. In the Wigner approach, free particles of mass m and spin s correspond to vector bundles $E^{\pm}_{m,s}$ over mass hyperboloids/light cones $\mathscr{V}^{\pm}_m$ in Minkowski energy-momentum space $T^*_x\mathcal{M}$. It is the Hilbert spaces of square-integrable cross-sections of these vector bundles $E^{\pm}_{m,s}$ which provide the irreducible unitary representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$.

The Wigner approach uses the method of induced group representations, applied to semi-direct product groups. Given a semi-direct product $G = H \circledS N$, the method of induced representation can obtain, up to unitary equivalence, all the irreducible, strongly continuous, unitary representations of the group G. Given that the local symmetry group of space-time is a semi-direct product $G = SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, the method of induced representation enables us to obtain, up to unitary equivalence, all the irreducible, strongly continuous, unitary representations of $G = SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$. In fact, the method of induced representation enables us to classify all the irreducible, strongly continuous, unitary representations of $G = SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, and to provide an explicit construction of one case from each unitary equivalence class, (Emch, [9], p503).

For a particle of mass m and spin s, the Wigner construction obtains a vector bundle $E^+_{m,s}$ over $\mathscr{V}^+_m$, and an irreducible unitary representation of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ upon the space $\Gamma_{L^2}(E^+_{m,s})$ of square-integrable cross-sections of $E^+_{m,s}$. This representation is unique up to unitary equivalence. The anti-particle is represented by the conjugate representation on the space $\Gamma_{L^2}(E^-_{m,s})$ of square-integrable cross-sections of the vector bundle $E^-_{m,s}$ over the backward mass hyperboloid/light cone $\mathscr{V}^-_m$. In other words, if the particle is represented by the Hilbert space $\mathscr{H}$, then the anti-particle is represented by the conjugate Hilbert space $\overline{\mathscr{H}}$. The two representations are related by an anti-unitary transformation.

Whilst the Wigner approach deals directly with the irreducible unitary representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, the configuration space approach requires two steps to arrive at such a representation. In the configuration space approach, for each possible spin s, one initially deals with a non-irreducible, mass-independent representation of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ upon an infinite-dimensional space. For spin s, there is a finite-dimensional vector space V_s, such that the mass-independent representation space can be either a set of vector-valued functions $\mathcal{F}(\mathcal{M}, V_s)$, or a space of cross-sections $\Gamma(\eta)$ of a vector bundle η over $\mathcal{M}$ with typical fibre V_s. $\mathcal{F}(\mathcal{M}, V_s)$ is $\Gamma(\eta)$ in the special case where η is the trivial bundle $\mathcal{M} \times V_s$.

One can also define, for each value of spin s, and possibly parity ϵ, a non-irreducible, mass-independent representation of $\overline{SL}(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ upon an infinite-dimensional space, where $\overline{SL}(2,\mathbb{C})$ is a $\mathbb{Z}_2$-extension of $SL(2,\mathbb{C})$. For spin s and parity ϵ, there is a finite-dimensional vector space $V_{s,\epsilon}$, such that the mass-independent representation space can be either a set $\mathcal{F}(\mathcal{M}, V_{s,\epsilon})$, or a set of cross-sections of a vector bundle η over $\mathcal{M}$ with typical fibre $V_{s,\epsilon}$.

These non-irreducible, mass-independent representations do not correspond to single particle species. Each space of vector-valued functions, or each space of vector bundle cross-sections, represents many different particle species. To obtain the mass m, spin-s irreducible unitary representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ in the configuration representation, one introduces linear differential equations, such as the Dirac equation or Klein-Gordon equation, which contain mass as a parameter. These differential equations are imposed upon the vector-valued functions or cross-sections in the non-irreducible, mass-independent, spin-s representation. Each individual particle species corresponds to the functions or cross-sections for a particular value of mass.

Now, the Fourier transform is well-defined for any cross-section or function which is 'rapidly decreasing' at spatial infinity. Distributions defined upon a space of rapidly decreasing cross-sections/functions are referred to as 'tempered' distributions. With a slight abuse of the mathematical convention, I will hereafter refer to rapidly decreasing cross-sections/functions themselves as tempered. Thus, the Fourier transform is well-defined for any tempered cross-section $\psi \in \mathscr{S}(\eta)$, or tempered V_s-valued function, $f(x) \in \mathscr{S}(\mathcal{M}, V_s)$. The mass-$m$, tempered solutions of the linear differential equations correspond, under a Fourier transform, to vector valued functions over $\mathscr{V}_m$, or the cross-sections of vector bundles over $\mathscr{V}_m$. The Fourier transform $\hat{f}(p)$ of a mass-m tempered function $f(x)$ defined on Minkowski space-time $\mathcal{M}$, is a function defined throughout Minkowski energy-momentum space $T^*_x\mathcal{M}$, but with support upon the mass hyperboloid or light cone $\mathscr{V}_m$. The same is true of the Fourier transform $\hat{\psi}(p)$ of a mass-m tempered cross-section $\psi(x)$. The mass-m tempered solutions can be split into positive-energy and negative-energy solutions. Under Fourier transform, the positive-energy mass-m solutions become vector-valued functions/cross-sections on $T^*_x\mathcal{M}$ with support on the 'forward-mass' hyperboloid $\mathscr{V}^+_m$. Under Fourier transform, the negative-energy mass-m solutions become vector-valued functions/cross-sections on $T^*_x\mathcal{M}$ with support on the 'backward-mass' hyperboloid $\mathscr{V}^-_m$.

Given a function $f(x)$ on Minkowski configuration space $\mathcal{M}$, the Fourier transform $\widehat{f}(p)$ on Minkowski energy-momentum space $T^*_x\mathcal{M}$ is defined to be

$$\widehat{f}(p) = \frac{1}{(2\pi)^2}\int e^{i\langle p,x\rangle} f(x) d^4x$$

or

$$\widehat{f}(p_0,\mathbf{p}) = \frac{1}{(2\pi)^2}\int e^{-i(\mathbf{p}\cdot\mathbf{x}-p_0 t)} f(x) d^3\mathbf{x}\, dt$$

Note that the indefinite Minkowski space inner product

$$\langle p,x\rangle = p_0 t - p_1 x_2 - p_2 x_2 - p_3 x_3$$

is being used to define the Fourier transform here. The inverse Fourier transform is defined to be

$$f(x) = \frac{1}{(2\pi)^2}\int e^{-i\langle p,x\rangle} \hat{f}(p) d^4p$$

or

$$f(\mathbf{x},t) = \frac{1}{(2\pi)^2}\int e^{i(\mathbf{p}\cdot\mathbf{x}-p_0 t)} \hat{f}(p) d^4p$$

The mass-m tempered solutions can be split into positive-energy and negative-energy solutions. Under Fourier transform, the positive-energy mass-m solutions become vector-valued functions/cross-sections on $T^*_x\mathcal{M}$ with support on the 'forward-mass' hyperboloid $\mathscr{V}^+_m$. Under Fourier transform, the negative-energy mass-m solutions become vector-valued functions/cross-sections on $T^*_x\mathcal{M}$ with support on the 'backward-mass' hyperboloid $\mathscr{V}^-_m$. For the sake of simplicity, let us concentrate upon the case of V_s-valued functions.

The positive energy solutions $f^+(x)$ for mass m are those which are the inverse Fourier transform of functions with support upon the forward mass-m hyperboloid. From the expression for an inverse Fourier transform,

$$f(\mathbf{x},t) = \frac{1}{(2\pi)^2}\int e^{i(\mathbf{p}\cdot\mathbf{x}-p_0 t)} \hat{f}(p) d^4p$$

and from the fact that $p_0(\mathbf{p}) = \omega(\mathbf{p}) = +(m^2 + \parallel \mathbf{p} \parallel^2)^{1/2}$ on a forward mass hyperboloid, it follows that a positive energy solution can be expressed as

$$\begin{aligned} f^+(\mathbf{x},t) &= \frac{1}{(2\pi)^2}\int_{T^*_x\mathcal{M}} e^{i(\mathbf{p}\cdot\mathbf{x}-p_0 t)} a(p)\, \theta(p_0)\delta(m^2-p^2)\, d^4p \\ &= \frac{1}{(2\pi)^2}\int_{\mathscr{V}^+_m} e^{i(\mathbf{p}\cdot\mathbf{x}-\omega(\mathbf{p})t)} a(p)\, d^3\mathbf{p}/\omega(\mathbf{p}) \end{aligned}$$

where $a(p)$ has support upon the forward mass hyperboloid $\mathscr{V}^+_m$, a subset of measure zero, but $a(p)\,\theta(p_0)\delta(m^2-p^2)$ is a tempered distribution in $\mathscr{S}'(T^*_x\mathcal{M})$, hence its inverse Fourier transform is well-defined.

In the natural measure d^4p upon Minkowski energy-momentum space $T^*_x\mathcal{M}$, each forward mass hyperboloid or light cone $\mathscr{V}^+_m$ is a subset of measure zero. Given that the Fourier transform of a tempered, mass-m, positive-energy solution on $\mathcal{M}$ is a function with support upon a subset $\mathscr{V}^+_m$, the inverse Fourier transform of this function with respect to d^4p would equal zero. The integral of a function over a set of measure zero, equals zero. Hence, one takes the inverse Fourier transform of the tempered distribution $a(p)\ \theta(p_0)\delta(m^2-p^2)$ instead. $\theta(p_0)$ is the Heaviside function, defined to be $+1$ when $p_0>0$, and 0 when $p_0<0$. Needless to say, $\delta(m^2-p^2)$ is the dirac delta function.

The negative energy solutions $f^-(x)$ for mass m are those which are the inverse Fourier transform of functions with support upon the backward mass-m hyperboloid. On a backward mass hyperboloid, the energy component is negative, so $p_0(\mathbf{p}) = -\omega(\mathbf{p}) = -(m^2+\parallel \mathbf{p} \parallel^2)^{1/2}$, hence a negative energy solution can be expressed as

$$\begin{aligned} f^-(\mathbf{x},t) &= \frac{1}{(2\pi)^2}\int_{T^*_x\mathcal{M}} e^{i(\mathbf{p}\cdot\mathbf{x}-p_0t)}c(p)\ \theta(-p_0)\delta(m^2-p^2)\ d^4p \\ &= \frac{1}{(2\pi)^2}\int_{\mathscr{V}^-_m} e^{i(\mathbf{p}\cdot\mathbf{x}+\omega(\mathbf{p})t)}c(p)\ d^3\mathbf{p}/\omega(\mathbf{p}) \end{aligned}$$

where $c(p)$ has support upon the backward mass hyperboloid $\mathscr{V}^-_m$, a subset of measure zero, but $c(p)\ \theta(-p_0)\delta(m^2-p^2)$ is a tempered distribution in $\mathscr{S}'(T^*_x\mathcal{M})$.

Supposing that there is a linear differential equation which contains mass as a parameter, and which can be imposed upon the V_s-valued functions, one can find subspaces $\mathcal{F}^+_m(\mathcal{M},V_s)$ of V_s-valued functions which are composed of mass-m, positive energy solutions to this differential equation. Furthermore, one can find subspaces $\mathscr{S}^+_m(\mathcal{M},V_s)$ of tempered V_s-valued functions which are composed of mass-m, positive energy solutions to this differential equation. For each such space, there is a yet further subspace consisting of functions with square-integrable Fourier transforms on $\mathscr{V}^+_m$. (Note that the square-integrability is defined with respect to a measure on $\mathscr{V}^+_m$, not a measure on Minkowski space-time $\mathcal{M}$). The completion of this topological vector space of V_s-valued functions on $\mathcal{M}$ takes one into the space $\mathscr{S}'(\mathcal{M},V_s)$ of V_s-valued tempered distributions on $\mathcal{M}$.[2] The completion

$$\mathscr{H}^+_{m,s} \subset \mathscr{S}'(\mathcal{M},V_s)$$

is a Hilbert space which provides the configuration representation for a particle of mass m and spin s. This Hilbert sub-space of $\mathscr{S}'(\mathcal{M},V_s)$ is equipped with an irreducible unitary representation of $SL(2,\mathbb{C})Ⓢ\mathbb{R}^{3,1}$, and is unitarily isomorphic to the Hilbert space for a mass m, spin s particle in the Wigner representation.

One can also find subspaces $\mathcal{F}^-_m(\mathcal{M},V_s)$ which are composed of mass-m, *negative* energy solutions to the relevant differential equation, and further subspaces $\mathscr{S}^-_m(\mathcal{M},V_s)$ of tempered, mass-m, negative energy solutions. The further subspace of functions with square-integrable Fourier transforms on $\mathscr{V}^-_m$, once

[2] Private communication with Veeravalli Varadarajan

completed, is unitarily isomorphic to the Hilbert space for a mass m, spin s *anti*-particle in the Wigner representation.

In terms of the Wigner representation, 'first quantization' is the process of obtaining a Hilbert space of cross-sections of a vector bundle over $\mathscr{V}_m^+$. In terms of the configuration representation, first quantization is the two-step process of obtaining a vector bundle/function space over $\mathcal{M}$, and then identifying a space of mass-m solutions. There are two mathematical directions one can go after first quantization:

One can treat the Hilbert space obtained, as the 'one-particle' state space, and one can use this Hilbert space to construct a Fock space. This is the process of 'second quantization'. One defines creation and annihilation operators upon the Fock space, and thence one defines scattering operators. One can use the scattering operators to calculate the transition amplitudes between incoming and outgoing free states of a system involved in a collision process. Calculation of these transition amplitudes requires the so-called 'regularization' and 'renormalization' of perturbation series, but these calculations do enable one to obtain empirically adequate predictions. Nevertheless, a Fock space is a space of states for a free system. In the configuration representation, the space of 1-particle states is a linear vector space precisely because it is a space of solutions to the *linear* differential equation for a free system.

Although one could use either the Wigner representation or the configuration representation, second quantization conventionally uses a Wigner representation for the one-particle Hilbert spaces.

The other mathematical direction one can go, which conventionally uses the configuration representation, is to treat first-quantization as an end in itself. In the fibre bundle approach, a mass m, spin s particle can be represented by the mass-m cross-sections of a spin-s bundle η. This mass-independent bundle η can, following Derdzinski, ([20]), be referred to as a *free-particle bundle*. One can associate a vector bundle δ with a gauge field, which can, again following Derdzinski, be referred to as an *interaction bundle*. One can take the free-particle bundle η, and with interaction bundle δ, one can construct an *interacting particle bundle* α. The mass-m cross-sections of this bundle represent the particle in the presence of the gauge field. This is the route of the first-quantized interacting theory. The first-quantized interacting theory is not empirically adequate, and it is not possible to subject the first-quantized interacting theory to second-quantization because the state space of an interacting system is not a linear vector space; in the configuration representation, the space of states for an interacting 1-particle system consists of vector bundle cross-sections which satisfy a *non-linear* differential equation. There is no Fock space for an interacting system.

E.Nelson stated that "First quantization is a mystery, but second quantization is a functor." In Andrzej Derdzinski's approach to particle physics, the first step to first quantization is a functor as well. Working with the isochronous Lorentz group $O^{\uparrow}(3,1)$ rather than $SO_0(3,1)$, Derdzinski introduces free-particle bundles η, which are 'natural bundles' in the sense that for each point $x \in \mathcal{M}$ in Minkowski space-time, there is a representation $O^{\uparrow}(T_x\mathcal{M}) \rightarrow Aut(\eta_x)$, ([20], p20-21). One therefore has a functor between the fibres of the tangent bundle over space-time, and the fibres of the free-particle bundle. Elementary particles, says Derdzinski, correspond to irreducible natural bundles, in the sense that the representation in each fibre is irreducible. The representations are permitted to be double-valued, hence they become irreducible representations of a double-cover $\overline{SL}(2,\mathbb{C})$ of $O^{\uparrow}(3,1)$. Although the Dirac spinor bundle σ, used in the first step to represent many elementary particles, possesses a reducible, direct sum representation of $SL(2,\mathbb{C})$ in each of its fibres, the corresponding representation of $\overline{SL}(2,\mathbb{C})$ is irreducible.

If first quantization consisted of only this functorial first step, then the functor would dictate that the relevant symmetry group would be the infinite-dimensional group of cross-sections of the automorphism bundle $Aut(\eta)$, the bundle of all automorphisms in each fibre of η. The first step is deceptive, however, and when the second step is taken into consideration, one obtains a Hilbert space $\mathscr{H}_{m,s}$, and a unitary representation of the finite-dimensional group $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ on $\mathscr{H}_{m,s}$. Such a Hilbert space is not invariant under the action of the infinite-dimensional group of cross-sections of $Aut(\eta)$. The original functor is not relevant here.

