GENERICITY AND ARBITRARINESS

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Abstract. We compare the notions of genericity and arbitrariness on the basis of the realist import of the method of forcing. We argue that Cohen's Theorem, similarly to Cantor's Theorem, can be considered a meta-theoretical argument in favor of the existence of uncountable collections. Then we discuss the effects of this meta-theoretical perspective on Skolem's Paradox. We conclude discussing how the connection between arbitrariness and genericity can offer arguments in favor of Forcing Axioms.

Introduction

Forcing was invented by Paul Cohen in 1963 and it is now considered the cornerstone of contemporary set theory. This technique was invented to produce independence results and, practically, allows one to extend a countable transitive model of ZFC to a larger one, that is forced to verify or falsify a given statement. The key ingredient of this construction is provided by sets that are generic with respect to a model, that is, collections with no particular properties besides those they have in virtue of being generic.

Although forcing has been extensively studied from a mathematical point of view, this technique still does not occupy a central role in the contemporary philosophical debate[1].

This paper intends to fill this gap proposing a philosophical analysis of the notion of genericity. It is interesting to notice that the need for such

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[1]Nonetheless, the plethora of different models of ZFC provided by forcing gave rise to new mathematical structures, called multiverses, that have been thoroughly investigated, with different philosophical aims and perspectives. A tentative sketch of this debate should include the works of Friedman ([11], [10]), Hamkins ([14], [15]), Magidor ([21]), Shelah ([26]), Steel ([6], [28]), Väänänen ([29]) and Woodin ([33], [34]).

clarification was felt quite early after the invention of forcing, in 1967, by Mostowski.

> Models constructed by Gödel and Cohen are important not only for the purely formal reasons that they enable us to obtain independence proofs, but also because they show us various possibilities which are open to us when we want to make more precise the intuitions underlying the notion of a set. Owing to Gödel's work we have a perfectly clear intuition of a set which is predicatively defined by means of a transfinite predicative process. No such clear interpretation has as yet emerged from Cohen's models because we possess as yet no intuition of generic sets; we only understand the relative notion of a set which is generic with respect to a given model.[2]

Without committing ourselves to an obscure notion of intuition, we intend to analyze the notion of genericity in connection with the notion of arbitrary set, that is considered a key notion in the foundations of set theory [7]. In arguing for an analogy between these two notions our aim is twofold: on the one hand we intend to give a more philosophical content to the technical notion of genericity, while on the other hand we will use this analogy to justify the axiomatic version of forcing constructions: Forcing Axioms.

The article is structured as follows. In §1 we present and discuss the connection between generic and arbitrary sets. In §1.1 we discuss the connection between Cantor's Theorem and Cohen's Theorem for what concerns their realist import and in §1.2 the effect that Cohen's Theorem has on Skolem Paradox. Finally §2 will offer a justification of Forcing Axiom in terms of the arbitrariness of generic sets.

1. Arbitrary and generic

The notion of arbitrary set is normally associated to the work of Dedekind, Cantor, and Zermelo and is intended to describe sets that are not characterized by a law, or a property, but whose existence can only be inferred indirectly, by results like *Cantor's Theorem* (CaT) on the uncountability of $\mathbb{R}$.

Conceptually, an arbitrary set is opposed to a set that is definable. But of course the notion of definability needs to be made relative to a context or a language, to avoid paradoxes like Richard's. This is not a problem for

[2] [24], p. 94.

the existence of arbitrary sets, since the language of ZFC is countable and consequently there are at most countable many definable real numbers[3].

Arbitrary sets are often associated to the so-called *quasi-combinatorial* conception of sets [3]—or *combinatorial maximality* [20]. The best way to explain this notion is using Bernays' words.

> [Platonism in analysis] abstracts from the possibility of giving definitions of sets, sequences, and functions. These notions are used in a "quasi-combinatorial" sense, by which I mean: in the sense of an analogy of the infinite to the finite.
>
> Consider, for example, the different functions which assign to each member of the finite series 1, 2, ..., n a number of the same series. There are n^n functions of this sort, and each of them is obtained by n independent determinations. Passing to the infinite case, we imagine functions engendered by an infinity of independent determinations which assign to each integer an integer, and we reason about the totality of these functions.
>
> In the same way, one views a set of integers as the result of infinitely many independent acts deciding for each number whether it should be included or excluded. We add to this the idea of the totality of these sets. Sequences of real numbers and sets of real numbers are envisaged in an analogous manner. From this point of view, constructive definitions of specific functions, sequences, and sets are only ways to pick out an object which exists independently of, and prior to, the construction.
>
> The axiom of choice is an immediate application of the quasi-combinatorial conception in question[4].

