# Three Ways of Being Non-Material

Vincenzo Crupi, Andrea Iacona

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This paper presents a novel unified account of three distinct non-material interpretations of 'if then': the suppositional interpretation, the evidential interpretation, and the strict interpretation. We will spell out and compare these three interpretations within a single formal framework which rests on fairly uncontroversial assumptions, in that it requires nothing but propositional logic and the probability calculus. As we will show, each of the three intrerpretations exhibits specific logical features that deserve separate consideration. In particular, the evidential interpretation as we understand it — a precise and well defined version of it which has never been explored before — significantly differs both from the suppositional interpretation and from the strict interpretation.

## 1 Preliminaries

Although it is widely taken for granted that indicative conditionals as they are used in ordinary language do not behave as material conditionals, there is little agreement on the nature and the extent of such deviation. Different theories tend to privilege different intuitions about conditionals, and there is no obvious answer to the question of which of them is the correct theory. In this paper, we will compare three interpretations of 'if then': the suppositional interpretation, the evidential interpretation, and the strict interpretation. These interpretations may be regarded either as three distinct meanings that ordinary speakers attach to 'if then', or as three ways of explicating a single indeterminate meaning by replacing it with a precise and well defined counterpart.

Here is a rough and informal characterization of the three interpretations. According to the suppositional interpretation, a conditional is acceptable when its consequent is credible enough *given* its antecedent. That is, on the supposition that its antecedent holds, there are good chances that its consequent holds. According to the evidential interpretation, a conditional is acceptable when its antecedent *supports* its consequent. This is to say that its antecedent provides a reason to accept its consequent, so that its consequent holds at least in part because its antecedent holds. Finally, according to the strict interpretation, a conditional is acceptable when its antecedent *necessitates* its consequent. That is, if its antecedent holds, then its consequent must hold as well.

Each of these three interpretations differs from the material interpretation that we find in logic textbooks, according to which a conditional amounts to the disjunction of its negated antecedent and its consequent. Assuming that $>$ stands for 'if then', we will take for granted that, although in some cases it is plausible to read $>$ as $\supset$, in other cases it is not. The three interpretations that will be considered are three coherent *non-material* readings of $>$. Each of them accords to some extent with the ordinary uses of 'if then', so is able to explain to some extent the linguistic evidence concerning conditionals.

In the history of the debate on conditionals, these three interpretations have emerged in various ways, and they have prompted different formal accounts of conditionals. The suppositional interpretation has been articulated by Adams and others by developing a logic based on a probabilistic semantics[1]. The strict interpretation, instead, is naturally treated in terms of standard modal logic. The evidential interpretation is harder to capture at the formal level. The idea of support is a slippery fish, and the same goes for its closest relatives, such as the idea of reason or the meaning of 'because'. This explains the heterogeneity and the multiplicity of the attempts that have been made to define a conditional with such a property[2].

In what follows, these three interpretations — in fact three precise and well defined counterparts of them — will be spelled out within a single formal framework that rests on fairly uncontroversial assumptions. Basically, all we need as a background theory is propostional logic and the probability calculus. What makes this framework insteresting, among other things, is that it provides a good basis for comparing the three interpretations and elucidating the logical relations between them. Of course, it is not the only possible way to do it, and it is not necessarily the best. But as far as we know, no alternative account has been developed so far.

In terms of novelty, the most important of the three formal accounts that will be outlined is the second. This account delineates a specific version of the evidential interpretation which has never been explored before. The third account is also important, for it provides a probabilistic reduction of the strict conditional, showing how the same logic can be obtained by a different semantics. The first account, instead, is already well understood, and will be outlined mainly for the sake of contrasting it with the other two.

The structure of the paper is as follows. Section 2 introduces a language that includes, in addition to the usual symbol $\supset$, three symbols $\Rightarrow, \triangleright, \prec$ that stand respectively for the suppositional conditional, the evidential conditional, and the strict conditional.[3] Section 3 outlines a set of principles of conditional logic and states some important relations between them. Sections 4-6 explain, for each of the three symbols, what kind of considerations can justify its use, and how it behaves with respect to the principles outlined. Section 7 adds some general reflections about the relations between the three interpretations considered. The remaining part of the paper is a technical appendix which contains the proofs of all the facts set out in the previous sections.

[1] Adams [1].

[2] McCall [25] describes some such attempts, which belong to the family of connexive logics. Another example is Rott [26], which contains a pioneering discussion of 'if' and 'because', relying on a variation of the belief revision formalism. The ranking-theoretic account offered in Spohn [29] explicitly involves the idea of the antecedent as providing a reason for the consequent. Finally, the approach to conditionals outlined in Douven [11] and Douven [12], which is the closest precedent of our analysis, employs the notion of evidential support in Bayesian epistemology.

[3] The symbol $\Rightarrow$ is used exactly as in Adams. The symbol $\triangleright$ is borrowed from Spohn [29]. The symbol $\prec$ is a tribute to the seminal work on strict implication in Lewis [22].

## 2 The language L

Let **P** be a propositional language whose vocabulary is constituted, as usual, by a set of sentence letters $p, q, r...$, the connectives $\sim, \supset, \wedge, \vee$, and the brackets $(,)$. **P** forms the basis of the symbolic apparatus set out in this section. From now on, we will call *propositional formulas* the formulas of **P**. Moreover, we will use the symbols $\vDash_{PL}$ and $\equiv_{PL}$ to indicate logical consequence and logical equivalence in **P**.

Let **L** be a language with the following vocabulary:

$p, q, r, ...$

$\sim, \supset, \wedge, \vee$

$\Box$

$\Rightarrow, \rhd, \dashv\!\!\ni$

$(,)$

The formulas of **L** are defined by induction as follows:

**Definition 1.**

*1 If* $\alpha \in \mathbf{P}$*, then* $\alpha \in \mathbf{L}$;

*2 if* $\alpha \in \mathbf{P}$*, then* $\Box\alpha \in \mathbf{L}$;

*3 if* $\alpha \in \mathbf{P}$ *and* $\beta \in \mathbf{P}$*, then* $\alpha \Rightarrow \beta \in \mathbf{L}$;

*4 if* $\alpha \in \mathbf{P}$ *and* $\beta \in \mathbf{P}$*, then* $\alpha \rhd \beta \in \mathbf{L}$;

*5 if* $\alpha \in \mathbf{P}$ *and* $\beta \in \mathbf{P}$*, then* $\alpha \dashv\!\!\ni \beta \in \mathbf{L}$;

*6 if* $\alpha \in \mathbf{L}$*, then* $\sim \alpha \in \mathbf{L}$.

Note that clause 2 rules out multiple occurrences of $\Box$ in the same formula. For example, $\Box(p \wedge \Box q)$ is not a formula of **L**. Similarly, clauses 3-5 rule out multiple occurrences of $\Rightarrow, \rhd, \dashv\!\!\ni$ in the same formula. For example, $p \Rightarrow (p \Rightarrow q)$ is not a formula of **L**. Moreover, since clause 6 is the only clause that applies to formulas of **L**, $\sim$ is the only connective whose scope can include the scope of $\Box, \Diamond, \Rightarrow, \rhd, \dashv\!\!\ni$. Although the expressive power of **L** is limited in some respects, its syntax is sufficient to express all the logical principles that are relevant for our current purposes.

The idea that underlies the semantics of **L** is the same that has inspired Adams and many others after him, the idea that the assertability of a sentence is a function of the probability of its propositional constituents, that is, the constituents that are adequately formalized in a standard propositional language.[4] As we shall see, a valuation function can be defined for the formulas of **L** in such a way that, for any formula $\alpha$, the value that the function assigns to $\alpha$ — which may be understood as the degree of assertability of $\alpha$ — depends on the probability of the propositional formulas that occur in $\alpha$.

For any probability function $P$ defined over **P**, the valuation function $V_P$ is defined as follows:

[4] Adams [1], Adams [2].

**Definition 2.**

*1 For every* $\alpha \in \mathbf{P}$, $V_P(\alpha) = P(\alpha)$*;*

*2* $V_P(\Box\alpha) = 1$ *iff* $P(\alpha) = 1$, *otherwise* $V_P(\Box\alpha) = 0$*;*

*3* $V_P(\alpha \Rightarrow \beta) = P(\beta|\alpha)$, *with* $V_P(\alpha \Rightarrow \beta) = 1$ *if* $P(\alpha) = 0$*;*

*4* $V_P(\alpha \rhd \beta) = \frac{P(\beta|\alpha) - P(\beta)}{1 - P(\beta)}$ *if* $P(\beta|\alpha) \geq P(\beta) > 0$, *with* $V_P(\alpha \rhd \beta) = 1$ *if* $P(\beta) = 1$ *or* $P(\alpha) = 0$, *otherwise* $V_P(\alpha \rhd \beta) = 0$*;*

*5* $V_P(\alpha \dashv\!\!\! 3\, \beta) = 1$ *if* $P(\beta|\alpha) = 1$, *with* $V_P(\alpha \dashv\!\!\! 3\, \beta) = 1$ *if* $P(\alpha) = 0$, *otherwise* $V_P(\alpha \dashv\!\!\! 3\, \beta) = 0$*;*

*6* $V_P(\sim \alpha) = 1 - V_P(\alpha)$.

Clause 1 says that $V_P$ assigns to the propositional formulas the same values as $P$. This means that the degree of assertability of any propositional formula amounts to its probability.

Clause 2 says that a formula $\Box\alpha$ takes either 1, the maximal value, or 0, the minimal value, depending on whether or not $P(\alpha) = 1$.

Clause 3 says that the value that $V_P$ assigns to $\alpha \Rightarrow \beta$ is $P(\beta|\alpha)$, the conditional probability of $\beta$ given $\alpha$, with the proviso that $V_P(\alpha \Rightarrow \beta) = 1$ if $P(\alpha) = 0$ (normally, $P(\beta|\alpha)$ would be undefined in that case). So, the degree of assertability of the conditional is the conditional probability of its consequent given its antecedent, as has been suggested by Adams.[5]

Clause 4 deserves more attention. It says that the value that $V_P$ assigns to $\alpha \rhd \beta$ is the degree of positive evidential support (if any) that the antecedent provides to the consequent according to $P$ (with the proviso that $V_P(\alpha \rhd \beta) = 1$ if $P(\beta) = 1$ or $P(\alpha) = 0$, in which case the evidential support measure loses mathematical meaning). This postulate conveys the key assumption that the antecedent has to "make a difference", namely, to contribute positively to the credibility of the consequent, because the measure employed to quantify $V_P(\alpha \rhd \beta)$ gives a definite positive value if and only if $P(\beta|\alpha) > P(\beta)$. The latter condition is a straigthforward characterization of positive probabilistic relevance, and also the standard qualitative definition of evidential support (or incremental confirmation) in a Bayesian framework. It is well known that several quantitative measures retain this fundamental idea, and here we will make no attempt to justify our specific choice, which has been spelled out and defended elsewhere. We will simply point out that there is a coherent sense in which the measure so defined characterizes positive evidential impact as the degree of partial entailment of $\beta$ by $\alpha$. In this sense, the degree of assertability of an evidential conditional may be understood as the degree of partial entailment — if any — from the antecedent to the consequent.[6]

Clause 5 says that the value that $V_P$ assigns to $\alpha \dashv\!\!\! 3\, \beta$ is 1 when $P(\beta|\alpha) = 1$, otherwise it is 0, with the proviso that $V_P(\alpha \dashv\!\!\! 3\, \beta) = 1$ if $P(\alpha) = 0$.

Clause 6 defines negation in the classical way, as it entails that the value of $\sim \alpha$ is 1 when the value of $\alpha$ is 0, and that the value of $\sim \alpha$ is 0 when the value

[5] Adams [2]. About the stipulation that $V_P(\alpha \Rightarrow \beta) = 1$ if $P(\alpha) = 0$, see Adams [3], p. 150.

[6] The details are in Crupi and Tentori [9] and Crupi and Tentori [10]. Related ideas are thoroughly discussed in Douven [11] and Douven [12].

of $\alpha$ is 1. In particular, when $V_P(\sim \Box \sim \alpha) = 1$, we get that $V_P(\Box \sim \alpha) = 0$, which means that $\sim \alpha$ is not necessary, hence that $\alpha$ is possible. This shows that $\Diamond\alpha$ can be defined in the usual way as $\sim \Box \sim \alpha$.

Note that clauses 3-5 entail that each of the three conditionals defined is fully assertable whenever $\alpha \vDash_{PL} \beta$. In fact, suppose that $\alpha \vDash_{PL} \beta$. Then, if $P(\alpha) = 0$, $V_P(\alpha \Rightarrow \beta) = V_P(\alpha \triangleright \beta) = V_P(\alpha \dashv\!3\, \beta) = 1$. If $P(\alpha) > 1$, then again $V_P(\alpha \Rightarrow \beta) = V_P(\alpha \triangleright \beta) = V_P(\alpha \dashv\!3\, \beta) = 1$, because $P(\beta|\alpha) = 1$. Note also that, no matter whether $>$ is replaced by $\Rightarrow$, $\triangleright$, or $\dashv\!3$, we have that $V_P(\top > \top) = V_P(\bot > \top) = V_P(\bot > \bot) = 1$, while $V_P(\top > \bot) = 0$.

Since the semantics of **L** is given in terms of assertability, validity in **L** will be defined accordingly. Following Adams, we will adopt the notion of *uncertainty*, understood as lack of assertability. Assuming that $U_P(\alpha) = 1 - V_P(\alpha)$, namely, that $U_P$ represents the degree of uncertainty of $\alpha$ relative to $P$, we will take validity to be a relation between the sum of the uncertainty of the premises and the uncertainty of the conclusion:

**Definition 3.**

$$\alpha_1, ...\alpha_n \vDash \beta \textit{ if and only if, for any } P,\ U_P(\alpha_1) + ... + U_P(\alpha_n) \geq U_P(\beta)$$

In other words, a valid argument is an argument in which the uncertainty of the conclusion cannot exceed the total uncertainty of the premises. As in the case of **P**, we will write $\alpha \equiv \beta$ to say that $\alpha \vDash \beta$ and $\beta \vDash \alpha$.

It is important to note that our formulation of the semantics of **L** in terms of assertability, and consequently our adoption of Adam's definition of validity, does not imply a skeptical attitude toward truth-conditions or logical consequence as classically understood. We regard the semantic framework provided here as neutral with respect to the question whether conditionals have truth-conditions. Moreover, definition 3 extensionally preserves the classical notion of logical consequence in at least two crucial respects. The first is that, as Adams has shown, all classically valid arguments expressible in **P** remain valid. That is, if $\alpha_1, ..., \alpha_n, \beta \in \mathbf{P}$ and $\alpha_1, ...\alpha_n \vDash_{PL} \beta$, then $\alpha_1, ...\alpha_n \vDash \beta$. We will use the label PL whenever we rely on this fact.[7] The second is that the following rule — *Substitution of Logical Equivalents* (SLE) — is valid:

**Fact 1.** *If $\alpha \equiv_{PL} \beta$, $\alpha$ occurs in $\gamma$, and $\gamma'$ is obtained from $\gamma$ by replacing $\alpha$ with $\beta$, then $\gamma \equiv \gamma'$.*

As will be shown in the appendix, SLE plays an important role in the proof of many technically useful results. This rule entails two rules for conditionals that are sometimes treated separately. One is Left Logical Equivalence: if $\alpha \equiv_{PL} \beta$, then $\alpha > \gamma$ is equivalent to $\beta > \gamma$. The second is Right Logical Equivalence: if $\beta \equiv_{PL} \gamma$, then $\alpha > \beta$ is equivalent to $\alpha > \gamma$.

From definitions 2 and 3 we also get two important results concerning the connection between $\Rightarrow, \triangleright, \dashv\!3$, namely, that $\triangleright$ is stronger than $\Rightarrow$ and $\dashv\!3$ is stronger than $\triangleright$.

**Fact 2.** $\alpha \triangleright \beta \vDash \alpha \Rightarrow \beta$ *but* $\alpha \Rightarrow \beta \nvDash \alpha \triangleright \beta$

**Fact 3.** $\alpha \dashv\!3\, \beta \vDash \alpha \triangleright \beta$ *but* $\alpha \triangleright \beta \nvDash \alpha \dashv\!3\, \beta$

[7] Adams [3], p. 38.

In general, this makes perfect sense. If $\alpha$ supports $\beta$, in that it provides a reason for accepting $\beta$, then it is reasonable to expect that $\beta$ is credible enough given $\alpha$. Similarly, if $\alpha$ necessitates $\beta$, then it is reasonable to expect that $\alpha$ supports $\beta$. In fact necessitation may be regarded as the strongest kind of support. Since $\Rightarrow$ is known to be stronger than $\supset$, from facts 2 and 3 we get that $\supset, \Rightarrow, \rhd, \dashv$ can be ordered along a scale of increasing strength. According to this scale, the material conditional is the weakest conditional, and each of the three non-material conditionals is characterized by a specific rank of strength that separates it from the material conditional.

## 3 Principles of conditional logic

Before dealing with the symbols $\Rightarrow, \rhd, \dashv$ one by one, it is useful to list thirty principles of conditional logic that we will discuss in connection with them, and spell out some important relations between these principles. In what follows, $\top$ and $\bot$ stand for tautology and contradiction, $\Box$ and $\Diamond$ stand for 'necessarily' and 'possibly', and the long arrow $\Longrightarrow$ indicates valid inference.[8]

1 *Super-Classicality* (SC): If $\alpha \vDash_{PL} \beta$, then $\alpha > \beta$ must hold
2 *Material Implication* (MI): $\alpha > \beta \Longrightarrow \alpha \supset \beta$
3 *Detachment* (DET): $\top > \alpha \Longrightarrow \alpha$
4 *Modus Ponens* (MP): $\alpha > \beta, \alpha \Longrightarrow \beta$
5 *Conjunction of Consequents* (CC): $\alpha > \beta, \alpha > \gamma \Longrightarrow \alpha > (\beta \wedge \gamma)$
6 *Disjunction of Antecedents* (DA): $\alpha > \gamma, \beta > \gamma \Longrightarrow (\alpha \vee \beta) > \gamma$
7 *Necessary Consequent* (NC): $\Box\alpha \Longrightarrow \beta > \alpha$
8 *Impossible Antecedent* (IA): $\Box \sim \alpha \Longrightarrow \alpha > \beta$
9 *Cautious Monotonicity* (CM): $\alpha > \beta, \alpha > \gamma \Longrightarrow (\alpha \wedge \beta) > \gamma$
10 *Negation Rationality* (NR): $\alpha > \gamma, \sim ((\alpha \wedge \sim \beta) > \gamma) \Longrightarrow (\alpha \wedge \beta) > \gamma$
11 *Rational Monotonicity* (RM): $\alpha > \gamma, \sim (\alpha > \sim \beta) \Longrightarrow (\alpha \wedge \beta) > \gamma$
12 *Right Weakening* (RW): If $\beta \vDash_{PL} \gamma$, then $\alpha > \beta \Longrightarrow \alpha > \gamma$
13 *Conversion* (CON): $\alpha \Longrightarrow \top > \alpha$
14 *Conjunctive Sufficiency* (CS): $\alpha \wedge \beta \Longrightarrow \alpha > \beta$
15 *Conditional Excluded Middle* (CEM): $\sim (\alpha > \beta) \Longrightarrow \alpha > \sim \beta$
16 *Limited Transitivity* (LT): $\alpha > \beta, (\alpha \wedge \beta) > \gamma \Longrightarrow \alpha > \gamma$
17 *Conditional Equivalence* (CE): $\alpha > \beta, \beta > \alpha, \beta > \gamma \Longrightarrow \alpha > \gamma$
18 *False Antecedent* (FA): $\sim \alpha \Longrightarrow \alpha > \beta$
19 *True Consequent* (TC): $\beta \Longrightarrow \alpha > \beta$
20 *Monotonicity* (M): $\alpha > \gamma \Longrightarrow (\alpha \wedge \beta) > \gamma$
21 *Transitivity* (T): $\alpha > \beta, \beta > \gamma \Longrightarrow \alpha > \gamma$
22 *Contraposition* (C): $\alpha > \beta \Longrightarrow \sim \beta > \sim \alpha$
23 *Conditional Proof* (CP): If $\Gamma, \alpha \vDash_{PL} \beta$, then $\Gamma \Longrightarrow \alpha > \beta$
24 *Empty Antecedent Strengthening* (EAS): $\top > \alpha \Longrightarrow \beta > \alpha$
25 *Prelinearity* (PRE): $\sim (\alpha > \beta) \Longrightarrow \beta > \alpha$
26 *Complementary Antecedent* (CA): $\sim (\alpha > \beta) \Longrightarrow \sim \alpha > \beta$
27 *Restricted Selectivity* (RS): If $\beta \vDash_{PL} \sim \gamma$, then $\Diamond\alpha, \alpha > \beta \Longrightarrow \sim (\alpha > \gamma)$
28 *Restricted Conditional Non-Contradiction* (RCN): $\Diamond\alpha, \alpha > \beta \Longrightarrow \sim (\alpha > \sim \beta)$
29 *Restricted Aristotle's Thesis* (RAT): $\Diamond\alpha \Longrightarrow \sim (\alpha > \sim \alpha)$
30 *Restricted Abelard's Thesis* (RAB): $\Diamond \sim \beta, \alpha > \beta \Longrightarrow \sim (\sim \alpha > \beta)$

[8]Some sources for our list are Arló-Costa and Egré [7], Douven [12], Huber [17], Unterhuber [30], Unterhuber [31].