Derdzinski defines a mass-independent configuration space vector bundle for each possible combination of spin and parity. The space of sections of such a configuration space vector bundle represents many different particle species, each corresponding to the cross-sections for a specific mass. In addition, the space of cross-sections includes both the positive energy and negative energy cross-sections for each mass value. Hence, the space of cross-sections of a configuration space bundle η includes the cross-sections for both the particles and anti-particles of each mass value.

Derdzinski, ([20], p17-20), derives the following vector bundles to represent free particles in the first step of the configuration space approach:

Spin	Parity	Charged bundle	Neutral bundle
0	+1	$\mathcal{M} \times \mathbb{C}$	$\mathcal{M} \times \mathbb{R}$
0	-1	$\Lambda^4 T^*\mathcal{M} \otimes \mathbb{C}$	$\Lambda^4 T^*\mathcal{M}$
$\frac{1}{2}$	+1	σ_L	-
$\frac{1}{2}$	-1	σ_R	-
$\frac{1}{2}$	-	σ	-
$s \geq 1, s = k \in \mathbb{Z}$	$(-1)^k$	$S^k_0 T^*\mathcal{M} \otimes \mathbb{C}$	$S^k_0 T^*\mathcal{M}$
$s \geq 1, s = k \in \mathbb{Z}$	$(-1)^{k+1}$	$S^k_0 T^*\mathcal{M} \otimes \Lambda^4 T^*\mathcal{M} \otimes \mathbb{C}$	$S^k_0 T^*\mathcal{M} \otimes \Lambda^4 T^*\mathcal{M}$
$s = k + \frac{1}{2}, k \geq 1, k \in \mathbb{Z}$	-	$\eta \subset S^k_0 T^*\mathcal{M} \otimes \sigma$	-

The cross-sections of $S^k_0 T^*\mathcal{M}$ consist of those cross-sections s of the k-fold symmetric tensor product of $T^*\mathcal{M}$ which satisfy the equation $\Delta s = 0$. The Laplacian Δ is defined with respect the pseudo-Riemannian metric g upon the manifold $\mathcal{M}$. $S^k_0 T^*_x\mathcal{M}$ is the space of pseudo-spherical harmonics in $T_x\mathcal{M}$.

In the case of a particle with $s = k + \frac{1}{2}$, for $k \geq 1$ and $k \in \mathbb{Z}$, the expression

$$\eta \subset S^k_0 T^*\mathcal{M} \otimes \sigma$$

denotes the subset of $S^k_0 T^*\mathcal{M} \otimes \sigma$ consisting of elements in the kernel of Clifford multiplication. In this context, Clifford multiplication c is a map

$$c : (\bigotimes^k T^*\mathcal{M}) \otimes \sigma \rightarrow (\bigotimes^{k-1} T^*\mathcal{M}) \otimes \sigma$$

To illustrate the configuration space approach, let us outline its application to the electron and the neutrino.

First note that the complex, finite-dimensional, irreducible representations of $SL(2,\mathbb{C})$ are indexed by the set of all ordered pairs (s_1, s_2), (Bleecker, [24], p77), with

$$(s_1, s_2) \in \frac{1}{2}\mathbb{Z}_+ \times \frac{1}{2}\mathbb{Z}_+$$

In other words, the irreducible representations of $SL(2,\mathbb{C})$ form a family $\mathscr{D}^{s_1,s_2}$, where s_1 and s_2 run independently over the set $\{0, 1/2, 1, 3/2, 2, ...\}$. The number $s_1 + s_2$ is called the spin of the representation. The standard representation of $SL(2,\mathbb{C})$ on $\mathbb{C}^2$ is the $\mathscr{D}^{1/2,0}$ representation, and the inequivalent conjugate representation is the $\mathscr{D}^{0,1/2}$ representation.

$\mathscr{D}^{0,1/2}A = A^{*-1}$ is a conjugate representation to $\mathscr{D}^{1/2,0}A = A$ in the sense that

$$\mathscr{L}A\mathscr{L}^{-1} = A^{*-1}, \quad A \in SL(2,\mathbb{C})$$

where $\mathscr{L}$ covers the parity reversal operation $\mathscr{P}$. However, an element A^{*-1} is only conjugate to an element A within the enlarged, two-component group $\overline{SL}(2,\mathbb{C})$. $\mathscr{L}$ does not belong to $SL(2,\mathbb{C})$, hence A^{*-1} is not conjugate to A within $SL(2,\mathbb{C})$.

From the standard representation of $SL(2,\mathbb{C})$ on $\mathbb{C}^2$, and its conjugate, one can construct all the complex, finite-dimensional, irreducible representations of $SL(2,\mathbb{C})$ as subspaces of tensor product representations. Given the tensor product representation

$$\left(\bigotimes^{2s_1} \mathscr{D}^{1/2,0}\right) \bigotimes \left(\bigotimes^{2s_2} \mathscr{D}^{0,1/2}\right)$$

on the tensor product space $(\mathbb{C}^2)^{\otimes 2s_1} \otimes (\mathbb{C}^2)^{\otimes 2s_2}$, the irreducible representation $\mathscr{D}^{s_1,s_2}$ is the subrepresentation on the symmetric subspace $(\mathbb{C}^2)^{\odot 2s_1} \otimes (\mathbb{C}^2)^{\odot 2s_2}$

Now one can define, for each possible spin s, an infinite-dimensional, mass-independent representation of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ upon a set of vector-valued functions $\mathcal{F}(\mathcal{M}, V_s)$. Letting $f(x)$ denote an element of $\mathcal{F}(\mathcal{M}, V_s)$, the representation is defined as

$$f(x) \to f'(x) = \mathscr{D}^{s_1,s_2}(A) \cdot f(\Lambda^{-1}(x-a))$$

To represent an electron, one starts with either a Dirac spinor bundle σ, with typical fibre $\mathbb{C}^4$, or with a space of $\mathbb{C}^4$-valued functions $\mathcal{F}(\mathcal{M}, \mathbb{C}^4)$. In this case, $\mathbb{C}^4$ can be considered to possess either $(1/2, 0)$, an irreducible representation of $\overline{SL}(2,\mathbb{C})$, or to possess $\tau = \mathscr{D}^{1/2,0} \oplus \mathscr{D}^{0,1/2}$, a reducible direct sum representation of $SL(2,\mathbb{C})$.

To represent a neutrino, one starts with either a left-handed Weyl spinor bundle σ_L, with typical fibre $\mathbb{C}^2$, or with a space of $\mathbb{C}^2$-valued functions $\mathcal{F}(\mathcal{M}, \mathbb{C}^2)$. To represent an anti-neutrino, one starts with either a right-handed Weyl spinor bundle σ_R, or, again, with a space of $\mathbb{C}^2$-valued functions $\mathcal{F}(\mathcal{M}, \mathbb{C}^2)$. In the case of the neutrino, $\mathbb{C}^2$ is considered to possess $\mathscr{D}^{1/2,0}$, the standard irreducible representation of $SL(2,\mathbb{C})$, while in the case of the anti-neutrino, $\mathbb{C}^2$ is considered to possess $\mathscr{D}^{0,1/2}$, the conjugate irreducible representation.

Note that both the Dirac spinor bundle σ and the Weyl spinor bundle are spin 1/2 vector bundles.

One cannot treat the electron and the neutrino so that they both correspond to finite-dimensional irreducible representations of the same group G. If one chooses $G = SL(2,\mathbb{C})$, then the neutrino corresponds to an irreducible representation, but the electron corresponds to a reducible representation. If one chooses $G = \overline{SL}(2,\mathbb{C})$, then the electron corresponds to an irreducible representation, but the neutrino does not correspond to any representation at all. Instead, it is the direct sum of the neutrino and anti-neutrino fibre spaces which provides an irreducible representation.

However, there are two points to bear in mind here: Firstly, the finite-dimensional representations of $SL(2,\mathbb{C})$ or $\overline{SL}(2,\mathbb{C})$ do not induce irreducible representations upon the corresponding infinite-dimensional function spaces or cross-section spaces, and this is true irrespective of whether the finite-dimensional representation is itself irreducible, or a direct sum, reducible representation. The first step in the configuration space approach obtains a non-irreducible, infinite-dimensional representation of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$. Secondly, it is only when the second step has been completed, when one has constructed a Hilbert space from a subspace of solutions to a differential equation, that one obtains the irreducible, infinite-dimensional representations for the electron and the neutrino, and it is at this stage that the electron and neutrino can be seen as irreducible, unitary representations of the same group, $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$.

In the case of the electron, one must distinguish the finite-dimensional representation of $SL(2,\mathbb{C})$ on $\mathbb{C}^4$ from the infinite-dimensional representation of $SL(2,\mathbb{C})$ on $\mathcal{F}(\mathcal{M}, \mathbb{C}^4)$ or $\Gamma(\sigma)$. Both representations are non-irreducible.

From the fact that the representation of $SL(2,\mathbb{C})$ on $\mathcal{F}(\mathcal{M}, \mathbb{C}^4)$ or $\Gamma(\sigma)$ is non-irreducible, it follows that the representation of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ on

$\mathcal{F}(\mathcal{M}, \mathbb{C}^4)$ or $\Gamma(\sigma)$, is also non-irreducible. In other words, the representation of the double cover of the restricted Poincare group on $\mathcal{F}(\mathcal{M}, \mathbb{C}^4)$ or $\Gamma(\sigma)$ is non-irreducible. However, one can identify subspaces which are invariant under the action of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ for each value of mass m, and for either positive energy or negative energy. One can form the direct integral of these subspaces, either

$$\left(\int_0^\infty \bigoplus_m \mathcal{F}_m^+(\mathcal{M}, \mathbb{C}^4)\right) \bigoplus \left(\int_0^\infty \bigoplus_m \mathcal{F}_m^-(\mathcal{M}, \mathbb{C}^4)\right)$$

or

$$\left(\int_0^\infty \bigoplus_m \Gamma_m^+(\sigma)\right) \bigoplus \left(\int_0^\infty \bigoplus_m \Gamma_m^-(\sigma)\right)$$

There is one invariant, positive energy direct summand for each value of mass $m \in (0, \infty)$, and one invariant, negative energy direct summand for each value of mass $m \in (0, \infty)$. Each set of $\psi \in \Gamma_m^+(\sigma)$ and $f(x) \in \mathcal{F}_m^+(\mathcal{M}, \mathbb{C}^4)$ consists of positive energy, mass-m solutions of the Dirac equation. However, neither $\Gamma_m^+(\sigma)$ nor $\mathcal{F}_m^+(\mathcal{M}, \mathbb{C}^4)$ provides the configuration space analogue of the irreducible unitary Wigner representation $\Gamma_{L^2}(E^+_{m,1/2})$. Instead, one must take the invariant subspace of positive energy, mass-m tempered cross-sections $\mathscr{S}_m^+(\sigma)$, or the invariant subspace of positive energy, mass-m tempered functions $\mathscr{S}_m^+(\mathcal{M}, V_s)$. From here, one takes the subspace of such functions which Fourier transform into square-integrable cross-sections/functions on the forward mass hyperboloid $\mathscr{V}_m^+$. One then completes this topological vector space to obtain a Hilbert space $\mathscr{H}^+_{m,s} \subset \mathscr{S}'(\mathcal{M}, V_s)$ which does possess a unitary irreducible representation of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$.

Whilst the Fourier transforms of the elements in $\mathscr{S}_m^+(\mathcal{M}, \mathbb{C}^4)$ are concentrated on the forward mass hyperboloid $\mathscr{V}_m^+$, the elements of $\mathscr{S}_m^+(\mathcal{M}, \mathbb{C}^4)$, are defined throughout Minkowski space-time. Positive-energy, mass-m solutions of the Dirac equation are defined throughout Minkowski space-time. Whilst each space of cross-sections $\Gamma(E^+_{m,1/2})$ in the Wigner representation is a space of cross-sections over a different base $\mathscr{V}_m^+$, each space of functions $\mathscr{S}_m^+(\mathcal{M}, \mathbb{C}^4)$ or cross-sections $\mathscr{S}_m^+(\sigma)$ in the configuration representation, is over the same base space $\mathcal{M}$. Hence, one can imagine $\bigoplus_m \mathscr{S}_m^+(\mathcal{M}, \mathbb{C}^4)$ as a stack of function spaces over $\mathcal{M}$, each function space $\mathscr{S}_m^+(\mathcal{M}, \mathbb{C}^4)$ in the stack containing the functions, defined throughout $\mathcal{M}$, which are tempered positive-energy solutions of the Dirac equation for mass m. Similarly, one can imagine $\bigoplus_m \mathscr{S}_m^+(\sigma)$ as a stack of cross-section spaces over $\mathcal{M}$.

An electron is represented by the Hilbert space $\mathscr{H}^+_{m_e,1/2} \subset \mathscr{S}'(\mathcal{M}, \mathbb{C}^4)$ constructed from the space of mass m_e, positive-energy tempered solutions $\mathscr{S}^+_{m_e}(\mathcal{M}, \mathbb{C}^4)$ of the Dirac equation (or the cross-sectional analogue). Similarly, a neutrino is actually represented by a Hilbert space $\mathscr{H}^+_{0,1/2} \subset \mathscr{S}'(\mathcal{M}, \mathbb{C}^2)$ constructed from the space of mass 0, positive-energy tempered solutions $\mathscr{S}_0^+(\mathcal{M}, \mathbb{C}^2)$ of the Weyl equation (or the cross-sectional analogue). Both of these spaces provide a unitary, irreducible, infinite-dimensional representation of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$. Hence, the electron and neutrino *can* be treated as irreducible representations of the same group in the configuration space approach.

3 Gauge fields

In the Standard Model, each gauge force field corresponds to a compact connected Lie Group G, called the gauge group. A gauge field with gauge group G can either be represented by a connection on a principal fibre bundle P with structure group G, or by a connection on a vector bundle δ equipped with a so-called 'G-structure'.

Given a complex vector bundle δ of fibre dimension n, any matrix sub-group $G \subset GL(n,\mathbb{C})$ acts freely, from the right, upon the set of bases in each fibre. Treating a basis as a row vector, $(e_1,...,e_n)$, it is mapped by $g \in G$ to another basis $(e'_1,...,e'_n)$ by matrix multiplication:

$$(e'_1,...,e'_n) = (e_1,...,e_n)g = (g_{j1}e_j,...,g_{jn}e_j)$$

Needless to say, one sums over repeated indices in this expression.

In general, the G-action upon the set of bases in each fibre will possess multiple orbits. The selection of one particular orbit of this G-action, in each fibre of δ, is called a G-structure in δ, (Derdzinski, [20], p81-82). Because G acts freely, it acts simply transitively within each orbit. A vector bundle δ equipped with a G-structure is sometimes referred to as a 'G-bundle'.

Geometrical objects in each fibre of a vector bundle, such as inner products and volume forms, can be used to select a G-structure. For example, if δ is a complex vector bundle of fibre dimension n, then the unitary group $U(n)$ acts freely upon the set of bases in each fibre. There are multiple orbits of the $U(n)$-action in each fibre, but if each fibre is equipped with a positive-definite Hermitian inner product, then the inner product singles out the orbit consisting of orthonormal bases. By stipulating that an inner product selects the orbit of orthonormal bases, one defines a bijection between inner products and $U(n)$-structures. For any orbit of the $U(n)$-action, there is an inner product with respect to which that orbit consists of orthonormal bases. Given any basis $(e_1,...,e_n)$, one can define an inner product which renders that basis an orthonormal basis by stipulating that the matrix of inner products between the vectors in the basis has the form *diag* $\{1,1,...,1\}$. Given any pair of vectors v, w, they can be expressed as $v = c_1e_1 + \cdots + c_ne_n$ and $w = a_1e_1 + \cdots + a_ne_n$ in this basis, and their inner product is now defined to be

$$\langle v, w\rangle = \langle c_1e_1 + \cdots + c_ne_n, a_1e_1 + \cdots + a_ne_n\rangle = c_1a_1 + \cdots + c_na_n$$

Once an inner product has been defined which renders $(e_1,...,e_n)$ orthonormal, all the other bases which can be obtained from $(e_1,...,e_n)$ under the action of $U(n)$ must themselves be orthonormal.

This bijection between inner products and $U(n)$-structures is, however, merely conventional.[3] Given the specification of an inner product, the convention is that a basis belongs to the $U(n)$-structure if the matrix of inner products between its constituent vectors has the form $diag\ \{1, 1, ..., 1\}$. Given the specification of an inner product, one could alternatively fix an arbitrary positive-definite Hermitian matrix, and stipulate that a basis belongs to the $U(n)$-structure if the matrix of inner products between its constituent vectors equals the chosen positive-definite Hermitian matrix. This would provide an alternative bijection between inner products and $U(n)$-structures.

Following Derdzinski, we shall refer to a complex vector bundle δ equipped with a G-structure as an *interaction bundle*.

