Completely Real?
A Critical Note on the Theorems by Colbeck and Renner

R. Hermens

Thursday 4th July, 2019

Abstract

In a series of papers Colbeck and Renner (2011, 2015a,b) claim to have shown that the quantum state provides a complete description for the prediction of future measurement outcomes. In this paper I argue that thus far no solid satisfactory proof has been presented to support this claim. Building on the earlier work of Landsman (2015) and Leegwater (2016) I highlight the implicit use of an assumption concerning the way unitary evolution is to be represented in any possible completion of quantum mechanics. I show that the assumption is quite crucial to the proof of the claim and argue that it is unwarranted. I further discuss a possible validation for a restricted version of this assumption that is based on considerations of measurement processes. I argue that also this restricted version is unsatisfactory.

1 Introduction

Can quantum-mechanical description of physical reality be considered complete? In the famous paper with this title, Einstein, Podolsky, and Rosen (1935) argued that the question should be answered in the negative. It was one of several arguments that Einstein devised and although it was presumably among his least favorites (Fine, 2017), it is still the most widely known. Around the same time, von Neumann (1927, 1932) presented a formal argument towards the opposite conclusion. Apart from starting from different assumptions[1], the adopted notions of completeness are also quite distinct (Elby, Brown, and Foster, 1993). While Einstein was concerned with whether the quantum mechanical description sufficed to give a physical *explanation* of the phenomena predicted, von Neumann adopted a more operational approach concerning the question whether the addition of hidden variables could allow for deviating *predictions* for the phenomena. The following question was considered. If we consider an ensemble of systems E described by a pure quantum state ψ, is it possible to decompose this ensemble into sub-ensembles $E_1, E_2, \ldots$ such that the predictions for the sub-ensembles are not equal to the predictions for the total ensemble? If not, then quantum mechanics may be considered complete.[2]

Colbeck and Renner's completeness theorem (Colbeck and Renner, 2011, 2015a,b) alludes to von Neumann's notion of completeness. In their own words, they show that "[u]nder the assumption that measurements can be chosen freely [...] no extension of quantum theory can give more information about the outcomes of future measurements than quantum theory itself" (Colbeck and Renner, 2011, p. 1). The assumption of "free choice" has since been identified as the conjunction of two more familiar assumptions: Parameter Independence and Setting Independence[3] (Ghirardi and Romano, 2013; Vona and Liang, 2014). Despite this clarification, there has been confusion about whether these two assumptions suffice, or if more assumptions that rely on the specific mathematical structure of quantum mechanics are needed. Landsman (2015) gave a critical assessment, arguing that on top of these explicit assumptions, the proof of the ψ-completeness theorem relies on no less than four further rather technical assumptions. A far more friendly conclusion was reached by Leegwater (2016) who gave a thorough reworking of Colbeck and Renner's original proof. However, Landsman's worries were not explicitly addressed by Leegwater, and the proof is not transparent enough to easily assess whether Landsman's criticism was indeed moot.

In this paper I argue that the general conclusion drawn by Colbeck and Renner is currently unwarranted. First, a formal statement of their claim in terms of the ontic models framework is given in section 2. The general strategy for proving the claim is to start with proving it for the special case of two systems in a maximally entangled state. This case is discussed in section 3. In section 4 I discuss how this result is supposed to generalize to the case of a single qubit. The crucial step needed to make that generalization is then scrutinized and criticized in section 5.

[1] See (Dieks, 2017) for a comprehensive account of von Neumann's proof.

[2] Actually, Von Neumann was specifically considering the possibility of dispersion-free sub-ensembles.

[3] Also known as Measurement Independence, λ-Independence, or No Conspiracy.

Disclaimer

This is a shortened version of a paper that is currently in preparation. In the extended version I also discuss a related ψ-ontology theorem by Colbeck and Renner and its relation to their completeness theorem. Although that theorem has been given a rigorous formulation and proof by Leifer (2014), that proof does not trivially extend to a proof of the completeness theorem. What is also missing in this version is an analysis of the extension of Colbeck and Renner's theorem for maximally entangled states, to measurements in the Schmidt-basis for pairs of systems in non-maximally entangled states. This is the part of their proof that relies on the notion of embezzlement.

2 Ontic models and ψ-completeness

Ontic models are a useful tool for studying non-classical features of quantum mechanics. They mimic the structure of a "classical" theory by associating with every system a set Λ of possible ontic states for the system. These ontic states determine how the system is to respond in the case of a measurement, in the sense that with each possible measurement procedure the state λ associates a probability distribution over possible measurement outcomes. When these probabilities are non-trivial, one may think of them encoding as dispositional properties of the system. Possible preparations of the system are associated with probability distributions over the state space (thus Λ is assumed to be a measurable space). These may be thought of as representing ignorance concerning the actual state of the system.

The use of ontic models is not trivially innocuous. Colbeck and Renner themselves use a framework that is more akin to the use of causal networks such as in the work of Wood and Spekkens (2015), so some notes are in order. One reason for considering ontic models to not be general enough, is that they implicitly assume Setting Independence: probability distributions over ontic states are taken to not depend on which measurements are or are not performed on the system. Indeed, this is a common loophole in no-go theorems that may be exploited in, for example, retrocausal approaches (Friederich and Evans, 2019). Although there are ways to generalize the framework to try to accommodate for this loophole[4], there is no need to go into this issue since Setting Independence is accepted as part of the assumptions for the Colbeck Renner theorems.

A reason for preferring ontic models is that it is conceptually clearer. In a causal network approach, all variables are treated on a par. Thus all the use of probability derives from a general probability distribution on some space that is large enough to model all relevant variables as random variables. This means that probability distributions not only specify the probabilities for states and measurement outcomes, but also for which measurements are to be performed. The peculiarity of such a state of affairs has also been considered troublesome by Seevinck and Uffink (2011, p. 438):

> Even quantum mechanics leaves the question what measurement is going to be performed on a system as one that is decided *outside* the theory, and does not specify how much more probable one measurement is than another. It thus seems reasonable not to require from the candidate theories that they describe such probabilities.