Apart from principles 27-30, which may be labelled *connexive principles*, the principles listed above hold for the material conditional.[9] This means that they hold if $>$ is replaced by $\supset$. As we shall see, the suppositional interpretation, the evidential interpretation, and the strict interpretation differ from the material interpretation — and from each other — with respect to these principles, because we get different results if we replace $>$ with $\Rightarrow$, $\triangleright$, or $\prec$.

The thirty principles listed above are related in various ways, so they cannot be accepted or rejected independently of each other. In particular, we will rely on the following facts, some of which are well known, which hold for any reading of $>$.

**Fact 4.** *If MI holds, then DET holds as well, given PL.*

**Fact 5.** *If MI holds, then MP holds as well, given PL.*

**Fact 6.** *If LT and SC hold, then RW holds as well.*

**Fact 7.** *If M and CON hold, then TC holds as well.*

**Fact 8.** *If CM and CEM hold, then RM holds as well.*

**Fact 9.** *If T and SC hold, then M holds as well.*[10]

**Fact 10.** *If C and RW hold, then M holds as well.*[11]

**Fact 11.** *If C and CC hold, then DA holds as well.*

**Fact 12.** *If SC, CC, CE hold, then LT holds as well.*[12]

**Fact 13.** *If CP holds, then FA and TC hold as well, given PL.*

**Fact 14.** *If CON and EAS hold, then TC holds as well.*

**Fact 15.** *If NC holds, and either C holds or RW, SC, CC hold, then IA holds as well.*

**Fact 16.** *If RS holds, then RCN holds as well.*

**Fact 17.** *If RCN and SC hold, then RAT holds as well.*

**Fact 18.** *if C and RS hold, then RAB holds as well.*

## 4 The suppositional conditional

So far we have shown how the semantics of **L**, which contains the symbols $\Rightarrow, \triangleright, \prec$, can be defined by adopting the standard notion of probability. Now we will dwell on each of the three symbols separately, in order to spell out its distinctive logical properties. This section focuses on $\Rightarrow$. Since the logic of the

[9]On connexive principles in general see McCall [25], pp. 415-417, Estrada Gonzáles and Ramírez-Cámara [14], pp. 346-348. The specific version of connexive principles adopted here, which involve a restriction to contingent antecedents or consequents, has been considered in Kapsner [20] and Unterhuber [31].

[10]Kraus, Lehmann, and Magidor [28], pp. 180-181.

[11]Kraus, Lehmann, and Magidor [28], pp. 180-181.

[12]Kraus, Lehmann, and Magidor [28], p. 179. Fact 12 also follows from a result given in Gärdenfors and Rott [15], p. 54.

suppositional conditional is well known, we will simply recall some established results and add some details that matter for our purposes.

To begin with, we will take for granted that principles 1-17 hold for $\Rightarrow$.[13] Of these principles, we will prove only NC and IA, which are seldom discussed in the orthodox treatment of the suppositional conditional because they involve modal operators.

**Fact 19.** $\Box\alpha \vDash \beta \Rightarrow \alpha$ (*Necessary Consequent* ✓)

**Fact 20.** $\Box \sim \alpha \vDash \alpha \Rightarrow \beta$ (*Impossible Antecedent* ✓)

Now let us consider principles 18-22. These principles do no hold for $\Rightarrow$.

**Fact 21.** $\sim \alpha \nvDash \alpha \Rightarrow \beta$ (*False Antecedent* $\times$)

**Fact 22.** $\beta \nvDash \alpha \Rightarrow \beta$ (*True Consequent* $\times$)

**Fact 23.** $\alpha \Rightarrow \gamma \nvDash (\alpha \wedge \beta) \Rightarrow \gamma$ (*Monotonicity* $\times$)

**Fact 24.** $\alpha \Rightarrow \beta, \beta \Rightarrow \gamma \nvDash \alpha \Rightarrow \gamma$ (*Transitivity* $\times$)

**Fact 25.** $\alpha \Rightarrow \beta \nvDash \sim \beta \Rightarrow \sim \alpha$ (*Contraposition* $\times$)

According to Adams, the failure of principles 18-22 speaks in favour of the suppositional view, the idea that $>$ is correctly represented as $\Rightarrow$. The main motivation for this view is the observation that conditionals typically do not behave as material conditionals, as it emerges when we reflect on apparently invalid arguments such as the following:

A1
(1) John will arrive at 10
―――
(2) If John does not arrive at 10, he will arrive at 11

A2
(1) John will arrive at 10
―――
(3) If John misses his plane in New York, he will arrive at 10

A3
(4) If Brown wins the election, Smith will retire
―――
(5) If Smith dies before the election and Brown wins it, Smith will retire

A4
(4) If Brown wins the election, Smith will retire
(6) If Smith dies before the election, Brown will win it
―――
(7) If Smith dies before the election, he will retire

A5
(8) If John makes a mistake, it is not a big mistake
―――
(9) If John makes a big mistake, it is not a mistake

Adams's point is that the truth-functional view, the idea that $>$ is correctly represented as $\supset$, is unable to explain the apparent invalidity of A1-A5. A1-A5 instantiate principles 18-22, which hold on the truth-functional view, that is, if we replace $>$ with $\supset$. The suppositional view, instead, invalidates A1-A5, as is shown by facts 21-25.[14]

Note that, while plain monotonicity fails for $\Rightarrow$, other weaker principles, that is, CM, RM, and NR, license similar kinds of inferences under additional

[13] Adams [3], chapter 7. See also Lehmann and Magidor [21].
[14] The examples A1-A4 are drawn from Adams [1], pp. 166-167.

conditions.[15] Similarly, while plain transitivity fails for $\Rightarrow$, LT and CE remain valid.

Now let us consider principles 23 and 24. These two principles also fail.

**Fact 26.** *Not: if* $\Gamma, \alpha \vDash_{PL} \beta$*, then* $\Gamma \vDash \alpha \Rightarrow \beta$ (*Conditional Proof* $\times$)[16]

**Fact 27.** $\top \Rightarrow \alpha \nvDash \beta \Rightarrow \alpha$ (*Empty Antecedent Strengthening* $\times$)

Principles 25 and 26 are further principles that hold for $\supset$ but not for $\Rightarrow$.

**Fact 28.** $\sim (\alpha \Rightarrow \beta) \nvDash \beta \Rightarrow \alpha$ (*Prelinearity* $\times$)

**Fact 29.** $\sim (\alpha \Rightarrow \beta) \nvDash \sim \alpha \Rightarrow \beta$ (*Complementary Antecedent* $\times$)

Facts 28 and 29 may be regarded as desirable results. Consider the following sentences:

(10) If Susan is red-haired, then she is a doctor

(11) If Susan is a doctor, then she is red-haired

Even though it seems right to deny (10), it seems wrong assert (11), against PRE. Similarly, consider the following sentences:

(12) If Queen Elizabeth is at home, then she is worrying where I am tonight

(13) If Queen Elizabeth is not at home, then she is worrying where I am tonight

Even though it seems right to deny (12), it seems wrong to assert (13), against CA.[17]

Finally, let us consider principles 27-30. RS, RCN, and RAT hold for $\Rightarrow$.

**Fact 30.** *If* $\beta \vDash_{PL} \sim \gamma$*, then* $\Diamond\alpha, \alpha \Rightarrow \beta \vDash \sim (\alpha \Rightarrow \gamma)$ (*Restricted Selectivity* ✓)

**Fact 31.** $\Diamond\alpha, \alpha \Rightarrow \beta \vDash \sim (\alpha \Rightarrow \sim \beta)$ (*Restricted Conditional Non-Contradiction* ✓)

**Fact 32.** $\Diamond\alpha \vDash \sim (\alpha \Rightarrow \sim \alpha)$ (*Restricted Aristotle's Thesis* ✓)

Facts 30-32 may be regarded as an advantage of $\Rightarrow$ over $\supset$, given that none of them hold for $\supset$. For example, there seems to be something wrong in the following sentence:

(14) If it is raining, then it is not raining

On the truth-functional view, (14) is true when it is not raining, so our pretheoretical judgement about (14) can be correct only when it is raining. But this is an odd thing to say. The impression of falsity that we get when we look at (14) has nothing to do with the weather, or so it appears. If we feel that there is something wrong in (14), it is not because we look out the window. It seems that (14) is wrong *no matter whether* it is raining or not. This is just what RAT entails. Similar considerations may be advanced in defence of RS and RCN.

It is important to note, however, that RAB does not hold for $\Rightarrow$.

[15] For discussions of Rational Monotonicity and Negation Rationality, see Kraus, Lehmann, and Magidor [28], p. 197, Lehmann and Magidor [21], and Bennett [4], p. 332.

[16] See Edgington [13], p. 176.

[17] The two examples are drawn respectively from MacColl [24] and Edgington [13], p. 171.

**Fact 33.** $\Diamond \sim \beta, \alpha \Rightarrow \beta \nvDash \sim (\sim \alpha \Rightarrow \beta)$ (*Restricted Abelard's Thesis* $\times$)

The following table summarizes what has been said so far about $\Rightarrow$.

| | | $\supset$ | $\Rightarrow$ |
|---|---|---|---|
| 1 | Super-Classicality | ✓ | ✓ |
| 2 | Material Implication | ✓ | ✓ |
| 3 | Detachment | ✓ | ✓ |
| 4 | Modus Ponens | ✓ | ✓ |
| 5 | Conjunction of Consequents | ✓ | ✓ |
| 6 | Disjunction of Antecedents | ✓ | ✓ |
| 7 | Necessary Consequent | ✓ | ✓ |
| 8 | Impossible Antecedent | ✓ | ✓ |
| 9 | Cautious Monotonicity | ✓ | ✓ |
| 10 | Negation Rationality | ✓ | ✓ |
| 11 | Rational Monotonicity | ✓ | ✓ |
| 12 | Right Weakening | ✓ | ✓ |
| 13 | Conversion | ✓ | ✓ |
| 14 | Conjunctive Sufficiency | ✓ | ✓ |
| 15 | Conditional Excluded Middle | ✓ | ✓ |
| 16 | Limited Transitivity | ✓ | ✓ |
| 17 | Conditional Equivalence | ✓ | ✓ |
| 18 | False Antecedent | ✓ | × |
| 19 | True Consequent | ✓ | × |
| 20 | Monotonicity | ✓ | × |
| 21 | Transitivity | ✓ | × |
| 22 | Contraposition | ✓ | × |
| 23 | Conditional Proof | ✓ | × |
| 24 | Empty Antecedent Strengthening | ✓ | × |
| 25 | Prelinearity | ✓ | × |
| 26 | Complementary Antecedent | ✓ | × |
| 27 | Restricted Selectivity | × | ✓ |
| 28 | Restricted Conditional Non-Contradiction | × | ✓ |
| 29 | Restricted Aristotle's Thesis | × | ✓ |
| 30 | Restricted Abelard's Thesis | × | × |

The logic of the suppositional conditional improves plain propositional logic in some respects, in that it invalidates some principles that hold for $\supset$ but may be perceived as counterintuitive, such as FA or TC, while it retains other principles that hold for $\supset$ and are widely accepted as correct, such as SC or MP. However, the behaviour of $\Rightarrow$ is not satisfactory in all respects, and this explains at least in part why the debate on conditionals has moved on after Adams. Here we will provide four observations, each of which points out a possible source of perplexity.

*Observation 1*: it is not obvious that CS should be preserved. Several critics have regarded this principle as an unsettling contamination of the truth-functional account in the logic of non-material conditionals, and we are inclined to agree with them.[18] The sheer fact that $\alpha$ and $\beta$ obtain seems not enough to

[18] Butcher [5], Bennett [4], pp. 239-240.

claim $\alpha > \beta$, unless some further connection holds between $\alpha$ and $\beta$. Accordingly, it seems that there are cases in which $\alpha > \beta$ can reasonably be denied even though $\alpha$ and $\beta$ hold. For example, one might reject (10) without committing oneself to claim that Susan is not actually a doctor and that she is not actually red-haired. Remarkably, Adams himself labels CS a "rather strange inference", and retreats on a Gricean escape to accommodate it. Note, however, that this is the same kind of move that is usually regarded as insufficient to relieve the truth-functional account from the counterintuitive effects of FA and TC.[19]

*Observation 2*: it is not obvious that CEM should be preserved. The intuitive status of this principle is notoriously controversial.[20] There seem to be cases in which it is correct to deny $\alpha > \beta$, while it is incorrect to assert $\alpha >\sim \beta$. For example, even if it may be right to deny (10), this does not make it right to assert the following conditional:

(15) If Susan is red-haired, then she is not a doctor

Adams assumes that $\sim (\alpha > \beta)$ simply means $\alpha >\sim \beta$, and other theorists of conditionals have agreed on this assumption.[21] But CEM can hardly be defended by appealing to the meaning of $>$, given that the whole debate on conditionals stems precisely from the fact that it is not entirely clear what 'if then' means.

*Observation 3*: it is not obvious that C is to be rejected. Although many theorists of conditionals follow Adams and think that C must fail, others are apt to think that C can coherently be preserved. The alleged counterexamples to C, such as A5, have been widely discussed, and there is no obvious way to handle them. In particular, one thing that has been noted is that such counterexamples imply that $\alpha > \beta$ conveys a corresponding "even if" claim.[22] For example, (8) conveys the claim that even if John makes a mistake, it is not a big mistake. This means that such counterexamples would loose their grip on any account of $>$ which rules out the "even if" reading.

*Observation 4*: the suppositional treatment of the connexive principles is not ideal. On the one hand, $\Rightarrow$ validates RS, RCN, and RAT, which are reasonable principles. On the other hand, however, it invalidates RAB, which seems equally reasonable. Consider the following conditionals:

(16) If is cold, then it is raining

(17) It is cold, then it is not raining

(18) If it is cold, then it is not raining

(19) If it is not cold, then it is not raining

Many people would naturally refrain from asserting (16) and (17) together, and there is no *prima facie* reason to think that the combination of (18) and (19) is less counterintuitive. So it is not clear why only the first combination should be rejected, by maintaining RCN but not RAB. Moreover, most connexivists

[19] Adams [3], p. 157.

[20] See for example Cross [8], and Williams [32].

[21] Adams [1], p. 181.

[22] Lycan [23], p. 34, Bennett [4], pp. 32 and 143-144.

assume, following Aristotle, that there is a close connection between the falsity of (14) and the falsity of the combination of (18) and (19).[23]

As the next two sections will show, the evidential interpretation and the strict interpretation provide different but equally motivated answers to the questions raised by these four observations. So, they may be regarded as interesting alternatives to the suppositional interpretation.

## 5 The evidential conditional

As we have seen, much of the appeal of the suppositional interpretation lies in the fact that $\Rightarrow$ invalidates some principles that hold for $\supset$ but may be perceived as counterintuitive, such as FA or TC, while it retains other principles that hold for $\supset$ and are widely accepted as correct, such as SC or MP. The evidential interpretation preserves this unquestionable virtue, although it significantly differs from the suppositional interpretation in some crucial respects, which are directly relevant to observations 1-4.

First of all, consider principles 1-6. These principles hold for $\rhd$, and the same goes for C:

**Fact 34.** *If* $\alpha \vDash_{PL} \beta$, *then* $\alpha \rhd \beta$ (*Superclassicality* ✓)

**Fact 35.** $\alpha \rhd \beta \vDash \alpha \supset \beta$ (*Material Implication* ✓)

**Fact 36.** $\top \rhd \alpha \vDash \alpha$ (*Detachment* ✓)

**Fact 37.** $\alpha \rhd \beta, \alpha \vDash \beta$ (*Modus Ponens* ✓)

**Fact 38.** $\alpha \rhd \beta, \alpha \rhd \gamma \vDash \alpha \rhd (\beta \wedge \gamma)$ (*Conjunction of Consequents* ✓)

**Fact 39.** $\alpha \rhd \beta \vDash \sim \beta \rhd \sim \alpha$ (*Contraposition* ✓)

**Fact 40.** $\alpha \rhd \gamma, \beta \rhd \gamma \vDash (\alpha \vee \beta) \rhd \gamma$ (*Disjunction of Antecedents* ✓)

As shown in the appendix, fact 39 is technically useful to connect facts 38 and 40. C marks a key difference between $\rhd$ and $\Rightarrow$, and constitutes one of the most interesting features of $\rhd$. The reason is that $\rhd$, unlike $\Rightarrow$, rules out the "even if" reading of conditionals. A suppositional conditional $\alpha \Rightarrow \beta$ can be highly assertable even though $\alpha$ is irrelevant to $\beta$, or is at odds with $\beta$: a high probability of $\beta$ is enough. Quite clearly, the target explicandum of $\alpha \Rightarrow \beta$ includes "even if $\alpha$, $\beta$". Instead, the assertability of an evidential conditional $\alpha \rhd \beta$ requires not only that $\beta$ is highly probable if $\alpha$ is assumed, but that it is so at least in part *because* $\alpha$ is assumed. In other words, "even if $\alpha$, $\beta$" is assertable when $\alpha \Rightarrow \beta$ but not $\alpha \rhd \beta$ is assertable.[24] Since the alleged counterexamples to C typically involve the "even if" reading of conditionals, as noted in observation 3, it makes sense that C holds for $\rhd$. For example, on the evidential interpretation (8) is hardly assertable, so A5 does not violate C.

Now let us consider principles 7-10. These principles hold for $\rhd$.

**Fact 41.** $\Box\alpha \vDash \beta \rhd \alpha$ (*Necessary Consequent* ✓)

**Fact 42.** $\Box \sim \alpha \vDash \alpha \rhd \beta$ (*Impossible Antecedent* ✓)

[23] Aristotle, *Prior Analytics* 57b3-14. This is the reading adopted in McCall [25].

[24] This is essentially the analysis of "even if" suggested in Douven [12], p. 119.

**Fact 43.** $\alpha \triangleright \beta, \alpha \triangleright \gamma \vDash (\alpha \wedge \beta) \triangleright \gamma$ (*Cautious Monotonicity* ✓)

**Fact 44.** $\alpha \triangleright \gamma, \sim ((\alpha \wedge \sim \beta) \triangleright \gamma) \vDash (\alpha \wedge \beta) \triangleright \gamma$ (*Negation Rationality* ✓)

From what has been said so far it turns out that principles 1-10 hold for $\triangleright$. Since the same principles hold for $\Rightarrow$, this shows that there is a considerable overlap between $\triangleright$ and $\Rightarrow$, and consequently between $\triangleright$, $\Rightarrow$, and $\supset$.

Now we will focus on the principles that do not hold for $\triangleright$, and thereby highlight some significant differences between $\triangleright$ and $\Rightarrow$. We have already seen that $\triangleright$, unlike $\Rightarrow$, validates C. Another difference is that $\triangleright$, unlike $\Rightarrow$, violates RM.

**Fact 45.** $\alpha \triangleright \gamma, \sim (\alpha \triangleright \sim \beta) \nvDash (\alpha \wedge \beta) \triangleright \gamma$ (*Rational Monotonicity* ×)

In the appendix, we prove this fact by means of an example. Suppose that we are interested in the blood type of a person named Sara. Let $\alpha$ be 'Sara's mother's blood type is A', let $\beta$ be 'Sara's father's blood type is B', and let $\gamma$ be 'Sara's blood type is A'. The following probability distribution arises from plausible background assumption and basic genetic theory:[25]

$$\begin{aligned}
P(\alpha \wedge \beta \wedge \gamma) &= 0,018 \\
P(\alpha \wedge \beta \wedge \sim \gamma) &= 0,052 \\
P(\alpha \wedge \sim \beta \wedge \gamma) &= 0,152 \\
P(\alpha \wedge \sim \beta \wedge \sim \gamma) &= 0,078 \\
P(\sim \alpha \wedge \beta \wedge \gamma) &= 0,003 \\
P(\sim \alpha \wedge \beta \wedge \sim \gamma) &= 0,160 \\
P(\sim \alpha \wedge \sim \beta \wedge \gamma) &= 0,127 \\
P(\sim \alpha \wedge \sim \beta \wedge \sim \gamma) &= 0,409
\end{aligned}$$

In the case described, the uncertainty of 'If Sara's mother's blood is type A, then Sara's blood type is A' is moderate. The uncertainty of 'It is not the case that, if Sara's mother's blood type is A, then Sara's father's blood type is not B' is null, because the antededent and the consequent of the negated conditional are statistically independent, so the negation of such evidential conditional is fully assertable. Instead, the uncertainty of 'If Sara's mother's blood type is A and Sara's father's blood type is B, then Sara's blood type is A' is maximal, because the probability of the consequent is not increased (in fact it is decreased) by the probability of the antecedent.