It is convenient to make explicit the basis of Bernays' argument, outlining three conceptual principles that we find here at play and that we can trace back to Cantor's view on sets.

[3]A different notion of definability is that with parameters, but since the class of all cardinals is provably larger that any given cardinal, and since a set of parameters is always fixed with its cardinality, there will always be sets that are not definable, even with parameters.

[4]In [3]. Quoted from the English translation, pp. 259–260.

(1) It is the possible to make infinitely many determinations at once[5], exactly as is the case of the Axiom of Choice (AC) and the Well-Ordering Theorem. The latter was shown to be equivalent to AC by Zermelo in 1904 [35], and was considered a law of thought by Cantor [4].
(2) We can see in Bernays' words an instance of what has been called *Cantor's finitism* by Hallett, in [13], that can be easily explained as the belief of a general uniformity between the finite and the transfinite. Specifically, this principle grants an analogy between the laws of arithmetics for natural numbers and the laws of transfinite arithmetics, the latter being an extension of the former, beyond ω.
(3) Finally, not only arbitrary objects are engendered by infinitely-many determinations, in analogy to the finite case, but "the totality of these sets" is also taken to exist. This is nothing else than the so-called *domain principle* [13], that is both a declaration of realism in mathematics and the ultimate reason that Cantor offers in defense of actual infinity. Simply put, the principle says that if a variable runs over a given domain, then that domain should exists in its entirety.

It is interesting to notice that the arbitrariness of an object is not given by the absence of a definition, but by the impossibility to offer one. This has the somewhat disturbing consequence of making impossible to characterize an arbitrary set. Even by showing the existence of arbitrary sets, CaT does not offer an example of such a set. Indeed, the classical diagonal argument shows the existence of a real number that does not belong to a countable list of real numbers, but that, being defined in term of a suitable modification of the diagonal, cannot be considered arbitrary.

A second important aspect of arbitrary sets worth noticing is their minimal contribution to the logical structure of the theory that proves their existence. Given that an arbitrary set is not determined by a definition or a law, there is no property that determines if and when an object belongs to it. This is why the only possible description of an arbitrary set A is given by sentences of the form "$x \in A$" or "$x \notin A$". In other words the only logical contribution that A can offer is given by the membership of its elements.

On the other hand the notion of generic set is formal and it is given by the theory of forcing.

[5]The temporal metaphor is often used in these contexts, but should not be taken too seriously, since it is only an dynamic picture that helps to humanize a context that is, however, completely static in its realism.

Definition 1.1. *Given a countable model M of ZFC* [6] *and a partial order (poset) $\mathbb{P}$ belonging to M, we say that a filter $G \subseteq \mathbb{P}$ is $(M, \mathbb{P})$-generic if, whenever $D \in M$ is a dense subset of $\mathbb{P}$, then $G \cap D \neq \varnothing$.*

Intuitively a set is generic, with respect to a model M and a poset $\mathbb{P}$, if it meets all requirements to be a subset of $\mathbb{P}$ from the perspective of M and nothing more. The elements of $\mathbb{P}$, called *conditions*, represent partial pieces of information that will eventually give the full description of the generic G. Moreover, the dense sets that belong to M represent the properties that a subset of $\mathbb{P}$ should eventually have, as considered from the perspective of M. For this reason, a generic set does not have a characteristic property that distinguishes it from all other elements of M. Consequently, when it happens not to belong to M—and this is the case when $\mathbb{P}$ is non-atomic—it is just because sentences like"$x \in G$" or "$x \notin G$" distinguishes it from all other elements of M.

Cohen showed that generic filters exist and we will call this result *Cohen's Theorem* (CoT). The proof of CoT is non-constructive. As a matter of fact, it shows the existence of a set that has non-empty intersection with each element of the cartesian product of the countably-many dense subsets of $\mathbb{P}$, that belong to M; these are the infinitely many determinations that engender G, without giving it any further property besides being generic. It is therefore no surprise that CoT is equivalent to the Baire Category Theorem for complete metric spaces, that in turn is equivalent to a form of the *Principle of Dependent Choice* (DC)[7]. In other terms CoT shows the existence of generic filters using the same principles that motivate a quasi-combinatorial approach to set theory and the existence of arbitrary sets.

Therefore the strong character of arbitrarity that generic sets seems to have can be easily explained by their non-constructive nature. In other words generic sets show elements of arbitrarity exactly because their existence is based on the same principles that ground the existence of arbitrary sets.