But perhaps more objectionable than demanding the well-definedness of probabilities for measurement settings, is that such an approach invites ambiguity concerning the role of probability. In the ontic models there is a separate role for both epistemic and ontological probabilities, but these get intertwined in a causal network approach when the measurement choices correspond to nodes in the network.

The prime constraint for ontic models, is that they can reproduce the quantum mechanical predictions for measurements on any quantum system. Here, with a quantum system we associate a finite-dimensional Hilbert space $\mathcal{H}$. For the set of possible measurements $\mathcal{M}$ we assume that any $M \in \mathcal{M}$ can be represented by a self-adjoint operator A, i.e., we only consider PVMs. For the set of possible preparations $\mathcal{P}$ we assume that any $P \in \mathcal{P}$ can be represented by a density operator ρ. If ρ is pure, we will often used a unit vector ψ to represent the state that satisfies $\rho = [\psi]$ where $[\psi]$ denotes the 1-dimensional projection on the line spanned by ψ. For the set of possible preparations $\mathcal{T}$ we assume that any $T \in \mathcal{T}$ can be represented by a unitary operator U. We allow for contextuality, i.e, the mappings $M \mapsto A$, $P \mapsto \rho$, $T \mapsto U$ will in general be many-to-one. The probabilities for measurement outcomes are given by the Born rule. That is, when P, T, M are represented by ρ, U, A respectively, then

$$\mathbb{P}(a|M, T, P) = \mathrm{Tr}\left(U\rho U^* P_a^A\right), \tag{1}$$

where P_a^A is the projection onto the eigenspace of A corresponding to the eigenvalue a.

[4] See for example (Hermens, 2019).

An ontic model for a quantum system consists of a measurable space of ontic states $(\Lambda_{\mathcal{H}}, \Sigma_{\mathcal{H}})$, where $\Lambda_{\mathcal{H}}$ is the set of ontic states and $\Sigma_{\mathcal{H}}$ is a σ-algebra of subsets of $\Lambda_{\mathcal{H}}$. Every measurement $M \in \mathcal{M}$, is associated with a Markov kernel p_M, called a response function, that associates with every $\lambda \in \Lambda_{\mathcal{H}}$ a probability distribution $\mathrm{p}_M(\,.\,|\lambda)$ over the possible measurement outcomes. Following Leegwater (2016), these probabilities will be called λ-probabilities. Every preparation $P \in \mathcal{P}$ is associated with a probability measure μ_P over the ontic states and every transformation $T \in \mathcal{T}$ is associated with a Markov kernel γ_T from $\Lambda_{\mathcal{H}}$ to itself. On average, the predictions of quantum mechanics are required to be reproduced:

$$\begin{aligned} \mathbb{P}(a|M,P) &= \int \mathrm{p}_M(a|\lambda)\,\mathrm{d}\mu_P(\lambda), \\ \mathbb{P}(a|M,T,P) &= \iint \mathrm{p}_M(a|\lambda)\gamma_T(\,\mathrm{d}\lambda|\lambda')\,\mathrm{d}\mu_P(\lambda'). \end{aligned} \tag{2}$$

Often, when there is no cause for confusion, the use of P, T, M will be replaced by their quantum mechanical representatives, resulting in more transparent equations like

$$\iint \mathrm{p}_A(a|\lambda)\gamma_U(\,\mathrm{d}\lambda|\lambda')\,\mathrm{d}\mu_\psi(\lambda') = \langle U\psi|P_a^A|U\psi\rangle\,. \tag{3}$$

In other cases, the quantum representatives will be added as subscripts. So M_A denotes a measurement procedure represented by the self-adjoint operator A in quantum mechanics.

It is worth noting that transformations can be reconsidered to be part of either the preparation procedure or the measurement procedure. Specifically, for any M_A, T_U we can introduce the response function $\mathrm{p}_{M_A\circ T_U}$ by

$$\mathrm{p}_{M_A\circ T_U}(a|\lambda) := \int \mathrm{p}_{M_A}(a|\lambda')\gamma_{T_U}(\,\mathrm{d}\lambda'|\lambda), \tag{4}$$

which corresponds to an operational procedure for a measurement that is represented quantum mechanically by the operator U^*AU. Likewise, for any T_U, P_ρ we can introduce the probability distribution $\mu_{T_U\circ P_\rho}$ by

$$\mu_{T_U\circ P_\rho}(\Delta) = \int \gamma_{T_U}(\Delta|\lambda)\,\mathrm{d}\mu_{P_\rho}(\lambda),\ \ \Delta \in \Sigma_{\mathcal{H}}, \tag{5}$$

which corresponds to an operational procedure for a preparation of the state represented by $U\rho U^*$. Finally, any two transformations $\gamma_{T_1}, \gamma_{T_2}$ may be stringed together to give the transformation "T_2 after T_1" given by

$$(\gamma_{T_2}\circ\gamma_{T_1})\,(\Delta|\lambda) = \int \gamma_{T_2}(\Delta|\lambda')\gamma_{T_1}(\,\mathrm{d}\lambda'|\lambda). \tag{6}$$

A straightforward ontic model for quantum mechanics is the one by Beltrametti and Bugajski (1995), which is basically quantum mechanics itself. The set of ontic states $\Lambda_{\mathcal{H}}$ is taken to be the set of pure quantum states. The λ-probabilities for a measurement M_A are given by the Born rule, i.e., for each $[\psi] \in \Lambda_{\mathcal{H}}$

$$\mathrm{p}_{M_A}(a|[\psi]) = \langle\psi|P_a^A|\psi\rangle \tag{7}$$

A preparation P_ψ of a pure state corresponds to the Dirac-distribution centered on $[\psi]$, while a preparation of a mixed state corresponds to an appropriate convex combination of such Dirac-distributions. So in general there are multiple distinct distributions μ_ρ corresponding to the same ρ. The quantum dynamics are just copied, i.e,

$$\gamma_U(\Delta|[\psi]) = \begin{cases} 1 & [U\psi] \in \Delta, \\ 0 & \text{otherwise}. \end{cases} \tag{8}$$

This model may rightfully be said to be trivial. In particular because it has the property that the λ-probabilities coincide with the quantum probabilities. This is the key idea to the formal notion of triviality.