The selection of a G-structure in a vector bundle δ is equivalent to the selection of a principal G-subbundle of the general linear frame bundle of δ. Given the selection of a matrix sub-group $G \subset GL(n, \mathbb{C})$, there are multiple principal G-subbundles of the general linear frame bundle. These correspond to the different orbits of the G-action upon the set of bases in each fibre of δ. For example, a vector bundle δ equipped with a $U(n)$-structure is equivalent to the selection of a principal $U(n)$-subbundle Q of the general linear frame bundle of δ. The $U(n)$-structure in δ is determined by a positive-definite Hermitian inner product in each fibre of δ. The corresponding principal bundle Q consists of the set of orthonormal bases in each fibre of δ.

All the principal G-subbundles of a general linear frame bundle are mutually isomorphic; they share the same collection of transition functions as the general linear frame bundle, and the same typical fibre, namely G. However, whilst all the principal G-subbundles of a general linear frame bundle are mutually isomorphic, the G-structure equipped vector bundles they correspond to are not mutually isomorphic. For example, although all the principal $U(n)$-subbundles of the general linear frame bundle of a complex vector bundle δ, are mutually isomorphic, the Hermitian vector bundles they correspond to are not isomorphic as Hermitian vector bundles. Whilst they all share the same vector bundle structure, namely δ, the inner product spaces of their respective fibres are not unitarily isomorphic. By design, the different principal G-subbundles of a general linear frame bundle are obtained by assigning different inner products and/or volume forms to the fibres of the complex vector bundle δ.

For any principal G-subbundle P of the principal $GL(n, \mathbb{C})$-bundle of all bases in δ, the vector bundle δ is isomorphic to the associated bundle $P \times_{\rho(G)} \mathbb{C}^n$, where $\rho(G)$ is the standard representation of G on $\mathbb{C}^n$. Let $[((e_1, ..., e_n), (c_1, ..., c_n))]$ denote an element from a fibre of the associated bundle $P \times_{\rho(G)} \mathbb{C}^n$. Given that $[((e_1, ..., e_n), (c_1, ..., c_n))] \equiv [((e_1, ..., e_n)g, g^{-1}(c_1, ..., c_n))]$ for all $g \in G$, there is a well-defined mapping from an element such as $[((e_1, ..., e_n), (c_1, ..., c_n))]$ to an element $c_1e_1 + \cdots + c_ne_n$ in the corresponding fibre of δ.

A vector bundle of fibre dimension n, which is expressed as an associated vector bundle $P \times_{\rho(G)} \mathbb{C}^n$ via the standard representation of $G \subset GL(n, \mathbb{C})$

[3] Private communication with Andrzej Derdzinski

on $\mathbb{C}^n$, is notable because it already has a distinguished G-structure, provided by the principal G-bundle to which it is associated. Hence, although the associated bundles $P \times_{\rho(G)} \mathbb{C}^n$ derived from the general linear frame bundle of δ are mutually isomorphic as vector bundles, in general they possess different G-structures. Given a complex vector bundle δ, one has the $GL(n, \mathbb{C})$-bundle of all linear frames F, and given the choice of a positive-definite Hermitian inner product in each fibre of δ, one has the $U(n)$-bundle of orthonormal frames Q. As vector bundles, these objects are isomorphic:

$$\delta \cong (F \times_{GL(n,\mathbb{C})} \mathbb{C}^n) \cong (Q \times_{U(n)} \mathbb{C}^n),$$

As G-bundles, however, these objects are distinct. Whilst $F \times_{GL(n,\mathbb{C})} \mathbb{C}^n$ has a natural $GL(n, \mathbb{C})$-structure, $Q \times_{U(n)} \mathbb{C}^n$ has a natural $U(n)$-structure.

In contrast with the case of an associated bundle, a vector bundle expressed as, say, a product bundle $\mathcal{M} \times V$, does not have a distinguished G-structure.

3.1 Classification of Principal G-Bundles

Whilst each gauge field corresponds to a particular compact and connected Lie group G, the choice of a particular G does not uniquely determine a principal fibre bundle P with structure group G, or a vector bundle δ with a G-structure. In other words, the choice of a gauge group does not uniquely determine the mathematical object upon which the representation of a gauge field is dependent.

In the case of a 4-dimensional manifold $\mathcal{M}$, it is possible, for any Lie group G, to classify all the principal G-bundles over $\mathcal{M}$. Given that space-time is represented by a 4-dimensional manifold, it is therefore possible to classify all the principal G-bundles over any space-time $\mathcal{M}$. Given that the specification of a principal G-bundle is equivalent to the specification of a vector bundle equipped with a G-structure, the classification of principal G-bundles over a space-time $\mathcal{M}$ is equivalent to a classification of all the possible interaction bundles over that space-time $\mathcal{M}$.

Suppose that G is a simply connected Lie group. In this case, the principal G-bundles over a four-dimensional manifold $\mathcal{M}$ are classified by the elements of the fourth cohomology group over the integers $H^4(\mathcal{M}; \mathbb{Z})$ of the manifold $\mathcal{M}$. In the event that $\mathcal{M}$ is compact and orientable, $H^4(\mathcal{M}; \mathbb{Z}) = \mathbb{Z}$, hence the principal G-bundles, for a simply connected Lie group G over a compact and orientable 4-manifold, are in one-to-one correspondence with the integers. In the event that $\mathcal{M}$ is either non-compact or non-orientable, $H^4(\mathcal{M}; \mathbb{Z}) = \{Id\}$. This means that for a simply connected Lie group G, all the principal G-bundles over a non-compact or non-orientable 4-manifold are trivial bundles, isomorphic to $\mathcal{M} \times G$.

In the special case where the simply connected Lie group G is a special unitary group $SU(n)$, the element of $H^4(\mathcal{M}; \mathbb{Z})$ which corresponds to a particular principal $SU(n)$-bundle, is the second Chern class of that bundle. For different principal $SU(n)$-bundles, the second Chern class of the bundle corresponds to different cohomology equivalence classes of the base manifold $\mathcal{M}$. The case of a special unitary group is of relevance to the Standard Model, where $SU(2)$ is involved with the electroweak force, and $SU(3)$ is the gauge group of the strong force.

Turning to non-simply connected Lie groups, take the case where G is a unitary group $U(n)$. In the case that $G = U(1)$, the set of inequivalent principal $U(1)$-bundles over any 4-manifold $\mathcal{M}$ is in one-to-one correspondence with the elements of the second cohomology group over the integers $H^2(\mathcal{M};\mathbb{Z})$. The element of $H^2(\mathcal{M};\mathbb{Z})$ which corresponds to a particular principal $U(1)$-bundle is the first Chern class of that bundle. This case is relevant to the electromagnetic force, which has gauge group $U(1)$.

In the case of $U(n)$, for $n > 1$, the set of inequivalent principal $U(n)$-bundles over any 4-manifold $\mathcal{M}$ is in one-to-one correspondence with the elements of $H^2(\mathcal{M};\mathbb{Z}) \oplus H^4(\mathcal{M};\mathbb{Z})$. The case of relevance to the Standard Model is $G = U(2)$, the gauge group of the electroweak force.

These results demonstrate that the choice of principal fibre bundle or interaction bundle is not determined by the gauge group. In the case of the electromagetic force, there are many principal $U(1)$-bundles $\{P_i : i \in H^2(\mathcal{M};\mathbb{Z})\}$ over a space-time $\mathcal{M}$, and for each different bundle P_i, the standard representation of $U(1)$ on $\mathbb{C}^1$ defines a different interaction bundle $\lambda_i = P_i \times_{U(1)} \mathbb{C}^1$ equipped with a $U(1)$-structure. Similarly, in the case of the electroweak force, there are many principal $U(2)$-bundles $\{Q_i : i \in H^2(\mathcal{M};\mathbb{Z}) \oplus H^4(\mathcal{M};\mathbb{Z})\}$ over a space-time $\mathcal{M}$, and for each different bundle Q_i, the standard representation of $U(2)$ on $\mathbb{C}^2$ defines a different interaction bundle $\iota_i = Q_i \times_{U(2)} \mathbb{C}^2$ equipped with a $U(2)$-structure.

In the case of the strong force, with simply connected gauge group $SU(3)$, a compact 4-manifold cannot possess a Lorentzian metric, hence if we accept that space-time is Lorentzian, and therefore non-compact, it follows that, up to isomorphism, the only principal $SU(3)$-bundle is $S = \mathcal{M} \times SU(3)$. There is therefore, up to isomorphism, only one vector bundle over space-time which can be equipped with an $SU(3)$-structure, namely $\rho = S \times_{SU(3)} \mathbb{C}^3 \cong \mathcal{M} \times \mathbb{C}^3$. One can select many different $SU(3)$-structures in ρ by assigning different inner products and volume forms to the fibres of ρ, and each such $SU(3)$-structure corresponds to a different principal $SU(3)$-subbundle of the general linear frame bundle of ρ, but each such principal $SU(3)$-bundle is isomorphic to $\mathcal{M} \times SU(3)$. Whilst it is not true to say that all $SU(3)$-interaction bundles are mutually isomorphic as oriented Hermitian vector bundles, they are all mutually isomorphic as vector bundles.

Because Minkowski space-time is contractible, all its cohomology groups are trivial. This entails that in the special case of the Standard Model over Minkowski space-time, all the interaction bundles are trivial.

3.2 Gauge connections

A gauge force field potential can be represented by a connection ∇ on a principal fibre bundle $(P, \pi, \mathcal{M}, G)$, where G is an m-dimensional Lie Group, π is the projection mapping $\pi : P \to \mathcal{M}$, and $\mathcal{M}$ is n-dimensional space-time. A connection on a bundle can be defined in terms of the assignment of a 'horizontal' subspace to the tangent vector space at each point of the bundle, (Torretti, [8], p269-271). At each point $p \in P$ in the total space of a principal fibre bundle, the vectors in the tangent vector space T_pP which are tangent to the fibre of P over $\pi(p)$, constitute an m-dimensional subspace called the space of vertical vectors V_p. There are an infinite number of possible n-dimensional subspaces of T_pP which, together with V_p, span the tangent vector space. Such an n-dimensional subspace is called a horizontal subspace, H_p, and is such that $T_pP = V_p \oplus H_p$. A connection on the principal fibre bundle P smoothly assigns a horizontal subspace to each point, $p \mapsto H_p$ in a manner which respects the right action R_g of each $g \in G$ on P. Thus, a connection satisfies the condition:

$$H_{pg} = R_{g*}(H_p)$$

Each connection ∇ corresponds to a Lie-algebra valued one-form ω on P. For each point $p \in P$, there is a natural isomorphism of V_p onto the Lie algebra $\mathfrak{g}$ of G, and the selection of a horizontal subspace H_p enables one to extend this mapping to the entire tangent vector space T_pP using the stipulation that ker $\omega_p = H_p$. At each point $p \in P$, a connection one-form is a mapping $\omega_p : T_pP \to \mathfrak{g}$. Such a Lie-algebra valued one-form satisfies the condition:

$$\omega_{pg}(R_{g*p}v) = ad(g^{-1})\omega_p(v)$$

Although each connection ∇ corresponds to a Lie-algebra valued one-form $\omega \in \Lambda^1(P, \mathfrak{g})$, not every Lie-algebra valued one-form on P corresponds to a connection on P. An element of $\Lambda^1(P, \mathfrak{g})$ must satisfy the condition above so that it respects the right action R_g for each $g \in G$, and must agree on V_p with the natural isomorphism to $\mathfrak{g}$.

A connection on a principal fibre bundle P enables one to define parallel transport between the fibres of P, to define parallel cross-sections of P, and to define a covariant derivative upon the cross-sections of P.

3.3 Choice of Gauge and Gauge Transformations

There is a correspondence between cross-sections and trivializations of a principal fibre bundle. A cross-section $\sigma : U \to P$ picks out an element $\sigma(x)$ from the fibre P_x over each point $x \in U \subset \mathcal{M}$ in a principal G-bundle, and thereby establishes an isomorphism between the points in each fibre and the elements of the structure group G. Each $p \in P_x$ is mapped to the unique $g \in G$ which is such that $p = \sigma(x)g$. This defines an isomorphism between $\pi^{-1}(U)$ and $U \times G$.

Picking out an element in each fibre of a principal fibre bundle P is equivalent to picking out a basis for each fibre of the associated vector bundles. Picking out a basis in each fibre of a vector bundle $\delta \cong P \times_{r(G)} V$ establishes an isomorphism between each fibre δ_x and the typical fibre V. If a cross-section $\sigma : U \rightarrow P$ picks out the basis $\sigma(x)$ in each fibre of $\delta \cong P \times_{r(G)} V$ over U, then each such fibre can be expressed as $\delta_x = \{[\sigma(x), v] : v \in V\}$ and there is the natural isomorphism $[\sigma(x), v] \mapsto v \in V$ between δ_x and the typical fibre V.

Hence, a section of a principal fibre bundle determines both a trivialization of the principal fibre bundle, and a trivialization of any associated vector bundle. If a global cross-section exists for the principal bundle, then the principal bundle and its associated vector bundles must be trivial. In the Standard Model over Minkowski space-time, the relevant principal G-bundles and their associated bundles are indeed trivial. However, there is no canonical trivialization; for each global cross-section of P there is a different global trivialization of P. Each different cross-section σ can pick out a different element $\sigma(x)$ from the fibre P_x over a point x, and establish a different isomorphism between the points in the fibre and the elements of the structure group G.

A cross-section σ of the principal G-bundle P of some gauge field is called a 'choice of gauge'. Assuming the cross-section σ is global, it determines a global trivialization ϕ_σ of the associated vector bundle, $\delta \cong P \times_{r(G)} V$, expressed as $\phi_\sigma : \delta \rightarrow \mathcal{M} \times V$. This establishes an isomorphism between each fibre δ_x and $x \times V$, hence the representation r of G on V induces a representation upon each fibre. Expressing each fibre as $\delta_x = \{[\sigma(x), v] : v \in V\}$, with respect to the cross-section $\sigma(x)$, the representation r_x of G on δ_x is given by

$$r_x(g) : [\sigma(x), v] \mapsto [\sigma(x), gv]$$

The representation of G on each fibre of δ induces a representation of G upon the space of sections $\Gamma(\delta)$. Although the representation of G on $\Gamma(\delta)$ changes with a different choice of gauge, it changes to an equivalent representation.

A gauge transformation can be defined in various ways, (Bleecker, [24], p46; Sternberg, [6], p114). For example, a gauge transformation of a principal fibre bundle $(P, \pi, \mathcal{M}, G)$ can be defined to be a diffeomorphism $f : P \rightarrow P$ which satisfies the conditions:

$$f(pg) = f(p)g \quad ; \quad p \in P,\ g \in G \qquad (i)$$

$$\pi(f(p)) = \pi(p) \qquad (ii)$$

Condition (i) means that a gauge transformation is an automorphism of the total space P. The stipulation that $f(pg) = f(p)g$ means that a gauge transformation f is G-equivariant. Condition (ii) means that the fibres of P remain fixed in a gauge transformation. The elements of each fibre are re-arranged, but the fibres themselves are not permuted. A gauge transformation re-arranges the elements in each fibre of the principal G-bundle in a way which preserves the G-relationships between the different elements of the fibre.

3.4 The Interaction bundle picture

In the interaction bundle picture, (Derdzinski, [20], p81-83), there is no need to introduce a principal fibre bundle P to define a gauge connection, a choice of gauge, or a gauge transformation. Instead, one deals with a vector bundle δ equipped with a G-structure, the *interaction bundle*. One introduces a bundle $G(\delta)$ of automorphisms of each fibre of δ, and a bundle $\mathfrak{g}(\delta)$ of endomorphisms of each fibre of δ. A cross-section of $G(\delta)$ specifies an automorphism of each fibre of δ, and a cross-section of $\mathfrak{g}(\delta)$ specifies an endomorphism of each fibre of δ. Given the G-structure in each fibre of δ, typically a Hermitian inner product, perhaps in tandem with a volume form, an automorphism or endomorphism of each fibre δ_x is a mapping which preserves this structure.

Each fibre of $G(\delta)$ is a Lie group, and each fibre of $\mathfrak{g}(\delta)$ is a Lie algebra. $G(\delta)$ is said to be a Lie group bundle, and $\mathfrak{g}(\delta)$ is said to be a Lie algebra bundle. Each fibre of $G(\delta)$ is isomorphic to the matrix Lie group $G \subset GL(n, \mathbb{C})$, and each fibre of $\mathfrak{g}(\delta)$ is isomorphic to the matrix Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n, C)$, but the isomorphisms are not canonical. It is necessary to fix a basis in a fibre δ_x to establish an isomorphism between $G(\delta)_x$ and $G \subset GL(n, \mathbb{C})$. Similarly, it is necessary to fix a basis in a fibre δ_x to establish an isomorphism between $\mathfrak{g}(\delta)_x$ and $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$.