[6] Notice that the fact of considering countable models, and not countable transitive model, is not a limitation here. Indeed, forcing constructions also work with countable models that are not transitive.

[7] See [32] and [12] for the proof of these facts and for other interesting equivalences.

It is this connection that we would like to discuss in the following pages[8], considering more closely the analogy between CaT and CoT.

1.1. **Cantor and Cohen.** Genericity and arbitrariness have not only in common their non-constructive character, but also a deep connections with a realist stand towards existence in set theory and, given the foundational character of the latter, also in mathematics.

A neat example of a realist aspect connected to quasi-combinatorialism is the standard interpretation of Cantor's diagonal argument for the uncountability of $\mathbb{R}$. Indeed, CoT is normally taken as an argument in favor of the existence of uncountable cardinalities and consequently of arbitrary sets. We briefly recall Cantor's argument in order to outline a parallel with forcing constructions.

Cantor's argument: Suppose there is a surjection $f : \omega \to \mathcal{P}(\omega)$. Then consider the set $A = \{n \in \omega : n \notin f(n)\}$. By surjectivity of f there is an a such that $A = f(a)$. But then $a \in A \iff a \notin f(a) \iff a \notin A$. Hence f does not exist.

Cohen's argument for showing how to extend a model with a generic filter has a very similar structure: if we assume that we cannot extend our domain (in this case a countable model M of ZFC, while before an enumeration of $\mathbb{R}$ in length ω), then we arrive to a contradiction by means of a diagonal argument that allows to find a new object outside the domain.

Cohen's argument: Let M be a countable transitive model of ZFC and let $\mathbb{P} \in M$ be a non-atomic poset. Suppose there is an filter $G \subseteq \mathbb{P}$, in M, that intersects every dense subsets of $\mathbb{P}$ in M. Then consider the set $E = \{p \in \mathbb{P} : p \notin G\}$. By density of E, there is an $e \in E \cap G$. But then $e \in G$, but as $e \in E$ it follows that $e \notin G$. Hence G does not exist in M.

Interestingly enough, this comparison has been already proposed in [22], in connection with an argument against the existence of uncountable cardinals. This argument belongs to a skeptical tradition, inaugurated by Skolem, that, although minority among set-theorists, represents a sort of

[8] In the literature it we find a similar suggestion in [8]: "Within mathematical logic, our theory of arbitrary objects leads to a modest formal development of its own. But it is also able to throw light on more traditional methods and results. It sometimes appears as if a mathematician is making significant use of an arbitrary or 'generic' object. Obvious examples are the use of generic sets in the independence proofs and the use of arbitrarily small quantities in analysis." p. 74. Unfortunately Fine seems not to have developed this interesting suggestion. We deserve for another occasion a discussion on the relationship between Bernays' notion of arbitrariness and Fine's, but we refer to [16] for a general discussion on arbitrary objects in mathematics.

Cartesian doubt on set-theoretical reality that has never been proved to be logically false. In what follows we won't provide such undoubtable argument but our strategy will be to push this doubt to its extreme consequences; to the point that an absurdity will be reached, even from a perspective like Skolem's. To this aim we briefly review [22].

In his paper Meadows notices that, among the many things that we can do with forcing, Cohen's method allows to extend a countable model M with filters that code bijections between uncountable cardinals (of M) and ω. Then he argues that, because of the existence of such filters, we might consider the existence of uncountable collections only an illusion.

> Observing this situation and given our claim there are not any really uncountable infinities, we might imagine ourselves as, so to speak, navigating an endless collection of these countable models in something like the generic multiverse we have described. While the *illusion of multiple infinite cardinalities* is witnessed *inside* each of the universes, we are free to move between them. [...] I would like to make the provocative suggestion that forcing is a kind of natural revenge or dual to Cantor's theorem: where Cantor gives us the transfinite, forcing tears it down[9].

We agree with Meadows that it is debatable whether CaT, alone, is able to ground an argument in favor of the existence of uncountable cardinals, without resting on further assumptions. However, we believe that the inverse analogy between CaT and CoT, suggested in [22], is misleading and rests on the confusion between the theoretical and the meta-theoretical levels. While CaT can support an argument in favor of the existence of uncountable cardinals within set the theory, on the contrary CoT deals with models of set theory; and if any conclusion can be drawn from it, that would be about the cardinality of these models. Therefore, we believe that in direct opposition with the above quotation we may argue that CoT can be seen as a positive argument for the uncountability of the universe of set theory, or, to put in a negative way, as a hint that countable models are only partial approximations of the universe of all sets.