Definition 1. An ontic model for a quantum system is said to be *trivial* w.r.t. a set of measurements $\mathcal{M}' \subset \mathcal{M}$ and preparations $\mathcal{P}' \subset \mathcal{P}$ if for every preparation $P_\rho \in \mathcal{P}'$ and every measurement $M_A \in \mathcal{M}'$ the λ-probabilities μ_P-almost surely coincide with the quantum mechanical probabilities:

$$\int \left|\mathrm{p}_{M_A}(a|\lambda) - \mathrm{Tr}\left(\rho P_a^A\right)\right|\,\mathrm{d}\mu_{P_\rho}(\lambda) = 0 \tag{9}$$

or, equivalently,

$$\mu_{P_\rho}\left(\left\{\lambda \in \Lambda_{\mathcal{H}} \,\middle|\, \mathrm{p}_{M_A}(a|\lambda) = \mathrm{Tr}\left(\rho P_a^A\right)\right\}\right) = 1. \tag{10}$$

Colbeck and Renner's completeness theorem may now be formulated as follows:

Claim 1. *For any quantum system where $\mathcal{M}$ covers all PVMs and $\mathcal{P}$ is arbitrary, every ontic model that satisfies Parameter Independence must be trivial w.r.t. $\mathcal{M}$ and $\mathcal{P}$.*

3 A partial proof for Claim 1

The simplest partial proof for Claim 1 concerns local measurements on a qubit pair in a maximally entangled state. The question is then of course how to generalize this to arbitrary states, arbitrary measurements and to higher dimensional Hilbert spaces. Here, I shall only be concerned with a specific step in the generalization process, leaving discussion of the other steps for an extended version of the paper. But first an unambiguous formulation of the partial claim is required.

The scenario is the familiar EPRB setup with associated Hilbert space $\mathcal{H}_2 = \mathbb{C}^2 \otimes \mathbb{C}^2$, and a preparation of the two qubits in the state

$$\psi_2 = \tfrac{1}{2}\sqrt{2}\,(e_1 \otimes e_1 + e_2 \otimes e_2)\,, \tag{11}$$

where e_1, e_2 is an arbitrary orthonormal basis for $\mathbb{C}^2$. One of the qubits is then send to Alice, and the other to Bob, who are assumed to be space-like separated.

Let $\mathcal{M}$ be a set of possible measurements that covers all PVMs. Let $\mathcal{M}_A$ denote the subset of possible measurements where only Alice performs a measurement (locally). These can be represented by self-adjoint operators of the form $A \otimes \mathbb{1}$. Symmetrically, let $\mathcal{B}$ to be the possible measurements where only Bob performs a measurements, which can thus be represented by operators of the form $\mathbb{1} \otimes B$. Finally, take $\mathcal{M}_{\mathrm{LOC}}$ to be the set of measurements where either Alice or Bob performs a measurement or both. So one may take $\mathcal{M}_{\mathrm{LOC}} \simeq \mathcal{M}_A \times \mathcal{M}_B$. Parameter Independence can now be formulated as

$$\begin{aligned} \mathrm{p}_{M_{A\otimes\mathbb{1}}}(a|\lambda) &= \sum_b \mathrm{p}_{M_{A\otimes B}}(a,b|\lambda),\\ \mathrm{p}_{M_{\mathbb{1}\otimes B}}(b|\lambda) &= \sum_a \mathrm{p}_{M_{A\otimes B}}(a,b|\lambda), \end{aligned} \tag{12}$$

for all $M_{A\otimes\mathbb{1}} \in \mathcal{M}_A, M_{\mathbb{1}\otimes B} \in \mathcal{M}_B, M_{A\otimes B} \in \mathcal{M}_{\mathrm{LOC}}$. Here the local operational procedures $M_{A\otimes\mathbb{1}}$ and $M_{\mathbb{1}\otimes B}$ should be the same as those represented by $M_{A\otimes B}$. We then have the following theorem.

Theorem 1. *Any ontic model for the qubit pair where $\mathcal{M}$ covers all local measurements and $\mathcal{P}$ covers the state ψ_2 that satisfies Parameter Independence must be trivial w.r.t. $\mathcal{M}_{\mathrm{LOC}}$ and ψ_2.*

A proof for this theorem can be found in the proof of Theorem 10.4 in (Leifer, 2014). A few remarks concerning possible generalizations are useful. First, it is relatively straightforward to generalize the theorem to cover local measurements on a pair of n-level systems in a maximally entangled state (see (ibid., Theorem 10.7)). Although this is still far removed from a proof of Claim 1, it deserves to be noted that the result is an improvement on earlier work. Stairs (1983) showed that for a qutrit pair in a maximally entangled state there is no value definite ontic model that satisfies Parameter Independence.[5] The work by Colbeck and Renner improves on this result on two accounts: the result also holds for qubits, and not only must ontic models be probabilistic, they must even follow the quantum probability rule. But like the proof of Stairs' theorem, the proof of Theorem 1 relies on the correlations between space-like separated measurements being perfect. It is then perhaps not too surprising that the next generalization of the theorem for arbitrary entangled states only works for local measurements whose corresponding self-adjoint operator is diagonal in some Schmidt basis. The final step to generalize to arbitrary measurements goes hand in hand with the generalization to measurements on a single system that is not entangled. However, for specific measurements on a single qubit the the final step can directly be applied and there is no need to go into the technical notion of embezzlement. This is follows the approach by Landsman (2015). In the next section I will explain how this step is supposed to work.