Fact 45 shows that there is at least one specific sense in which the evidential conditional is *less* monotonic than the suppositional conditional. This is clear if we consider that, in the case decribed above, the inference would go through if 'if then' were read as $\Rightarrow$. For $U_P(\alpha \Rightarrow \gamma) = 1 - V_P(\alpha \Rightarrow \gamma) = 1 - P(\gamma|\alpha) = 0,44$, and $U_P(\sim (\alpha \Rightarrow \sim \beta)) = 1 - V_P(\sim (\alpha \Rightarrow \sim \beta) = 1 - P(\beta|\alpha) = 1 - P(\beta) = 0,77$, so $U_P(\alpha \Rightarrow \gamma) + U_P(\sim (\alpha \Rightarrow \sim \beta) = 1,21$, while $U_P((\alpha \wedge \beta) \Rightarrow \gamma) = 1 - V_P((\alpha \wedge \beta) \Rightarrow \gamma) = 1 - P(\gamma|\alpha \wedge \beta) = 0,75$. The main difference between the two interpretations

[25] The underlying hypothetical distribution of blood phenotypes 0, A, B, AB is 40%, 30%, 23%, 7%. Assuming a Hardy-Weinberg model, the corresponding genotype distribution for AA, BB, 00, AB, A0, B0 is 4%, 3%, 40%, 7%, 26%, 20% respectively (figures rounded). All other figures are implied given random mating (another standard background condition) and a basic Mendelian model of inheritance, with alleles A and B dominant and 0 recessive.

concern the second premise, whose uncertainty is null for $\triangleright$ but quite high for $\Rightarrow$.

Two direct corollaries of fact 45 are that M and RW do not hold for $\triangleright$.

**Fact 46.** $\alpha \triangleright \gamma \nvDash (\alpha \wedge \beta) \triangleright \gamma$ (*Monotonicity* $\times$)

**Fact 47.** *Not: if* $\beta \vDash_{PL} \gamma$, *then* $\alpha \triangleright \beta \vDash \alpha \triangleright \gamma$ (*Right Weakening* $\times$)

Fact 46 shows that $\triangleright$ is exactly like $\Rightarrow$ as far as M is concerned. Instead, fact 47 shows that $\triangleright$ and $\Rightarrow$ differ with respect to RW, which is quite interesting. RW is one of the most entrenched and technically powerful rules of traditional logics for conditionals. Yet, as puzzling as it may seem at first sight, the failure of RW is a very natural outcome for evidential conditionals. As Rott has pointed out, "it is the *hallmark* of difference-making conditionals that they do not satisfy Right Weakening", so that one can naturally accept, for instance, "if you pay an extra fee, your letter will be delivered by express" but not "if you pay an extra fee, your letter will be delivered", precisely because the letter will be delivered even if no extra fee is paid.[26] Moreover, note that, at least since the debate between Hempel and Carnap, it is clear that evidential support must fail the so-called "special consequence condition".[27] In fact if $\alpha$ and $\beta$ are probabilistically independent propositional formulas, and $0 < P(\alpha)P(\beta) < 1$, then $\alpha$ provides evidential support to $\alpha \wedge \beta$ but not to $\beta$, that is, $P(\alpha \wedge \beta|\alpha) > P(\alpha \wedge \beta)$ while $P(\beta|\alpha) = P(\beta)$, in spite of the fact that $\alpha \wedge \beta \vDash_{PL} \beta$.

From the failure of M and RW we can also conclude that principles 16, 17, and 21 do not hold for $\triangleright$.

**Fact 48.** $\alpha \triangleright \beta, \beta \triangleright \gamma \nvDash \alpha \triangleright \gamma$ (*Transitivity* $\times$)

**Fact 49.** $\alpha \triangleright \beta, (\alpha \wedge \beta) \triangleright \gamma \nvDash \alpha \triangleright \gamma$ (*Limited Transitivity* $\times$)

**Fact 50.** $\alpha \triangleright \beta, \beta \triangleright \alpha, \beta \triangleright \gamma \nvDash \alpha \triangleright \gamma$ (*Conditional Equivalence* $\times$)

Another important difference between $\Rightarrow$ and $\triangleright$ concerns principles 14 and 15. These two principles do not hold for $\triangleright$.

**Fact 51.** $\alpha \wedge \beta \nvDash \alpha \triangleright \beta$ (*Conjunction Sufficiency* $\times$)

**Fact 52.** $\sim (\alpha \triangleright \beta) \nvDash \alpha \triangleright \sim \beta$ (*Conditional Excluded Middle* $\times$)

As it emerges from facts 51 and 52, the evidential interpretation differs from the suppositional interpretation in that it provides opposite answers to the questions raised in observations 1 and 2. First, according to the evidential interpretation there are cases in which $\alpha > \beta$ can reasonably be denied even though $\alpha$ and $\beta$ hold. For example, if one denies (10), one does so because 'Susan is red-haired' provides no positive evidential support for 'Susan is a doctor', independently of Susan's actual hair colour or profession. Second, according to the evidential interpretation there are cases in which it is correct to deny $\alpha > \beta$ while it is incorrect to assert $\alpha >\sim \beta$. For example, it is perfectly consistent to deny both (10) and (15), for in both cases the antecedent provides no positive evidential support for the consequent.

As far as principles 18, 19, 23, 25, and 26 are concerned, the evidential conditional behaves exactly like the suppositional conditional, and unlike the material conditional.

[26]Rott [27], p. 7.

[27]Hempel [16], Carnap [6]

**Fact 53.** $\sim \alpha \nvDash \alpha \triangleright \beta$ (*False Antecedent* $\times$)

**Fact 54.** $\beta \nvDash \alpha \triangleright \beta$ (*True Consequent* $\times$)

**Fact 55.** *Not: if* $\Gamma, \alpha \vDash_{PL} \beta$*, then* $\Gamma \vDash \alpha \triangleright \beta$ (*Conditional Proof* $\times$)

**Fact 56.** $\sim (\alpha \triangleright \beta) \nvDash \beta \triangleright \alpha$ (*Prelinearity* $\times$)

**Fact 57.** $\sim (\alpha \triangleright \beta) \nvDash \sim \alpha \triangleright \beta$ (*Complementary Antecedent* $\times$)

Two further differences between $\Rightarrow$ and $\triangleright$ concern principles 13 and 24. While $\Rightarrow$ validates the former but not the latter, $\triangleright$ validates the latter but not the former.

**Fact 58.** $\top \triangleright \alpha \vDash \beta \triangleright \alpha$ (*Empty Antecedent Strengthening* ✓)

**Fact 59.** $\alpha \nvDash \top \triangleright \alpha$ (*Conversion* $\times$)

Finally, consider the connexive principles. All these principles hold for $\triangleright$.

**Fact 60.** *If* $\beta \vDash_{PL} \sim \gamma$*, then* $\Diamond\alpha, \alpha \triangleright \beta \vDash \sim (\alpha \triangleright \gamma)$ (*Restricted Selectivity* ✓)

**Fact 61.** $\Diamond\alpha, \alpha \triangleright \beta \vDash \sim (\alpha \triangleright \sim \beta)$ (*Restricted Conditional Non-Contradiction* ✓)

**Fact 62.** $\Diamond\alpha \vDash \sim (\alpha \triangleright \sim \alpha)$ (*Restricted Aristotle's Thesis* ✓)

**Fact 63.** $\Diamond \sim \beta, \alpha \triangleright \beta \vDash \sim (\sim \alpha \triangleright \beta)$ (*Restricted Abelard's Thesis* ✓)

Facts 60-63 show that the evidential interpretation provides a coherent treatment of the connexive principles, which does not produce the anomaly pointed out in observation 4. As far as the evidential interpretation is concerned, the combination of (18) and (19) can be treated just in the same way as the combination of (16) and (17), in accordance with RAB.

The following table summarizes what has been said so far about $\triangleright$.

| | | $\supset$ | $\Rightarrow$ | $\rhd$ |
|---|---|---|---|---|
| 1 | Super-Classicality | ✓ | ✓ | ✓ |
| 2 | Material Implication | ✓ | ✓ | ✓ |
| 3 | Detachment | ✓ | ✓ | ✓ |
| 4 | Modus Ponens | ✓ | ✓ | ✓ |
| 5 | Conjunction of Consequents | ✓ | ✓ | ✓ |
| 6 | Disjunction of Antecedents | ✓ | ✓ | ✓ |
| 7 | Necessary Consequent | ✓ | ✓ | ✓ |
| 8 | Impossible Antecedent | ✓ | ✓ | ✓ |
| 9 | Cautious Monotonicity | ✓ | ✓ | ✓ |
| 10 | Negation Rationality | ✓ | ✓ | ✓ |
| 11 | Rational Monotonicity | ✓ | ✓ | × |
| 12 | Right Weakening | ✓ | ✓ | × |
| 13 | Conversion | ✓ | ✓ | × |
| 14 | Conjunctive Sufficiency | ✓ | ✓ | × |
| 15 | Conditional Excluded Middle | ✓ | ✓ | × |
| 16 | Limited Transitivity | ✓ | ✓ | × |
| 17 | Conditional Equivalence | ✓ | ✓ | × |
| 18 | False Antecedent | ✓ | × | × |
| 19 | True Consequent | ✓ | × | × |
| 20 | Monotonicity | ✓ | × | × |
| 21 | Transitivity | ✓ | × | × |
| 22 | Contraposition | ✓ | × | ✓ |
| 23 | Conditional Proof | ✓ | × | × |
| 24 | Empty Antecedent Strengthening | ✓ | × | ✓ |
| 25 | Prelinearity | ✓ | × | × |
| 26 | Complementary Antecedent | ✓ | × | × |
| 27 | Restricted Selectivity | × | ✓ | ✓ |
| 28 | Restricted Conditional Non-Contradiction | × | ✓ | ✓ |
| 29 | Restricted Aristotle's Thesis | × | ✓ | ✓ |
| 30 | Restricted Abelard's Thesis | × | × | ✓ |

The first third of the table shows that $\rhd$ agrees with $\supset$ and $\Rightarrow$ on principles 1-10. This is a distinctive feature of the evidential interpretation as we understand it, a feature that makes $\rhd$ particularly appealing. The most important discussion of evidential conditionals so far has been given by Douven.[28] Our analysis of evidential conditionals differs from Douven's in at least two crucial respects. First, the logic generated by Douven's approach is rather weak, as it fails five among principles 1-10, that is, MP, CC, DA, CM, and NR.[29] Second, unlike Douven's, our framework provides an unified account of the suppositional conditional and the evidential conditional.

The remaining part of the table displays the main contrasts between $\rhd$ and $\Rightarrow$. As we have seen, the evidential interpretation differs from the suppositional interpretation with respect to each of the four issues raised in observations 1-4, as it invalidates CS and CEM, validates C, and provides a uniform treatment of the connexive principles. These results may also be regarded as positive features of the evidential interpretation.

[28] Douven [11], Douven [12].

[29] See Douven [12], theorem 5.2.1, p. 130.

Since the evidential interpretation retains the main virtues of the suppositional interpretation, in that it validates some highly plausible principles which hold for $\supset$, such as principles 1-10, it invalidates some intuitively questionable principles which hold for $\supset$, such as principles 18, 19, 23, 25, 26, and validates some apparently correct principles which do not hold for $\supset$, such as principles 27-29, the evidential interpretation may be regarded as a coherent alternative to the suppositional interpretation.

## 6 The strict conditional

The strict interpretation is another coherent alternative to the suppositional interpretation, as it relies on an understanding of $>$ that significantly differs from that adopted in the evidential interpretation. On the reading of $>$ represented by $\strictif$, to assert $\alpha > \beta$ is to assert that $\alpha$ necessitates $\beta$, in the sense that it cannot be the case that $\alpha$ but not $\beta$.[30] The strict interpretation agrees with the evidential interpretation in at least two important respects. First, it preserves all the classical principles preserved by the evidential interpretation. Second, it offers the same kind of responses to the questions raised in observations 1-4, including the treatment of the connexive principles. However, as we will see, the logic of the strict conditional differs from the logic of the evidential conditional in other crucial respects.

Let us start with principles 1-6. These principles hold for $\strictif$, and the same goes for C:

**Fact 64.** *If* $\alpha \vDash_{PL} \beta$, *then* $\alpha \strictif \beta$ (*Superclassicality* ✓)

**Fact 65.** $\alpha \strictif \beta \vDash \alpha \supset \beta$ (*Material Implication* ✓)

**Fact 66.** $\top \rhd \alpha \vDash \alpha$ (*Detachment* ✓)

**Fact 67.** $\alpha \rhd \beta, \alpha \vDash \beta$ (*Modus Ponens* ✓)

**Fact 68.** $\alpha \strictif \beta, \alpha \strictif \gamma \vDash \alpha \strictif (\beta \wedge \gamma)$ (*Conjunction of Consequents* ✓)

**Fact 69.** $\alpha \strictif \beta \vDash \sim \beta \strictif \sim \alpha$ (*Contraposition* ✓)

**Fact 70.** $\alpha \strictif \gamma, \beta \strictif \gamma \vDash (\alpha \vee \beta) \strictif \gamma$ (*Disjunction of Antecedents* ✓)

Principles 7 and 8 also hold for $\strictif$.

**Fact 71.** $\Box\alpha \vDash \beta \strictif \alpha$ (*Necessary Consequent* ✓)

**Fact 72.** $\Box \sim \alpha \vDash \alpha \rhd \beta$ (*Impossible Antecedent* ✓)

The crucial difference between $\rhd$ and $\strictif$ is that $\strictif$ is fully transitive and fully monotonic, in that principles 20 and 21 hold for $\strictif$.

**Fact 73.** $\alpha \strictif \beta, \beta \strictif \gamma \vDash \alpha \strictif \gamma$ (*Transitivity* ✓)

**Fact 74.** $\alpha \strictif \gamma \vDash (\alpha \wedge \beta) \strictif \gamma$ (*Monotonicity* ✓)

[30] Iacona [18] outlines some general arguments for the strict interpretation.

Although T and M may not accord with the evidential interpretation, they make sense on the strict interpretation. If $\alpha$ necessitates $\beta$, and $\beta$ necessitates $\gamma$, then clearly $\alpha$ necessitates $\gamma$. Similarly, if $\alpha$ necessitates $\gamma$, then clearly $\alpha \wedge \beta$ necessitates $\gamma$. In the literature on conditionals, there has been plenty of discussion about the alleged counterexamples to T and M, such as A3 and A4, and there is no widespread agreement about them. The strict interpretation may be combined with some accounts of these cases that explain away the apparent violation of T and M.[31]

From facts 73 and 74 we obtain some important corollaries: principles 11, 12, 16, and 17 hold for $\dashv$, while they do not hold for $\triangleright$.

**Fact 75.** $\alpha \dashv \beta, \beta \dashv \alpha, \beta \dashv \gamma \vDash \alpha \dashv \gamma$ (*Conditional Equivalence* ✓)

**Fact 76.** $\alpha \dashv \beta, (\alpha \wedge \beta) \dashv \gamma \vDash \alpha \dashv \gamma$ (*Limited Transitivity* ✓)

**Fact 77.** *If* $\beta \vDash_{PL} \gamma$*, then* $\alpha \dashv \beta \vDash \alpha \dashv \gamma$ (*Right Weakening* ✓)

**Fact 78.** $\alpha \dashv \gamma, \sim (\alpha \dashv \sim \beta) \vDash (\alpha \wedge \beta) \dashv \gamma$ (*Rational Monotonicity* ✓)

The divergence between $\triangleright$ and $\dashv$ emerges clearly if we focus on RM. As we have seen, the evidential interpretation differs from the suppositional interpretation with respect to the case of Sara. In that case the evidential interpretation makes the argument invalid because it entails that the second premise is certain: 'It is not the case that, if Sara's mother's blood type is A, then Sara's father's blood type is not B'. This is quite plausible, since the antecedent and the consequent of the negated conditional are statistically independent. The suppositional interpretation assigns to the second premise a high degree of uncertainty nonetheless, and this is how the validity of RM is preserved. The strict interpretation also makes the argument valid, but for a different reason, namely, that it raises the uncertainty of the first premise. More precisely, in this case we have that $V_P(\alpha \dashv \gamma) = 0$, $V_P(\sim (\alpha \dashv \sim \beta)) = 1$, and $V_P((\alpha \wedge \beta) \dashv \gamma) = 0$, so $U_P(\alpha \dashv \gamma) + U_P(\sim (\alpha \dashv \sim \beta)) = U_P((\alpha \wedge \beta) \dashv \gamma) = 1$. So the strict interpretation agrees with the evidential interpretation on the certainty of the second premise, but it preserves RM because it poses higher constraints on the assertability of the first premise.

Obviously, since principle 20 holds for $\dashv$, the same goes for principles 9, 10, and 24, which are weaker. In this respect, $\dashv$ agrees with $\triangleright$.

**Fact 79.** $\alpha \dashv \beta, \alpha \dashv \gamma \vDash (\alpha \wedge \beta) \dashv \gamma$ (*Cautious Monotonicity* ✓)

**Fact 80.** $\alpha \dashv \gamma, \sim ((\alpha \wedge \sim \beta) \dashv \gamma) \vDash (\alpha \wedge \beta) \dashv \gamma$ (*Negation Rationality* ✓)

**Fact 81.** $\top \dashv \alpha \vDash \beta \dashv \alpha$ (*Empty Antecedent Strengthening* ✓)

The agreement between $\dashv$ and $\triangleright$ also concerns principles 13, 14, 15, 18, 19, 23, 25, and 26, which are all invalidated by $\dashv$.

**Fact 82.** $\sim \alpha \nvDash \alpha \dashv \beta$ (*False Antecedent* ×)

**Fact 83.** $\beta \nvDash \alpha \dashv \beta$ (*True Consequent* ×)

**Fact 84.** $\alpha \nvDash \top \dashv \alpha$ (*Conversion* ×)

[31] Iacona [18] outlines such an account.

**Fact 85.** $\alpha \wedge \beta \nvDash \alpha \prec \beta$ (*Conjunctive Sufficiency* $\times$)

**Fact 86.** $\sim (\alpha \prec \beta) \nvDash \alpha \prec\sim \beta$ (*Conditional Excluded Middle* $\times$)

**Fact 87.** *Not: if* $\Gamma, \alpha \vDash_{PL} \beta$*, then* $\Gamma \vDash \alpha \prec \beta$ (*Conditional Proof* $\times$)

**Fact 88.** $\sim (\alpha \prec \beta) \nvDash \beta \prec \alpha$ (*Prelinearity* $\times$)

**Fact 89.** $\sim (\alpha \prec \beta) \nvDash\sim \alpha \prec \beta$ (*Complementary Antecedent* $\times$)

FA, TC, CON, CS, CEM, CP, PRE, and CA do not hold for $\prec$ for the same reason for which they do not hold for $\rhd$, namely, that the connection between antecedent and consequent that is implied by $\prec$ would not be preserved if these principles were valid.

As far as the connexive principles are concerned, $\prec$ behaves exactly like $\rhd$.