It is important to stress that we are not arguing that CaT and CoT are able to *prove* the existence of uncountable sets (respectively, models), but only that if we accept that CaT gives us the transfinite within set theory, then CoT does not tear it down, but on the contrary it gives us the transfinite from a model-theoretic perspective. On this view CoT would thus harmonize theory and meta-theory: if we accept that there are uncountable sets, then we have models that can accommodate them.

[9] [22], p. 206-207.

Discussing CaT Meadows suggests that the non-existence of a bijection between $\mathbb{N}$ and $\mathbb{R}$ could be ascribed to the range of quantification where this bijection is supposed to live: “The key and contentious idea is that there is no powerset of the naturals to quantify over. There is no theory than can quantify over all the subsets of the natural numbers.”, [22], p. 196. Therefore the set-theoretical Cartesian doubt is here shaped in the form of keeping open the possibility that the universe of all sets is just countable. This perspective—perfectly opposite to what we consider to be the meta-theoretical consequence of CoT—can also be found in the words of Dana Scott, from the preface of Bell’s book on Boolean-valued models [2], with the same misleading mixture of theoretical and meta-theoretical levels.

> Perhaps we would be pushed in the end to say that all sets are countable (and that the continuum is not even a set) when at last all cardinals are absolutely destroyed[10].

To oppose the inverse analogy between CaT and CoT suggested above and to argue in favor of their close connection with a realist perspective, we can also notice that it is possible to prove CaT directly from CoT. We can reason as follows[11].

Let $\mathbb{C}$ be Cohen forcing in the form of $\omega^{<\omega}$: i.e., the set of finite sequences of natural numbers, with the order given by reverse inclusion: i.e., for $p, q \in \mathbb{C}$, we say that $p \leq q$ whenever $q \subseteq p$. Now consider a countable collection of real numbers, that can be seen as a countable $X \subseteq \omega^{<\omega}$, say $X = \{x_n : n \in \omega\}$. Moreover consider a model M of ZFC that contains $\mathbb{C}$ and X—this is possible by standard model-theoretic arguments. Now, for $n \in \omega$, consider

$$D_n = \{p \in \mathbb{C} : p \neq x_n \restriction dom(p)\}$$

and

$$D^*_n = \{p \in \mathbb{C} : n \in dom(p)\}$$

that not only belong to M, by absoluteness of the definition and since $\mathbb{C} \in M$, but are also dense in $\mathbb{C}$. Now, CoT provides a filter G that intersects, among all dense subsets of $\mathbb{C}$ that belong to M, also every D_n and D^*_n, for $n \in \omega$. Therefore, $\bigcup G$ is an element of ω^ω, that, by a density argument, is different from any single x_n. In other terms $\bigcup G \in \omega^\omega \setminus X$ and by the arbitrary choice of X, we can infer that $|\mathbb{R}| = 2^{\aleph_0} > \aleph_0 = |\mathbb{N}|$.

Therefore forcing allows to prove CaT, by showing the existence of a new generic real that does not belong to any countable enumeration. Notice

[10] [2], p. xv

[11] The argument of the next paragraph is taken from [25].

that, for the sake of our presentation, this proof is stronger than the classical one, since the new real is now generic, contrary to the easy definition that a real receive in a diagonal construction.

Indeed, diagonal arguments show that however large is the class of objects we consider, that will never be all there is. This idea can be described in two complementary ways: positively, saying that our domain of discourse is always extendible with new objects, while negatively, saying that our language is too weak to describe all there is.

On the other hand, arguments based on the existence of arbitrary sets have stronger philosophical consequences, being grounded on a realist perspective on mathematical existence. These arguments say not only that the existence of objects is prior to our determinations, but also that there are things that cannot be linguistically determined.

1.2. **Skolem and Cohen.** So far we argued in favor of the similarities between CaT and CoT and we disentangled the mixture of theoretical and meta-theoretical considerations that one can draw from these results. But coming back to the original Skolem's concern, does CoT tells us something more than CaT to counter this Cartesian doubt?

As already suggested neither CaT nor CoT can, alone, assure the existence of uncountable sets. Exactly because, as Jané put it: "[T]hat no countable list of number collections is complete does not imply that *the complete list* of all number collections is uncountable, since there may be not such list", [17], p. 145.