4 Claim 1 for a Single Qubit

The idea of a completeness proof for single quantum system seems peculiar given that non-trivial ontic models for arbitrary n-level quantum systems have been around since the work of Bell (1966) and Gudder (1970). Trivially, these models may be assumed to satisfy Parameter Independence since there is no second system in play with which it could interact. But this lack of a description of interaction is also just a deficiency of these models. In principle, one can imagine that under certain minimal assumptions on interactions, Parameter Independence becomes applicable and ontic models necessarily have to become trivial. This seems at least to be the aim of Colbeck and Renner, and so it is useful to look if their strategy works for the simplest possible case: a single qubit.

[5] In the present formulation, value definiteness can be understood as the assumption that the λ-probabilities are 0,1-valued. Stair's theorem is now perhaps more widely known as the Free Will Theorem by Conway and Kochen (2006, 2009).

The Hilbert space for a single qubit is $\mathcal{H}_1 = \mathbb{C}^2$. Consider a preparation of the system corresponding to the pure state

$$\psi_1 = \tfrac{1}{2}\sqrt{2}(e_1 + e_2), \tag{13}$$

for some orthonormal basis $\{e_1, e_2\}$. The focus is on a possible measurement represented by the self-adjoint operator $A = a_1[e_1] + a_2[e_2]$ $(a_1 \neq a_2)$. It is straightforward to devise an ontic model for the qubit that is non-trivial with respect to this measurement for this state. That is, a model in which

$$\int \left|\mathrm{p}_A(a_i|\lambda) - \tfrac{1}{2}\right| \mathrm{d}\mu_{\psi_1}(\lambda) \neq 0. \tag{14}$$

This is in striking contrast to the consequence of Theorem 1 that says that for any ontic model for the qubit pair we have

$$\int \left|\mathrm{p}_{A\otimes\mathbb{1}}(a_i|\lambda) - \tfrac{1}{2}\right| \mathrm{d}\mu_{\psi_2}(\lambda) = 0. \tag{15}$$

Could it be possible to argue from this that any ontic model for the single qubit in which (14) is the case should be rejected?

The step needed to make the connection is most explicit in the work of (Leegwater, 2016, §8). Instead of looking at $A \otimes \mathbb{1}$ the focus is on $\mathbb{1} \otimes B$ where $B = b_1[e_1] + b_2[e_2]$. By Theorem 1 the relevant analogue of (15) also holds for measurements represented by this operator. The step is then that since "by definition"[6]

$$\mathbb{P}(a_i|A, \psi_1) = \mathbb{P}(b_i|\mathbb{1} \otimes B, \psi_2) \tag{16}$$

"the same relation holds when considering λ-probabilities:"

$$\mathrm{p}_A^{\psi_1}(a_i|, \lambda) = \mathrm{p}_{\mathbb{1}\otimes B}^{\psi_2}(b_i|\lambda). \tag{17}$$

Presumably, (17) is supposed to imply that (14) cannot hold in any ontic model for a qubit that also allows the qubit to be coupled to another qubit in such a way that all the relevant predictions of quantum mechanics can be reproduced and such that the ontic model satisfies Parameter Independence. Whether such is the case of course depends on what (17) exactly expresses. This is not entirely trivial as a formal definition of the expression is lacking. As a first step in fleshing out what it expresses let us assume that the inference from (16) to (17) is valid if and only if the inference from

$$\mathbb{P}(a_i|A, \psi_1) = \mathbb{P}(a_i|A \otimes \mathbb{1}, \psi_2) \tag{18}$$

to

$$\mathrm{p}_A^{\psi_1}(a_i|\lambda) = \mathrm{p}_{A\otimes\mathbb{1}}^{\psi_2}(a_i|\lambda) \tag{19}$$

is valid. This is reasonable since if one inference holds, then, by Theorem 1, the other holds as well.

In quantum mechanics (18) holds as a consequence of the mathematical structure of the theory and how it deals with composing joint systems out of individual systems. But it is important to note that the equation expresses a numerical equivalence of two probabilities that are defined in separate models. The objects A and ψ_1 strictly belong to the single qubit model while $A \otimes \mathbb{1}$ and ψ_2 belong to the qubit pair model. The fact that the same symbol $\mathbb{P}$ is used on both sides of the equation does not mean that the equation establishes the equality of two values a single function takes on for two distinct arguments; the two functions are distinct. But no such relation is a priori available to link ontic models of bipartite systems to ontic models for their parts, and so it is not clear how one can have an equality between objects from distinct models. A second problem is that the objects on both side of (19) also are not well-defined. As these problems are intertwined I will deal with them simultaneously.

A part of the problem stems from trying to analyze (19) in the language of ontic models, while Colbeck, Renner and Leegwater take a different approach. On their approach, both ψ_1 and λ are possible values for random variables, as is the measurement setting and the measurement outcome. Then $\mathrm{p}_A^{\psi_1}(a_i|\lambda)$ expresses the probability of obtaining the outcome a_i given that the quantum state is ψ_1 and the measurement is A and something else, dubbed λ. It may be possible that they intend the random variable that determines the quantum state to not just range over states for a specific Hilbert space, but also across different Hilbert spaces. Then both ψ_1 and ψ_2 are possible values and so there is a well-defined probability that the system will be a single qubit and a well-defined probability that it will be a qubit pair, determined by the single function $\mathbb{P}$. But this idea will be avoided here.

[6]The notation of the equations has been adjusted to fit the notation in this paper.