**Fact 90.** *If* $\beta \vDash_{PL}\sim \gamma$*, then* $\Diamond\alpha, \alpha \prec \beta \vDash\sim (\alpha \prec \gamma)$ (*Restricted Selectivity* ✓)

**Fact 91.** $\Diamond\alpha, \alpha \prec \beta \vDash\sim (\alpha \prec\sim \beta)$ (*Restricted Conditional Non-Contradiction* ✓)

**Fact 92.** $\Diamond\alpha \vDash\sim (\alpha \prec\sim \alpha)$ (*Restricted Aristotle's Thesis* ✓)

**Fact 93.** $\Diamond \sim \beta, \alpha \prec \beta \vDash\sim (\sim \alpha \prec \beta)$ (*Restricted Abelard's Thesis* ✓)

From facts 90-93 it turns out that the strict interpretation provides a coherent treatment of the connexive principles, just like the evidential interpretation. The following table summarizes what has been said so far about $\prec$ and provides an overall picture of our results.

| | | $\supset$ | $\Rightarrow$ | $\triangleright$ | ⥽ |
|---|---|---|---|---|---|
| 1 | Super-Classicality | ✓ | ✓ | ✓ | ✓ |
| 2 | Material Implication | ✓ | ✓ | ✓ | ✓ |
| 3 | Detachment | ✓ | ✓ | ✓ | ✓ |
| 4 | Modus Ponens | ✓ | ✓ | ✓ | ✓ |
| 5 | Conjunction of Consequents | ✓ | ✓ | ✓ | ✓ |
| 6 | Disjunction of Antecedents | ✓ | ✓ | ✓ | ✓ |
| 7 | Necessary Consequent | ✓ | ✓ | ✓ | ✓ |
| 8 | Impossible Antecedent | ✓ | ✓ | ✓ | ✓ |
| 9 | Cautious Monotonicity | ✓ | ✓ | ✓ | ✓ |
| 10 | Negation Rationality | ✓ | ✓ | ✓ | ✓ |
| 11 | Rational Monotonicity | ✓ | ✓ | × | ✓ |
| 12 | Right Weakening | ✓ | ✓ | × | ✓ |
| 13 | Conversion | ✓ | ✓ | × | × |
| 14 | Conjunctive Sufficiency | ✓ | ✓ | × | × |
| 15 | Conditional Excluded Middle | ✓ | ✓ | × | × |
| 16 | Limited Transitivity | ✓ | ✓ | × | ✓ |
| 17 | Conditional Equivalence | ✓ | ✓ | × | ✓ |
| 18 | False Antecedent | ✓ | × | × | × |
| 19 | True Consequent | ✓ | × | × | × |
| 20 | Monotonicity | ✓ | × | × | ✓ |
| 21 | Transitivity | ✓ | × | × | ✓ |
| 22 | Contraposition | ✓ | × | ✓ | ✓ |
| 23 | Conditional Proof | ✓ | × | × | × |
| 24 | Empty Antecedent Strengthening | ✓ | × | ✓ | ✓ |
| 25 | Prelinearity | ✓ | × | × | × |
| 26 | Complementary Antecedent | ✓ | × | × | × |
| 27 | Restricted Selectivity | × | ✓ | ✓ | ✓ |
| 28 | Restricted Conditional Non-Contradiction | × | ✓ | ✓ | ✓ |
| 29 | Restricted Aristotle's Thesis | × | ✓ | ✓ | ✓ |
| 30 | Restricted Abelard's Thesis | × | × | ✓ | ✓ |

As the table shows, $\triangleright$ and ⥽ agree in several important respects. The difference between ⥽ and $\triangleright$ lies in the fact that ⥽ validates five principles that do not hold for $\triangleright$, namely, RM, RW, LT, CE, M, and T. By contrast, there is no principle that is validated by $\triangleright$ but not by ⥽.

As a matter of fact, ⥽ is exactly as strong as the necessitation of $\supset$, as the following equivalence holds.

**Fact 94.** $\alpha$ ⥽ $\beta \equiv \Box(\alpha \supset \beta)$

This makes the strict interpretation particularly interesting, because it shows that, insofar as conditionals are adequately formalized as strict conditionals, we can express their logical properties in **L**, instead of employing a modal language defined in terms of possible worlds. Of course, modal logic works perfectly well, and many people take possible worlds to be entirely acceptable theoretical entities, or at least no more problematic than probabilities. However, some people — including some friends of ours — are apt to believe that probabilities are theoretically kosher in some sense in which possible worlds are not, and accordingly tend to dislike modal languages. If you belong to the second category, then here there is something for you. You can have the logic of the strict conditional, but without possible worlds.

## 7 Final remarks

In the foregoing sections we have spelled out three non-material interpretations of 'if then' — the suppositional interpretation, the evidential interpretation, and the strict interpretation — by elucidating some important logical properties of the symbols $\Rightarrow$, $\rhd$, $\strictif$. This last section provides some final remarks about the relations between these symbols.

As facts 2 and 3 show, $\rhd$ is stronger than $\Rightarrow$ and $\strictif$ is stronger than $\rhd$. It is important to note, however, that this does not mean that the logic of each of the three non-material conditionals is an extension of the logic of the conditional that precedes it. Although it may be reasonable to conjecture that the logic of $\strictif$ is an extension of the logic of $\rhd$, it is certainly not the case that the logic of $\rhd$ is an extension of the logic of $\Rightarrow$. The fact is that $\Rightarrow$ and $\rhd$ have different logics, neither of which is an extension of the other.

The distinction between the suppositional interpretation and the evidential interpretation deserves careful consideration in our view. These two interpretations may be regarded as two alternative and complementary ways to depart from the material interpretation *and* abandon full monotonicity. Consider M first. As fact 10 shows, the rejection of M forces the failure of at least one among RW and C: the logic of $\Rightarrow$ retains the former, while the logic of $\rhd$ retains the latter, and each option finds a coherent theoretical motivation in the corresponding interpretation of 'if then'. A very similar pattern arises from the rejection of TC, which is a distinctive "paradox" of the material conditional, because this rejection imposes a choice between CON and EAS, as is shown by fact 14.

More generally, each of the three interpretations considered has interesting logical implications, and finds some support in the ordinary use of 'if then'. As pointed out at the beginning, these three interpretations may be regarded either as three distinct meanings that ordinary speakers attach to 'if then', or as three ways of explicating a single indeterminate meaning by replacing it with a precise and well defined counterpart. The second option leaves open the question of whether there is a unique correct analysis of conditionals. Some theorists of conditionals work under the assumption that there is such an analysis, while others are inclined to think that different formal accounts of conditionals may be equally correct. We believe that the contents presented here are to a large extent neutral with respect to this divide. If there is a unique correct analysis of conditionals, then the results presented in the foregoing sections may shed some light on such analysis. On the other hand, if different formal accounts of conditionals are equally correct, then the distinction between $\Rightarrow$, $\rhd$, and $\strictif$ suggests one definite way to carve the space of the possible options.

## Appendix

In what follows we will adopt three methodological conventions. First, from now on the letters $\alpha, \beta, \gamma$ will be used to refer to propositional formulas, so we will no longer specify that they belong to $\mathbf{P}$. Second, the letter $P$ will be used to refer to an arbitrary probability function, so we will use it without further explanations. Third, since definition 3 says that $\alpha_1, ...\alpha_n \vDash \beta$ if and only if $U_P(\alpha_1) + ... + U_P(\alpha_n) \geq U_P(\beta)$, in order to prove that $\alpha_1, ...\alpha_n \vDash \beta$ it will suffice to show that $U_P(\beta) = 0$, or that $U_P(\alpha_i) = 1$ for at least some $\alpha_i$ in $\alpha_1, ...\alpha_n$.

**Fact 1**: If $\alpha \equiv_{PL} \beta$, $\alpha$ occurs in $\gamma$, and $\gamma'$ is obtained from $\gamma$ by replacing $\alpha$ with $\beta$, then $\gamma \equiv \gamma'$.

*Proof.* The proof is by induction on the complexity of $\gamma$, assuming that $\alpha \equiv_{PL} \beta$, that $\alpha$ occurs in $\gamma$, that $\gamma'$ is obtained from $\gamma$ by replacing $\alpha$ with $\beta$, and that $P$ is any probability function. The basis of the induction is the case in which $\gamma \in \mathbf{P}$. In this case $\gamma \equiv_{PL} \gamma'$, therefore $\gamma \equiv \gamma'$. In the inductive step we assume that the result to be proved holds for any formula of complexity less than or equal to $n$, and that $\gamma$ is a formula of complexity $n + 1$. The possible cases are five.
*Case 1*: $\gamma$ has the form $\Box\delta$. In this case $\gamma' = \Box\delta'$. Since $\delta \in \mathbf{P}$, $\delta \equiv_{PL} \delta'$. So, $P(\delta) = P(\delta')$. By clause 2 of definition 2, it follows that $V_P(\gamma) = V_P(\gamma')$, hence that $U_P(\gamma) = U_P(\gamma')$. Therefore, $\gamma \equiv \gamma'$.
*Case 2*: $\gamma$ has the form $\delta \Rightarrow \phi$. In this case either $\gamma' = \delta' \Rightarrow \phi$, or $\gamma' = \delta \Rightarrow \phi'$. Suppose that $\gamma' = \delta' \Rightarrow \phi$. Since $\delta \in \mathbf{P}$, $\delta \equiv_{PL} \delta'$. So, $P(\delta) = P(\delta')$, and $P(\phi \wedge \delta) = P(\phi \wedge \delta')$. It follows that $P(\phi|\delta) = P(\phi|\delta')$ whenever $P(\delta) > 0$. By clause 3 of definition 2 this entails that $V_P(\gamma) = V_P(\gamma')$, hence that $U_P(\gamma) = U_P(\gamma')$. The reasoning is similar if $\gamma' = \delta \Rightarrow \phi'$. Therefore $\gamma \equiv \gamma'$.
*Case 3*: $\gamma$ has the form $\delta \rhd \phi$. This case is like case 2 but relies on clause 4 of definition 2.
*Case 4*: $\gamma$ has the form $\delta \dashv\!\!\!\!\!\; \ni \phi$. This case is like 2 but relies on clause 5 of definition 2.
*Case 5*: $\gamma$ has the form $\sim \delta$. In this case $\gamma' = \sim \delta'$. Since $\delta$ has complexity $n$, by the inductive hypothesis $\delta \equiv \delta'$, so $V_P(\delta) = V_P(\delta')$. By clause 6, $V_P(\sim \delta) = 1 - V_P(\delta)$ and $V_P(\sim \delta') = 1 - V_P(\delta')$, so $V_P(\sim \delta) = V_P(\sim \delta')$. It follows that $U_P(\sim \delta) = U_P(\sim \delta')$, hence that $\gamma \equiv \gamma'$. □

**Fact 2**: $\alpha \rhd \beta \vDash \alpha \Rightarrow \beta$ but $\alpha \Rightarrow \beta \nvDash \alpha \rhd \beta$

*Proof.* In order to prove that $\alpha \rhd \beta \vDash \alpha \Rightarrow \beta$, three cases must be considered.
*Case 1*: $P(\alpha) = 0$ or $P(\beta) = 1$. In this case $V_P(\alpha \Rightarrow \beta) = 1$, so $U_P(\alpha \Rightarrow \beta) = 0$. Therefore, $U_P(\alpha \rhd \beta) \geq U_P(\alpha \Rightarrow \beta)$.
*Case 2*: $P(\alpha) > 1$, $P(\beta) < 1$, and $P(\beta|\alpha) < P(\beta)$. In this case $V_P(\alpha \rhd \beta) = 0$, hence $U_P(\alpha \rhd \beta) = 1$. Therefore, $U_P(\alpha \rhd \beta) \geq U_P(\alpha \Rightarrow \beta)$.

*Case 3*: $P(\alpha) > 1$, $P(\beta) < 1$, and $P(\beta|\alpha) \geq P(\beta)$. In this case we have that

$$\begin{gathered}
0 \leq P(\beta)(P(\alpha) - P(\alpha \wedge \beta)) \\
P(\alpha \wedge \beta) - P(\alpha \wedge \beta) \leq P(\alpha)P(\beta) - P(\beta)P(\alpha \wedge \beta) \\
P(\alpha \wedge \beta) - P(\alpha)P(\beta) \leq P(\alpha \wedge \beta) - P(\beta)P(\alpha \wedge \beta) \\
\frac{P(\alpha \wedge \beta)}{P(\alpha)} - P(\beta) \leq \frac{P(\alpha \wedge \beta) - P(\beta)P(\alpha \wedge \beta)}{P(\alpha)} \\
P(\beta|\alpha) - P(\beta) \leq \frac{P(\alpha \wedge \beta)(1 - P(\beta))}{P(\alpha)} \\
\frac{P(\beta|\alpha) - P(\beta)}{1 - P(\beta)} \leq \frac{P(\alpha \wedge \beta)}{P(\alpha)}
\end{gathered}$$

This means that $V_P(\alpha \triangleright \beta) \leq V_P(\alpha \Rightarrow \beta)$, so that $U_P(\alpha \triangleright \beta) \geq U_P(\alpha \Rightarrow \beta)$. To prove that $\alpha \Rightarrow \beta \not\models \alpha \triangleright \beta$ it suffices to note that it can happen that $0 < P(\beta|\alpha) \leq P(\beta)$. In this case $V_P(\alpha \Rightarrow \beta) > V_P(\alpha \triangleright \beta)$, so $U_P(\alpha \Rightarrow \beta) < U_P(\alpha \triangleright \beta)$. □

**Fact 3**: $\alpha \strictif \beta \models \alpha \triangleright \beta$ but $\alpha \triangleright \beta \not\models \alpha \strictif \beta$

*Proof.* In order to prove that $\alpha \strictif \beta \models \alpha \triangleright \beta$, three cases must be considered.
*Case 1*: $P(\alpha) = 0$ or $P(\beta) = 1$. In this case $V_P(\alpha \triangleright \beta) = 1$, so $U_P(\alpha \triangleright \beta) = 0$. Therefore, $U_P(\alpha \strictif \beta) \geq U_P(\alpha \triangleright \beta)$.
*Case 2*: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) < 1$. In this case $V_P(\alpha \strictif \beta) = 0$, so $U_P(\alpha \strictif \beta) = 1$. Therefore, $U_P(\alpha \strictif \beta) \geq U_P(\alpha \triangleright \beta)$.
*Case 3*: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) = 1$. In this case $P(\beta|\alpha) - P(\beta) = 1 - P(\beta)$, so $V_P(\alpha \triangleright \beta) = 1$ and $U_P(\alpha \triangleright \beta) = 0$. Therefore, $U_P(\alpha \strictif \beta) \geq U_P(\alpha \triangleright \beta)$.
To prove that $\alpha \triangleright \beta \not\models \alpha \strictif \beta$ it suffices to note that it can happen that $P(\beta) < P(\beta|\alpha) < 1$. In this case $V_P(\alpha \triangleright \beta) > V_P(\alpha \strictif \beta)$, so $U_P(\alpha \triangleright \beta) < U_P(\alpha \strictif \beta)$. □

**Fact 4**: If MI holds, then DET holds as well, given PL.

*Proof.*

| | | |
|---|---|---|
| 1 | $\top > \alpha$ | A |
| 2 | $\top \supset \alpha$ | 1 MI |
| 3 | $\top$ | PL |
| 4 | $\alpha$ | 2,3 PL |

□

**Fact 5**: If MI holds, then MP holds as well, given PL.

*Proof.*

| | | |
|---|---|---|
| 1 | $\alpha > \beta$ | A |
| 2 | $\alpha$ | A |
| 3 | $\alpha \supset \beta$ | 1 MI |
| 4 | $\beta$ | 2,3 PL |

□

**Fact 6**: If LT and SC hold, then RW holds as well.

*Proof.*

| | | |
|---|---|---|
| 1 | $\alpha > \beta$ | A |
| 2 | $(\alpha \wedge \beta) > \gamma$ | SC [assuming that $\beta \vDash_{PL} \gamma$] |
| 3 | $\alpha > \gamma$ | 1,2 LT |

□

**Fact 7**: If M and CON hold, then TC holds as well.

*Proof.*

| | | |
|---|---|---|
| 1 | $\beta$ | A |
| 2 | $\top > \beta$ | 1 CON |
| 3 | $(\top \wedge \alpha) > \beta$ | 2 M |
| 4 | $\alpha > \beta$ | 3 SLE |

□

**Fact 8**: If CM and CEM hold, then RM holds as well.

*Proof.*

| | | |
|---|---|---|
| 1 | $\alpha > \gamma$ | A |
| 2 | $\sim (\alpha > \sim \beta)$ | A |
| 3 | $\alpha > \beta$ | 2 CEM |
| 4 | $(\alpha \wedge \beta) > \gamma$ | 1,3 CM |

□

**Fact 9**: If T and SC hold, then M holds as well.

*Proof.*

| | | |
|---|---|---|
| 1 | $\alpha > \beta$ | A |
| 2 | $(\alpha \wedge \gamma) > \alpha$ | SC |
| 3 | $(\alpha \wedge \gamma) > \beta$ | 1,2 T |

□

**Fact 10**: If C and RW hold, then M holds as well.

*Proof.*

| | | |
|---|---|---|
| 1 | $\alpha > \gamma$ | A |
| 2 | $\sim \gamma > \sim \alpha$ | 1 C |
| 3 | $\sim \gamma > (\sim \alpha \vee \sim \beta)$ | 2 RW |
| 4 | $\sim (\sim \alpha \vee \sim \beta) > \sim\sim \gamma$ | 3 C |
| 5 | $(\alpha \wedge \beta) > \gamma$ | 4 SLE |

□

**Fact 11**: If C and CC hold, then DA holds as well.

*Proof.*

| | | |
|---|---|---|
| 1 | $\alpha > \gamma$ | A |
| 2 | $\beta > \gamma$ | A |
| 3 | $\sim \gamma > \sim \alpha$ | 1 C |
| 4 | $\sim \gamma > \sim \beta$ | 2 C |
| 5 | $\sim \gamma > (\sim \alpha \wedge \sim \beta)$ | 3,4 CC |
| 6 | $\sim (\sim \alpha \wedge \sim \beta) > \sim\sim \gamma$ | 5 C |
| 7 | $(\alpha \vee \beta) > \gamma$ | 6 SLE |

□

**Fact 12**: If SC, CC, CE hold, then LT holds as well.

*Proof.*

| | | |
|---|---|---|
| 1 | $\alpha > \beta$ | A |
| 2 | $(\alpha \wedge \beta) > \gamma$ | A |
| 3 | $\alpha > \alpha$ | SC |
| 4 | $(\alpha \wedge \beta) > \alpha$ | SC |
| 5 | $\alpha > (\alpha \wedge \beta)$ | 1,3 CC |
| 6 | $\alpha > \gamma$ | 2,4,5 CE |

□

**Fact 13**: If CP holds, then FA and TC hold as well, given PL.

*Proof.* Assume that CP holds. Since $\beta, \alpha \vDash_{PL} \beta$, we get that $\beta \Longrightarrow \alpha > \beta$. So TC holds. Similarly, since $\alpha, \sim \alpha \vDash_{PL} \beta$, we get that $\sim \alpha \Longrightarrow \alpha > \beta$. So FA holds. □

**Fact 14**: If CON and EAS hold, then TC holds as well.

*Proof.*

| | | |
|---|---|---|
| 1 | $\beta$ | A |
| 2 | $\top > \beta$ | 1 CON |
| 3 | $\alpha > \beta$ | 2 EAS |

□

**Fact 15**: If NC holds, and either C holds or RW, SC, CC hold, then IA holds as well.

*Proof.*

| | | |
|---|---|---|
| 1 | $\Box \sim \alpha$ | A |
| 2 | $\sim \beta > \sim \alpha$ | 1 NC |
| 3 | $\sim\sim \alpha > \sim\sim \beta$ | 2 C |
| 4 | $\alpha > \beta$ | 3 SLE |

| | | |
|---|---|---|
| 1 | $\Box \sim \alpha$ | A |
| 2 | $\alpha > \sim \alpha$ | 1 NC |
| 3 | $\alpha > (\sim \alpha \vee \beta)$ | 2 RW |
| 4 | $\alpha > \alpha$ | SC |
| 5 | $\alpha > (\alpha \vee \beta)$ | 4 RW |
| 6 | $\alpha > (\sim \alpha \vee \beta) \wedge (\alpha \vee \beta)$ | 3,5 CC |
| 7 | $\alpha > \beta$ | 7 SLE |

□

**Fact 16**: If RS holds, then RCN holds as well.

*Proof.*

| | | |
|---|---|---|
| 1 | $\Diamond \alpha$ | A |
| 2 | $\alpha > \beta$ | A |
| 3 | $\sim (\alpha > \sim \beta)$ | 1,2 RS [because $\beta \vDash_{PL} \sim\sim \beta$] |

□

**Fact 17**: If RCN and SC hold, then RAT holds as well.

*Proof.*

| | | |
|---|---|---|
| 1 | $\Diamond\alpha$ | A |
| 2 | $\alpha > \alpha$ | SC |
| 3 | $\sim (\alpha >\sim \alpha)$ | 1,2 RCN |

□

**Fact 18**: if C and RS hold, then RAB holds as well.

*Proof.* First note that, if C and RS hold, then from $\Diamond \sim \beta$ and $\alpha > \beta$ we obtain $\sim (\sim \beta > \alpha)$:

| | | |
|---|---|---|
| 1 | $\Diamond \sim \beta$ | A |
| 2 | $\alpha > \beta$ | A |
| 3 | $\sim \beta >\sim \alpha$ | 2 C |
| 4 | $\sim (\sim \beta > \alpha)$ | 1,3 RS [because $\sim \alpha \vDash_{PL} \sim \alpha$] |