However, we think that CoT, and the meta-theoretical realism that this result suggests, offers a more compelling argument against Skolem's relativism about set theoretic notions. To this aim we summarize Skolem Paradox: that feeling of unease that comes from the apparent clash between CaT and *Löwenheim-Skolem Theorem* (LST). Indeed, LST allows to shows that there are countable models of ZFC, that thus verify CaT. Therefore there are models that think that some of their sets are uncountable, while them being countable.

The standard answer consists in keeping separate the internal perspective of the model, and the external one: i.e., the theoretical and the meta-theoretical levels. Following this strategy the apparent contradiction is broken down by noting that not all bijections between ω and a given set belong to a countable model. Henceforth, the "solution" to the paradox is found in the impossibility for a countable model to capture the uncountably many bijections between ω and a given set. Although technically irreproachable this answer has the weakness of relying on the uncountability of $2^{\aleph_0}$. Exactly the point of Skolem's disagreement.

Skolem never though about Skolem Paradox as a paradox. Indeed there is nothing paradoxical in it for a person distrustful of the existence of uncountable collections. On the contrary, he thought that LST was a clear sign of an inevitable relativity of set theoretical notions.

The scope, the target and the structure of Skolem's argument has been intensively discussed. Since the vast literature on this subject does not always cohere and because our aim here is not exegetic, we will try not to side with any particular interpretation, keeping our discussion on the less controversial aspects of Skolem's criticism towards set theory.

In discussing the relativity of set-theoretical notions we have to be clear that Skolem is not considering the obvious form of relativism that originates from the simple observation that different models represent set theoretical notions in different ways. On the contrary Skolem's form of relativism rests on two remarks: the first is that the axiomatic method is the best we have to deal with set theory, since we do not have a consistent notion of set from naïve set theory; the second is that the only models we can build, without assuming the existence of uncountable collections from the outset, are the countable ones. In [17] Jané has precisely and concisely described Skolem's relativism in the following way.

> Recall that Skolem does not argue against the absolute meaning of certain set-theoretical notions because such notions take different forms in different models, but rather because he has been given no reason to believe that there is even one model in which these notions take their presumed right form; in all the models that he is able to build, sets deemed to be uncountable are indeed countable[12].

In other terms, Skolem believed that the notion of uncountable collection is relative simply because the only models of axiomatic set theory we can build are countable. Skolem is thus expressing two clear points: the axiomatic method is trustful and we have no axiomatic reason to believe in the existence of uncountable collections.

Now, what can CoT say in this respect? As we showed in §1.1 both CaT and CoT allow to show the incompleteness of a countable set by means of a diagonal argument. However there is an important difference with respect to the nature of the set that is shown to be incomplete: in the case of CaT a given set of reals, while in the case of CoT a model of set theory.

While it is true that the fact that all countable lists are incomplete does not imply that the complete list is uncountable, nonetheless it does show, at least, that the list is not all there is. This simple observation gains value when applied to a collection that is supposed to be complete in some

[12][17], p. 148.

respect, like a model of set theory. Although the existence of sets outside a countable model does not directly imply that the universe of all sets is uncountable, however it tells us that the countable model we consider has to be understood as a toy model: a model that does not faithfully represent the correct interpretation of the axioms of set theory.

Moreover, we may find a significant structural difference between CaT and CoT. While CaT proceeds by contradiction and concludes that a countable list of all real numbers does not exist, CoT does not aim at contradicting the existence of a countable model of ZFC.

So we found ourselves in an awkward dilemma. We assumed that set theory is consistent, but all models we can build cannot be considered adequate semantic realizations of its axioms. So either there are adequate models that are not countable, or there are no adequate models of set theory at all. Remember that Skolem's skepticism held a quite different alternative: either we assume that there are uncountable collections from the outset—what Skolem strongly doubted—or we have to consider set-theoretic notions relative to a model of axiomatic set theory.

The two options offered by CoT are definitely more extreme than the ones offered by Skolem. Indeed, if we exclude the possibility that there are uncountable collections, then we have to conclude that set theory has no models adequate to its axioms. Notice that, in Skolem's view, the latter is not a consequence of the non existence of uncountable sets, exactly because of the relativity of set-theoretical notions. Indeed, for Skolem there was no absolute notion with respect to which a countable model was unfaithful. However, CoT forces us to declare a countable model inadequate to model the axioms of set theory, even in the absence of an absolute notion of uncountability. Let me stress the importance of this point. While the standard solution of Skolem Paradox declares countable models unfaithful only in the case of the existence of uncountable sets, CoT forces us to consider every countable models inadequate to model the axioms of set theory, even in the absence of an absolute notion of uncountability.