Given an interaction bundle δ equipped with a G-structure, there is a corresponding principal fibre bundle P, with structure group G, whose fibres consist of the bases in each fibre of δ selected by the G-structure. A connection on δ consists of the selection of a horizontal subspace in each fibre of the tangent bundle $T\delta$, just as a connection on P consists of the selection of a horizontal subspace in each fibre of the tangent bundle TP. A connection upon a principal fibre bundle determines a connection upon any associated bundle, hence a connection on P determines a connection on $\delta \cong P \times_G \mathbb{C}^n$. One can define a G-connection on δ to be a connection on δ which is induced by a connection upon the principal fibre bundle P, where P is the principal fibre bundle that corresponds to the G-structure in δ. Alternatively, one can define a G-connection on δ to be a connection which is such that the objects defining the G-structure are parallel with respect to the covariant derivative of the connection.

The space of connections on P, and the space of G-connections on δ are both affine spaces. In the case of the affine space of connections on P, the translation space is $\Lambda^1(P, \mathfrak{g})$, otherwise thought of as the space of cross-sections of $T^*P \otimes (P \times \mathfrak{g})$. In the case of the space of G-connections on δ, there is no analogue of the space of cross-sections of $T^*P \otimes (P \times \mathfrak{g})$. However, the affine space of G-connections on δ does have a translation space, namely the cross-sections of $T^*\mathcal{M} \otimes \mathfrak{g}(\delta)$. If we consider the space of G-connections on δ as the cross-sections of an affine bundle $\mathscr{C}(\delta)$, the translation space bundle of $\mathscr{C}(\delta)$ is $T^*\mathcal{M} \otimes \mathfrak{g}(\delta)$. One can equate the space of G-connections on δ with the space of connections upon the principal fibre bundle P, and these two affine spaces duly have isomorphic translation spaces.

A choice of gauge corresponds to a cross-section σ of the principal fibre bundle P, otherwise thought of as a collection of trivializing sections $\psi_1, .., \psi_n$

of δ which respect the G-structure. e.g. if the G-structure consists of an inner product, then the trivializing sections should be orthonormal. A choice of gauge does two things:

- It selects a base connection ω_0, and thereby (i) renders the space of connections on P canonically isomorphic to its translation space $\Lambda^1(P, \mathfrak{g})$, and (ii) renders the space of G-connections on δ canonically isomorphic to its translation space, the space of cross-sections of $T^*\mathcal{M} \otimes \mathfrak{g}(\delta)$, (Derdzinski, [20], p91).
- It renders $\mathfrak{g}(\delta)$ canonically isomorphic with $\mathcal{M} \times \mathfrak{g}$.

In sum, a choice of gauge renders the space of connections canonically isomorphic with $T^*\mathcal{M} \otimes (\mathcal{M} \times \mathfrak{g})$. In other words, a choice of gauge enables one to treat a connection on P, or a G-connection on δ, as a Lie-algebra valued one-form on the base space $\mathcal{M}$. A choice of gauge corresponds to the selection of a basis in each fibre δ_x, and this establishes an isomorphism between each fibre $\mathfrak{g}(\delta)_x$ and $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$. In other words, a choice of gauge establishes a correspondence between endomorphisms of δ_x and $n \times n$ complex matrices.

To see why a choice of gauge selects a base connection ω_0, think of the choice of gauge σ as selecting a principal sub-bundle of P with structure group $\{Id\}$. i.e. a principal fibre bundle for which each fibre consists of a single element, the basis selected by $\sigma = \psi_1, .., \psi_n$. This principal fibre bundle P_σ has a unique connection. The tangent vector space at each point of the total space has the same dimension as the base space. The vertical subspace is the zero vector, and there is only one choice for the horizontal subspace, the entire tangent vector space. Given that the Lie algebra of $\{Id\}$ is $\{Id\}$, the Lie-algebra valued one-form which specifies this unique connection maps the entire tangent vector space to $\{Id\}$. The vertical subspace (the zero vector) is mapped to the Lie-algebra $\{Id\}$, as required, and the horizontal subspace (the whole of the tangent space) must belong to the kernel, and is therefore mapped to $\{Id\}$, also as required.

One can injectively map the sub-bundle P_σ back into the principal fibre bundle P corresponding to the G-structure in δ. Under the differential map of this injection, the images of the horizontal subspaces on P_σ provide horizontal subspaces $H_{\sigma(x)}$ on P at each point in the codomain of the cross-section σ. Given $H_{\sigma(x)}$ one can then use the right action of G to define horizontal subspaces at all the other points of the fibres of P. One defines

$$H_{\sigma(x)g} = R_{g*}(H_{\sigma(x)})$$

and this determines a connection upon P. This base connection on P then induces a base G-connection on δ.

Given a choice of gauge σ which selects a base connection ω_0 on P, there is a canonical isomorphism between the affine space of connections on P and the translation space $\Lambda^1(P, \mathfrak{g})$. To each connection one-form ω, there is a unique $\tau \in \Lambda^1(P, \mathfrak{g})$ such that

$$\omega = \omega_0 + \tau$$

The space of $\mathfrak{g}$-valued one-forms $\Lambda^1(P, \mathfrak{g})$, with the group structure derived from its vector space structure, acts transitively and simply transitively upon the affine space of connection one-forms. Hence, the selection of a base connection determines a canonical isomorphism between the space of connections and $\Lambda^1(P, \mathfrak{g})$. It remains true that not every element of $\Lambda^1(P, \mathfrak{g})$ is a connection one-form, but every connection one-form is an element of $\Lambda^1(P, \mathfrak{g})$. Given a base connection ω_0, and any $\tau \in \Lambda^1(P, \mathfrak{g})$, the sum $\omega_0 + \tau$ is guaranteed to be a connection one-form. This should not be taken to mean that every element of $\Lambda^1(P, \mathfrak{g})$ is a connection one-form.

In the interaction bundle δ picture, a gauge transformation is a cross-section of $G(\delta)$. Hence, a gauge transformation selects, at each point x, an automorphism α_x of the fibre δ_x. A gauge transformation is a bundle automorphism which respects the G-structure in each fibre.

A cross-section of $G(\delta)$ also acts upon the Lie algebra bundle of endomorphisms $\mathfrak{g}(\delta)$. At each point x, the automorphism α_x acts adjointly, as an inner automorphism upon $\mathfrak{g}(\delta)_x$, mapping an endomorphism T to $\alpha_x T \alpha_x^{-1}$. A gauge transformation changes a cross-section of $\mathfrak{g}(\delta)$, hence it changes a cross-section of $T^*\mathcal{M} \otimes \mathfrak{g}(\delta)$. Gauge transformations therefore act upon the space of gauge connections.

4 Interactions

Recall that in the first-quantized interacting theory, the interaction bundles and free-particle bundles are conventionally bundles over Minkowski configuration space. The principal fibre bundle and interaction bundle of a gauge field are, by convention, bundles over Minkowski configuration space, and a mass-independent vector bundle, associated with the free particles of spin s, is also, by convention, a bundle over Minkowski configuration space.

Recall also that a specific particle species, of mass m and spin s, is represented by a Hilbert space constructed from the positive energy, mass-m, tempered cross-sections of a spin-s vector bundle. The space of positive energy, mass-m tempered cross-sections is a dense subspace of the Hilbert space, so for convenience, we can consider a free particle of mass m and spin s to be represented by the positive energy, mass-m, tempered cross-sections $\mathscr{S}_m^+(\eta)$ of a spin-s free-particle vector bundle η.

A cross-section ϕ representing a free particles of mass m and spin s, must satisfy free field equations, (Derdzinski, [20], p84),

$$\mathscr{P}(x, \phi(x), (\nabla^\eta \phi)(x), (\nabla^{\eta^2} \phi)(x), ...) = 0$$

for some value of mass m. ∇^η here is the Levi-Civita connection on η.

Free gauge fields, represented by G-connections ∇^δ on an interaction bundle δ, must satisfy the free-field Yang-Mills equations, (Derdzinski, [20], p84),

$$div\ R^{\nabla^\delta} = 0$$

R^{∇^δ} is the curvature two-form of the connection ∇^δ.

The G-connections on δ correspond to smooth cross-sections of an affine bundle $\mathscr{C}(\delta)$. The space of G-connections on δ which satisfy the free-field Yang-Mills equations correspond to a subspace of this cross-section space.

An interacting particle of mass m and spin s is represented by a positive energy, mass-m solution of a ∇^δ-dependent differential equation imposed upon the cross-sections of a spin-s interacting-particle bundle α. The connection ∇^δ is a connection upon the interaction bundle δ. The spin-s interacting particle bundle α is a construction from the spin-s free-particle bundle η, and the interaction bundle δ. In the simplest case, if the free-particle bundle is η, then with the interaction represented by δ switched on, the interacting-particle bundle will be the tensor product $\alpha = \eta \otimes \delta$.

Recall that for the Standard Model over an arbitrary space-time, a gauge group G does not, in general, determine a unique interaction bundle δ, hence, in general, a spin-s particle interacting with a group-G gauge field does not have a unique interacting-particle bundle, even if one assumes the simplest type of interacting particle bundle $\eta \otimes \delta$. Instead, one has a family of interaction bundles δ_i, and a consequent family of interacting-particle bundles $\eta \otimes \delta_i$.

However, given that the interaction bundles and spinor bundles over Minkowski space-time are trivial bundles, and given that the interacting-particle bundles are constructed from tensor product and direct sum combinations of the interaction and spinor bundles, it follows that in the special case of the Standard Model over Minkowski space-time, the interacting-particle bundles are trivial bundles.

Given that each interaction bundle over Minkowski space-time is isomorphic to a product bundle, $\delta \cong \mathcal{M} \times \mathbb{C}^n$, each trivialization, (i.e. choice of gauge), establishes an isomorphism between each fibre δ_x and the typical fibre $\mathbb{C}^n$. Hence, each trivialization (choice of gauge) induces a representation ρ of G upon the space of sections $\Gamma(\delta)$, and therefore induces a representation $Id_\eta \otimes \rho$ of the gauge group G upon the space of cross-sections $\Gamma(\alpha)$ of the spin-s interacting particle bundle $\alpha = \eta \otimes \delta$. In this sense, one can say that a spin-s interacting particle, represented by the cross-sections ψ of $\alpha = \eta \otimes \delta$, transforms according to a representation of G. However, the action of G here corresponds only to a global gauge transformation. The more general case of a local gauge transformation corresponds to a cross-section of $G(\delta)$, which selects a choice of gauge and an element of the matrix group G at each point, to provide an automorphism of δ. The (infinite-dimensional) group of all such automorphisms $\mathcal{G} = \Gamma(G(\delta))$ acts upon the space of sections $\Gamma(\delta)$, thence it acts upon the space of sections $\Gamma(\alpha)$. Hence, one can say that a spin-s interacting particle transforms under the action of this infinite-dimensional group.

Note that $\mathcal{G}$ is an infinite-dimensional Lie group, and as an infinite-dimensional manifold it possesses a locally compact topology. Hence $\mathcal{G}$ itself is not a compact group. An interacting particle does not transform under an infinite-dimensional representation of a compact Lie group, rather, it transforms under the action of an infinite-dimensional Lie group upon an infinite-dimensional space.

Whilst a free-particle corresponds to an irreducible representation of the 'external' symmetry group, a particle with a gauge force field switched on transforms under the same external symmetry group, and the infinite-dimensional group of gauge transformations.

Whilst a free particle corresponds to a unitary, irreducible, infinite-dimensional representation of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, a particle interacting with a gauge field of gauge group G does *not* correspond to a unitary, irreducible, infinite-dimensional representation of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1} \times G$. One could find, and classify, all the infinite-dimensional, unitary, irreducible representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1} \times G$, as an extension of the Wigner classification. All the irreducible representations of compact groups are finite-dimensional, so one could set about taking all the tensor products of the unitary, irreducible, infinite-dimensional representation of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ with the unitary, irreducible, *finite*-dimensional representations of G, to obtain all the infinite-dimensional, unitary, irreducible representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1} \times G$.[4] However, these vector space representations do not correspond with the state spaces of interacting particles, which are non-linear.

The 'two-step' approach to specifying a particle, in which one first introduces a vector bundle, and then specifies that the particle corresponds to a special set of cross-sections satisfying a differential equation, is vindicated by the nature of the state space for an interacting particle. If one takes an exclusively Hilbert space approach to free particles, based upon the unitary irreducible representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, then the transition to interacting particles is difficult to understand, given that they are not unitary irreducible representations of anything.

An interacting particle ψ does not transform under a representation of $SL(2,\mathbb{C}) \times G$ or a representation of $(SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}) \times G$. Instead, an interacting particle ψ transforms under a group action of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, and a group action of $\mathcal{G} = \Gamma(G(\delta))$. However, there *is* a representation of $SL(2,\mathbb{C}) \times G$ upon the typical fibre of the interacting particle bundle $\eta \otimes \delta$. The typical fibre of the spin-s free-particle bundle η will carry a spin-s representation of $Spin(3,1) \equiv SL(2,C)$, and the typical fibre of the interaction bundle δ will carry a representation of the gauge group G of the interaction in question, so the typical fibre of the tensor product $\eta \otimes \delta$ must carry a representation of the product group $SL(2,C) \times G$.

Similarly, a gauge field connection pull-down A does not transform under a representation of $SL(2,\mathbb{C}) \times G$ or a representation of $(SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}) \times G$. Instead, a gauge field connection pull-down A transforms under a representation/group action of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, and a representation/group action of $\mathcal{G} = \Gamma(G(\delta))$. However, there *is* a representation of $SL(2,\mathbb{C}) \times G$ upon $\mathbb{R}^4 \otimes \mathfrak{g}$, the typical fibre of the translation space bundle $T^*\mathcal{M} \otimes \mathfrak{g}(\delta)$.

[4] Private communication with Heinrich Saller

A crucial difference between an interacting particle and a gauge field is that the representation of $SL(2,\mathbb{C})\times G$ upon the typical fibre of the interacting particle bundle $\eta\otimes\delta$ uses the standard representation of G, whilst the representation of $SL(2,\mathbb{C})\times G$ upon $\mathbb{R}^4\otimes\mathfrak{g}$, the typical fibre of the translation space bundle $T^*\mathcal{M}\otimes\mathfrak{g}(\delta)$, uses the adjoint representation of G.

Note also that there are two different group actions of $\mathcal{G}$ here. In the case of an interacting particle, there is an action of $\mathcal{G}$ upon $\Gamma(\delta)$, whilst in the case of the gauge field, there is an action of $\mathcal{G}$ upon $\Gamma(\mathfrak{g}(\delta))$.

Given the representation of $SL(2,\mathbb{C})\times G$ upon the typical fibre $\mathbb{R}^4\otimes\mathfrak{g}$ of the gauge field translation space bundle $T^*\mathcal{M}\otimes\mathfrak{g}(\delta)$, the selection of a basis in $\mathfrak{g}$, or the restriction of the representation to $SL(2,\mathbb{C})\times Id$, enables one to decompose this representation as a direct sum

$$\bigoplus^{dim\ \mathfrak{g}}\mathbb{R}^4$$

i.e. one decomposes the representation into a direct sum of $dim\ \mathfrak{g}$ copies of the representation of $SL(2,\mathbb{C})$ on $\mathbb{R}^4$.

Recall that a choice of gauge renders $\mathscr{C}(\delta)$, the affine bundle housing the G-connections on δ, canonically isomorphic with $T^*\mathcal{M}\otimes(\mathcal{M}\times\mathfrak{g})$. The selection of a basis in $\mathfrak{g}$ then enables one to decompose $\mathcal{M}\times\mathfrak{g}$ as the direct sum

$$\bigoplus^{dim\ \mathfrak{g}}(\mathcal{M}\times\mathbb{R}^1),$$

and thereby enables one to decompose $T^*\mathcal{M}\otimes(\mathcal{M}\times\mathfrak{g})$ as the direct sum, (Derdzinski, [20], p91):

$$\bigoplus^{dim\ \mathfrak{g}}T^*\mathcal{M}$$

Given that $S^1_0T^*\mathcal{M}=T^*\mathcal{M}$, this is the configuration space bundle for $dim\ \mathfrak{g}$ 'real vector bosons', neutral particles of spin 1 and parity -1. Recall that a spin-s configuration space bundle possesses, upon its typical fibre, either the $\mathscr{D}^{s_1,s_2}$ complex, finite-dimensional, irreducible representation of $SL(2,\mathbb{C})$, for $s=s_1+s_2$, or a direct sum of such representations. Given that $T^*\mathcal{M}$ is a real vector bundle, it cannot possess upon its typical fibre a member of the $\mathscr{D}^{s_1,s_2}$ family of complex representations, but it does possess the real representation of $SL(2,\mathbb{C})$ which complexifies to the $\mathscr{D}^{1/2,1/2}$ representation. In this sense, $T^*\mathcal{M}$ is a spin-1 configuration space bundle.