Another important distinction is that Colbeck, Renner and Leegwater do not presuppose that λ provides all relevant information concerning the system and so "adding" ψ to λ may give more information about the possible outcomes. This explains the occurrence of quantum states in (19). Informally speaking, the quantum state may also be taken to give relevant information within the use of an ontic model. As it specifies the preparation of the system, it may indicate what behavior of λ may be considered to be "typical". So the notation in (19) may be interpreted as providing a convenient way of specifying a constraint on the possible λ's. A way to make this precise, is to assume (like Landsman (2015)) that equations adopting quantum states express equation that hold μ_{ψ_1}-almost surely, i.e.,

$$\mathrm{p}_A^{\psi_1}(a_i|\lambda) = f(\lambda) \iff \int |\mathrm{p}_A(a_i|\lambda) - f(\lambda)| \, \mathrm{d}\mu_{\psi_1}(\lambda) = 0. \tag{20}$$

This helps in clarifying how to interpret both sides of (19). It does not help yet in interpreting what equality between the two means. A problem with (19) is that it refers to two distinct quantum states (never mind that they also belong to distinct Hilbert spaces). Landsman (ibid.) doubles down on the almost surely interpretation here and proposes the definition

$$\begin{gathered} \mathrm{p}_A^{\psi_1}(a_i|\lambda) = \mathrm{p}_{A\otimes\mathbb{1}}^{\psi_2}(a_i|\lambda) \\ \iff \\ \forall f : \mathrm{p}_A^{\psi_1}(a_i|\lambda) = f(\lambda) \text{ iff } \mathrm{p}_{A\otimes\mathbb{1}}^{\psi_2}(a_i|\lambda) = f(\lambda). \end{gathered} \tag{21}$$

A first problem here is that, given any pair of ontic models for the two considered systems, it is to be expected that the λ's on both sides of the equation of (19) do not refer to the same thing. But even when that issue is resolved, (21) is an unreasonably strong assumption, as it implies that if (19) holds, then $p_A(a_i|\lambda) = p_{A\otimes\mathbb{1}}(a_i|\lambda)$ both μ_{ψ_1}-almost surely and μ_{ψ_2}-almost surely. Moreover, distributions for μ_{ψ_1} and μ_{ψ_2} completely overlap on the region of Λ where $\mathrm{p}_A(a_i|\lambda)$ is non-zero.[7] A solution to the problem can be found in the work of Leegwater (2016), and it overlaps with the solution to the double use of λ for physically distinct systems. In the discussion of some of the notation used similar to that occurring in (19), Leegwater notes

> here λ still refers to the variable assigned to system A when it was in the state $|\psi\rangle_A$ [...] λ always refers to the original system A, and there is only one measure $\mu(\lambda)$ that is considered. [p.21]

The measure μ refers to a probability distribution over Λ that may depend on $|\psi\rangle_A$ as well as other factors. Given that there is only one measure considered, which is related to a particular quantum state, all other quantum states should be understood as being arrived at after interactions with a system prepared according to $|\psi\rangle_A$.

Translating this to the present discussion, we find that $\mathrm{p}_{A\otimes\mathbb{1}}^{\psi_2}(a_i|\lambda)$ refers to a single qubit that is prepared in the quantum state ψ_1 and having ontic state λ, then is being coupled to a second qubit with unknown state and then a unitary transformation is performed on the joint system such that the resulting quantum state is ψ_2. To deal with the coupling, one needs a rule to relate an ontic model for a single system to that of a combined system. One way to do this, is to adopt the Preparation Independence assumption from the PBR-theorem. I will however follow a more economic approach based on (Leifer, 2014, §8.2): appending an ancilla is modeled by a Markov kernel from the ontic state space of the single system $\Lambda_{\mathcal{H}_1}$ to the ontic state space of the combined system $\Lambda_{\mathcal{H}_2}$.

For sake of definiteness, let us assume that the second qubit was prepared according to the quantum state ϕ. Appending this system to the first qubit is then modeled by a Markov kernel γ_ϕ in such a way that the resulting measure on $\Lambda_{\mathcal{H}_2}$ given by $\int_{\Lambda_{\mathcal{H}_1}} \gamma_\phi(\,.\,|\lambda) \, \mathrm{d}\mu_{\psi_1}(\lambda)$ models a preparation of the state $\psi_1 \otimes \phi$ for the joint system. Now let U be a unitary operator that takes $\psi_1 \otimes \phi$ to ψ_2 and let γ_U be a Markov kernel on $\Lambda_{\mathcal{H}_2}$ that models it. We can now give a proper reformulation of (19):

$$\mathrm{p}_A(a_i|\lambda) = \iint \mathrm{p}_{A\otimes\mathbb{1}}(a_i|\lambda'')\gamma_U(\,\mathrm{d}\lambda''|\lambda')\gamma_\phi(\,\mathrm{d}\lambda'|\lambda) \tag{22}$$

μ_{ψ_1}-almost surely. In the next section I argue that the validity of this equality is non-trivial and should be distrusted.

[7] This may be seen as follows. The first claim follows from evaluating (21) for the choice $f(\lambda) = p_A(a_i|\lambda)$. The second claim follows with a proof from contradiction. If there exists a $\Delta \subset \Lambda$ such that $\mu_{\psi_2}(\Delta) > 0 = \mu_{\psi_1}(\Delta)$ and on which $\mathrm{p}_{A\otimes\mathbb{1}}(a_i|\lambda)$ is non-zero, then for any f such that $\mathrm{p}_A^{\psi_1}(a_i|\lambda) = f(\lambda)$ and $\mathrm{p}_{A\otimes\mathbb{1}}^{\psi_2}(a_i|\lambda) = f(\lambda)$ one can define an f' that differs from f only on Δ in such a way that $\mathrm{p}_{A\otimes\mathbb{1}}^{\psi_2}(a_i|\lambda) = f'(\lambda)$ no longer holds. So (19) would fail.