Second, note that if $\Gamma, \alpha \vDash \beta$, then $\Gamma, \sim \beta \vDash \sim \alpha$. This can be seen as follows. $\alpha_1, ...\alpha_n \vDash \beta$ if and only if $U_P(\alpha_1) + ... + U_P(\alpha_n) \geq U_P(\beta)$. But $U_P(\alpha_1) + ... + U_P(\alpha_n) \geq U_P(\beta)$ if and only if $U_P(\alpha_1) + ... + U_P(\alpha_{n-1}) + (1 - U_P(\beta)) \geq 1 - U_P(\alpha_n)$, that is, if and only if $U_P(\alpha_1) + ... + U_P(\alpha_{n-1}) + U_P(\sim \beta) \geq U_P(\sim \alpha_n)$. Therefore, if $\alpha_1, ...\alpha_n \vDash \beta$, then $\alpha_1, ...\alpha_{n-1}, \sim \beta \vDash \sim \alpha_n$. Now, from $\sim (\sim \beta > \alpha)$, given this rule, we obtain $\sim (\sim \alpha > \beta)$, because $\sim \alpha > \beta \vDash \sim \beta > \alpha$ by C and SLE. □

**Fact 19**: $\Box\alpha \vDash \beta \Rightarrow \alpha$

*Proof.* Let $\alpha, \beta \in \mathbf{P}$ and let $P$ be any probability function. Two cases must be considered.
*Case 1*: $P(\alpha) = 1$. In this case $V_P(\beta \Rightarrow \alpha) = 1$ no matter whether $P(\beta) = 0$ or $P(\beta) > 0$, so $U_P(\beta \Rightarrow \alpha) = 0$. Therefore, $U_P(\Box\alpha) \geq U_P(\beta \Rightarrow \alpha)$.
*Case 2*: $P(\alpha) < 1$. In this case $V_P(\Box\alpha) = 0$, so $U_P(\Box\alpha) = 1$. Therefore, $U_P(\Box\alpha) \geq U_P(\beta \Rightarrow \alpha)$. □

**Fact 20**: $\Box \sim \alpha \vDash \alpha \Rightarrow \beta$

*Proof.* Since RW, SC, and CC hold for $\Rightarrow$, from facts 15 and 19 we get that $\Box \sim \alpha \vDash \alpha \Rightarrow \beta$. □

**Fact 21**: $\sim \alpha \nvDash \alpha \Rightarrow \beta$

*Proof.* Suppose that $0 < P(\beta) < 1$, and posit $\alpha =\sim \beta$. Then, $U_P(\sim \alpha) = 1 - V_P(\sim \alpha) = 1 - P(\sim \alpha) = 1 - P(\beta) < 1$, but $U_P(\alpha \Rightarrow \beta) = 1 - V_P(\alpha \Rightarrow \beta) = 1 - P(\beta|\alpha) = 1 - P(\beta| \sim \beta) = 1$. Therefore, $U_P(\sim \alpha) < U_P(\alpha \Rightarrow \beta)$. □

**Fact 22**: $\beta \nvDash \alpha \Rightarrow \beta$

*Proof.* Suppose that $0 < P(\alpha) < 1$, and posit $\beta =\sim \alpha$. Then, $U_P(\beta) = 1 - V_P(\beta) = 1 - P(\beta) = 1 - P(\sim \alpha) < 1$, but $U_P(\alpha \Rightarrow \beta) = 1 - V_P(\alpha \Rightarrow \beta) = 1 - P(\beta|\alpha) = 1 - P(\sim \alpha|\alpha) = 1$. Therefore, $U_P(\beta) < U_P(\alpha \Rightarrow \beta)$. □

**Fact 23**: $\alpha \Rightarrow \gamma \nvDash (\alpha \wedge \beta) \Rightarrow \gamma$

*Proof.* Suppose that $\alpha \Rightarrow \gamma \vDash (\alpha \wedge \beta) \Rightarrow \gamma$. Since CON holds for $\Rightarrow$, by fact 7 we get that $\beta \vDash \alpha \Rightarrow \beta$, contrary to fact 22. $\square$

**Fact 24**: $\alpha \Rightarrow \beta, \beta \Rightarrow \gamma \nvDash \alpha \Rightarrow \gamma$

*Proof.* Suppose that $\alpha \Rightarrow \beta, \beta \Rightarrow \gamma \vDash \alpha \Rightarrow \gamma$. Since SC holds for $\Rightarrow$, by fact 9 we get that $\alpha \Rightarrow \gamma \vDash (\alpha \wedge \beta) \Rightarrow \gamma$, contrary to fact 23. $\square$

**Fact 25**: $\alpha \Rightarrow \beta \nvDash \sim \beta \Rightarrow \sim \alpha$

*Proof.* Suppose that $\alpha \Rightarrow \beta \vDash \sim \beta \Rightarrow \sim \alpha$. Since RW holds for $\Rightarrow$, by fact 10 we get that $\alpha \Rightarrow \gamma \vDash (\alpha \wedge \beta) \Rightarrow \gamma$, contrary to fact 23. $\square$

**Fact 26**: Not: if $\Gamma, \alpha \vDash_{PL} \beta$, then $\Gamma \vDash \alpha \Rightarrow \beta$.

*Proof.* Suppose that CP holds for $\Rightarrow$. Then by fact 13 we get that FA and TC also hold for $\Rightarrow$, contrary to facts 21 and 22. $\square$

**Fact 27**: $\top \Rightarrow \alpha \nvDash \beta \Rightarrow \alpha$

*Proof.* Suppose that $\top \Rightarrow \alpha \vDash \beta \Rightarrow \alpha$. Since CON holds for $\Rightarrow$, by fact 14 we get that $\beta \vDash \alpha \Rightarrow \beta$, contrary to fact 22. $\square$

**Fact 28**: $\sim (\alpha \Rightarrow \beta) \nvDash \beta \Rightarrow \alpha$

*Proof.* Suppose that $0 < P(\alpha) < 1$, and posit $\beta = \sim \alpha$. Then $U_P(\sim (\alpha \Rightarrow \beta)) = 1 - V_P(\sim (\alpha \Rightarrow \beta)) = 1 - (1 - V_P(\alpha \Rightarrow \beta)) = V_P(\alpha \Rightarrow \beta) = P(\sim \alpha|\alpha) = 0$, whereas $U_P(\beta \Rightarrow \alpha) = 1 - V_P(\beta \Rightarrow \alpha) = 1 - P(\alpha|\beta) = 1 - P(\alpha| \sim \alpha) = 1$. Therefore, $U_P(\sim (\alpha \Rightarrow \beta)) < U_P(\beta \Rightarrow \alpha)$. $\square$

**Fact 29**: $\sim (\alpha \Rightarrow \beta) \nvDash \sim \alpha \Rightarrow \beta$

*Proof.* Suppose that $P(\alpha) > 0$ and $P(\sim \alpha \wedge \sim \gamma) > 0$, and posit $\beta = \sim \alpha \wedge \gamma$. Then $U_P(\sim (\alpha \Rightarrow \beta)) = 1 - V_P(\sim (\alpha \Rightarrow \beta)) = 1 - (1 - V_P(\alpha \Rightarrow \beta)) = V_P(\alpha \Rightarrow \beta) = P(\beta|\alpha) = P(\sim \alpha \wedge \gamma|\alpha) = 0$, but $U_P(\sim \alpha \Rightarrow \beta) = 1 - V_P(\sim \alpha \Rightarrow \beta) = 1 - P(\sim \alpha \wedge \gamma| \sim \alpha) = 1 - P(\gamma| \sim \alpha) > 0$. Therefore, $U_P(\sim (\alpha \Rightarrow \beta)) < U_P(\sim \alpha \Rightarrow \beta)$. $\square$

**Fact 30**: If $\beta \vDash_{PL} \sim \gamma$, then $\Diamond\alpha, \alpha \Rightarrow \beta \vDash \sim (\alpha \Rightarrow \gamma)$

*Proof.* Assume that $\beta \vDash_{PL} \sim \gamma$. Two cases must be considered.
*Case 1*: $P(\alpha) = 0$. In this case $U_P(\Diamond\alpha) = 1 - V_P(\Diamond\alpha) = 1 - V_P(\sim \Box \sim \alpha) = 1 - (1 - V_P(\Box \sim \alpha) = V_P(\Box \sim \alpha) = 1$. Therefore, $U_P(\Diamond\alpha) + U_P(\alpha \Rightarrow \beta) \geq U_P(\sim (\alpha \Rightarrow \gamma))$.
*Case 2*: $P(\alpha) > 0$. In this case, since $\beta \vDash_{PL} \sim \gamma$, we have that $P(\beta|\alpha) + P(\gamma|\alpha) \leq 1$, so that $1 - P(\beta|\alpha) \geq P(\gamma|\alpha)$. Given that $U_P(\alpha \Rightarrow \beta) = 1 - V_P(\alpha \Rightarrow \beta) = 1 - P(\beta|\alpha)$, and that $U_P(\sim (\alpha \Rightarrow \gamma)) = 1 - V_P(\sim (\alpha \Rightarrow \gamma)) = 1 - (1 - V_p(\alpha \Rightarrow \gamma)) = 1 - (1 - P(\gamma|\alpha)) = P(\gamma|\alpha)$, we get that $U_P(\alpha \Rightarrow \beta) \geq U_P(\sim (\alpha \Rightarrow \gamma))$. Therefore, $U_P(\Diamond\alpha) + U_P(\alpha \Rightarrow \beta) \geq U_P(\sim (\alpha \Rightarrow \gamma))$. $\square$

**Fact 31**: $\Diamond\alpha, \alpha \Rightarrow \beta \vDash \sim (\alpha \Rightarrow \sim \beta)$

*Proof.* From facts 16 and 30. $\square$

**Fact 32**: $\Diamond\alpha \vDash \sim (\alpha \Rightarrow \sim \alpha)$

*Proof.* From facts 17 and 31. $\square$

**Fact 33**: $\Diamond \sim \beta, \alpha \Rightarrow \beta \not\models \sim (\sim \alpha \Rightarrow \beta)$

*Proof.* Suppose that $P(\beta) > 0$ and $P(\sim \beta) > 0$, and posit $\alpha = \beta \wedge \sim \beta$. Then $U_P(\Diamond \sim \beta) + U_P(\alpha \Rightarrow \beta) = U_P(\Diamond \sim \beta) + U_P((\beta \wedge \sim \beta) \Rightarrow \beta) = 1 - V_P(\Diamond \sim \beta) + 1 - V_P((\beta \wedge \sim \beta) \Rightarrow \beta)) = 1 - V_P(\sim \Box\beta) + 1 - V_P(\bot \Rightarrow \beta) = 1 - (1 - V_P(\Box\beta)) + 1 - V_P(\bot \Rightarrow \beta)) = 1 - (1-0) + (1-1) = 0$, and $U_P(\sim (\sim \alpha \Rightarrow \beta)) = 1 - V_P(\sim (\sim \alpha \Rightarrow \beta)) = 1 - (1 - V_P(\sim \alpha \Rightarrow \beta)) = 1 - (1 - V_P(\sim (\beta \wedge \sim \beta) \Rightarrow \beta)) = 1 - (1 - V_P(\sim \bot \Rightarrow \beta)) = 1 - (1 - V_P(\top \Rightarrow \beta)) = 1 - (1 - P(\beta|\top)) = P(\beta|\top) = P(\beta) > 0$. Therefore, $U_P(\Diamond \sim \beta) + U_P(\alpha \Rightarrow \beta) < U_P(\sim (\sim \alpha \Rightarrow \beta))$. $\square$

**Fact 34**: If $\alpha \models_{PL} \beta$, then $\models \alpha \triangleright \beta$

*Proof.* Assume that $\alpha \models_{PL} \beta$. Two cases must be considered.
*Case 1*: $P(\beta) = 1$. In this case, $V_P(\alpha \triangleright \beta) = 1$, so $U_P(\alpha \triangleright \beta) = 0$.
*Case 2*: $P(\beta) < 1$. In this case, if $P(\alpha) = 0$, then again $V_P(\alpha \triangleright \beta) = 1$, so $U_P(\alpha \triangleright \beta) = 0$. If $P(\alpha) > 0$, we have that $P(\beta|\alpha) = 1$ because $\alpha \models_{PL} \beta$. It follows that $P(\beta|\alpha) - P(\beta) = 1 - P(\beta)$, so that $V_P(\alpha \triangleright \beta) = 1$ and $U_P(\alpha \triangleright \beta) = 0$. $\square$

**Fact 35**: $\alpha \triangleright \beta \models \alpha \supset \beta$

*Proof.* Three cases must be considered.
*Case 1*: $P(\alpha) = 0$ or $P(\beta) = 1$. In this case $V_P(\alpha \supset \beta) = 1$, so $U_P(\alpha \supset \beta) = 0$. Therefore, $U_P(\alpha \triangleright \beta) \geq U_P(\alpha \supset \beta)$.
*Case 2*: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) > P(\beta)$. In this case we have that

$$\begin{gathered}
P(\sim \beta|\alpha)P(\alpha)P(\sim \beta) \leq P(\sim \beta|\alpha) \\
P(\beta|\alpha) + P(\sim \beta|\alpha)P(\alpha)P(\sim \beta) \leq P(\beta|\alpha) + P(\sim \beta|\alpha) \\
P(\beta|\alpha) + P(\sim \beta|\alpha)P(\alpha)P(\sim \beta) \leq P(\beta) + P(\sim \beta) \\
P(\beta|\alpha) - P(\beta) \leq P(\sim \beta) - P(\sim \beta|\alpha)P(\alpha)P(\sim \beta) \\
\frac{P(\beta|\alpha) - P(\beta)}{P(\sim \beta)} \leq 1 - P(\sim \beta|\alpha)P(\alpha) \\
\frac{P(\beta|\alpha) - P(\beta)}{1 - P(\beta)} \leq 1 - P(\alpha \wedge \sim \beta) \\
\frac{P(\beta|\alpha) - P(\beta)}{1 - P(\beta)} \leq P(\sim (\alpha \wedge \sim \beta))
\end{gathered}$$

This means that $V_P(\alpha \triangleright \beta) \leq V_P(\alpha \supset \beta)$. Therefore, $U_P(\alpha \triangleright \beta) \geq U_P(\alpha \supset \beta)$.
*Case 3*: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) \leq P(\beta)$. In this case $V_P(\alpha \triangleright \beta) = 0$, so $U_P(\alpha \triangleright \beta) = 1$. Therefore, $U_P(\alpha \triangleright \beta) \geq U_P(\alpha \supset \beta)$. $\square$

**Fact 36**: $\top \triangleright \alpha \models \alpha$

*Proof.* From facts 4 and 35. $\square$

**Fact 37**: $\alpha \triangleright \beta, \alpha \models \beta$

*Proof.* From facts 5 and 35. $\square$

**Fact 38**: $\alpha \triangleright \beta, \alpha \triangleright \gamma \models \alpha \triangleright (\beta \wedge \gamma)$

*Proof.* First, note that if $P(\alpha) = 0$, then $U_P(\alpha \triangleright (\beta \wedge \gamma)) = 1 - V_P(\alpha \triangleright (\beta \wedge \gamma)) = 1 - 1 = 0$, so $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P(\alpha \triangleright (\beta \wedge \gamma))$. Second, note that if $P(\beta) = 1$, then $V_P(\alpha \triangleright \gamma) = V_P(\alpha \triangleright (\beta \wedge \gamma))$, so $U_P(\alpha \triangleright \gamma) = U_P(\alpha \triangleright (\beta \wedge \gamma))$. Therefore, $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P(\alpha \triangleright (\beta \wedge \gamma))$. The same conclusion follows if $P(\gamma) = 1$. Third, note that if $P(\beta \wedge \gamma) = 1$, then $P(\beta) = 1$ and $P(\gamma) = 1$, so $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P(\alpha \triangleright (\beta \wedge \gamma))$ for the reasons just explained. Now let us reason under the assumption that $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\gamma) < 1$. Three cases must be considered.

*Case 1*: $P(\beta|\alpha) \leq P(\beta)$ or $P(\gamma|\alpha) \leq P(\gamma)$. In this case $V_P(\alpha \triangleright \beta) = 0$ or $V_P(\alpha \triangleright \gamma) = 0$, and consequently $U_P(\alpha \triangleright \beta) = 1$ or $U_P(\alpha \triangleright \gamma) = 1$. Therefore, $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P(\alpha \triangleright (\beta \wedge \gamma))$.

*Case 2*: $P(\beta|\alpha) > P(\beta)$, $P(\gamma|\alpha) > P(\gamma)$, and $P(\beta \wedge \gamma|\alpha) > P(\beta \wedge \gamma)$. We know that $P(\beta \wedge \sim \gamma)P(\sim \gamma)P(\sim \beta|\alpha) + P(\sim \beta \wedge \gamma)P(\sim \beta)P(\sim \gamma|\alpha) + P(\sim \gamma)P(\sim \beta)P(\sim \beta \wedge \sim \gamma|\alpha) \geq 0$. Therefore, $P(\beta \wedge \sim \gamma)(P(\beta \wedge \sim \gamma) + P(\sim \beta \wedge \sim \gamma))(P(\sim \beta \wedge \gamma|\alpha) + P(\sim \beta \wedge \sim \gamma|\alpha)) + P(\sim \beta \wedge \gamma)(P(\sim \beta \wedge \gamma) + P(\sim \beta \wedge \sim \gamma)(P(\beta \wedge \sim \gamma|\alpha) + P(\sim \beta \wedge \sim \gamma|\alpha) + (P(\beta \wedge \sim \gamma) + P(\sim \beta \wedge \sim \gamma))(P(\sim \beta \wedge \gamma) + P(\sim \beta \wedge \sim \gamma))P(\sim \beta \wedge \sim \gamma|\alpha) \geq 0$. From this, by means of purely algebraic steps, we get what follows.[32]

$$\frac{P(\sim \beta \wedge \gamma|\alpha) + P(\sim \beta \wedge \sim \gamma|\alpha)}{P(\sim \beta \wedge \gamma) + P(\sim \beta \wedge \sim \gamma)} + \frac{P(\beta \wedge \sim \gamma|\alpha) + P(\sim \beta \wedge \sim \gamma|\alpha)}{P(\beta \wedge \sim \gamma) + P(\sim \beta \wedge \sim \gamma)} \geq \frac{P(\beta \wedge \sim \gamma|\alpha) + P(\sim \beta \wedge \gamma|\alpha) + P(\sim \beta \wedge \sim \gamma|\alpha)}{P(\beta \wedge \sim \gamma) + P(\sim \beta \wedge \gamma) + P(\sim \beta \wedge \sim \gamma)}$$

$$\frac{P(\sim \beta|\alpha)}{P(\sim \beta)} + \frac{P(\sim \gamma|\alpha)}{P(\sim \gamma)} \geq \frac{P(\sim \beta \vee \sim \gamma|\alpha)}{P(\sim \beta \vee \sim \gamma)}$$

$$\frac{P(\sim \beta|\alpha)}{P(\sim \beta)} + \frac{P(\sim \gamma|\alpha)}{P(\sim \gamma)} \geq \frac{P(\sim (\beta \wedge \gamma)|\alpha)}{P(\sim (\beta \wedge \gamma))}$$

$$\frac{1 - P(\beta|\alpha)}{P(\sim \beta)} + \frac{1 - P(\gamma|\alpha)}{P(\sim \gamma)} \geq \frac{1 - P(\beta \wedge \gamma|\alpha)}{P(\sim (\beta \wedge \gamma))}$$

$$\frac{P(\sim \beta) - P(\beta|\alpha) + 1 - P(\sim \beta)}{P(\sim \beta)} + \frac{P(\sim \gamma) - P(\gamma|\alpha) + 1 - P(\sim \gamma)}{P(\sim \gamma)} \geq \frac{P(\sim (\beta \wedge \gamma)) - P(\beta \wedge \gamma|\alpha) + 1 - P(\sim (\beta \wedge \gamma))}{P(\sim (\beta \wedge \gamma))}$$

[32] Let $a = P(\beta \wedge \sim \gamma)$, $b = P(\beta \wedge \sim \gamma|\alpha)$, $c = P(\sim \beta \wedge \gamma)$, $d = P(\sim \beta \wedge \gamma|\alpha)$, $e = P(\sim \beta \wedge \sim \gamma)$, $f = P(\sim \beta \wedge \sim \gamma|\alpha)$.