The differential equations for a spin 1, parity -1 bundle, (Derdzinski, [20], p19), consist of the Klein-Gordon equation,

$$\Box\psi=m^2\psi$$

and the divergence condition

$$div\ \psi=0$$

Under a choice of gauge, the cross-sections of the affine bundle $\mathscr{C}(\delta) \cong \bigoplus^{dim\ \mathfrak{g}} T^*\mathcal{M}$ which satisfy the free-field Yang-Mills equations, correspond to the mass 0 solutions to these equations. This is easiest to see in the case of electromagnetism, where a choice of gauge selects an isomorphism $\mathscr{C}(\lambda) \cong T^*\mathcal{M}$ which maps a connection ∇ to a real vector potential A. With the Lorentz choice of gauge, the Maxwell equations upon a real vector potential,

$$\Box A = 0, \quad div\ A = 0,$$

clearly correspond to the differential equations for a spin 1, parity -1 of mass 0.

Hence, under a choice of gauge, from the space of $U(1)$ connections satisfying the free-field Maxwell equations, one can construct a space which is the inverse Fourier transform of the space of single photon spates $\Gamma_{L^2}(E^+_{0,1})$ in the Wigner representation.

The 'gauge bosons', or 'interaction carriers' of a gauge field are the spin 1, mass 0, Wigner-representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, which inverse Fourier transform into spaces constructed from tempered, mass 0 cross-sections of the spin 1 bundles $T^*\mathcal{M}$. These spin 1 bundles belong to a decomposition $\bigoplus^{dim\ \mathfrak{g}} T^*\mathcal{M}$ of the affine bundle $\mathscr{C}(\delta)$ housing the G-connections on δ. At least, this is the case under formal symmetry breaking. We will presently see that for gauge fields which undergo spontaneous symmetry breaking, the decomposition changes.

Given that a choice of gauge renders the affine bundle $\mathscr{C}(\delta)$ isomorphic to the translation space bundle $T^*\mathcal{M} \otimes \mathfrak{g}(\delta)$, and given that a choice of Lie algebra basis then enables one to decompose the translation space bundle into separate interaction carrier bundles, one might refer to the translation space bundle as the interaction carrier bundle. In the case of the strong force, with $G = SU(3)$, one has $dim\ SU(3) = 8$, therefore one has 8 strong force interaction carriers, the gluons. In the case of the electroweak force, with $G = U(2)$, one has $dim\ U(2) = 4$, therefore one has 4 interaction carriers: the photon γ, the $W^\pm$ particles, and the Z^0 particle.

Note that whilst the interaction carriers can be defined by irreducible, infinite-dimensional representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ alone in the Wigner representation, cross-sections of the interaction carrier bundle $T^*\mathcal{M}\otimes\mathfrak{g}(\delta)$ transform under both $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ and $\mathcal{G}$. The space of single-photon states in the Wigner representation is the Fourier transform of a space of $U(1)$-connections *modulo gauge transformations*. This is the reason that the gauge bosons in the Wigner representation do not transform under the group of gauge transformations. Note also that it is only under symmetry breaking that the interaction carrier bundle breaks into a direct sum of bundles housing the inverse Fourier transforms of the Wigner representations.

Note that there is some distortion of meaning when people say that the interaction carriers of a gauge field 'belong to' the adjoint representation of the gauge group G. The interaction carriers of a gauge field belong to an infinite-dimensional representation of $(SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}) \times \mathcal{G}$, which is certainly not the same thing as the finite-dimensional adjoint representation of G. To reiterate, it is the representation of $SL(2,\mathbb{C}) \times G$ upon the typical fibre of $T^*\mathcal{M} \otimes \mathfrak{g}(\delta)$ which uses the finite-dimensional adjoint representation of G, tensored with a finite-dimensional representation of $SL(2,\mathbb{C})$ on $\mathbb{R}^4$.

Thus, in the case of the strong force, the gluons belong to an infinite-dimensional representation of $(SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}) \times \mathcal{G}$, with $\mathcal{G} = \Gamma(SU(\rho))$. However, the representation of $SL(2,\mathbb{C}) \times SU(3)$ upon the typical fibre of the translation bundle $T^*\mathcal{M} \otimes \mathfrak{su}(\rho)$ *does* use the eight-dimensional adjoint representation of $SU(3)$, tensored with a finite-dimensional representation of $SL(2,\mathbb{C})$ on $\mathbb{R}^4$. In the case of the electroweak force, the interaction carriers of the unified electroweak force belong to an infinite-dimensional representation of $(SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}) \times \mathcal{G}$, with $\mathcal{G} = \Gamma(U(\iota))$. One has a representation of $SL(2,\mathbb{C}) \times U(2)$ upon the typical fibre of the translation bundle $T^*\mathcal{M} \otimes \mathfrak{u}(\iota)$, and this representation *does* use the four-dimensional adjoint representation of $U(2)$.

In addition to the formal symmetry breaking which obtains the 8 interaction carriers of the strong force, one can apply formal symmetry breaking to the interacting quark bundles to obtain quark 'colours', (Derdzinski, [20], p100). The free-particle bundle for any quark flavour is the Dirac spinor bundle σ. With the strong interaction 'switched on', the interacting particle bundle for any quark flavour is a tensor product $\sigma \otimes \rho$. As before, ρ is an interaction bundle for the strong force, a complex vector bundle, of fibre dimension 3, equipped with a Hermitian inner product and compatible volume form in each fibre. Recall that, in general, the tensor product of two vector spaces, $\mathscr{H} \otimes \mathscr{K}$, is isomorphic to the n-fold direct sum of $\mathscr{H}$ with itself, where $n = \; dim \; \mathscr{K}$, (Kadison and Ringrose, [26], p140). This entails that $\sigma \otimes \rho$ is isomorphic to $\sigma \oplus \sigma \oplus \sigma$. The specific choice of isomorphism depends upon the choice of a basis $\{v_i : i = 1, 2, 3\}$ in each fibre of ρ. A 'choice of gauge' for the strong force corresponds to the selection of a basis $\{v_i : i = 1, 2, 3\}$ in each fibre of ρ which is compatible with the $SU(3)$-structure in each fibre. Hence, a choice of gauge corresponds to a cross-section of the principal bundle P_ρ of all oriented, orthonormal bases in the fibres of ρ. Thus, with a choice of gauge, $\sigma \otimes \rho$ can be decomposed into a direct sum $\oplus_{i=1}^{i=3} \sigma_i$ of sub-bundles $\sigma_i \subset \sigma \otimes \rho$, each of which is isomorphic with σ.

At each point x, the fibre of each direct summand σ_i is the span of the set of simple tensors $\{w \otimes v_i : w \in \sigma_x\}$. With the choice of basis $\{v_i : i = 1, 2, 3\}$ in ρ_x, it follows that $v \in \rho_x$ can be expressed as $c_1 v_1 + c_2 v_2 + c_3 v_3$, and that

$$\begin{aligned} w \otimes v &= w \otimes c_1 v_1 + c_2 v_2 + c_3 v_3 \\ &= w \otimes c_1 v_1 + w \otimes c_2 v_2 + w \otimes c_3 v_3 \\ &= c_1 w \otimes v_1 + c_2 w \otimes v_2 + c_3 w \otimes v_3 \end{aligned}$$

Given that the fibre of each direct summand σ_i over x is the span of the set of simple tensors $\{w \otimes v_i\}$, it follows that we have the mapping

$$w \otimes v = c_1 w \otimes v_1 + c_2 w \otimes v_2 + c_3 w \otimes v_3 \mapsto (c_1 w, c_2 w, c_3 w) \in (\oplus_{i=1}^{i=3} \sigma_i)_x$$

Another way of thinking about it is that a choice of gauge determines an orthogonal direct sum decomposition of each fibre of ρ:

$$\rho_x = \mathbb{C}^1 v_1 \oplus \mathbb{C}^1 v_2 \oplus \mathbb{C}^1 v_3$$

This, in turn, determines an orthogonal direct sum decomposition of each fibre of the tensor product $\sigma \otimes \rho$:

$$\begin{aligned}(\sigma \otimes \rho)_x &= \sigma_x \otimes (\mathbb{C}^1 v_1 \oplus \mathbb{C}^1 v_2 \oplus \mathbb{C}^1 v_3) \\ &= (\sigma_x \otimes \mathbb{C}^1 v_1) \oplus (\sigma_x \otimes \mathbb{C}^1 v_2) \oplus (\sigma_x \otimes \mathbb{C}^1 v_3)\end{aligned}$$

Now $\sigma_x \otimes \mathbb{C}^1 v_i \cong \sigma_x$, hence

$$(\sigma_x \otimes \mathbb{C}^1 v_1) \oplus (\sigma_x \otimes \mathbb{C}^1 v_2) \oplus (\sigma_x \otimes \mathbb{C}^1 v_3) \cong \sigma_x \oplus \sigma_x \oplus \sigma_x$$

The three summands of the direct sum $\sigma \oplus \sigma \oplus \sigma$, are the three so-called 'colour sectors' of a quark flavour. Each different cross-section of the principal bundle P_ρ selects a different decomposition of $\sigma \otimes \rho$ into the three colour sectors. Selecting such a cross-section is also referred to as 'formal symmetry breaking'. Before the decomposition, one has a representation of $SL(2,\mathbb{C}) \times SU(3)$ upon the typical fibre of $\sigma \otimes \rho$. After the decomposition, the only element of $SU(3)$ which preserves the selection of a basis in each fibre of ρ is Id. Hence, after the decomposition, one merely has a (reducible) representation of $SL(2,\mathbb{C}) \times Id \cong SL(2,\mathbb{C})$ on the typical fibre of $(\sigma \otimes \rho) \cong \sigma \oplus \sigma \oplus \sigma$. One can say that the direct sum decomposition is obtained by restricting the representation of $SL(2,\mathbb{C}) \times SU(3)$ to a representation of $SL(2,\mathbb{C})$, the double cover of the restricted Lorentz group. One says that the $SU(3)$-symmetry has been broken. This symmetry breaking is referred to as 'formal' because it doesn't correspond to a physical process.

A direct sum of free-particle bundles can be thought of as the free-particle bundle which represents the generalization of the free-particles represented by the individual bundles. It is in this sense that a quark with the strong force switched on, can be thought of as a generalization of three quarks. Using the metaphorical language of quark colours, a quark with the strong force switched on can be thought of as a generalization of a red quark, a green quark, and a blue quark. However, the fact that the tensor product bundle $\sigma \otimes \rho$ represents an *interacting*-quark, and the individual bundles in the direct sum decomposition are bundles that would represent a *free*-quark, indicates the artifice of such thinking.

There is a significant difference between the strong force and the electroweak force. Whilst the space of connections for the strong force decomposes as

$$\mathscr{C}(\rho) \cong \bigoplus^{dim\ \mathfrak{su}(3)} T^*\mathcal{M}$$

under *formal* symmetry breaking, the space of connections for the electroweak force decomposes as

$$\mathscr{C}(\iota) \cong \mathscr{C}(\lambda) \oplus (T^*\mathcal{M} \otimes \lambda) \oplus T^*\mathcal{M},$$

an orthogonal direct sum decomposition under *spontaneous* symmetry breaking, (Derdzinski, [20], p104-111).

Under *formal* symmetry breaking, the affine bundle $\mathscr{C}(\iota)$ of electroweak connections decomposes into $T^*\mathcal{M} \oplus T^*\mathcal{M} \oplus T^*\mathcal{M} \oplus T^*\mathcal{M}$. A choice of gauge, i.e. a cross-section of P_ι, has the dual effect of rendering $\mathscr{C}(\iota)$ canonically isomorphic with the translation bundle $T^*\mathcal{M} \otimes \mathfrak{u}(\iota)$, and rendering the bundle of skew-adjoint endomorphisms $\mathfrak{u}(\iota)$ canonically isomorphic with the product bundle $\mathcal{M} \times \mathfrak{u}(2)$. A choice of gauge therefore renders $\mathscr{C}(\iota)$ canonically isomorphic with $T^*\mathcal{M} \otimes (\mathcal{M} \times \mathfrak{u}(2))$. A choice of basis in the Lie algebra $\mathfrak{u}(2)$ renders $\mathcal{M} \times \mathfrak{u}(2)$ canonically isomorphic with $\oplus^4(\mathcal{M} \times \mathbb{R}^1)$. In turn, this renders $T^*\mathcal{M} \otimes (\mathcal{M} \times \mathfrak{u}(2))$ canonically isomorphic with $\oplus^4 T^*\mathcal{M}$. Formal symmetry breaking therefore suggests that there are four interaction carriers for the unified electroweak force, each of which is represented by cross-sections of the bundle $T^*\mathcal{M}$, the bundle for a real vector boson, a neutral particle of spin 1 and parity -1.

Under spontaneous symmetry breaking, one obtains a different decomposition of the affine bundle $\mathscr{C}(\iota)$ of electroweak connections. Instead of using a choice of gauge and a choice of Lie algebra basis to determine a decomposition, one uses a constant length cross-section ψ_0 of ι to select the decomposition. However, the decomposition selected is not uniquely determined by the cross-section ψ_0...

Given a choice of ψ_0, the cross-section decomposes each fibre of ι into $\mathbb{C}\psi_0 + \psi_0^\perp$. This, in turn, selects a sub-bundle $W(\iota)$ of $\mathfrak{u}(\iota)$ consisting of endomorphisms in each fibre which are such that $a\psi_0 \in \psi_0^\perp$ and $a(\psi_0^\perp) \subset \mathbb{C}\psi_0$. Each fibre of $W(\iota)$ is a 2-dimensional real vector space. The choice of ψ_0 alone determines a decomposition of $\mathfrak{u}(\iota)$ into $W(\iota) + W^\perp(\iota)$.

Each fibre of the bundle of skew-adjoint endomorphisms $\mathfrak{u}(\iota)$ can be equipped with a positive-definite metric $\langle\ ,\ \rangle_{p_0,q_0}$, the choice of which is determined by two positive real numbers p_0, q_0. These two numbers are related to the value of the Weinberg angle θ by $\tan^2\theta = p_0/q_0$.

If the fibre metric in $\mathfrak{u}(\iota)$ is fixed, and a choice of ψ_0 is fixed, the direct sum decomposition of $\mathfrak{u}(\iota)$ into $W(\iota) + W^\perp(\iota)$ is orthogonal, but no finer decomposition is determined by the combination of ψ_0 and $\langle\ ,\ \rangle_{p_0,q_0}$. To obtain a decomposition which is consistent with the four observed interaction carriers of the *broken* electroweak force, one must use empirical considerations to select an orthogonal decomposition of $W^\perp(\iota)$ into a pair of real line bundles. With these considerations, one defines $\gamma(\iota)$ as the endomorphisms in each fibre of ι

which are such that $a\psi_0 = 0$, and one defines $Z(\iota) = \gamma^\perp(\iota)$. One then obtains the orthogonal direct sum decomposition of $\mathfrak{u}(\iota)$:

$$\mathfrak{u}(\iota) = \gamma(\iota) + W(\iota) + Z(\iota)$$

It follows that the translation bundle $T^*\mathcal{M} \otimes \mathfrak{u}(\iota)$ decomposes as

$$T^*\mathcal{M} \otimes \mathfrak{u}(\iota) = T^*\mathcal{M} \otimes \gamma(\iota) + T^*\mathcal{M} \otimes W(\iota) + T^*\mathcal{M} \otimes Z(\iota)$$

Now, the cross-section ψ_0 selects an affine sub-bundle $\mathscr{C}_{\psi_0}(\iota)$ consisting of all the $U(2)$-connections on ι which make ψ_0 parallel. This affine bundle has $T^*\mathcal{M} \otimes \gamma(\iota)$ as its translation space bundle. Because the translation bundle of the affine bundle $\mathscr{C}_{\psi_0}(\iota)$ is $T^*\mathcal{M} \otimes \gamma(\iota)$, the translation bundle of the following affine bundle

$$\mathscr{C}_{\psi_0}(\iota) + T^*\mathcal{M} \otimes W(\iota) + T^*\mathcal{M} \otimes Z(\iota),$$

is

$$T^*\mathcal{M} \otimes \gamma(\iota) + T^*\mathcal{M} \otimes W(\iota) + T^*\mathcal{M} \otimes Z(\iota).$$

This is simply the translation bundle $T^*\mathcal{M} \otimes \mathfrak{u}(\iota)$ of $\mathscr{C}(\iota)$ under the orthogonal direct sum decomposition obtained above. Hence, the affine bundle $\mathscr{C}_{\psi_0}(\iota) + T^*\mathcal{M} \otimes W(\iota) + T^*\mathcal{M} \otimes Z(\iota)$ and the affine bundle $\mathscr{C}(\iota)$ possess the same translation bundle. Given that an affine bundle can be rendered isomorphic with its translation bundle, affine bundles with isomorphic translation bundles must be isomorphic affine bundles. Hence, we have obtained an orthogonal affine bundle decomposition:

$$\mathscr{C}(\iota) \cong \mathscr{C}_{\psi_0}(\iota) + T^*\mathcal{M} \otimes W(\iota) + T^*\mathcal{M} \otimes Z(\iota)$$

To complete this spontaneous symmetry breaking decomposition, one must note some further isomorphisms. One has the affine bundle isomorphism $\mathscr{C}_{\psi_0}(\iota) \cong \mathscr{C}(\lambda)$, obtained by defining $\lambda = \psi_0^\perp$, and by restricting to $\lambda = \psi_0^\perp$ those connections on ι which make ψ_0 parallel. One then has the isomorphisms $W(\iota) \cong \lambda$ and $Z(\iota) \cong (\mathcal{M} \times \mathbb{R})$. One then obtains the final decomposition

$$\mathscr{C}(\iota) \cong \mathscr{C}(\lambda) + T^*\mathcal{M} \otimes \lambda + T^*\mathcal{M}$$

The $T^*\mathcal{M}$ summand corresponds to the Z^0 particle, a strictly neutral, spin 1 particle, but $(T^*\mathcal{M} \otimes \lambda)$ is the interacting particle bundle for $W^\pm$, a spin 1 particle with the charge of an electron. The affine bundle $\mathscr{C}(\lambda)$ represents the photon γ. Formal symmetry breaking has $\mathscr{C}(\lambda) \cong T^*\mathcal{M}$, hence the photon is a strictly neutral, spin 1 particle.