As a proud opponent of uncountable totalities, Skolem would probably opt for the second horn of the dilemma offered by CoT, but this would be highly problematic, because it would show intrinsic limitations of the axiomatic method.

In other words what we can call *Cohen's Dilemma* shows that either we accept the existence of uncountable collections, or that first order logic is not powerful enough to deal with set theory, and therefore—from a perspective far from Skolem's—with mathematics.

As we hinted before Skolem's criticism of the foundational role of set theory did not correspond to a similar distrust of the axiomatic method in general, leaving Cohen's Dilemma problematic even for Skolem.

> The fact that axiomatizing leads to relativism has been sometimes considered to the weak spot of the axiomatic method. There is no reasons for this. The analysis of mathematical though, the fixation of fundamental hypotheses and the ways of reasoning can only be an advantage for science. It is not a weakness of an axiomatic method that it cannot yield what is impossible.[13]

Interestingly enough, Cohen himself, in a partisan description of Skolem's position, seems to advocate the second horn of Cohen's dilemma.

> So, let me say that I will ascribe to Skolem a view, not explicitly stated by him, that there is a reality to mathematics, but axioms cannot describe it. Indeed one goes further and says that there is no reason to think that any axiom system can adequately describe it.[14]

To summarize, we did not attempted to propose a straightforward argument in favor of the existence of uncountable cardinals based on CaT and CoT, but we only outlined their parallelism and not their opposite complementarity. If CaT gives us the uncountable within set theory, CoT gives it at a meta-theoretical level. Moreover, the alternatives suggested by CoT are more extreme than those offered by Skolem Paradox: if we cast doubts upon the existence of the uncountable we do not only conclude that set-theoretical notions are relative, but also that their models are inadequate to realize the axioms of set theory.

We also tried to clarify the common conceptual origin of both notions of genericity and arbitrariness, finding that they both hinge on a form of realism described by Bernays as quasi-combinatorialism. Moreover we discussed the connections of these ideas with the main goal of set theory: to give mathematical dignity to the notion of infinity in mathematics. We now would like to tackle the problem of how a formal approach could work in clarifying these basic notions of set theory. In other words we would like to ask to what extent a connection between genericity and arbitrariness is able to justify axioms able to formally capture the idea of sets whose existence is prior to, and independent of, our constructions.

2. On the justification of Forcing Axioms

So far we analyzed sets that are generic with respect to a model of ZFC. But the connection we found between genericity and arbitrariness did not originate by the fact of being generic with respect to a model, but instead

[13] [27], p. 470.
[14] [5], p. 2417.

by the fact of being determined in a non-constructive way by intersecting countable many dense sets.

Then, if we drop the requirement that the dense sets belong to a model, we get the notion of $\mathcal{D}$-generic, for a family $\mathcal{D}$ of dense subsets of a poset $\mathbb{P}$. It is unclear if this is the notion of genericity that Mostowski had in mind when he suggested a more general analysis of genericity, but it clearly goes in the direction of freeing this notion form its dependence on models.

Following this line of reasoning, we can consider $\mathcal{D}$-generic sets as sets with strong conceptual similarities with arbitrary sets and a forcing construction as *a non-constructive way to produce new mathematical objects.* Consequently, we may inquire about the possibility to devise new axiomatic principles that, as AC gives choice functions, would produce $\mathcal{D}$-generic sets. But this is exactly what Forcing Axioms do.

2.1. **Forcing Axioms.** Intuitively, Forcing Axioms tell us that the universe of all sets has been saturated by means of the possibilities offered by forcing constructions. As a consequence we could think of a structure satisfying forcing axioms as one obtained after many applications of forcing. Indeed this is the rough idea behind their relative consistency proof[15].

Mathematically, forcing axioms can be presented in several different ways. We give their standard definition in terms of posets.

Definition 2.1. *Let Γ be a collection of partial orders and κ e regular cardinal, then the Forcing Axiom $FA(\Gamma, \kappa)$ states: for any partial order $\mathbb{P}$ in Γ and for any collection $\mathcal{D}$ of dense subsets of $\mathbb{P}$, with $|\mathcal{D}| \leq \kappa$, there is a filter $G \subseteq \mathbb{P}$ that intersects every $D \in \mathcal{D}$ (i.e. a $\mathcal{D}$-generic filter).*

Notice that CoT shows that it is always possible to find a generic filter that intersects a countable family of dense sets. Therefore, it is a theorem of ZFC that $\mathrm{FA}(\Omega, \aleph_0)$ holds, with Ω being the (proper) class of all possible posets.