$$a(a+e)(d+f) + c(c+e)(b+f) + (a+e)(c+e)f \geq 0$$

$$a(a+e)d + a(a+e)f + c(c+e)b + c(c+e)f + a(c+e)f + e(c+e)f \geq 0$$

$$a^2d + aed + a^2f + acf + aef + aef + cef + e^2f + c^2b + ceb + c^2f + cef \geq 0$$

$$a^2d + acd + aed + aed + ced + e^2d + a^2f + acf + aef + aef + cef + e^2f + acb + c^2b + ceb + aeb + ceb + e^2b$$
$$+acf + c^2f + cef + aef + cef + e^2f \geq acb + ceb + aeb + e^2b + acd + ced + aed + e^2d + acf + cef + aef + e^2f$$

$$(a+e)(a+c+e)(d+f) + (c+e)(a+c+e)(b+f) \geq (c+e)(a+e)(b+d+f)$$

$$\frac{d+f}{c+e} + \frac{b+f}{a+e} \geq \frac{b+d+f}{a+c+e}$$

$$\frac{P(\sim \beta) - P(\beta|\alpha) + P(\beta)}{P(\sim \beta)} + \frac{P(\sim \gamma) - P(\gamma|\alpha) + P(\gamma)}{P(\sim \gamma)}$$
$$\geq \frac{P(\sim (\beta \wedge \gamma)) - P(\beta \wedge \gamma|\alpha) + P(\beta \wedge \gamma)}{P(\sim (\beta \wedge \gamma))}$$
$$1 - \frac{P(\beta|\alpha) - P(\beta)}{P(\sim \beta)} + 1 - \frac{P(\gamma|\alpha) - P(\gamma)}{P(\sim \gamma)} \geq 1 - \frac{P(\beta \wedge \gamma|\alpha) - P(\beta \wedge \gamma)}{P(\sim (\beta \wedge \gamma))}$$

This means that $1 - V_P(\alpha \triangleright \beta) + 1 - V_P(\alpha \triangleright \gamma) \geq 1 - V_P(\alpha \triangleright (\beta \wedge \gamma))$, so that $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P(\alpha \triangleright (\beta \wedge \gamma))$.
*Case 3*: $P(\beta|\alpha) > P(\beta)$, $P(\gamma|\alpha) > P(\gamma)$, and $P(\beta \wedge \gamma|\alpha) \leq P(\beta \wedge \gamma)$. In this case we have that

$$\frac{P(\beta \wedge \gamma|\alpha) - P(\beta \wedge \gamma)}{P(\sim (\beta \wedge \gamma))} \leq 0$$

From this and the last line of the reasoning set out in case 2 we obtain that

$$1 - \frac{P(\beta|\alpha) - P(\beta)}{P(\sim \beta)} + 1 - \frac{P(\gamma|\alpha) - P(\gamma)}{P(\sim \gamma)} \geq 1$$

This is to say that $1 - V_P(\alpha \triangleright \beta) + 1 - V_P(\alpha \triangleright \gamma) \geq 1$, hence that $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq 1$. Therefore, $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P(\alpha \triangleright (\beta \wedge \gamma))$. □

**Fact 39**: $\alpha \triangleright \beta \vDash \sim \beta \triangleright \sim \alpha$

*Proof.* Three cases must be considered.
*Case 1*: $P(\alpha) = 0$ or $P(\beta) = 1$. In this case $P(\sim \alpha) = 1$ or $P(\sim \beta) = 0$, so $V_P(\sim \beta \triangleright \sim \alpha) = 1$. It follows that $U_P(\sim \beta \triangleright \sim \alpha) = 0$, hence that $U_P(\alpha \triangleright \beta) \geq U_P(\sim \beta \triangleright \sim \alpha)$.
*Case 2*: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) \leq P(\beta)$. In this case $V_P(\alpha \triangleright \beta) = 0$, so $U_P(\alpha \triangleright \beta) = 1$. Therefore, $U_P(\alpha \triangleright \beta) \geq U_P(\sim \beta \triangleright \sim \alpha)$.
*Case 3*: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) > P(\beta)$. In this case, by the probability calculus we have that $P(\sim \alpha| \sim \beta) > P(\sim \alpha)$, so that

$$V_P(\alpha \triangleright \beta) = \frac{P(\beta|\alpha) - P(\beta)}{1 - P(\beta)} = \frac{P(\sim \beta) - P(\sim \beta|\alpha)}{P(\sim \beta)} = 1 - \frac{P(\sim \beta|\alpha)}{P(\sim \beta)}$$

$$V_P(\sim \beta \triangleright \sim \alpha) = \frac{P(\sim \alpha| \sim \beta) - P(\sim \alpha)}{1 - P(\sim \alpha)} = \frac{P(\alpha) - P(\alpha| \sim \beta)}{P(\alpha)} = 1 - \frac{P(\alpha| \sim \beta)}{P(\alpha)}$$

But

$$1 - \frac{P(\sim \beta|\alpha)}{P(\sim \beta)} = 1 - \frac{P(\alpha| \sim \beta)}{P(\alpha)}$$

Therefore, $V_P(\alpha \triangleright \beta) = V_P(\sim \beta \triangleright \sim \alpha)$, and consequently $U_P(\alpha \triangleright \beta) \geq U_P(\sim \beta \triangleright \sim \alpha)$. □

**Fact 40** $\alpha \triangleright \gamma, \beta \triangleright \gamma \vDash (\alpha \vee \beta) \triangleright \gamma$

*Proof.* From facts 11 and 39. □

**Fact 41**: $\Box\alpha \vDash \beta \triangleright \alpha$

*Proof.* Two cases must be considered.
*Case 1*: $P(\alpha) < 1$. In this case $V_P(\Box\alpha) = 0$, so $U_P(\Box\alpha) = 1$. Therefore, $U_P(\Box\alpha) \geq U_P(\beta \triangleright \alpha)$.
*Case 2*: $P(\alpha) = 1$. In this case $V_P(\beta \triangleright \alpha) = 1$, so $U_P(\beta \triangleright \alpha) = 0$. Therefore, $U_P(\Box\alpha) \geq U_P(\beta \triangleright \alpha)$. □

**Fact 42**: $\Box \sim \alpha \vDash \alpha \triangleright \beta$

*Proof.* From facts 15 and 41. □

**Fact 43**: $\alpha \triangleright \beta, \alpha \triangleright \gamma \vDash (\alpha \wedge \beta) \triangleright \gamma$

*Proof.* First, note that if $P(\alpha \wedge \beta) = 0$ or $P(\gamma) = 1$, then $V_P((\alpha \wedge \beta) \triangleright \gamma) = 1$, so $U_P((\alpha \wedge \beta) \triangleright \gamma) = 0$. Therefore, $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P((\alpha \wedge \beta) \triangleright \gamma)$. Second, note that if $P(\beta) = 1$ or $P(\alpha \wedge \sim \beta) = 0$, then $V_P(\alpha \triangleright \gamma) = V_P((\alpha \wedge \beta) \triangleright \gamma)$, so $U_P(\alpha \triangleright \gamma) = U_P((\alpha \wedge \beta) \triangleright \gamma)$. Therefore, $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P((\alpha \wedge \beta) \triangleright \gamma)$. Now let us reason under the assumption that $P(\beta) < 1$, $P(\gamma) < 1$, $P(\alpha \wedge \beta) > 0$, $P(\alpha \wedge \sim \beta) > 0$. Three cases are possible.
*Case 1*: $P(\beta|\alpha) \leq P(\beta)$ or $P(\gamma|\alpha) \leq P(\gamma)$. In this case $V_P(\alpha \triangleright \beta) = 0$ or $V_P(\alpha \triangleright \gamma) = 0$, which means that $U_P(\alpha \triangleright \beta) = 1$ or $U_P(\alpha \triangleright \gamma) = 1$. It follows that $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P((\alpha \wedge \beta) \triangleright \gamma)$.
*Case 2*: $P(\beta|\alpha) > P(\beta)$ and $P(\gamma|\alpha) > P(\gamma)$, but $P(\gamma|\alpha \wedge \beta) \leq P(\gamma)$. In this case $P(\alpha \wedge \beta|\gamma) \leq P(\alpha \wedge \beta) \leq P(\alpha \wedge \beta| \sim \gamma)$, and we have that

$$P(\alpha \wedge \sim \beta| \sim \gamma) + P(\alpha| \sim \beta) \geq P(\alpha| \sim \beta)P(\sim \beta)$$
$$P(\alpha \wedge \sim \beta| \sim \gamma) + P(\alpha| \sim \beta) \geq P(\alpha \wedge \sim \beta)$$
$$P(\alpha \wedge \beta| \sim \gamma) + P(\alpha \wedge \sim \beta| \sim \gamma) + P(\alpha| \sim \beta) \geq P(\alpha \wedge \beta) + P(\alpha \wedge \sim \beta)$$
$$\frac{P(\alpha| \sim \beta)}{P(\alpha \wedge \beta) + P(\alpha \wedge \sim \beta)} + \frac{P(\alpha \wedge \beta| \sim \gamma) + P(\alpha \wedge \sim \beta| \sim \gamma)}{P(\alpha \wedge \beta) + P(\alpha \wedge \sim \beta)} \geq 1$$
$$\frac{P(\alpha| \sim \beta)}{P(\alpha)} + \frac{P(\alpha| \sim \gamma)}{P(\alpha)} \geq 1$$
$$\frac{P(\sim \beta|\alpha)}{P(\sim \beta)} + \frac{P(\sim \gamma|\alpha)}{P(\sim \gamma)} \geq 1$$
$$\frac{1 - P(\beta|\alpha)}{P(\sim \beta)} + \frac{1 - P(\gamma|\alpha)}{P(\sim \gamma)} \geq 1$$
$$\frac{P(\sim \beta) + P(\beta) - P(\beta|\alpha)}{P(\sim \beta)} + \frac{P(\sim \gamma) + P(\gamma) - P(\gamma|\alpha)}{P(\sim \gamma)} \geq 1$$
$$1 + \frac{P(\beta) - P(\beta|\alpha)}{P(\sim \beta)} + 1 + \frac{P(\gamma) - P(\gamma|\alpha)}{P(\sim \gamma)} \geq 1$$
$$1 - \frac{P(\beta|\alpha) - P(\beta)}{1 - P(\beta)} + 1 - \frac{P(\gamma|\alpha) - P(\gamma)}{1 - P(\gamma)} \geq 1$$

This means that $1 - V_P(\alpha \triangleright \beta) + 1 - V_P(\alpha \triangleright \gamma) \geq 1$, hence that $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P((\alpha \wedge \beta) \triangleright \gamma)$.

*Case 3*: $P(\beta|\alpha) > P(\beta)$, $P(\gamma|\alpha) > P(\gamma)$, and $P(\gamma|\alpha \wedge \beta) > P(\gamma)$. In this case $P(\alpha \wedge \beta|\gamma) > P(\alpha \wedge \beta) > P(\alpha \wedge \beta| \sim \gamma)$, and we have

$$P(\alpha \wedge \beta) > P(\alpha \wedge \beta| \sim \gamma)$$
$$1 > \frac{P(\alpha \wedge \beta| \sim \gamma)}{P(\alpha \wedge \beta)}$$
$$\frac{P(\alpha \wedge \sim \beta| \sim \gamma)}{P(\alpha \wedge \sim \beta)} + 1 > \frac{P(\alpha \wedge \beta| \sim \gamma)}{P(\alpha \wedge \beta)}$$
$$\frac{P(\alpha \wedge \sim \beta)}{P(\sim \beta)P(\alpha \wedge \sim \beta)} + \frac{P(\alpha \wedge \sim \beta| \sim \gamma)}{P(\alpha \wedge \sim \beta)} > \frac{P(\alpha \wedge \beta| \sim \gamma)}{P(\alpha \wedge \beta)}$$
$$\frac{P(\alpha| \sim \beta)}{P(\alpha \wedge \sim \beta)} + \frac{P(\alpha \wedge \sim \beta| \sim \gamma)}{P(\alpha \wedge \sim \beta)} > \frac{P(\alpha \wedge \beta| \sim \gamma)}{P(\alpha \wedge \beta)}$$
$$P(\alpha| \sim \beta)P(\alpha \wedge \beta) + P(\alpha \wedge \sim \beta| \sim \gamma)P(\alpha \wedge \beta) > P(\alpha \wedge \beta| \sim \gamma)P(\alpha \wedge \sim \beta)$$
$$P(\alpha| \sim \beta)P(\alpha \wedge \beta) + P(\alpha \wedge \beta| \sim \gamma)P(\alpha \wedge \beta) + P(\alpha \wedge \sim \beta| \sim \gamma)P(\alpha \wedge \beta)$$
$$> P(\alpha \wedge \beta| \sim \gamma)P(\alpha \wedge \beta) + P(\alpha \wedge \beta| \sim \gamma)P(\alpha \wedge \sim \beta)$$
$$P(\alpha| \sim \beta)P(\alpha \wedge \beta) + P(\alpha \wedge \beta)(P(\alpha \wedge \beta| \sim \gamma) + P(\alpha \wedge \sim \beta| \sim \gamma)) > P(\alpha \wedge \beta| \sim \gamma)(P(\alpha \wedge \beta) + P(\alpha \wedge \sim \beta))$$
$$\frac{P(\alpha| \sim \beta)}{P(\alpha \wedge \beta) + P(\alpha \wedge \sim \beta)} + \frac{P(\alpha \wedge \beta| \sim \gamma) + P(\alpha \wedge \sim \beta| \sim \gamma)}{P(\alpha \wedge \beta) + P(\alpha \wedge \sim \beta)} > \frac{P(\alpha \wedge \beta| \sim \gamma)}{P(\alpha \wedge \beta)}$$
$$\frac{P(\alpha| \sim \beta)}{P(\alpha)} + \frac{P(\alpha| \sim \gamma)}{P(\alpha)} > \frac{P(\alpha \wedge \beta| \sim \gamma)}{P(\alpha \wedge \beta)}$$
$$\frac{P(\sim \beta|\alpha)}{P(\sim \beta)} + \frac{P(\sim \gamma|\alpha)}{P(\sim \gamma)} > \frac{P(\sim \gamma|\alpha \wedge \beta)}{P(\sim \gamma)}$$
$$\frac{1 - P(\beta|\alpha)}{P(\sim \beta)} + \frac{1 - P(\gamma|\alpha)}{P(\sim \gamma)} > \frac{1 - P(\gamma|\alpha \wedge \beta)}{P(\sim \gamma)}$$
$$\frac{P(\sim \beta) + P(\beta) - P(\beta|\alpha)}{P(\sim \beta)} + \frac{P(\sim \gamma) + P(\gamma) - P(\gamma|\alpha)}{P(\sim \gamma)} > \frac{P(\sim \gamma) + P(\gamma) - P(\gamma|\alpha \wedge \beta)}{P(\sim \gamma)}$$
$$1 + \frac{P(\beta) - P(\beta|\alpha)}{P(\sim \beta)} + 1 + \frac{P(\gamma) - P(\gamma|\alpha)}{P(\sim \gamma)} > 1 + \frac{P(\gamma) - P(\gamma|\alpha \wedge \beta)}{P(\sim \gamma)}$$
$$1 - \frac{P(\beta|\alpha) - P(\beta)}{1 - P(\beta)} + 1 - \frac{P(\gamma|\alpha) - P(\gamma)}{1 - P(\gamma)} > 1 - \frac{P(\gamma|\alpha \wedge \beta) - P(\gamma)}{1 - P(\gamma)}$$

This means that $1 - V_P(\alpha \triangleright \beta) + 1 - V_P(\alpha \triangleright \gamma) > 1 - V_P((\alpha \wedge \beta) \triangleright \gamma)$. Therefore, $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P((\alpha \wedge \beta) \triangleright \gamma$. □

**Fact 44**: $\alpha \triangleright \gamma, \sim((\alpha \wedge \sim \beta) \triangleright \gamma) \vDash (\alpha \wedge \beta) \triangleright \gamma$

*Proof.* First, note that if $P(\alpha \wedge \beta) = 0$, then $V_P((\alpha \wedge \beta) \triangleright \gamma) = 1$, hence $U_P((\alpha \wedge \beta) \triangleright \gamma) = 0$. Therefore, $U_P(\alpha \triangleright \gamma) + U_P(\sim((\alpha \wedge \sim \beta) \triangleright \gamma)) \geq U_P((\alpha \wedge \beta) \triangleright \gamma)$. Second, note that if $P(\alpha \wedge \sim \beta) = 0$, then $U_P(\sim((\alpha \wedge \sim \beta) \triangleright \gamma)) = 1 - V_P(\sim((\alpha \wedge \sim \beta) \triangleright \gamma)) = 1 - (1 - (V_P((\alpha \wedge \sim \beta) \triangleright \gamma))) = 1 - (1 - 1) = 1$. Therefore, $U_P(\alpha \triangleright \gamma) + U_P(\sim((\alpha \wedge \sim \beta) \triangleright \gamma)) \geq U_P((\alpha \wedge \beta) \triangleright \gamma)$. Now let us reason under the assumption that $P(\alpha \wedge \beta) > 0$ and $P(\alpha \wedge \sim \beta) > 0$. Three cases are possible.
*Case 1*: $P(\gamma) = 1$. In this case $V_P((\alpha \wedge \beta) \triangleright \gamma) = 1$, so $U_P((\alpha \wedge \beta) \triangleright \gamma) = 0$. Therefore, $U_P(\alpha \triangleright \gamma) + U_P(\sim((\alpha \wedge \sim \beta) \triangleright \gamma)) \geq U_P((\alpha \wedge \beta) \triangleright \gamma)$.

*Case 2*: $P(\gamma) < 1$ and $P(\gamma|\alpha) \leq P(\gamma)$. In this case $V_P(\alpha \triangleright \gamma) = 0$, so $U_P(\alpha \triangleright \gamma) = 1$. Therefore, $U_P(\alpha \triangleright \gamma) + U_P(\sim ((\alpha \wedge \sim \beta) \triangleright \gamma)) \geq U_P((\alpha \wedge \beta) \triangleright \gamma)$.
*Case 3*: $P(\gamma) < 1$ and $P(\gamma|\alpha) > P(\gamma)$. In this case we can assume that $P(\gamma|\alpha \wedge \beta) \geq P(\gamma|\alpha) \geq P(\gamma|\alpha \wedge \sim \beta)$ with no loss of generality, and we have:

$$\begin{gathered}
P(\gamma|\alpha \wedge \beta) \geq P(\gamma|\alpha) \\
P(\gamma|\alpha \wedge \beta) - P(\gamma) \geq P(\gamma|\alpha) - P(\gamma) \\
\frac{P(\gamma|\alpha \wedge \beta) - P(\gamma)}{P(\sim \gamma)} \geq \frac{P(\gamma|\alpha) - P(\gamma)}{P(\sim \gamma)} \\
1 - \frac{P(\gamma|\alpha \wedge \beta) - P(\gamma)}{P(\sim \gamma)} \leq 1 - \frac{P(\gamma|\alpha) - P(\gamma)}{P(\sim \gamma)} \\
1 - \frac{P(\gamma|\alpha \wedge \beta) - P(\gamma)}{1 - P(\gamma)} \leq 1 - \frac{P(\gamma|\alpha) - P(\gamma)}{1 - P(\gamma)} \\
1 - V_P((\alpha \wedge \beta) \triangleright \gamma) \leq 1 - V_P(\alpha \triangleright \gamma) \\
V_P(\sim ((\alpha \wedge \beta) \triangleright \gamma)) \leq V_P(\sim (\alpha \triangleright \gamma)) \\
1 - V_P(\sim ((\alpha \wedge \beta) \triangleright \gamma)) \geq 1 - V_P(\sim (\alpha \triangleright \gamma)) \\
U_P(\sim ((\alpha \wedge \beta) \triangleright \gamma)) \geq U_P(\sim (\alpha \triangleright \gamma)) \\
U_P(\sim ((\alpha \wedge \beta) \triangleright \gamma)) + U_P(\sim ((\alpha \wedge \sim \beta) \triangleright \gamma)) \geq U_P(\sim (\alpha \triangleright \gamma)) \\
1 + U_P(\sim ((\alpha \wedge \beta) \triangleright \gamma)) + U_P(\sim ((\alpha \wedge \sim \beta) \triangleright \gamma)) \geq 1 + U_P(\sim (\alpha \triangleright \gamma)) \\
1 - U_P(\sim (\alpha \triangleright \gamma)) + U_P(\sim ((\alpha \wedge \sim \beta) \triangleright \gamma)) \geq 1 - U_P(\sim ((\alpha \wedge \beta) \triangleright \gamma)) \\
1 - (1 - V_P(\sim (\alpha \triangleright \gamma))) + U_P(\sim ((\alpha \wedge \sim \beta) \triangleright \gamma)) \geq 1 - (1 - V_P(\sim ((\alpha \wedge \beta) \triangleright \gamma))) \\
1 - V_P(\sim\sim (\alpha \triangleright \gamma)) + U_P(\sim ((\alpha \wedge \sim \beta) \triangleright \gamma)) \geq 1 - V_P(\sim\sim ((\alpha \wedge \beta) \triangleright \gamma)) \\
1 - V_P(\alpha \triangleright \gamma) + U_P(\sim ((\alpha \wedge \sim \beta) \triangleright \gamma)) \geq 1 - V_P((\alpha \wedge \beta) \triangleright \gamma)
\end{gathered}$$