When an interaction is 'switched on', one must deal with pairs (ψ, ∇^δ), where ψ is a cross-section of the interacting-particle bundle α, and ∇^δ is a connection on the interaction bundle δ, (Derdzinski, [20], p84). Such pairs must satisfy coupled field equations, consisting of (i) the interacting field equation upon the cross-sections ψ of α, and (ii) the coupled Yang-Mills equation upon the curvature R^{∇^δ} of the connection ∇^δ on δ:

$$\mathscr{P}(x, \psi(x), ((\nabla^\eta \otimes \nabla^\delta)\psi)(x), ((\nabla^\eta \otimes \nabla^\delta)^2\psi)(x), ...) = 0$$

$$div\ R^{\nabla^\delta} = C_0 J(\psi)$$

The move from η to α, and the move from the use of ∇^η in the free field equation, to the use of $(\nabla^\eta \otimes \nabla^\delta)$ in the interacting field equation, is often referred to as the 'minimal coupling substitution'.

These coupled equations are non-linear, entailing that the set of all pairs (ψ, ∇^δ) which solve the coupled equations does not possess a linear vector space structure. Given a choice of gauge which renders $\mathscr{C}(\delta)$ canonically isomorphic with $T^*\mathcal{M} \otimes (\mathcal{M} \times \mathfrak{g})$, the set of all pairs (ψ, ∇^δ) which solve the coupled equations constitutes a non-linear subset of $\Gamma(\alpha) \times \Gamma(T^*\mathcal{M} \otimes (\mathcal{M} \times \mathfrak{g}))$. The interacting field equation imposed upon the cross-sections ψ of α is linear, but contains ∇^δ within its very definition. If one fixes ∇^δ, then the space of cross-sections ψ of α which solve the interacting field equation, is a linear vector space, but this should not be considered as the state space for the interacting particle. Each different ψ entails a different current $J(\psi)$, and a different ∇^δ solving the coupled Yang-Mills equation $div\ R^{\nabla^\delta} = C_0 J(\psi)$ with respect to this current. This feeds back to the definition of the covariant derivative in the interacting-particle equation, hence one cannot treat the solutions ψ and ∇^δ separately.

5 Composite systems

A bound and stable collection of particles can be referred to as a composite system. In such a bound state, the particles involved tend to neutralize each other's ability to interact with the environment, so bunches of more than one interacting particle can be described by the cross-sections of a free-particle bundle, (Derdzinski, [20], p86-88).

Consider a collection of n particles, represented individually by cross-sections of the vector bundles $\alpha_1, ..., \alpha_n$. If the collection is not necessarily considered to form a bound system, it is represented collectively by cross-sections of the cartesian product vector bundle $\alpha_1 \times \cdots \times \alpha_n$, (Derdzinski, [20], p22). The base space of this cartesian product bundle is the cartesian product $\mathcal{M}_1 \times \cdots \times \mathcal{M}_n$ of the individual base spaces. However, the typical fibre of the cartesian product vector bundle is understood to be the tensor product $V_1 \otimes \cdots \otimes V_n$ of the individual typical fibres, and **not** the cartesian product of the individual typical fibres.

To emphasize, the vector bundle for the collection of n-particles is not the n-fold tensor product bundle $\alpha_1 \otimes \cdots \otimes \alpha_n$, but the n-fold cartesian product $\alpha_1 \times \cdots \times \alpha_n$. This is actually consistent with the quantum theoretical principle that the state space of a n-particle system is (a subspace of) the n-fold tensor product of the individual particle state spaces. Given that the state space of an interacting particle is not a linear vector space, let us suppose that each of the vector bundles $\alpha_1, ..., \alpha_n$ is a free-particle bundle, and that the state space of each particle k is a vector subspace of the set of cross-sections $\Gamma(\alpha_k)$. Consider the simple case where each bundle α_k is trivial, and the set of cross-sections is isomorphic to $\mathcal{F}(\mathcal{M}, \mathbb{C}^m)$, the space of $\mathbb{C}^m$-valued functions on $\mathcal{M}$. Two isomorphisms need to be considered here. Firstly,

$$\mathcal{F}(\mathcal{M}, \mathbb{C}^m) \cong \mathcal{F}(\mathcal{M}) \otimes \mathbb{C}^m$$

and then the fact that

$$\bigotimes^n \mathcal{F}(\mathcal{M}) \cong \mathcal{F}(\mathcal{M}^n)$$

It follows from these isomorphisms that

$$\begin{aligned}\bigotimes^n \mathcal{F}(\mathcal{M}, \mathbb{C}^m) &\cong \bigotimes^n (\mathcal{F}(\mathcal{M}) \otimes \mathbb{C}^m) \\ &\cong (\bigotimes^n \mathcal{F}(\mathcal{M})) \otimes (\bigotimes^n \mathbb{C}^m) \\ &\cong \mathcal{F}(\mathcal{M}^n) \otimes (\bigotimes^n \mathbb{C}^m) \\ &\cong \mathcal{F}(\mathcal{M}^n, (\bigotimes^n \mathbb{C}^m))\end{aligned}$$

Given the assumption that $\Gamma(\alpha_k) \cong \mathcal{F}(\mathcal{M}, \mathbb{C}^m)$,

$$\Gamma(\alpha_1) \otimes \cdots \otimes \Gamma(\alpha_n) \cong \bigotimes^n \mathcal{F}(\mathcal{M}, \mathbb{C}^m) \cong \mathcal{F}(\mathcal{M}^n, (\bigotimes^n \mathbb{C}^m))$$

This demonstrates, under the assumptions made here, that the states of the n-particle system are represented by cross-sections of a vector bundle which has a cartesian product $\mathcal{M}^n$ as base space, and a tensor product $\otimes^n \mathbb{C}^m$ as typical fibre. In the case where $\Gamma(\alpha_k) \cong \mathcal{F}(\mathcal{M}, \mathbb{C}^m)$,

$$\mathcal{F}(\mathcal{M}^n, (\bigotimes^n \mathbb{C}^m)) \cong \Gamma(\alpha_1 \times \cdots \times \alpha_n))$$

hence

$$\Gamma(\alpha_1) \otimes \cdots \otimes \Gamma(\alpha_n) \cong \Gamma(\alpha_1 \times \cdots \times \alpha_n)) \neq \Gamma(\alpha_1 \otimes \cdots \otimes \alpha_n))$$

The vector bundle for a collection of n particles which *are* bound together to form a composite system, is the n-fold tensor product $\alpha_1 \otimes \cdots \otimes \alpha_n \subset \alpha_1 \times \cdots \times \alpha_n$. In other words, one restricts the base space to the subset of $\mathcal{M}_1 \times \cdots \times \mathcal{M}_n$ consisting of n-tuples $(x_1, ..., x_n)$ in which $x_1 = x_2 = \cdots = x_n$. This is naturally isomorphic to $\mathcal{M}$, (Derdzinski, [20], p22). From the fact that $\alpha_1 \otimes \cdots \otimes \alpha_n \subset \alpha_1 \times \cdots \times \alpha_n$, it follows that $\Gamma(\alpha_1 \otimes \cdots \otimes \alpha_n) \subset \Gamma(\alpha_1 \times \cdots \times \alpha_n)$. Hence, the fact that the bound states of an n-particle system are cross-sections of $\alpha_1 \otimes \cdots \otimes \alpha_n$, remains consistent with the principle that the state space of a n-particle system is (a subspace of) the n-fold tensor product of the individual particle state spaces. Note, however, that particles require interactions to bind together, so the α_k will be interacting particle bundles in this case.

A composite system can possess orbital angular momentum about its centre of mass, and as a consequence, if a composite system has an orbital angular momentum of $\ell\hbar$, its states belong to $S^\ell_0 T^*\mathcal{M} \otimes \alpha_1 \otimes \cdots \otimes \alpha_n$, (Derdzinski, [20], p12-13). (The $S^\ell_0 T^*\mathcal{M}$ factor for the orbital angular momentum of a composite system should be distinguished from the $S^k_0 T^*\mathcal{M}$ factor for an elementary system with intrinsic spin). There is then a surjective bundle morphism onto a free-particle bundle η, which represents the bound states of the composite system as if it were a free elementary particle:

$$S^\ell_0 T^*\mathcal{M} \otimes \alpha_1 \otimes \cdots \otimes \alpha_n \to \eta$$

A composite system of spin s and mass m can be represented, in the configuration space approach, by the cross-sections of a spin-s free-particle bundle which provide mass m solutions to the relevant differential equation. The Hilbert space constructed from these cross-sections is, under Fourier transform, the Hilbert space for a spin s, mass m particle in the Wigner approach. Hence, the irreducible unitary representations of the local symmetry group, $SL(2,\mathbb{C})$ Ⓢ $\mathbb{R}^{3,1}$, can be used to represent not only free elementary particles, but also stable and bound collections of elementary particles. A composite system of spin s and mass m can be represented by Wigner's spin s, mass m, unitary, irreducible representation of $SL(2,\mathbb{C})$ Ⓢ $\mathbb{R}^{3,1}$. One might conclude from this that the unitary, irreducible representations of $SL(2,\mathbb{C})$ Ⓢ $\mathbb{R}^{3,1}$ specify not merely the possible free elementary particles which can exist in a universe, but all the possible free stable particles which can exist in a universe, whether they be elementary or composite. One might also conclude that the irreducibility of a representation of $SL(2,\mathbb{C})$ Ⓢ $\mathbb{R}^{3,1}$ does not entail the elementarity of the corresponding particle.

To resist this conclusion, one can argue that stable and bound collections of elementary particles can only be idealised as free particles, given that interactions are necessary to hold the parts together, and given that the constituent particles do not exactly neutralize each other's ability to interact with the environment. Whilst a free elementary particle is a physical idealisation, it is a perfectly consistent concept. In contrast, a free composite particle is a conceptual idealisation. A free composite particle is only a consistent concept if one idealises away any residual effects of the interactions which make the system a composite system.

6 Baryons, Mesons and Hadron symmetries

The interacting *elementary* particles in each fermion generation can be partitioned into so-called 'multiplets' by finite-dimensional irreducible representations of the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$. However, *composite* particles can also be partitioned into multiplets by group representations. In particular, composite systems which participate in the strong force, referred to as 'hadrons', can be partitioned into hadron multiplets, (Derdzinski, [20], p138-154). Hadrons are divided into quark-antiquark pairs, called mesons, and quark triples, called baryons. The set of hadrons are partitioned into multiplets by representations of $SU(n)$, for each $2 \leq n \leq 6$. These symmetries are referred to as hadron symmetries. The partition is different for each value of n.

The set of 6 quark flavours can also be partitioned into multiplets by representations of $SU(n)$, for each $2 \leq n \leq 6$. These symmetries are referred to as flavour symmetries. Once again, the partition is different for each value of n. In the case of $n = 3$, it is important to distinguish this symmetry group from the $SU(3)_c$ colour symmetry group.

In general, if a hadron or a quark is represented by a free-particle bundle η, then one obtains a hadron or quark multiplet by taking the tensor product $\eta \otimes (\mathcal{M} \times W)$, where W is a complex vector space possessing an $SU(n)$-structure, for $2 \leq n \leq 6$. The number of hadrons or quarks in the multiplet equals the dimension of W.

For each n, it is the lightest n quarks which belong to the $SU(n)$ quark flavour n-plet, with the remaining quarks each belonging to $SU(n)$ singlets. Given that the free-particle bundle of each quark flavour is the Dirac spinor bundle σ, the $SU(n)$ quark flavour n-plet is housed by the vector bundle $\sigma \otimes (\mathcal{M} \times \mathbb{C}^n)$, where $\mathbb{C}^n$ possesses the $SU(n)$-structure which corresponds to the standard representation of $SU(n)$. Given a choice of oriented, orthonormal basis in $\mathbb{C}^n$, this vector bundle can then be decomposed into an n-fold direct sum of σ.

For example, for $n = 3$, the u, d, s quarks belong to the $SU(3)$ quark flavour triplet. The free-particle bundle for this triplet is $\sigma \otimes (\mathcal{M} \times \mathbb{C}^3)$. Given the choice of an oriented, orthonormal basis in $\mathbb{C}^3$, this bundle decomposes into $\sigma \oplus \sigma \oplus \sigma$.

Consider the baryons and mesons composed of the u, d, s quarks and their antiquarks. Given the interacting particle bundle for an individual quark, $\sigma \otimes \rho$, a meson is represented by a free-particle bundle η satisfying the following map

$$S^\ell_0 T^* \mathcal{M} \otimes \sigma \otimes \rho \otimes \overline{\sigma} \otimes \overline{\rho} \rightarrow \eta$$

and a baryon is represented by a free-particle bundle η satisfying the following map

$$S^\ell_0 T^* \mathcal{M} \otimes \sigma \otimes \rho \otimes \sigma \otimes \rho \otimes \sigma \otimes \rho \rightarrow \eta$$

The $SU(3)$ meson multiplets are the multiplets of mesons composed of the u, d, s quarks and antiquarks. To obtain the vector bundles which represent these meson multiplets, one begins by forming the u, d, s interacting quark multiplet bundle $\sigma \otimes \rho \otimes (\mathcal{M} \times \mathbb{C}^3)$ and the u, d, s interacting antiquark multiplet bundle $\overline{\sigma} \otimes \overline{\rho} \otimes (\mathcal{M} \times \overline{\mathbb{C}^3})$. Then one forms the 2-fold tensor product bundle

$$\sigma \otimes \rho \otimes (\mathcal{M} \times \mathbb{C}^3) \otimes \overline{\sigma} \otimes \overline{\rho} \otimes (\mathcal{M} \times \overline{\mathbb{C}^3})$$

Adding the orbital angular momentum space, and re-arranging, one obtains

$$S_0^\ell T^* \mathcal{M} \otimes \sigma \otimes \rho \otimes \overline{\sigma} \otimes \overline{\rho} \otimes (\mathcal{M} \times \mathbb{C}^3 \otimes \overline{\mathbb{C}^3})$$

Using the surjective bundle morphism for mesons,

$$S_0^\ell T^* \mathcal{M} \otimes \sigma \otimes \rho \otimes \overline{\sigma} \otimes \overline{\rho} \to \eta$$

one then obtains

$$\eta \otimes (\mathcal{M} \times \mathbb{C}^3 \otimes \overline{\mathbb{C}^3})$$

Next, one looks for a decomposition of the representation of $SU(3)$ on $\mathbb{C}^3 \otimes \overline{\mathbb{C}^3}$ into irreducible direct summands. For each irreducible direct summand W, the vector bundle $\eta \otimes (\mathcal{M} \times W)$ represents a meson multiplet, with the dimension of W being the number of mesons in the multiplet. In the case of $\mathbb{C}^3 \otimes \overline{\mathbb{C}^3}$, it decomposes into the direct sum of the 8-dimensional adjoint representation of $SU(3)$, and the 1-dimensional trivial representation. Hence, there are 8 mesons composed of u, d, s quarks in one $SU(3)$ octet, and one u, d, s meson in an $SU(3)$ singlet.