On the other hand it is not hard to find a poset $\mathbb{P}$ and a family $\mathcal{D}$, of size $\aleph_1$, for which there is no $\mathcal{D}$-generic filter $G \subseteq \mathbb{P}$. For example consider $Coll(\omega, \omega_1)$, i.e., the poset of finite partial functions from ω to ω_1, ordered by reverse inclusion. It is easy to see that, for every $\alpha \in \omega_1$ and $n \in \omega$, the set $D_\alpha = \{p : \alpha \in ran(p)\}$ and the set $D^*_n = \{p : n \in dom(p)\}$ are dense in $Coll(\omega, \omega_1)$. But now, if there was a filter $G \subseteq Coll(\omega, \omega_1)$ that intersected all D_α, for $\alpha \in \omega_1$, and all D^*_n, for $n \in \omega$, then $\bigcup G$ would be a bijective function from ω to ω_1; which is clearly impossible.

[15] See [31] for an expanded presentation of Forcing Axioms and their role in foundations of set theory.

However, many independence proofs depends on forcing constructions given by intersecting $\aleph_1$-many dense subsets of a suitable poset [23]. Consequently one of the driving questions which led the research in set theory during the past decades has been to isolate the largest class of posets Γ for which $\mathrm{FA}(\Gamma, \aleph_1)$ can possibly hold. This, indeed, would offer a Forcing Axiom able to settle a vast number of mathematical problems at once.

Shelah, Magidor and Foreman [9] isolated a property of posets $\mathbb{P}$, called[16] *stationary set preserving* and noted $\mathsf{SSP}(\mathbb{P})$, which is provably in ZFC a necessary condition in order for $\mathrm{FA}(\{\mathbb{P}\}, \aleph_1)$ to hold; but they were also able to show that this can also be a sufficient condition. Indeed, if a supercompact cardinal exists[17], then there is a model of ZFC such that the following holds.

(MM) For $\mathbb{P}$ a poset, $\mathrm{FA}(\{\mathbb{P}\}, \aleph_1)$ if and only if $\mathsf{SSP}(\mathbb{P})$.

This principle is known in the literature as *Martin's Maximum*, and is the strongest possible Forcing Axiom for the intersection of $\aleph_1$-many dense sets.

2.2. **Forcing Axioms and AC.** The connection between genericity and arbitrariness that we argued at a conceptual level finds a mathematical counterpart in the possibility to present AC as a collection of Forcing Axioms.

Definition 2.2. *A poset $\mathbb{P}$ is $<\lambda$-closed if whenever $\{p_i : i \in \kappa\}$ is a family, of size $\kappa < \lambda$, of decreasing elements of $\mathbb{P}$: i.e., $p_i \leq p_j$ whenever $i \leq j$, then there is a $p \in \mathbb{P}$, such that $p \leq p_i$, for $i \in \kappa$.*

Let Γ_λ denote the class of posets which are $<\lambda$-closed and let Ω_λ denote the largest possible class for which $\mathrm{FA}(\Omega_\lambda, \lambda)$ holds. Goldblatt in [12], and more recently Todorčević, noted the following interesting result.

Theorem 2.3. *(Goldblatt, Todorčević)* AC *is equivalent to the assertion that, for all cardinals λ,* $\mathrm{FA}(\Gamma_\lambda, \lambda)$ *holds.*

Now, as we noted before, CoT is equivalent to DC, that is equivalent to $\mathrm{FA}(\Gamma_{\aleph_0}, \aleph_0)$ [12], [32]. Notice that, by transitivity of the order relation, every poset is $<\aleph_0$-closed. Therefore $\mathrm{FA}(\Gamma_{\aleph_0}, \aleph_0)$ is nothing else then

[16] $C \subseteq \omega_1$ is a club if it is unbounded in ω_1 and contains the supremum of all its countable subsets; $S \subseteq \omega_1$ is stationary if it meets all the club subsets of ω_1. $\mathsf{SSP}(\mathbb{P})$ holds if, after forcing with $\mathbb{P}$ over a model M of ZFC, every $S \subseteq \omega_1$ that was stationary in M is still stationary in every generic extension of M.

[17] For reason of space we cannot give a definition of supercompact cardinals, but refer the interested reader to the standard reference on large cardinals [19].

$\mathrm{FA}(\Omega_{\aleph_0}, \aleph_0)$ and $\Omega_{\aleph_0}$ is the class of all posets. Consequently $\mathrm{FA}(\Omega_{\aleph_0}, \aleph_0)$ expresses the same content of CoT.