5 Unitary Processes and Measurements

It is not obvious from the work from Colbeck, Renner and Leegwater what should be taken to be the main argument for the inference from (18) to (22). To get a handle on what it takes for (22) to hold, I show that it can be derived with the help of two assumptions:

- **Ancilla Independence.** λ-probabilities for single systems arise as averages of λ-probabilities for local measurements on a joint system:

$$\mathrm{p}_{M_A}(a|\lambda) = \int \mathrm{p}_{M_{A\otimes\mathbb{1}}}(a|\lambda')\gamma_\phi(\,\mathrm{d}\lambda'|\lambda), \tag{23}$$

  where p_{M_A} and $\mathrm{p}_{M_{A\otimes\mathbb{1}}}$ are taken to represent the same operational procedure of a measurement performed on the first system.

- **Unitary Faithfulness.** Probabilities that are invariant under a unitary operation in quantum mechanics are also invariant at the ontic level:

$$\mathrm{Tr}\left(\rho P_a^A\right) = \mathrm{Tr}\left(U\rho U^* P_a^A\right) \implies \mathrm{p}_{M_A}(a|\lambda) = \int \mathrm{p}_{M_A}(a|\lambda')\gamma_{T_U}(\,\mathrm{d}\lambda'|\lambda)\ \mu_{P_\rho}\text{-a.s.}. \tag{24}$$

With these assumptions (22) can indeed be derived:

$$\begin{aligned}
\int_{\Lambda_{\mathcal{H}_1}} & \left| \mathrm{p}_A(a_i|\lambda) - \iint_{\Lambda_{\mathcal{H}_2}} \mathrm{p}_{A\otimes\mathbb{1}}(a_i|\lambda'')\gamma_U(\,\mathrm{d}\lambda''|\lambda')\gamma_\phi(\,\mathrm{d}\lambda'|\lambda) \right| \mathrm{d}\mu_{\psi_1}(\lambda) \\
&= \int_{\Lambda_{\mathcal{H}_1}} \left| \int_{\Lambda_{\mathcal{H}_2}} \mathrm{p}_{A\otimes\mathbb{1}}(a_i|\lambda')\gamma_\phi(\,\mathrm{d}\lambda'|\lambda) - \iint_{\Lambda_{\mathcal{H}_2}} \mathrm{p}_{A\otimes\mathbb{1}}(a_i|\lambda'')\gamma_U(\,\mathrm{d}\lambda''|\lambda')\gamma_\phi(\,\mathrm{d}\lambda'|\lambda) \right| \mathrm{d}\mu_{\psi_1}(\lambda) \\
&\leq \int_{\Lambda_{\mathcal{H}_1}} \int_{\Lambda_{\mathcal{H}_2}} \left| \mathrm{p}_{A\otimes\mathbb{1}}(a_i|\lambda') - \int_{\Lambda_{\mathcal{H}_2}} \mathrm{p}_{A\otimes\mathbb{1}}(a_i|\lambda'')\gamma_U(\,\mathrm{d}\lambda''|\lambda') \right| \gamma_\phi(\,\mathrm{d}\lambda'|\lambda)\,\mathrm{d}\mu_{\psi_1}(\lambda) \\
&= \int_{\Lambda_{\mathcal{H}_2}} \left| \mathrm{p}_{A\otimes\mathbb{1}}(a_i|\lambda') - \int_{\Lambda_{\mathcal{H}_2}} \mathrm{p}_{A\otimes\mathbb{1}}(a_i|\lambda'')\gamma_U(\,\mathrm{d}\lambda''|\lambda') \right| \mathrm{d}\mu_{\psi_1\otimes\phi}(\lambda) = 0.
\end{aligned} \tag{25}$$

A possible motivation for Ancilla Independence is the idea that the act of “appending an ancilla” does not itself disturb the initial system but merely concerns a re-description of the initial system. On this reading Ancilla Independence is the criterion that a model for a joint system should be at least as expressive as a model for any of its components. I do not have a tentative motivation for Unitary Faithfulness. This is problematic, as it has immediate consequences that are quite crucial when it comes to proving completeness. Note that completeness boils down to the idea that response functions are dispersion free under preparations of pure quantum states. Now suppose $\langle\psi|P_a^A|\psi\rangle = \langle U\psi|P_a^A|U\psi\rangle$, Unitary Faithfulness then has the following consequence for the dispersion of any corresponding response function:

$$\begin{aligned}
\mathrm{Var}_\psi(\mathrm{p}_A(a|\,.\,)) &= \int \mathrm{p}_A(a|\lambda)^2\,\mathrm{d}\mu_\psi(\lambda) - \left(\int \mathrm{p}_A(a|\lambda)\,\mathrm{d}\mu_\psi(\lambda)\right)^2 \\
&= \int \left(\int \mathrm{p}_A(a|\lambda')\gamma_U(\,\mathrm{d}\lambda'|\lambda)\right)^2 \mathrm{d}\mu_\psi(\lambda) - \left(\iint \mathrm{p}_A(a|\lambda')\gamma_U(\lambda'|\lambda)\,\mathrm{d}\mu_\psi(\lambda)\right)^2 \\
&\leq \iint \mathrm{p}_A(a|\lambda')^2\gamma_U(\,\mathrm{d}\lambda'|\lambda)\,\mathrm{d}\mu_\psi(\lambda) - \left(\iint \mathrm{p}_A(a|\lambda')\gamma_U(\lambda'|\lambda)\,\mathrm{d}\mu_\psi(\lambda)\right)^2 = \mathrm{Var}_{U\psi}(\mathrm{p}_A(a|\,.\,))
\end{aligned} \tag{26}$$

It follows that if one can show that for some quantum state $U\psi$ the response function is dispersion free for a particular measurement, then the response function must also be dispersion free for any other quantum state ψ whenever the operational probabilities for the measurement are equal. Since this gets at the heart of what the Colbeck-Renner theorem states, this implicit assumption should have at least been explicitly stated and preferably be well-motivated. Neither appears to be the case though.