This means that $U_P(\alpha \triangleright \gamma) + U_P(\sim ((\alpha \wedge \sim \beta) \triangleright \gamma)) \geq U_P((\alpha \wedge \beta) \triangleright \gamma)$. □

**Fact 45**: $\alpha \triangleright \gamma, \sim (\alpha \triangleright \sim \beta) \not\models (\alpha \wedge \beta) \triangleright \gamma$

*Proof.* Let us assume what follows:

$$\begin{aligned}
P(\alpha \wedge \beta \wedge \gamma) &= 0,018 \\
P(\alpha \wedge \beta \wedge \sim \gamma) &= 0,052 \\
P(\alpha \wedge \sim \beta \wedge \gamma) &= 0,152 \\
P(\alpha \wedge \sim \beta \wedge \sim \gamma) &= 0,078 \\
P(\sim \alpha \wedge \beta \wedge \gamma) &= 0,003 \\
P(\sim \alpha \wedge \beta \wedge \sim \gamma) &= 0,160 \\
P(\sim \alpha \wedge \sim \beta \wedge \gamma) &= 0,127 \\
P(\sim \alpha \wedge \sim \beta \wedge \sim \gamma) &= 0,409
\end{aligned}$$

Thus we have $P(\alpha) = P(\gamma) = 0,30$, and $P(\beta) = 0,23$. Moreover, $P(\gamma|\alpha) = 0,56$, $P(\gamma|\alpha \wedge \beta) = 0,25$, and $P(\beta|\alpha) = P(\beta) = 0,23$. In this case, we have

$$U_P(\alpha \triangleright \gamma) = 1 - V_P(\alpha \triangleright \gamma) = 1 - \frac{P(\gamma|\alpha) - P(\gamma)}{1 - P(\gamma)} = 1 - \frac{0,56 - 0,30}{1 - 0,30} = 1 - 0,38 = 0,62$$

$$U_P(\sim (\alpha \triangleright \sim \beta)) = 1 - V_P(\sim (\alpha \triangleright \sim \beta)) = 1 - (1 - V_P(\alpha \triangleright \sim \beta)) = 1 - (1 - 0) = 0$$

$$U_P((\alpha \wedge \beta) \triangleright \gamma) = 1 - V_P((\alpha \wedge \beta) \triangleright \gamma) = 1 - 0 = 1$$

Therefore, $U_P(\alpha \triangleright \gamma) + U_P(\sim (\alpha \triangleright \sim \beta) < U_P((\alpha \wedge \beta) \triangleright \gamma)$. □

**Fact 46**: $\alpha \triangleright \gamma \nvDash (\alpha \wedge \beta) \triangleright \gamma$

*Proof.* Suppose that $\alpha \triangleright \gamma \vDash (\alpha \wedge \beta) \triangleright \gamma$. Then, $\alpha \triangleright \gamma, \sim (\alpha \triangleright \sim \beta) \vDash (\alpha \wedge \beta) \triangleright \gamma$, contrary to fact 45. □

**Fact 47**: Not: if $\beta \vDash_{PL} \gamma$, then $\alpha \triangleright \beta \vDash \alpha \triangleright \gamma$

*Proof.* Suppose that, if $\beta \vDash_{PL} \gamma$, then $\alpha \triangleright \beta \vDash \alpha \triangleright \gamma$. Then, by facts 10 and 39 we get that $\alpha \triangleright \gamma \vDash (\alpha \wedge \beta) \triangleright \gamma$, contrary to fact 46. □

**Fact 48**: $\alpha \triangleright \beta, \beta \triangleright \gamma \nvDash \alpha \triangleright \gamma$

*Proof.* Suppose that $\alpha \triangleright \beta, \beta \triangleright \gamma \vDash \alpha \triangleright \gamma$. Then, by facts 9 and 34 we get that $\alpha \triangleright \beta \vDash (\alpha \wedge \beta) \triangleright \gamma$, contrary to fact 46. □

**Fact 49**: $\alpha \triangleright \beta, (\alpha \wedge \beta) \triangleright \gamma \nvDash \alpha \triangleright \gamma$

*Proof.* Suppose that $\alpha \triangleright \beta, (\alpha \wedge \beta) \triangleright \gamma \vDash \alpha \triangleright \gamma$. Then, by facts 6 and 34 we get that, if $\beta \vDash_{PL} \gamma$, then $\alpha \triangleright \beta \vDash \alpha \triangleright \gamma$, contrary to fact 47. □

**Fact 50**: $\alpha \triangleright \beta, \beta \triangleright \alpha, \beta \triangleright \gamma \nvDash \alpha \triangleright \gamma$

*Proof.* Suppose that $\alpha \triangleright \beta, \beta \triangleright \alpha, \beta \triangleright \gamma \vDash \alpha \triangleright \gamma$. Then, by facts 12, 34, and 38 we get that $\alpha \triangleright \beta, (\alpha \wedge \beta) \triangleright \gamma \vDash \alpha \triangleright \gamma$, contrary to fact 49. □

**Fact 51**: $\alpha \wedge \beta \nvDash \alpha \triangleright \beta$

*Proof.* Suppose that $P(\alpha \wedge \beta) > 0$, and that $\alpha$ and $\beta$ are probabilistically independent, so that $P(\alpha \wedge \beta) = P(\alpha)P(\beta)$. Then $U_P(\alpha \wedge \beta) = 1 - V_P(\alpha \wedge \beta) = 1 - P(\alpha \wedge \beta) < 1$. However, $U_P(\alpha \triangleright \beta) = 1 - V_P(\alpha \triangleright \beta) = 1 - 0 = 1$, because $P(\beta|\alpha) = P(\beta)$. Therefore, $U_P(\alpha \wedge \beta) < U_P(\alpha \triangleright \beta)$. □

**Fact 52**: $\sim (\alpha \triangleright \beta) \nvDash \alpha \triangleright \sim \beta$

*Proof.* Suppose that $\sim (\alpha \triangleright \beta) \vDash \alpha \triangleright \sim \beta$. Then, by facts 8 and 43 we get that $\alpha \triangleright \gamma, \sim (\alpha \triangleright \sim \beta) \vDash (\alpha \wedge \beta) \triangleright \gamma$, contrary to fact 45. □

**Fact 53**: $\sim \alpha \nvDash \alpha \triangleright \beta$

*Proof.* Suppose that $0 < P(\beta) < 1$, and posit $\alpha = \sim \beta$. Then $U_P(\sim \alpha) = 1 - V_P(\sim \alpha) = 1 - P(\sim \alpha) = 1 - P(\beta) < 1$. But $U_P(\alpha \triangleright \beta) = 1 - V_P(\alpha \triangleright \beta) = 1 - 0 = 1$, because $P(\beta|\alpha) = P(\beta| \sim \beta) = 0$. Therefore, $U_P(\sim \alpha) < U_P(\alpha \triangleright \beta)$. □

**Fact 54**: $\beta \nvDash \alpha \triangleright \beta$

*Proof.* Suppose that $0 < P(\alpha) < 1$, and posit $\beta = \sim \alpha$. Then $U_P(\beta) = 1 - V_P(\beta) = 1 - P(\sim \alpha) < 1$. But $U_P(\alpha \triangleright \beta) = 1 - V_P(\alpha \triangleright \beta) = 1 - 0 = 1$, because $P(\beta|\alpha) = P(\sim \alpha|\alpha) = 0$. Therefore, $U_P(\beta) < U_P(\alpha \triangleright \beta)$. □

**Fact 55**: Not: if $\Gamma, \alpha \vDash_{PL} \beta$, then $\Gamma \vDash \alpha \triangleright \beta$

*Proof.* Like that of fact 26. □

**Fact 56**: $\sim(\alpha \triangleright \beta) \nvDash \beta \triangleright \alpha$

*Proof.* Suppose that $0 < P(\alpha) < 1$, and posit $\beta = \sim \alpha$. Then $U_P(\sim(\alpha \triangleright \beta)) = 1 - V_P(\sim(\alpha \triangleright \beta)) = 1 - (1 - V_P(\alpha \triangleright \beta)) = 1 - (1 - 0) = 0$, because $P(\beta|\alpha) = P(\sim \alpha|\alpha) = 0$. But $U_P(\beta \triangleright \alpha) = 1 - V_P(\beta \triangleright \alpha) = 1 - 0 = 1$, because $P(\alpha|\beta) = P(\alpha| \sim \alpha) = 0$. Therefore, $U_P(\sim(\alpha \triangleright \beta)) < U_P(\beta \triangleright \alpha)$. □

**Fact 57**: $\sim(\alpha \triangleright \beta) \nvDash \sim \alpha \triangleright \beta$

*Proof.* Suppose that $P(\alpha) > 0$ and that $P(\sim \alpha \wedge \sim \gamma) > 0$, and posit $\beta = \sim \alpha \wedge \gamma$. Then $U_P(\sim(\alpha \triangleright \beta)) = 1 - V_P(\sim(\alpha \triangleright \beta)) = 1 - (1 - V_P(\alpha \triangleright \beta)) = 1 - (1 - 0) = 0$, because $P(\beta|\alpha) = P(\sim \alpha \wedge \gamma|\alpha) = 0$. But $U_P(\sim \alpha \triangleright \beta) = 1 - V_P(\sim \alpha \triangleright \beta) > 0$, because we have

$$P(\beta| \sim \alpha) = P(\sim \alpha \wedge \gamma| \sim \alpha) = \frac{P(\sim \alpha \wedge \gamma)}{P(\sim \alpha)} = \frac{P(\sim \alpha \wedge \gamma)}{P(\sim \alpha \wedge \gamma) + P(\sim \alpha \wedge \sim \gamma)} < 1$$

Therefore, $U_P(\sim(\alpha \triangleright \beta)) < U_P(\sim \alpha \triangleright \beta)$. □

**Fact 58**: $\top \triangleright \alpha \vDash \beta \triangleright \alpha$

*Proof.* Let $\alpha, \beta \in \mathbf{P}$, and consider any probability function $P$. Two cases are possible.
*Case 1*: $P(\alpha) < 1$. In this case $P(\alpha|\top) = P(\alpha)$, and $U_P(\top \triangleright \alpha) = 1 - V_P(\top \triangleright \alpha) = 1 - 0 = 1$. Therefore, $U_P(\top \triangleright \alpha) \geq U_P(\beta \triangleright \alpha)$.
*Case 2*: $P(\alpha) = 1$. In this case, $U_P(\top \triangleright \alpha) = 1 - 1 = 0$, and $U_P(\beta \triangleright \alpha) = 1 - V_P(\beta \triangleright \alpha) = 1 - 1 = 0$. Therefore, $U_P(\top \triangleright \alpha) \geq U_P(\beta \triangleright \alpha)$. □

**Fact 59**: $\alpha \nvDash \top \triangleright \alpha$

*Proof.* Suppose that $\alpha \vDash \top \triangleright \alpha$. Then, by facts 14 and 58 we get that $\beta \vDash \alpha \triangleright \beta$, contrary to fact 54. □

**Fact 60**: If $\beta \vDash_{PL} \sim \gamma$, then $\Diamond\alpha, \alpha \triangleright \beta \vDash \sim(\alpha \triangleright \gamma)$

*Proof.* Assume that $\beta \vDash_{PL} \sim \gamma$. First, note that if $P(\alpha) = 0$, then $V_P(\Diamond\alpha) = 0$, so $U_P(\Diamond\alpha) = 1$. Therefore, $U_P(\Diamond\alpha) + U_P(\alpha \triangleright \beta) \geq U_P(\sim(\alpha \triangleright \gamma))$. Second, note that if $P(\beta) = 1$, then $P(\sim \gamma) = 1$, because $\beta \vDash_{PL} \sim \gamma$, so $P(\gamma) = 0$ and $V_P(\alpha \triangleright \gamma) = 0$, which means that $V_P(\sim(\alpha \triangleright \gamma)) = 1$, hence that $U_P(\sim(\alpha \triangleright \gamma)) = 0$. Therefore, $U_P(\Diamond\alpha) + U_P(\alpha \triangleright \beta) \geq U_P(\sim(\alpha \triangleright \gamma))$. Third, note that if $P(\gamma) = 1$, then $P(\sim \beta) = 1$, because $\beta \vDash_{PL} \sim \gamma$, so $P(\beta) = 0$. It follows that $V_P(\alpha \triangleright \beta) = 0$, hence that $U_P(\alpha \triangleright \beta) = 1$. Therefore, $U_P(\Diamond\alpha) + U_P(\alpha \triangleright \beta) \geq U_P(\sim(\alpha \triangleright \gamma))$. Now let us reason under the assumption that $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\gamma) < 1$. Three cases are possible.
*Case 1*: $P(\beta|\alpha) \leq P(\beta)$. In this case $V_P(\alpha \triangleright \beta) = 0$, so $U_P(\alpha \triangleright \beta) = 1$. Therefore, $U_P(\Diamond\alpha) + U_P(\alpha \triangleright \beta) \geq U_P(\sim(\alpha \triangleright \gamma))$.
*Case 2*: $P(\gamma|\alpha) \leq P(\gamma)$. In this case $V_P(\alpha \triangleright \gamma) = 0$, so $V_P(\sim(\alpha \triangleright \gamma)) = 1$ and $U_P(\sim(\alpha \triangleright \gamma)) = 0$. Therefore, $U_P(\Diamond\alpha) + U_P(\alpha \triangleright \beta) \geq U_P(\sim(\alpha \triangleright \gamma))$.

*Case 3*: $P(\beta|\alpha) > P(\beta)$ and $P(\gamma|\alpha) > P(\gamma)$. In this case, since $\beta \vDash_{PL} \sim \gamma$, we have that $1 \geq P(\beta \vee \gamma|\alpha) = P(\beta|\alpha) + P(\gamma|\alpha)$. Moreover, $P(\beta|\alpha) \geq V_P(\alpha \triangleright \beta)$, because

$$
\begin{gathered}
P(\beta)(1 - P(\beta|\alpha)) \geq 0 \\
P(\beta) - P(\beta)P(\beta|\alpha) \geq 0 \\
P(\beta) + (-P(\beta))P(\beta|\alpha) \geq 0 \\
P(\beta) + (-(1 - P(\sim \beta)))P(\beta|\alpha) \geq 0 \\
P(\beta) + (P(\sim \beta) - 1)P(\beta|\alpha) \geq 0 \\
P(\beta|\alpha)P(\sim \beta) - P(\beta|\alpha) + P(\beta) \geq 0 \\
P(\beta|\alpha)P(\sim \beta) \geq P(\beta|\alpha) - P(\beta) \\
P(\beta|\alpha) \geq \frac{P(\beta|\alpha) - P(\beta)}{P(\sim \beta)}
\end{gathered}
$$

For the same reasons, $P(\gamma|\alpha) \geq V_P(\alpha \triangleright \gamma)$. Thus, given that $1 \geq P(\beta \vee \gamma|\alpha) = P(\beta|\alpha) + P(\gamma|\alpha)$, we have that

$$
\begin{gathered}
1 \geq V_P(\alpha \triangleright \beta) + V_P(\alpha \triangleright \gamma) \\
1 - V_P(\alpha \triangleright \beta) \geq V_P(\alpha \triangleright \gamma) \\
1 - V_P(\alpha \triangleright \beta) \geq 1 - (1 - V_P(\alpha \triangleright \gamma)) \\
1 - V_P(\alpha \triangleright \beta) \geq 1 - (V_P(\sim (\alpha \triangleright \gamma)) \\
U_P(\alpha \triangleright \beta) \geq U_P(\sim (\alpha \triangleright \gamma))
\end{gathered}
$$

This entails that $U_P(\Diamond\alpha) + U_P(\alpha \triangleright \beta) \geq U_P(\sim (\alpha \triangleright \gamma))$. □

**Fact 61**: $\Diamond\alpha, \alpha \triangleright \beta \vDash \sim (\alpha \triangleright \sim \beta)$

*Proof.* From facts 16 and 60. □

**Fact 62**: $\Diamond\alpha \vDash \sim (\alpha \triangleright \sim \alpha)$

*Proof.* From facts 17, 34, and 61. □

**Fact 63**: $\Diamond \sim \beta, \alpha \triangleright \beta \vDash \sim (\sim \alpha \triangleright \beta)$

*Proof.* From facts 18, 39, and 60. □

**Fact 64**: If $\alpha \vDash_{PL} \beta$, then $\vDash \alpha \multimap \beta$

*Proof.* Assume that $\alpha \vDash_{PL} \beta$. Then $V_P(\alpha \multimap \beta) = 1$ no matter whether $P(\alpha) > 0$ or $P(\alpha) = 0$, so $U_P(\alpha \multimap \beta) = 1 - V_P(\alpha \multimap \beta) = 1 - 1 = 0$. □

**Fact 65**: $\alpha \multimap \beta \vDash \alpha \supset \beta$

*Proof.* Two cases must be considered.

*Case 1*: $P(\beta|\alpha) = 1$. In this case $V_P(\alpha \supset \beta) = 1$, because we have that

$$
\begin{gathered}
P(\sim \beta|\alpha)P(\alpha) \leq P(\sim \beta|\alpha) \\
P(\beta|\alpha) + P(\sim \beta|\alpha)P(\alpha) \leq P(\beta|\alpha) + P(\sim \beta|\alpha) \\
P(\beta|\alpha) + P(\sim \beta|\alpha)P(\alpha) \leq 1 \\
P(\beta|\alpha) \leq 1 - P(\sim \beta|\alpha)P(\alpha) \\
P(\beta|\alpha) \leq 1 - P(\alpha \wedge \sim \beta) \\
P(\beta|\alpha) \leq 1 - P(\sim (\alpha \supset \beta)) \\
P(\beta|\alpha) \leq P(\alpha \supset \beta)
\end{gathered}
$$

This entails that $U_P(\alpha \supset \beta) = 0$. Therefore, $U_P(\alpha \mathrel{-\!\!3} \beta) \geq U_P(\alpha \supset \beta)$.
*Case 2*: $P(\beta|\alpha) < 1$. In this case $V_P(\alpha \mathrel{-\!\!3} \beta) = 0$, so $U_P(\alpha \mathrel{-\!\!3} \beta) = 1$. Therefore, $U_P(\alpha \mathrel{-\!\!3} \beta) \geq U_P(\alpha \supset \beta)$. □

**Fact 66**: $\top \triangleright \alpha \vDash \alpha$

*Proof.* From facts 4 and 65. □

**Fact 67**: $\alpha \triangleright \beta, \alpha \vDash \beta$

*Proof.* From facts 5 and 65. □

**Fact 68**: $\alpha \mathrel{-\!\!3} \beta, \alpha \mathrel{-\!\!3} \gamma \vDash \alpha \mathrel{-\!\!3} (\beta \wedge \gamma)$

*Proof.* First, note that if $P(\alpha) = 0$, then $V_P(\alpha \mathrel{-\!\!3} (\beta \wedge \gamma)) = 1$, so $U_P(\alpha \mathrel{-\!\!3} (\beta \wedge \gamma)) = 0$. Therefore, $U_P(\alpha \mathrel{-\!\!3} \beta) + U_P(\alpha \mathrel{-\!\!3} \gamma) \geq U_P(\alpha \mathrel{-\!\!3} (\beta \wedge \gamma))$. Now we will reason under the assumption that $P(\alpha) > 0$. Two cases must be considered.
*Case 1*: $P(\beta|\alpha) < 1$ or $P(\gamma|\alpha) < 1$. In this case $V_P(\alpha \mathrel{-\!\!3} \beta) = 0$ or $V_P(\alpha \mathrel{-\!\!3} \gamma) = 0$, which means that $U_P(\alpha \mathrel{-\!\!3} \beta) = 1$ or $U_P(\alpha \mathrel{-\!\!3} \gamma) = 1$. Therefore, $U_P(\alpha \mathrel{-\!\!3} \beta) + U_P(\alpha \mathrel{-\!\!3} \gamma) \geq U_P(\alpha \mathrel{-\!\!3} (\beta \wedge \gamma))$.
*Case 2*: $P(\beta|\alpha) = 1$ and $P(\gamma|\alpha) = 1$. In this case $P(\alpha \wedge \beta) = P(\alpha)$. Since $P(\alpha) = P(\alpha \wedge \beta) + P(\alpha \wedge \sim \beta)$, then $P(\alpha \wedge \sim \beta) = 0$. Moreover, since $P(\alpha \wedge \sim \beta) = P(\gamma \wedge \sim \beta \wedge \alpha) + P(\sim \gamma \wedge \sim \beta \wedge \alpha)$, it follows that $P(\gamma \wedge \sim \beta \wedge \alpha) = P(\sim \gamma \wedge \sim \beta \wedge \alpha) = 0$. A similar reasoning leads from the premise that $P(\alpha \wedge \gamma) = P(\alpha)$ to the conclusion that $P(\beta \wedge \sim \gamma \wedge \alpha) = P(\sim \beta \wedge \sim \gamma \wedge \alpha) = 0$. Consequently, $P(\sim (\beta \wedge \gamma) \wedge \alpha) = P((\sim \beta \vee \sim \gamma) \wedge \alpha)) = P(\sim \beta \wedge \sim \gamma \wedge \alpha) + P(\beta \wedge \sim \gamma \wedge \alpha) + P(\sim \beta \wedge \gamma \wedge \alpha) = 0$, so that $P(\alpha) = P((\beta \wedge \gamma) \wedge \alpha)) + P(\sim (\beta \wedge \gamma) \wedge \alpha) = P((\beta \wedge \gamma) \wedge \alpha)$, which implies that $P(\beta \wedge \gamma|\alpha) = 1$. Thus, $V_P(\alpha \mathrel{-\!\!3} (\beta \wedge \gamma)) = 1$, hence $U_P(\alpha \mathrel{-\!\!3} (\beta \wedge \gamma)) = 0$. Therefore, $U_P(\alpha \mathrel{-\!\!3} \beta) + U_P(\alpha \mathrel{-\!\!3} \gamma) \geq U_P(\alpha \mathrel{-\!\!3} (\beta \wedge \gamma))$. □