Moreover, it is easy to show that if a poset $\mathbb{P}$ is $< \aleph_1$-closed, then $\mathsf{SSP}(\mathbb{P})$ holds [18]. Therefore, MM is just the best possible strengthening of $\mathrm{FA}(\Gamma_{\aleph_1}, \aleph_1)$ and MM is just $\mathrm{FA}(\Omega_{\aleph_1}, \aleph_1)$. In other terms, the strongest Forcing Axiom for $\aleph_1$ maximizes the class of objects whose existence can be proved with the non-constructive tools that AC offers at the level of $\aleph_1$.

2.3. **Justification of Forcing Axioms.** We got to the point where we can offer a thorough justification of Forcing Axioms in terms of the quasi-combinatorialism expressed by Bernays. His argument was based on three principles, then we now analyze singularly, showing that they offer a sufficient ground for the acceptance of Forcing Axioms as natural extensions of ZFC [30].

First of all, Bernays argued that it is possible to offer infinite determinations that engender new objects in a non-constructive way, since these objects exist independently of, and prior to, our constructions. This is a declaration of realism that justifies the existence of arbitrary sets. As we argued in these pages, generic sets bear a deep conceptual similarity to arbitrary sets, based on their mutual non-constructive character. Moreover, the close connection between Forcing Axioms and AC shows that if we accept the kind of objects that choice principles provide, we should also accept the objects that forcing constructions produce. Therefore, Forcing Axioms and AC should be considered on a par, for what concern the type of mathematical objects they show to exist.

The second principle that backed Bernays' quasi-combinatorial perspective was something akin to Cantor's finitism. Now, the analogy between the finite and the transfinite, and the uniformity of the laws that govern transfinite numbers, is what allows to extend principles that are valid down below the hierarchy of sets to higher levels. Therefore the same arguments used to extend the laws of arithmetic, from the finite to the transfinite case, can be used to justify the extension of a principle valid in ZFC, like $\mathrm{FA}(\Omega_{\aleph_0}, \aleph_0)$, to the next cardinal: that is $\mathrm{FA}(\Omega_{\aleph_1}, \aleph_1)$. Indeed, on the same ground on which we accept an axiom like DC, that shows a maximality principle with respect to existence of arbitrary objects given by $\aleph_0$-many determinations, we should accept MM as a maximality principle with respect to existence of arbitrary objects given by $\aleph_1$-many determinations.

The third pillar on which Bernays' argument rested was the domain principle. This principle allows to consider the totality of sets of a given kind, both arbitrary and non-arbitrary, as a new complete set. This is the ground on which axioms like the Powerset Axiom are justified. Indeed, we see a subset of a set as ranging over the collection of all possible subsets, therefore, by the domain principle, we should accept the existence of this domain of variability. However, this principle does not help much in capturing, mathematically, the notion of powerset. As Ferreirós put it: "[T]he axiom of Powersets postulates a somehow *maximal* set of subsets of any given S, with the maximality remaining fuzzy—or perhaps better, with it remaining an ideal horizon that might even be impossible to make fully concrete in mathematical terms", [7], p. 384.

With respect to this third point, the justification of Forcing Axioms takes into considerations their effects more then their nature[18]. Indeed, Forcing Axioms allow to go beyond ZFC in making mathematically more precise the notion of arbitrary set. In the specific case of the Powerset Axiom, an axiom like MM is able to make more concrete this fuzzy horizon of *all* subsets of $\mathbb{N}$. As a matter of fact MM decides the cardinality of the continuum in the following sense.

Theorem 2.4. *(Foreman, Magidor and Shelah,* [9]*) If* MM *holds then* $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$.

Therefore Forcing Axioms allow to make concrete the imprecise notion of $\mathcal{P}(\mathbb{N})$, thus not only accepting the domain principle, but permitting a better determination of its application to arbitrary subsets of natural numbers.

In conclusion we can say that Forcing Axioms are well-justified from a quasi-combinatorial perspective on mathematical existence and that they consist of maximality principles aimed at making the universe of sets as rich of arbitrary sets as possible. Moreover, Forcing Axioms are able to determine mathematically this richness, giving a determinate size to the powerset of *all* natural numbers. In this sense MM and similar principles should be seen as natural axioms for set theory; axioms that grant the possibility of going beyond the limits of ZFC, giving more precise determinations of the necessarily imprecise notion of arbitrary set.

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[18]We see here a concrete example of a mixture of intrinsic and extrinsic reasons in the process of justification; see [1] for a general discussion on this topic.

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