The $SU(3)$ baryon multiplets are the multiplets of baryons composed of the u, d, s quarks. To obtain the vector bundles which represent these baryon multiplets, one begins by forming the u, d, s interacting quark multiplet bundle $\sigma \otimes \rho \otimes (\mathcal{M} \times \mathbb{C}^3)$, and then one forms the 3-fold tensor product bundle

$$\sigma \otimes \rho \otimes (\mathcal{M} \times \mathbb{C}^3) \otimes \sigma \otimes \rho \otimes (\mathcal{M} \times \mathbb{C}^3) \otimes \sigma \otimes \rho \otimes (\mathcal{M} \times \mathbb{C}^3)$$

Adding the orbital angular momentum space, and re-arranging, one obtains

$$S_0^\ell T^* \mathcal{M} \otimes \sigma \otimes \rho \otimes \sigma \otimes \rho \otimes \sigma \otimes \rho \otimes (\mathcal{M} \times \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$$

Using the surjective bundle morphism for baryons,

$$S_0^\ell T^* \mathcal{M} \otimes \sigma \otimes \rho \otimes \sigma \otimes \rho \otimes \sigma \otimes \rho \to \eta$$

one then obtains

$$\eta \otimes (\mathcal{M} \times \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$$

Next, one looks for a direct sum decomposition of the representation of $SU(3)$ on $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ into irreducible direct summands. For each irreducible direct summand W, the vector bundle $\eta \otimes (\mathcal{M} \times W)$ represents a baryon multiplet, with the dimension of W being the number of baryons in the multiplet. In the case of $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, it decomposes into the direct sum of the 10-dimensional space $S^3(\mathbb{C}^3)$ of symmetric tensors, two copies of the 8-dimensional adjoint representation, and the 1-dimensional space of antisymmetric tensors $\Lambda^3(\mathbb{C}^3)$. Hence, there are 10 baryons composed of u, d, s quarks in one $SU(3)$ decuplet, two $SU(3)$ octets containing u, d, s baryons, and one u, d, s baryon in an $SU(3)$ singlet.

The hadron symmetry group $SU(2)$ is referred to variously as the isotopic spin group, the isospin group, and the isobaryic spin group. The hadron symmetry group $SU(2)$ should be distinguished from the weak isospin group $SU(2)_L$. The neutron and proton belong to an $SU(2)$ hadron symmetry doublet. Although each nucleon is a baryon, and can therefore be housed in the three-fold tensor product $\sigma \otimes \rho \otimes \sigma \otimes \rho \otimes \sigma \otimes \rho$, because each nucleon is a bound state it can be treated as if it is a free elementary particle, housed in a free particle bundle η, and because it is a spin-1/2 particle, the free particle bundle is a Dirac spinor bundle σ. The neutron-proton doublet is then housed in the vector bundle $\sigma \otimes (\mathcal{M} \times \mathbb{C}^2) \cong \sigma \oplus \sigma$, where $\mathbb{C}^2$ is equipped with an $SU(2)$-structure. It requires the choice of an oriented, orthonormal basis with respect to this $SU(2)$ structure in $\mathbb{C}^2$, to select a decomposition into $\sigma \oplus \sigma$, and to thereby select a neutron sub-bundle and a proton sub-bundle.

7 Does an elementary particle have a unique intrinsic state?

J.M.G.Fell has argued that an elementary particle has only one 'intrinsic' state, (Fell and Doran, [27], p29-32).[5] I will argue below that this claim is not consistent with the mathematical objects used to represent an elementary particle.

Recall that each free particle corresponds to a unitary representation of the local, external (space-time) symmetry group $SL(2, \mathbb{C})$ Ⓢ $\mathbb{R}^{3,1}$. In the 'passive' approach to external symmetries, $SL(2, \mathbb{C})$ Ⓢ $\mathbb{R}^{3,1}$ acts upon the set of (local) inertial reference frames. Each $g \in SL(2, \mathbb{C})$ Ⓢ $\mathbb{R}^{3,1}$ maps a reference frame σ to a reference frame $g\sigma$. For each type of free particle, the group element g is represented by a unitary linear operator T_g on a Hilbert space. If v is the state of a system as observed from a reference frame σ, then $w = T_g v$ will be the state of the system as observed from the reference frame $g\sigma$. Fell argues that if v and w are a pair of unit vectors in a Hilbert space such that $w = T_g v$ for some $g \in SL(2, \mathbb{C})$ Ⓢ $\mathbb{R}^{3,1}$, then "in a sense," v and w, "(or rather the rays through them) describe the same 'intrinsic state'...for the transition from one state to the other can be exactly duplicated by a change in the standpoint of the observer," ([27], p30-31).

The state of a physical object is the set of all properties possessed by that object. Let us agree to define an intrinsic property of an object to be a property which the object possesses independently of its relationships to other objects, and let us also agree to define an extrinsic property of an object to be a property which the object possesses depending upon its relationships with other objects. If the value of a quantity possessed by an object can change under a change of reference frame, then the value of that quantity must be an extrinsic property of the object, not an intrinsic property. The value of such a quantity must be

[5] Private communication with R.S.Doran

a relationship between the object and a reference frame, and under a change of reference frame, that relationship can change.

When the intrinsic state of an object doesn't change, it means that the intrinsic properties of the object don't change. The extrinsic properties of an object, its relationships with other objects, in particular its relationships with a reference frame, can change even if the intrinsic properties of the object don't change. Hence, the intrinsic state of an object can remain unchanged even though the overall state of the object, taking into account its extrinsic properties, does change.

To claim that an object has only one intrinsic state, as Fell claims for an elementary particle, means that it can only possess one particular set of intrinsic properties.

Fell assumes that the irreducibility of a representation is the defining characteristic of an elementary particle representation, and argues that the group action is "essentially" transitive upon the state space of such a representation. He argues, therefore, that an elementary particle has only one 'intrinsic' state. "It can never undergo any intrinsic change. Any change which it *appears* to undergo (change in position, velocity, etc.) can be 'cancelled out' by an appropriate change in the frame of reference of the observer. Such a material system is called an *elementary system* or an *elementary particle*. The word 'elementary' reflects our preconception that, if a physical system undergoes an intrinsic change, it must be that the system is 'composite', and that the change consists in some rearrangement of the 'elementary parts'," ([27], p31). Fell implies that when an elementary system is observed to undergo a change within some reference frame σ, it is the particle's relationship to the reference frame σ which changes, not any of the particle's intrinsic, non-relational properties.

The first objection to this argument is that the irreducible representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ can be used to represent stable, composite systems as well as elementary systems. If irreducibility entails only one intrinsic state, then stable, composite systems would also have only one intrinsic state.

The argument that irreducibility itself entails only one intrinsic state is flawed anyway. This argument only has plausibility if one thinks in terms of classical particle mechanics. In quantum theory, there is no reason why the irreducibility of a particle representation should entail that there is only one intrinsic state. The field-like aspects of particles in quantum theory make for an infinite-dimensional state space. This is true in non-relativistic quantum mechanics, first-quantized relativistic quantum theory and second-quantized relativistic quantum theory. Because a free elementary particle is represented in the first-quantized theory by an *infinite-dimensional* irreducible representation of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, the finite-dimensional space-time symmetry group cannot act transitively upon the state space. There are many changes in the state of an elementary particle which cannot be cancelled out by a change in observational standpoint. In fact, there is an uncountable infinity of such changes! This is essentially because the state of an elementary particle is represented by a field-like object, a cross-section of a vector bundle, and one can change the value of the cross-section in an independent fashion at different points of space-time.

A change of reference frame, in the restrictive, Special Relativistic sense mandated by $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, is a more rigid, global transformation. $SL(2,\mathbb{C})$ acts transitively[6] upon the set of one-dimensional subspaces in the typical fibre of a free particle bundle η, but the transformation

$$f \mapsto f' = \mathscr{D}^{s_1,s_2}(A) \cdot f(\Lambda^{-1}(x-a))$$

permits only a global SL-symmetry in each fibre, and a global shift in reference frame, a global shift in the field values assigned to coordinate quadruples. The idea that an elementary particle has only one intrinsic state is destroyed by the infinite-dimensional nature of particle representations in quantum theory.

Mathematically, it is quite possible to introduce an infinite-dimensional group of external symmetries. Each fibre of a free particle bundle η is equipped with an $SL(2,\mathbb{C})$ structure, hence one has an automorphism bundle $SL(\eta)$, consisting of all the automorphisms in each fibre of η. The typical fibre of $SL(\eta)$ is isomorphic to $SL(2,\mathbb{C})$. The space of cross-sections $\mathcal{E} = \Gamma(SL(\eta))$ is the group of vertical bundle automorphisms of η. $\mathcal{E}$ provides an infinite-dimensional group which acts upon the cross-sections of the free-particle bundle η. Given a cross-section $\psi(x)$ of η, and an element $a(x)$ of $\mathcal{E}$, the cross-section is simply mapped to $a(x)\psi(x)$. It seems reasonable to call $\mathcal{E} = \Gamma(SL(\eta))$ a group of external (space-time) symmetries because it provides a double cover of $\Gamma(SO_0(T\mathcal{M}))$, the infinite-dimensional group of local oriented Lorentz transformations. This is the group of vertical automorphisms of the oriented Lorentz frame bundle. The latter consists of all the orthonormal bases $\{e_\mu : \ \mu = 0,1,2,3\}$ of the tangent spaces at all the points of the manifold $\mathcal{M}$, such that each e_0 is a future-pointing, timelike vector, and such that each $\{e_i : \ i = 1,2,3\}$ is a right-handed triple of spacelike vectors. This principal fibre bundle has the restricted Lorentz group $SO_0(1,3)$ as its structure group. A cross-section of the automorphism bundle $SO_0(T\mathcal{M})$ selects a linear isometry of the tangent space at each point, and thereby maps an oriented Lorentz frame at each point into another oriented Lorentz frame.

To reiterate, whilst $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ does act upon cross-sections of η as well as the base space, Minkowski space-time $\mathcal{M}$, it does not act transitively upon the space of cross-sections. Given that $SL(2,\mathbb{C})$ acts transitively upon the set of one-dimensional subspaces in the typical fibre of η, and given that the choice of SL-symmetry can be locally varying in the infinite-dimensional group $\mathcal{E} = \Gamma(SL(\eta))$, one needs to take a combination of this group with a group of transformations of the base space $\mathcal{M}$, to obtain a group which does act transitively upon the space of cross-sections. Consider $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, treated purely as point transformations of the base space. In this sense, $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ consists of the 'active' counterparts of the group of transformations between inertial reference frames. The combination of $\mathcal{E}$ with this group acts transitively upon the set of cross-sections in η representing free particle states. Hence, if $\mathcal{E}$ and $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ were both physical symmetry groups of a free elementary particle, then a free elementary particle would only have one intrinsic state.

[6]Private communication with Shlomo Sternberg

Recall, however, that a free particle bundle η houses many different particle species. The various $\mathscr{H}_{m,s}$ which are constructed out of cross-sections of η are not invariant under the action of the infinite-dimensional group $\mathcal{E}$. Whilst one particular particle may be represented by the space constructed from the mass m, positive-energy solutions of a differential equation in η, the group $\mathcal{E}$ is more than capable of mapping such cross-sections into objects which solve that differential equation for a different mass value, or which don't solve the equation at all. The automorphism group of each $\mathscr{H}_{m,s}$ is the unitary group $\mathscr{U}(\mathscr{H}_{m,s})$, into which $SL(2,\mathbb{C})$ Ⓢ $\mathbb{R}^{3,1}$ is mapped, as manifested under Fourier transform in the Wigner representation. $\mathcal{E}$ is not a group of automorphisms of any $\mathscr{H}_{m,s}$, even if it is the group of vertical automorphisms of η.

Note that Fell includes changes of velocity, i.e. accelerations, amongst the things which can be cancelled out by a change of reference frame. This implies that Fell is not merely thinking of the transformations between inertial reference frames provided by $SL(2,\mathbb{C})$ Ⓢ $\mathbb{R}^{3,1}$, but general coordinate transformations. It also implies that he considers an interacting elementary particle to only have one intrinsic state. By definition, a free particle cannot undergo acceleration, hence a representation of $SL(2,\mathbb{C})$ Ⓢ $\mathbb{R}^{3,1}$ is quite adequate to define a free particle.

In the case of an interacting, first-quantized, elementary fermion, one forms, in the simplest case, an interacting particle bundle $\eta \otimes \delta$, and in contrast with the free-particle case, one *does* use the infinite-dimensional group of vertical automorphisms of δ as a physical symmetry group. This is a significant difference between external symmetries and internal symmetries. Recall that the internal symmetry group is the *infinite-dimensional* group of cross-sections $\mathcal{G} = \Gamma(G(\delta))$ of an automorphism bundle $G(\delta)$. This means that any change in the internal degrees of freedom of an interacting particle, even if the change occurs in an independent fashion at different points in space-time, can be cancelled out by an internal symmetry (gauge transformation). This allows the group of internal symmetries to act transitively upon the infinite-dimensional space of internal states of an interacting particle. The gauge groups $SU(3)$, $U(2)$, $SU(2)$, and $U(1)$ act transitively[7] upon the set of one-dimensional subspaces in the typical fibres of the relevant interaction bundles, and because an internal symmetry is, in each case, a locally varying cross-section of the corresponding $G(\delta)$, the infinite-dimensional group $\mathcal{G} = \Gamma(G(\delta))$ of internal symmetries acts transitively upon the space of internal states of an interacting elementary particle. It is the external degrees of freedom which prevent an elementary particle, free or interacting, from having only one intrinsic state.

To reiterate, an interacting elementary particle can undergo accelerations, so in addition to $\mathcal{E}$ and $\mathcal{G}$, one would require general coordinate transformations to cancel out all possible changes. Given that the base space $\mathcal{M}$ is Minkowski space-time, one can assume that all the physical reference frames correspond to global charts. The general coordinate transformations between physical reference frames in Minkowski space-time form an infinite-dimensional subgroup

[7]Private communication with Shlomo Sternberg

of $Diff(\mathbb{R}^4)$. The active counterparts of these particular coordinate transformations form an infinite-dimensional subgroup of $Diff(\mathcal{M})$. The groups of vertical bundle automorphisms, $\mathcal{E}$ and $\mathcal{G}$, can be combined with this subgroup of $Diff(\mathcal{M})$. For an interacting elementary particle to have only one intrinsic state, $\mathcal{E}$, $\mathcal{G}$ and this subgroup of $Diff(\mathcal{M})$, would all have to be physical symmetry groups. The fact that this subgroup of $Diff(\mathcal{M})$ is not a physical symmetry group entails that an acceleration is an intrinsic change of state.

One can define the physical symmetry group to be the group under which the intrinsic properties of an object remain unchanged. Conversely, one can define intrinsic properties to be those properties which remain unchanged under the action of the physical symmetry group, and extrinsic properties to be those properties which change under the action of the physical symmetry group.

Fell claims that a composite object can possess different intrinsic properties at different times, but he also appears to hold an 'endurantist' notion of the identity of an object over time. The endurantist position holds that the same object is capable of possessing a property at one time, and not possessing that property at another time. Now, one can argue that properties which are capable of being possessed by an object at one time, and not being possessed at another time, are properties which are possessed in relation to certain times, (Weatherson, [13], Section 1.1). If moments of time correspond to the state of other objects in the universe, then one might argue that properties which are possessed by an object in relation to certain times, must be extrinsic properties. Under this argument, then, the endurantist position entails that all properties capable of change must be extrinsic properties. Under the endurantist view, one might have to concede that the changing properties of all objects, composite or elementary, are extrinsic properties.

There is, however, an alternative 'perdurantist' view, which holds that an object has temporal parts, and different temporal parts can possess different properties. Under the perdurantist view, the different temporal parts can possess different intrinsic properties.

In perdurantism, the ascription of a property to an object at a particular time corresponds to the ascription of a property to a temporal part of a 4-dimensional object. The proposition 'x possesses F at time t' means that 'x' is a 4-dimensional object which has a temporal part 't' possessing the property 'F'. Quentin Smith describes the notion of temporal parts in these terms: "If an object x is a whole of temporal parts, then x is composed of distinct particulars, each of which exists at one instant only, such that whatever property x is said to have at a certain time is [possessed by] the particular (temporal part) that exists at that time," ([28], p84). With the notion of temporal parts, an object can be defined to undergo change if "one temporal part of x possesses a certain property F at one time and...another temporal part of x does not possess F at another time," ([28], p84). Smith contrasts the 'temporal parts' notion of change with the notion that "the particular that possesses the property at one time is identical with the particular that does not possess the property at another time," ([28], p84).

One might claim that any object, composite or elementary, can possess different intrinsic properties at different times. This, and Fell's claim that only a composite object can possess different intrinsic properties at different times, may both be inconsistent with endurantism, but consistent with perdurantism.

To render the notion of variable intrinsic properties consistent with endurantism, may require one of the following lines of attack: one might argue that an object can possess an 'internal' clock, hence the claim that an object can only possess a changing property in relation to certain times does not entail that such properties are only possessed by an object depending upon its relationships with other objects. In addition, one might argue that a changing property can be an intrinsic property even if the times at which it is possessed by an object are relationships between that object and other objects in the universe. The intrinsic-ness of a property, one might argue, is not affected by the relationships which are necessary to define the times at which it is possessed.