A possible explanation is that perhaps the inference from (18) to (22) is not assumed to hold in general, but only under circumstances that make it more plausible. An appeal to measurement procedures could perhaps do the trick. According to quantum theory, instead of directly measuring A on the qubit in the state ψ_1, one may equivalently first entangle it with a second system to obtain the joint state ψ_2 and measure instead B on the second system. If we assume this to be a legitimate procedure generally also for the ontic model, we are just assuming Unitary Faithfulness. But we may instead focus on the case where system B is a measurement device designed to measure A on the qubit. This is indeed the scenario within which the crucial inference is considered in (Colbeck and Renner, 2011, p.4-5) and (Leegwater, 2016, §8).

Arguably, an appeal to what actually happens in a measurement process is not a very elegant strategy in a theory that is infamous for its measurement problem. The transition from ψ_1 to ψ_2 is of course well-known as part of the von Neumann measurement scheme. Although that scheme is idealized, that will not concern us here. Indeed, Colbeck and Renner also consider situations with interaction with the environment.

The well-known problem is that the state ψ_2, on its own, does not signify a situation in which a measurement outcome is obtained. Ontic models are usually not designed to resolve this problem. The Beltrametti-Bugajski model illustrates this nicely. More generally, given any ontic model for a single qubit, a peculiarity is that probabilities for outcomes of measurements are well-defined without any mentioning of how the qubit would interact with a measurement apparatus. Ideally, the λ-probability $\mathrm{p}_M(a|\lambda)$ encodes the probability with which the system, upon interaction with an appropriate measurement apparatus, would evolve towards a joint state in which the measurement apparatus can be taken to be in a well-defined pointer state displaying the outcome of the measurement. Explicitly, if $\Delta_a^M \subset \Lambda_{\mathcal{H}_2}$ denotes the set of states of the joint system in which the apparatus displays the outcome a, then we may assume

$$\mathrm{p}_M(a|\lambda) = \gamma_M\left(\Delta_a^M|\lambda\right), \tag{27}$$

where γ_M is a Markov kernel that models the interaction with the measurement apparatus.

The question is now how this measurement process γ_M relates to the von Neumann-type process described by $\gamma_U \circ \gamma_\phi$. In a spontaneous collapse theory, the macroscopic superposition ψ_2 is extremely unlikely to obtain and γ_M contains a collapse way before the unitary evolution U is completed. Consequently, an appeal to measurements is irrelevant for the question whether Claim 1 applies to such theories and a general assumption like Unitary Faithfulness is required.

To be fair, in the first paper Colbeck and Renner (2011) did assume that "all processes within quantum theory can be considered as unitary evolutions". Although this explicit statement is dropped in any of the following papers, perhaps their result should be taken to apply only to no-collapse ontic models. In this case we may have that $\gamma_M = \gamma_U \circ \gamma_\phi$. A solution to the measurement problem in the form of satisfying (27) is then only possible if the ontic model is non-trivial.[8] Bohmian mechanics is the most familiar theory in this style. But although it violates Parameter Independence, it may serve as a source of inspiration if the logic behind measurement-based argument for Claim 1 can be closed.

It is worthwhile to quickly recap what the argument now has become. The procedure to measure A on a qubit, involves hooking it up to a measurement device, evolving the joint system to ψ_2 and then reading of the value of B from the device. Reading of this value constitutes a measurement of B on the measurement device. Because in a no-collapse theory this is roughly what it means to measure A on a qubit, the outcome statistics for B on the joint system should equal the outcome statistics for A on the singe qubit. But, if in addition the criteria for Theorem 1 are in place, then the outcome statistics for B should be equal to the quantum statistics and so also the statistics for A should equal the quantum statistics.

Applicability of Theorem 1 requires Parameter Independence. Thus we need to assume that it is possible to postpone the B-measurement until qubit and apparatus are again spatially separated. Moreover, at this stage it should also be possible to refrain from performing a B-measurement, and instead perform any other measurement and, in addition, to perform any other measurement on the qubit as well. If not, the chained Bell-inequalities upon which the proof of Theorem 1 rests cannot come of the ground.

This scenario seems quite unlikely when one thinks about how spin measurements work in Bohmian mechanics (Norsen, 2014). In the Stern-Gerlach setup, the spin degree of freedom becomes encoded in the trajectory of the silver atom. A position measurement then yields the outcome of the spin measurement. But there is of course no sense in which at that point one can still also measure the spin in another direction whilst not affecting the particle position. Perhaps then the right system to figure as apparatus is the screen. While the particles of the screen become entangled with the silver atom, it is hard to imagine how they may be separated again from the silver atom in such a way that still different spin measurements can be performed on the atom, as well as different measurements on the screen. One may complaint that in principle the separation of the two systems once in the state ψ_2 should be possible and that in principle the required measurements are possible. But I do not see how that argument would amount to anything more substantial than the idea that in principle Unitary Faithfulness should just be true.

[8] Arguably, taking an Everett-type approach to solving the measurement problem takes one out of the frameworks of ontic models and causal networks, as these frameworks in a way presuppose a single world universe.