**Fact 69**: $\alpha \mathrel{-\!\!3} \beta \vDash \sim \beta \mathrel{-\!\!3} \sim \alpha$

*Proof.* Three cases must be considered.
*Case 1*: $P(\alpha) = 0$ or $P(\beta) = 1$. In this case $P(\sim \alpha) = 1$ or $P(\sim \beta) = 0$, so $V_P(\sim \beta \mathrel{-\!\!3} \sim \alpha) = 1$ and $U_P(\sim \beta \mathrel{-\!\!3} \sim \alpha) = 0$. Therefore, $U_P(\alpha \mathrel{-\!\!3} \beta) \geq U_P(\sim \beta \mathrel{-\!\!3} \sim \alpha)$.
*Case 2*: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) < 1$. In this case $V_P(\alpha \mathrel{-\!\!3} \beta) = 0$, so $U_P(\alpha \mathrel{-\!\!3} \beta) = 1$. Therefore, $U_P(\alpha \mathrel{-\!\!3} \beta) \geq U_P(\sim \beta \mathrel{-\!\!3} \sim \alpha)$.
*Case 3*: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) = 1$. In this case $P(\alpha \wedge \beta) = P(\alpha)$. Since $P(\alpha) = P(\alpha \wedge \beta) + P(\alpha \wedge \sim \beta)$, we get that $P(\alpha \wedge \sim \beta) = 0$, hence that

$P(\alpha| \sim \beta) = 0$. It follows that $P(\sim \alpha| \sim \beta) = 1$, so that $V_P(\sim \beta \strictif\sim \alpha) = 1$, which means that $U_P(\sim \beta \strictif\sim \alpha) = 0$. Therefore, $U_P(\alpha \strictif \beta) \geq U_P(\sim \beta \strictif\sim \alpha)$. □

**Fact 70**: $\alpha \strictif \gamma, \beta \strictif \gamma \vDash (\alpha \vee \beta) \strictif \gamma$

*Proof.* From facts 11 and 69. □

**Fact 71**: $\Box\alpha \vDash \beta \strictif \alpha$

*Proof.* Two cases must be considered.
*Case 1*: $P(\alpha) < 1$. In this case $V_P(\Box\alpha) = 0$, hence $U_P(\Box\alpha) = 1$. Therefore, $U_P(\Box\alpha) \geq U_P(\alpha \strictif \beta)$.
*Case 2*: $P(\alpha) = 1$. In this case, if $P(\beta) = 0$, then $V_P(\beta \strictif \alpha) = 1$, and if $P(\beta) > 0$, then $P(\alpha|\beta) = 1$, given that $P(\alpha \wedge \beta) = P(\beta)$, so again $V_P(\beta \strictif \alpha) = 1$. It follows that $U_P(\beta \strictif \alpha) = 0$, hence that $U_P(\Box\alpha) \geq U_P(\alpha \strictif \beta)$. □

**Fact 72**: $\Box \sim \alpha \vDash \alpha \rhd \beta$

*Proof.* From facts 15 and 71. □

**Fact 73**: $\alpha \strictif \beta, \beta \strictif \gamma \vDash \alpha \strictif \gamma$

*Proof.* First, note that if $P(\alpha) = 0$, then $V_P(\alpha \strictif \gamma) = 1$, hence $U_P(\alpha \strictif \gamma) = 0$. Therefore, $U_P(\alpha \strictif \beta) + U_P(\beta \strictif \gamma) \geq U_P(\alpha \strictif \gamma)$. Second, note that the same holds if $P(\alpha) > 0$ but $P(\beta) = 0$, because $V_P(\alpha \strictif \beta) = 0$, hence $U_P(\alpha \strictif \beta) = 1$. Now let us reason under the assumption that $P(\alpha) > 0$ and $P(\beta) > 0$. Two cases are possible.
*Case 1*: $P(\beta|\alpha) < 1$ or $P(\gamma|\beta) < 1$. In this case $V_P(\alpha \strictif \beta) = 0$ or $V_P(\beta \strictif \gamma) = 0$, hence $U_P(\alpha \strictif \beta) = 1$ or $U_P(\beta \strictif \gamma) = 1$. Therefore, $U_P(\alpha \strictif \beta) + U_P(\beta \strictif \gamma) \geq U_P(\alpha \strictif \gamma)$.
*Case 2*: $P(\beta|\alpha) = 1$ and $P(\gamma|\beta) = 1$. In this case we have that $P(\alpha \wedge \beta) = P(\alpha)$. Since $P(\alpha) = P(\alpha \wedge \beta) + P(\alpha \wedge \sim \beta)$, it follows that $P(\alpha \wedge \sim \beta) = 0$. Moreover, since $P(\alpha \wedge \sim \beta) = P(\gamma \wedge \sim \beta \wedge \alpha) + P(\sim \gamma \wedge \sim \beta \wedge \alpha)$, it follows that $P(\sim \gamma \wedge \sim \beta \wedge \alpha) = 0$. A similar reasoning leads from the premise that $P(\beta \wedge \gamma) = P(\beta)$ to the conclusion that $P(\beta \wedge \sim \gamma \wedge \alpha) = 0$. Thus we have that $P(\sim \gamma \wedge \alpha) = P(\sim \gamma \wedge \beta \wedge \alpha) + P(\sim \gamma \wedge \sim \beta \wedge \alpha) = 0$. So $P(\alpha) = P(\alpha \wedge \gamma) + P(\alpha \wedge \sim \gamma) = P(\alpha \wedge \gamma)$, which entails that $P(\gamma|\alpha) = 1$. It follows that $V_P(\alpha \strictif \gamma) = 1$, so that $U_P(\alpha \strictif \gamma) = 0$. Therefore, $U_P(\alpha \strictif \beta) + U_P(\beta \strictif \gamma) \geq U_P(\alpha \strictif \gamma)$. □

**Fact 74**: $\alpha \strictif \gamma \vDash (\alpha \wedge \beta) \strictif \gamma$

*Proof.* From facts 9, 64, and 73. □

**Fact 75**: $\alpha \strictif \beta, \beta \strictif \alpha, \beta \strictif \gamma \vDash \alpha \strictif \gamma$

*Proof.* Directly from fact 73. □

**Fact 76**: $\alpha \strictif \beta, (\alpha \wedge \beta) \strictif \gamma \vDash \alpha \strictif \gamma$

*Proof.* From facts 12, 64, 68, and 75. □

**Fact 77**: If $\beta \vDash_{PL} \gamma$, then $\alpha \strictif \beta \vDash \alpha \strictif \gamma$

*Proof.* From facts 6, 64, and 76. □

**Fact 78**: $\alpha \dashrightarrow \gamma, \sim(\alpha \dashrightarrow\sim \beta) \vDash (\alpha \wedge \beta) \dashrightarrow \gamma$

*Proof.* Directly from fact 74. □

**Fact 79**: $\alpha \dashrightarrow \beta, \alpha \dashrightarrow \gamma \vDash (\alpha \wedge \beta) \dashrightarrow \gamma$

*Proof.* Directly from fact 74. □

**Fact 80**: $\alpha \dashrightarrow \gamma, \sim((\alpha \wedge \sim \beta) \dashrightarrow \gamma) \vDash (\alpha \wedge \beta) \dashrightarrow \gamma$

*Proof.* Directly from fact 74. □

**Fact 81**: $\top \dashrightarrow \alpha \vDash \beta \dashrightarrow \alpha$

*Proof.* Directly from fact 74, given that $\beta$ is equivalent to $\top \wedge \beta$. □

**Fact 82**: $\sim \alpha \nvDash \alpha \dashrightarrow \beta$

*Proof.* Suppose that $0 < P(\beta) < 1$, and posit $\alpha =\sim \beta$. Then $U_P(\sim \alpha) = 1 - V_P(\sim \alpha) = 1 - P(\sim \alpha) < 1$. But $U_P(\alpha \dashrightarrow \beta) = 1 - V_P(\alpha \dashrightarrow \beta) = 1$, because $P(\beta|\alpha) = P(\beta| \sim \beta) = 0$. Therefore, $U_P(\sim \alpha) < U_P(\alpha \dashrightarrow \beta)$. □

**Fact 83**: $\beta \nvDash \alpha \dashrightarrow \beta$

*Proof.* Suppose that $0 < P(\alpha) < 1$, and posit $\beta =\sim \alpha$. Then $U_P(\beta) = 1 - V_P(\beta) = 1 - P(\sim \alpha) < 1$. But $U_P(\alpha \dashrightarrow \beta) = 1 - V_P(\alpha \dashrightarrow \beta) = 1$, because $P(\beta|\alpha) = P(\sim \alpha|\alpha) = 0$. Therefore, $U_P(\beta) < U_P(\alpha \dashrightarrow \beta)$. □

**Fact 84**: $\alpha \nvDash \top \dashrightarrow \alpha$

*Proof.* From facts 14, 81, and 83. □

**Fact 85**: $\alpha \wedge \beta \nvDash \alpha \dashrightarrow \beta$

*Proof.* Suppose that $P(\alpha \wedge \beta) > 0$ and $P(\alpha \wedge \sim \beta) > 0$. Then $U_P(\alpha \wedge \beta) = 1 - V_P(\alpha \wedge \beta) = 1 - P(\alpha \wedge \beta) < 1$, because $P(\alpha \wedge \beta) > 0$. But $U_P(\alpha \dashrightarrow \beta) = 1 - V_P(\alpha \dashrightarrow \beta) = 1$, because $P(\alpha \wedge \sim \beta) > 0$, thus $P(\alpha \wedge \beta) < P(\alpha)$ and $P(\beta|\alpha) < 1$. Therefore, $U_P(\alpha \wedge \beta) < U_P(\alpha \dashrightarrow \beta)$. □

**Fact 86**: $\sim(\alpha \dashrightarrow \beta) \nvDash \alpha \dashrightarrow\sim \beta$

*Proof.* Suppose that $P(\alpha \wedge \beta) > 0$ and $P(\alpha \wedge \sim \beta) > 0$. Then $U_P(\sim(\alpha \dashrightarrow \beta)) = 1 - V_P(\sim(\alpha \dashrightarrow \beta)) = 1 - (1 - V_P(\alpha \dashrightarrow \beta)) = 1 - (1 - 0) = 0$, because $P(\alpha \wedge \sim \beta) > 0$, so $P(\alpha \wedge \beta) < P(\alpha)$ and consequently $P(\beta|\alpha) < 1$. But $U_P(\alpha \dashrightarrow\sim \beta) = 1 - V_P(\alpha \dashrightarrow\sim \beta) = 1 - 0 = 1$, because $P(\alpha \wedge \beta) > 0$, so $P(\alpha \wedge \sim \beta) < P(\alpha)$ and consequently $P(\sim \beta|\alpha) < 1$. Therefore, $U_P(\sim(\alpha \dashrightarrow \beta)) < U_P(\alpha \dashrightarrow\sim \beta)$. □

**Fact 87**: Not: if $\Gamma, \alpha \vDash_{PL} \beta$, then $\Gamma \vDash \alpha \dashrightarrow \beta$

*Proof.* Like that of facts 26 and 55. □

**Fact 88**: $\sim(\alpha \dashrightarrow \beta) \nvDash \beta \dashrightarrow \alpha$

*Proof.* Suppose that $0 < P(\alpha) < 1$, and posit $\beta =\sim \alpha$. Then $U_P(\sim (\alpha \dashrightarrow \beta)) = 1 - V_P(\sim (\alpha \dashrightarrow \beta)) = 1 - (1 - V_P(\alpha \dashrightarrow \beta)) = 1 - (1 - 0) = 0$, because $P(\beta|\alpha) = P(\sim \alpha|\alpha) = 0$. But $U_P(\beta \dashrightarrow \alpha) = 1 - V_P(\beta \dashrightarrow \alpha) = 1 - 0 = 1$, because $P(\alpha|\beta) = P(\alpha| \sim \alpha) = 0$. Therefore, $U_P(\sim (\alpha \dashrightarrow \beta)) < U_P(\beta \dashrightarrow \alpha)$. □

**Fact 89**: $\sim (\alpha \dashrightarrow \beta) \nvDash \sim \alpha \dashrightarrow \beta$

*Proof.* Suppose that $P(\alpha) > 0$ and $P(\sim \alpha \wedge \sim \gamma) > 0$, and posit $\beta =\sim \alpha \wedge \gamma$. Then $U_P(\sim (\alpha \dashrightarrow \beta)) = 1 - V_P(\sim (\alpha \dashrightarrow \beta)) = 1 - (1 - V_P(\alpha \dashrightarrow \beta)) = 1 - (1 - 0) = 0$, because $P(\beta|\alpha) = P(\sim \alpha \wedge \gamma|\alpha) = 0$. But $U_P(\sim \alpha \dashrightarrow \beta) = 1 - V_P(\sim \alpha \dashrightarrow \beta) = 1 - 0 = 1$, because $P(\beta| \sim \alpha) = P(\sim \alpha \wedge \gamma| \sim \alpha) = P(\gamma| \sim \alpha)$, and

$$P(\gamma| \sim \alpha) = \frac{P(\sim \alpha \wedge \gamma)}{P(\sim \alpha)} = \frac{P(\sim \alpha \wedge \gamma)}{P(\sim \alpha \wedge \gamma) + P(\sim \alpha \wedge \sim \gamma)} < 1$$

Therefore, $U_P(\sim (\alpha \dashrightarrow \beta)) < U_P(\sim \alpha \dashrightarrow \beta)$. □

**Fact 90**: If $\beta \vDash_{PL} \sim \gamma$, then $\Diamond\alpha, \alpha \dashrightarrow \beta \vDash \sim (\alpha \dashrightarrow \gamma)$

*Proof.* Two cases must be considered.
*Case 1*: $P(\alpha) = 0$. In this case $V_P(\Diamond\alpha) = 0$, which entails that $U_P(\Diamond\alpha) = 1$. Therefore, $U_P(\Diamond\alpha) + U_P(\alpha \dashrightarrow \beta) \geq U_P(\sim (\alpha \dashrightarrow \gamma))$.
*Case 2*: $P(\alpha) > 0$. In this case, since $\beta \vDash_{PL} \sim \gamma$, we have that $1 \geq P(\beta \vee \gamma|\alpha) = P(\beta|\alpha) + P(\gamma|\alpha)$. It follows that either $V_P(\alpha \dashrightarrow \beta) = 1$ and $V_P(\alpha \dashrightarrow \gamma) = 0$, with the consequence that $U_P(\alpha \dashrightarrow \beta) = 1 - V_P(\alpha \dashrightarrow \beta) = 0$ and $U_P(\sim (\alpha \dashrightarrow \gamma)) = 1 - V_P(\sim (\alpha \dashrightarrow \gamma)) = 1 - (1 - V_P(\alpha \dashrightarrow \gamma)) = 0$, or $V_P(\alpha \dashrightarrow \beta) = 0$ and $V_P(\alpha \dashrightarrow \gamma) = 1$, with the consequence that $U_P(\alpha \dashrightarrow \beta) = 1 - V_P(\alpha \dashrightarrow \beta) = 1$ and $U_P(\sim (\alpha \dashrightarrow \gamma)) = 1 - V_P(\sim (\alpha \dashrightarrow \gamma)) = 1 - (1 - V_P(\alpha \dashrightarrow \gamma)) = 1$. In both cases, $U_P(\Diamond\alpha) + U_P(\alpha \dashrightarrow \beta) \geq U_P(\sim (\alpha \dashrightarrow \gamma))$. □

**Fact 91**: $\Diamond\alpha, \alpha \dashrightarrow \beta \vDash \sim (\alpha \dashrightarrow \sim \beta)$

*Proof.* From facts 16 and 90. □

**Fact 92**: $\Diamond\alpha \vDash \sim (\alpha \dashrightarrow \sim \alpha)$

*Proof.* From facts 17, 64, and 91. □

**Fact 93**: $\Diamond \sim \beta, \alpha \dashrightarrow \beta \vDash \sim (\sim \alpha \dashrightarrow \beta)$

*Proof.* From facts 18, 69, and 90. □

**Fact 94**: $\alpha \dashrightarrow \beta \equiv \Box(\alpha \supset \beta)$

*Proof.* In order to prove that $\alpha \dashrightarrow \beta \vDash \Box(\alpha \supset \beta)$, three cases must be considered.
*Case 1*: $P(\alpha) = 0$ or $P(\beta) = 1$. In this case $P(\alpha \supset \beta) = 1$, so $V_P(\Box(\alpha \supset \beta)) = 1$ and $U_P(\Box(\alpha \supset \beta)) = 0$. Therefore, $U_P(\alpha \dashrightarrow \beta) \geq U_P(\Box(\alpha \supset \beta))$.
*Case 2*: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) = 1$. Since $P(\beta|\alpha) \leq P(\alpha \supset \beta)$, as has been shown in the proof of fact 65, in this case $P(\alpha \supset \beta) = 1$, hence $V_P(\Box(\alpha \supset \beta)) = 1$ and $U_P(\Box(\alpha \supset \beta)) = 0$. Therefore, $U_P(\alpha \dashrightarrow \beta) \geq U_P(\Box(\alpha \supset \beta)$.
*Case 3*: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) < 1$, In this case $V_P(\alpha \dashrightarrow \beta) = 0$ and $U_P(\alpha \dashrightarrow \beta) = 1$, so $U_P(\alpha \dashrightarrow \beta) \geq U_P(\Box(\alpha \supset \beta)$.

In order to prove that $\Box(\alpha \supset \beta) \vDash \alpha \dashv\!\!\! \rightarrow \beta$, it suffices to note what follows. In cases 1 and 2, $V_P(\alpha \dashv\!\!\! \rightarrow \beta) = 1$ and $U_P(\alpha \dashv\!\!\! \rightarrow \beta) = 0$. In case 3, $P(\alpha \supset \beta) < 1$, because $P(\beta|\alpha) < 1$ and we have that

$$\begin{gathered}
P(\sim \beta|\alpha)\frac{P(\alpha)}{P(\alpha)} = P(\sim \beta|\alpha) \\
P(\beta|\alpha) + P(\sim \beta|\alpha)\frac{P(\alpha)}{P(\alpha)} = P(\beta|\alpha) + P(\sim \beta|\alpha) \\
P(\beta|\alpha) + P(\sim \beta|\alpha)\frac{P(\alpha)}{P(\alpha)} = 1 \\
P(\beta|\alpha) = 1 - \frac{P(\sim \beta|\alpha)P(\alpha)}{P(\alpha)} \\
P(\beta|\alpha) = 1 - \frac{P(\alpha \wedge \sim \beta)}{P(\alpha)} \\
P(\beta|\alpha)P(\alpha) = P(\alpha) - P(\alpha \wedge \sim \beta) \\
P(\beta|\alpha)P(\alpha) + P(\sim \alpha) = P(\alpha) + P(\sim \alpha) - P(\alpha \wedge \sim \beta) \\
P(\beta|\alpha)P(\alpha) + P(\sim \alpha) = 1 - P(\alpha \wedge \sim \beta) \\
P(\beta|\alpha)P(\alpha) + P(\sim \alpha) = P(\sim \alpha \vee \beta) \\
P(\beta|\alpha)P(\alpha) + P(\sim \alpha) = P(\alpha \supset \beta)
\end{gathered}$$

It follows that $V_P(\Box(\alpha \supset \beta)) = 0$, hence that $U_P(\Box(\alpha \supset \beta)) = 1$. Therefore, $U_P(\Box(\alpha \supset \beta)) \geq U_P(\alpha \dashv\!\!\! \rightarrow \beta)$. □

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