8 What is an interacting elementary particle in the Standard Model?

To reiterate, whilst a *free* elementary particle in our universe corresponds to an infinite-dimensional, irreducible unitary representation of the 'external' space-time symmetry group $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, an *interacting* elementary particle transforms under the external symmetry group $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, **and** an infinite-dimensional group of gauge transformations $\mathcal{G}$. The latter is associated with a compact, connected Lie group G, the 'gauge group' or 'internal symmetry group' of the interaction(s) in question. The spectrum of interacting elementary particles which can exist, and the mathematical definition of what an interacting elementary particle is, changes with different gauge groups. All the non-gravitational interactions in our universe are collected together, and partially unified, in the so-called 'Standard Model'. This Standard Model has a gauge group $G = SU(3) \times SU(2) \times U(1)$ which includes the strong force, and unifies the electromagnetic and weak interactions. This gauge group defines which interacting elementary particles are consistent with the electroweak-unified Standard Model: they are those which belong to interacting particle bundles or interaction carrier bundles possessing a finite-dimensional irreducible representation of $SL(2,\mathbb{C}) \times SU(3) \times SU(2) \times U(1)$ upon their typical fibres. However, because our universe has undergone electroweak symmetry breaking, the gauge group has broken from $SU(3) \times SU(2) \times U(1)$ into $SU(3) \times U(1)_Q$, and the interacting elementary particles with which we are most familiar, actually correspond to bundles that possess a finite-dimensional representation of $SL(2,\mathbb{C}) \times SU(3) \times U(1)_Q$ upon their typical fibres.

The spectrum of interacting elementary particles, and the definition of what an interacting elementary particle is, changes again in a Grand Unified Theory (GUT), where the gauge group unifies the strong and electroweak interactions.

For example, in the $Spin(10)$ GUT, an interacting elementary *fermion* corresponds to an interacting particle bundle which possesses a finite-dimensional irreducible representation of $SL(2,\mathbb{C}) \times Spin(10)$ upon its typical fibre. For the collection of distinct elementary fermions identifiable in each fermion generation of today's universe, there is only one corresponding elementary fermion in the $Spin(10)$ GUT. Such an elementary fermion is represented by an interacting particle bundle $\sigma_L \otimes \zeta$, (Derdzinski, [20], p127). The interaction bundle ζ is a complex vector bundle of fibre dimension 16, which possesses the irreducible spinorial representation of $Spin(10)$ upon its typical fibre, whilst σ_L is the left-handed Weyl spinor bundle, possessing the $(1/2, 0)$ irreducible representation of $SL(2,\mathbb{C})$ upon its typical fibre. $\sigma_L \otimes \zeta$ is a complex vector bundle of fibre dimension 32, which is capable of representing an entire fermion generation, such as (e, ν_e, u, d), as a single elementary fermion. The fermions which are considered to possess distinct identities after GUT symmetry breaking and electroweak symmetry breaking, are merely considered to be different states of a single fermion in the $Spin(10)$ GUT. After symmetry breaking, changes between such states are no longer possible, and different types of elementary fermions are identifiable. GUTs permit transmutations between the quarks and leptons in a fermion generation because they are represented as merely quark states and lepton states of a single type of elementary fermion. After GUT symmetry breaking, quarks and leptons become elementary fermions with distinct identities.

Supersymmetry attempts to take this process a step further, postulating a Supergroup as the gauge group, so that bosons and fermions can be treated as merely different states of the same type of elementary particle.

In the Standard Model of the particle world in our universe, a select collection of finite-dimensional irreducible representations of $SL(2,\mathbb{C}) \times SU(3) \times SU(2) \times U(1)$ are said to define the elementary particle multiplets. A particle multiplet in our universe can be represented by an interacting particle bundle α or interaction carrier bundle $T^*\mathcal{M} \otimes \mathfrak{g}(\delta)$ which possesses a finite-dimensional irreducible representation of $SL(2,\mathbb{C}) \times SU(3) \times SU(2) \times U(1)$ upon its typical fibre. The gauge bosons and each generation of interacting elementary fermions are partitioned into multiplets by a collection of finite-dimensional irreducible representations of $SU(3) \times SU(2) \times U(1)$, each of which is tensored with a finite-dimensional irreducible representation of $SL(2,\mathbb{C})$.[8]

This select collection of finite dimensional irreducible representations of $SL(2,\mathbb{C}) \times SU(3) \times SU(2) \times U(1)$ define the set of actual interacting elementary particles, boson or fermion, consistent with the electroweak-unified Standard Model. These irreducible representations only correspond to multiplets when interpreted in terms of the spectrum of interacting elementary particles consistent with the electroweak-broken Standard Model. It is the gauge bosons and

[8] Because these representations of $SU(3) \times SU(2) \times U(1)$ have a common $\mathbb{Z}_6$-kernel, the gauge group of the Standard Model is arguably $SU(3) \times SU(2) \times U(1)/\mathbb{Z}_6$. See McCabe, [1], Chapter 5, for a full account.

interacting elementary fermions which exist after electroweak SSB which are partitioned into multiplets by the finite-dimensional irreducible representations of $SL(2,\mathbb{C}) \times SU(3) \times SU(2) \times U(1)$. Also note that whilst the elementary interacting particles consistent with the electroweak-unified Standard Model, correspond to *irreducible* representations of $SL(2,\mathbb{C})$ tensored with irreducible representations of the gauge group $SU(3) \times SU(2) \times U(1)$, the elementary interacting particles consistent with the electroweak-broken Standard Model, correspond to *reducible or irreducible* representations of $SL(2,\mathbb{C})$ tensored with irreducible representations of the gauge group $SU(3) \times U(1)_Q$.

With the tacit understanding that each of the following representations is tensored with an irreducible representation of $SL(2,\mathbb{C})$, the particles in the first fermion generation are partitioned into multiplets by a select collection of finite-dimensional irreducible representations of $SU(3) \times SU(2) \times U(1)$ in the following way, (Baez, [14] and [15]; Sch*ü*cker, [29], p30-31):

- The neutrino and the 'left-handed' part of the state-space of the electron (ν_L, e_L), transform according to the (1,2,-1) irreducible representation of $SU(3) \times SU(2) \times U(1)$. i.e. The tensor product of the trivial representation of $SU(3)$ with the 2-dimensional standard representation of $SU(2)$ with the 1-dimensional representation of $U(1)$ with hypercharge -1.
- The left-handed part of the state-spaces of the up quark and down quark (u_L, d_L) transform according to the (3,2,1/3) representation. i.e. The tensor product of the standard representation of $SU(3)$ with the 2-dimensional standard representation of $SU(2)$ with the 1-dimensional representation of $U(1)$ with hypercharge 1/3.
- The right-handed part of the state-space of the electron e_R transforms according to the (1,1,-2) representation.
- The right-handed part of the state-space of the up quark u_R transforms according to the (3,1,4/3) representation.
- The right-handed part of the state-space of the down quark d_R transforms according to the (3,1,-2/3) representation.

The list of particle multiplets here is based upon the assumption that the neutrino is massless.

The irreducible representations of $U(1)$ are specified here by the hypercharge y, which comes in integer multiples of 1/3. However, because the irreducible representations of $U(1)$ are indexed by the integers, one needs to take $3y$ to obtain the index of the $U(1)$ representation. In other words, the hypercharge y representation of $U(1)$ maps $e^{i\theta}$ to $e^{i3y\theta}$. The standard representation of $SU(3)$ is denoted here by a '3' to signify the dimension of the representation, and the standard representation of $SU(2)$ is denoted by a '2', again to signify the dimension of the representation.

Each irreducible representation of $SU(3) \times SU(2) \times U(1)$ corresponds to at least one vector bundle which possesses that representation upon its typical fibre.

The only two free-particle bundles used in the Standard Model multiplets are σ_L and σ_R, the left-handed and right-handed Weyl spinor bundles, respectively. These bundles possess upon their typical fibres the $(1/2, 0)$ and $(0, 1/2)$ complex, finite-dimensional, irreducible representations of $SL(2, \mathbb{C})$. The interacting-particle bundles which correspond to elementary fermion multiplets in the Standard Model, are obtained by tensoring a Weyl spinor bundle with an interaction bundle that possesses an irreducible finite-dimensional representation of $SU(3) \times SU(2) \times U(1)$. Hence, to specify which elementary fermion multiplets exist in the Standard Model, it is first necessary to specify which finite-dimensional irreducible representations of $SU(3) \times SU(2) \times U(1)$ are selected for use.

Given that the interacting elementary particles with which we are most familiar are the interacting elementary particles which exist after electroweak symmetry breaking, when the gauge group has changed from $SU(3) \times SU(2) \times U(1)$ to $SU(3) \times U(1)_Q$, the interacting elementary *fermions* with which we are most familiar correspond to interacting particle bundles which possess a representation of $SL(2, \mathbb{C}) \times SU(3) \times U(1)_Q$ upon their typical fibres. The representation of $SL(2, \mathbb{C})$ upon the typical fibre of such bundles is often a reducible direct sum representation, corresponding to the Dirac spinor bundle $\sigma = \sigma_L + \sigma_R$.

As ever, it must be emphasized that the state spaces of interacting fermions and gauge bosons are not finite-dimensional, nor are they the vector space representations, reducible or irreducible, of any group. The finite-dimensional irreducible representations of the Standard Model gauge group, $SU(3) \times SU(2) \times U(1)$, either correspond to representations upon the typical fibres of interacting-particle bundles, or to adjoint representations upon the typical fibres of interaction carrier bundles, whilst the state spaces of interacting fermions and gauge bosons are constructed from cross-section spaces of these bundles.

If an interaction bundle δ possesses a finite-dimensional representation of $SU(3) \times SU(2) \times U(1)$ upon its typical fibre, then given a free-particle bundle η equipped with a finite-dimensional representation of $SL(2, \mathbb{C})$ upon its typical fibre, the interacting particle bundle α constructed from δ and η will possess a finite-dimensional representation of $SL(2, \mathbb{C}) \times SU(3) \times SU(2) \times U(1)$ upon its typical fibre. If the representation of $SU(3) \times SU(2) \times U(1)$ is irreducible, if the representation of $SL(2, \mathbb{C})$ is irreducible, and if the interacting particle bundle is the tensor product $\alpha = \eta \otimes \delta$, then the representation of $SL(2, \mathbb{C}) \times SU(3) \times SU(2) \times U(1)$ upon the typical fibre of α will also be irreducible.

If the state space, or part of the state space, of an interacting fermion is represented by a set of cross-sections of the interacting particle bundle $\alpha = \eta \otimes \delta$, then these states of the interacting particle, these cross-sections of α, transform under a group action of $SL(2, \mathbb{C})$ Ⓢ $\mathbb{R}^{3,1}$, and under the action of the infinite-dimensional group of gauge transformations $\mathcal{G} = \Gamma(G(\delta))$, with $G = SU(3) \times SU(2) \times U(1)$.

An elementary fermion multiplet in the electroweak-unified Standard Model, typically contains parts of the state spaces of one or more of the elementary fermions which exist after electroweak symmetry breaking. In addition, different parts of the state space of an elementary fermion after electroweak symmetry breaking can correspond to different irreducible representations of $SL(2,\mathbb{C}) \times SU(3) \times SU(2) \times U(1)$.

It is sometimes said that the particles in a multiplet can transform into each other under a gauge transformation. If true, this would mean that the finite-dimensional irreducible representations of $SL(2,\mathbb{C}) \times SU(3) \times SU(2) \times U(1)$ determine which particles, or which parts of the state spaces of particles, can transform into each other under a gauge transformation. However, the different sets of cross-sections which represent the different particles within a fermion multiplet, are often defined by the requirement that they satisfy differential equations with respect to different values of mass. This implies that the states of the different particles within a multiplet cannot transform into each other under a gauge transformation. The state space parts contained in a multiplet may be cross-sections of the same vector bundle, but they are disjoint subsets of the space of bundle cross-sections.

The direct sum of all the interacting-particle bundles of the elementary fermion multiplets, possesses upon its typical fibre a reducible representation of $SL(2,\mathbb{C}) \times SU(3) \times SU(2) \times U(1)$. This reducible representation is the direct sum of the irreducible representations corresponding to each multiplet. The direct sum interacting particle bundle represents a generalized particle, a sort of amalgam of all the elementary fermions whose state space parts were contained in the multiplets.

Although it is possible for two elementary particles, or parts of their state spaces, to correspond to the same finite-dimensional irreducible representation of $SU(3) \times SU(2) \times U(1)$, but to different finite-dimensional irreducible representations of $SL(2,\mathbb{C})$, this does not occur in our universe.

The anti-particle multiplets correspond to the conjugate/dual irreducible representations of $SU(3) \times SU(2) \times U(1)$. If a particle multiplet uses the index k irreducible representation of $U(1)$, then the anti-particle multiplet uses the index $-k$ irreducible representation. In terms of the irreducible representations of $SU(n)$, for $n \geq 2$, with spins $(s_1, ..., s_{n-1})$, the conjugate representation is the $(s_{n-1}, ..., s_1)$ representation, (Derdzinski, [20], p134). In general, this means that the conjugate representation is an inequivalent irreducible representation, but note, as an exception, that the conjugate representation is equivalent in the special case of $SU(2)$. The inner product in each representation space establishes an equivalence between the conjugate representation and the dual representation.

Given an interaction bundle δ which possesses an irreducible representation of $SU(3) \times SU(2) \times U(1)$, the conjugate bundle $\overline{\delta}$ possesses the conjugate/dual representation.

Given an interacting particle bundle $\eta \otimes \delta$, the interacting anti-particle bundle is $\overline{\eta} \otimes \overline{\delta}$. The interacting particle bundles which represent the elementary fermion multiplets of the Standard Model are obtained by tensoring either left-handed σ_L, or right-handed σ_R Weyl spinor bundles with interaction bundles that possess irreducible representations of $SU(3) \times SU(2) \times U(1)$. Now, σ_L and σ_R are mutually conjugate bundles i.e. $\overline{\sigma}_R = \sigma_L$. Hence, given a particle multiplet bundle $\sigma_L \otimes \delta$, the anti-particle multiplet is represented by $\sigma_R \otimes \overline{\delta}$, and given a particle multiplet bundle $\sigma_R \otimes \delta$, the anti-particle multiplet is represented by $\sigma_L \otimes \overline{\delta}$.

Given that each irreducible representation of $SU(3) \times SU(2) \times U(1)$ is possessed by the typical fibre of at least one vector bundle, one can form the direct sum of the particle multiplet interaction bundles to obtain an interaction bundle $\kappa = \oplus_i \delta_i$ which possesses the reducible direct sum representation $(1, 2, -1) \oplus (3, 2, 1/3) \oplus (1, 1, -2) \oplus (3, 1, 4/3) \oplus (3, 1, -2/3)$ upon its typical fibre. One can then form the following interacting particle bundles: $\sigma_L \otimes \kappa$, $\sigma_R \otimes \kappa$, $\sigma_L \otimes \overline{\kappa}$, and $\sigma_R \otimes \overline{\kappa}$. Given that κ is a direct sum of interaction bundles $\kappa = \oplus_i \delta_i$, it follows that

$$\sigma_L \otimes \kappa = \oplus_i (\sigma_L \otimes \delta_i)$$

and

$$\sigma_R \otimes \kappa = \oplus_i (\sigma_R \otimes \delta_i)$$

and

$$\sigma_L \otimes \overline{\kappa} = \oplus_i (\sigma_L \otimes \overline{\delta}_i)$$

and

$$\sigma_R \otimes \overline{\kappa} = \oplus_i (\sigma_R \otimes \overline{\delta}_i)$$

Note, however, that within the multiplets of a fermion generation, both right and left-handed Weyl spinor bundles are tensored with the bundles that possess the irreducible representations of $SU(3) \times SU(2) \times U(1)$. Some of the particle multiplets are represented by interaction bundles tensored with σ_L, and some are represented by interaction bundles tensored with σ_R. Similarly, within the multiplets of an anti-fermion generation, both right and left-handed Weyl spinor bundles are tensored with the bundles that possess the irreducible conjugate representations of $SU(3) \times SU(2) \times U(1)$. Hence, some of the particle anti-multiplets are represented by interaction bundles tensored with σ_L, and some are represented by interaction bundles tensored with σ_R. It is therefore incorrect to represent a fermion generation with either $\sigma_L \otimes \kappa$ or $\sigma_R \otimes \kappa$, and it is incorrect to represent an anti-fermion generation with either $\sigma_L \otimes \overline{\kappa}$ or $\sigma_R \otimes \overline{\kappa}$.

To conclude, and to return to the main question of this section, an interacting elementary particle is the type of thing which is represented in the first-quantized, electroweak-unified Standard Model, to be a cross-section of a bundle over space-time possessing a finite-dimensional irreducible representation of $SL(2, \mathbb{C}) \times SU(3) \times SU(2) \times U(1)$ upon its typical fibre.

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