What lies at the heart of the problem here, is that in Bohmian mechanics, particles do not have spin properties in any classical sense. Necessarily so, given the Kochen-Specker theorem. Instead, spin properties correspond to dispositional properties and the physical quantities are only meaningful in the context of an experimental setup that can be used to reveal them (Goldstein, 2017, §10). This is also the way to think about response functions in ontic models: they encode dispositional properties that can come to the for in a measurement scenario. But once brought about, it is not per se meaningful to say that dispositional properties before measurement encoded by other response functions are still there. This idea is of course very Bohrian in spirit. In fact, Bohr's response to Einstein, Podolsky and Rosen, when taken out of context, seems strangely apt to qualify the issue regarding the stage when ψ_2 obtains, if one changes just one word:

> But even at this stage there is essentially the question of *an influence on the very conditions which define the possible types of predictions regarding the future behavior of the system.* Since these conditions constitute an inherent element of the description of any phenomenon to which the term "physical reality" can properly be attached, we see that the argumentation of the mentioned authors does not justify their conclusion that the quantum-mechanical description is essentially [complete]. (Bohr, 1935, p.700)

References

Bell, J. S. (1966). "On the Problem of Hidden Variables in Quantum Mechanics". *Review of Modern Physics* 38 (3), pp. 447–452.

Beltrametti, E. G. and S. Bugajski (1995). "A classical extension of quantum mechanics". *Journal of Physics A: Mathematical and General* 28 (12), pp. 3329–3343.

Bohr, N. (1935). "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?" *Physical Review* 48 (8), pp. 696–702.

Colbeck, R. and R. Renner (2011). "No extension of quantum theory can have improved predictive power". *Nature communications* 2, p. 411. arXiv: `1005.5173 [quant-ph]`.

— (2015a). "On the Sufficiency of the Wavefunction". *The Message of Quantum Science*. Ed. by P. Blanchard and J. Fröhlich. Berlin: Springer.

— (2015b). "The Completeness of Quantum Theory for Predicting Measurement Outcomes". *Quantum Theory: Informational Foundations and Foils*. Ed. by G. Chiribella and R. W. Spekkens. Dordrecht: Springer. arXiv: `1208.4123 [quant-ph]`.

Conway, J. and S. Kochen (2006). "The Free Will Theorem". *Foundations of Physics* 36 (10), pp. 1441–1473. arXiv: `quant-ph/0604079`.

— (2009). "The Strong Free Will Theorem". *Notices of the AMS* 56 (2), pp. 226–232. arXiv: `arXiv:0807.3286`.

Dieks, D. (2017). "Von Neumann's impossibility proof: Mathematics in the service of rhetorics". *Studies in History and Philosophy of Modern Physics* 60, pp. 136–148. arXiv: `1801.09305 [physics.hist-ph]`.

Einstein, A., B. Podolsky, and N. Rosen (1935). "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?" *Physical Review* 47 (10), pp. 777–780.

Elby, A., H. R. Brown, and S. Foster (1993). "What Makes a Theory Physically "Complete"?" *Foundations of Physics* 23 (7).

Fine, A. (2017). "The Einstein-Podolsky-Rosen Argument in Quantum Theory". *The Stanford Encyclopedia of Philosophy*. Ed. by E. N. Zalta. URL: `https://plato.stanford.edu/archives/win2017/entries/qt-epr/`.

Friederich, S. and P. W. Evans (2019). "Retrocausality in Quantum Mechanics". *The Stanford Encyclopedia of Philosophy*. Ed. by E. N. Zalta. URL: `https://plato.stanford.edu/archives/sum2019/entries/qm-retrocausality/`.

Ghirardi, G. and R. Romano (2013). "About Possible Extensions of Quantum Theory". *Foundations of Physics* 43, pp. 881–894. arXiv: `1301.5040 [quant-ph]`.

Goldstein, S. (2017). "Bohmian Mechanics". *The Stanford Encyclopedia of Philosophy*. Ed. by E. N. Zalta. URL: `https://plato.stanford.edu/archives/sum2017/entries/qm-bohm/`.

Gudder, S. P. (1970). "On hidden variable theories". *Journal of Mathematical Physics* 11, pp. 431–436.

Hermens, R. (2019). "An Operationalist Perspective on Setting Dependence". *Foundations of Physics* 49.3, pp. 260–282.

Landsman, K. (2015). "On the Colbeck-Renner theorem". *Journal of Mathematical Physics* 56, p. 122103.

Leegwater, G. (2016). "An impossibility theorem for parameter independent hidden variable theories". *Studies in History and Philosophy of Modern Physics* 54, pp. 18–34. arXiv: `1705.08341 [quant-ph]`.

Leifer, M. S. (2014). "Is the quantum state real? An extended review of ψ-ontology theorems". *Quanta* 3.1, pp. 67–155.

Neumann, J. von (1927). "Wahrscheinlichkeitstheoretischer Aufbau der Quantenmechanik". *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, pp. 245–272.

— (1932). *Mathematische Grundlagen der Quantenmechanik*. Berlin: Springer.

Norsen, T. (2014). "The pilot-wave perspective on spin". *American Journal of Physics* 82, pp. 337–348. arXiv: 1305.1280 [quant-ph].

Seevinck, M. P. and J. Uffink (2011). "Not Throwing out the Baby with the Bathwater: Bell's Condition of Local Causality Mathematically 'Sharp and Clean'". *Explanation, Prediction, and Confirmation*. Ed. by D. Dieks, W. J. Gonzalez, S. Hartmann, T. Uebel, and M. Weber. Dordrecht: Springer, pp. 425–450.

Stairs, A. (1983). "Quantum Logic, Realism, and Value Definiteness". *Philosophy of Science* 50 (4), pp. 578–602.

Vona, N. and Y.-C. Liang (2014). "Bell's theorem, accountability and nonlocality". *Journal of Physics A: Mathematical and Theoretical* 47.42, p. 424026. arXiv: 1307.4365 [quant-ph].

Wood, C. J. and R. W. Spekkens (2015). "The lesson of causal discovery algorithms for quantum correlations: causal explanations of Bell-inequality violations require fine-tuning". *New Journal of Physics* 17, p. 033002. arXiv: 1208.4119 [quant-ph].