

Title: ‘Ramseyfying’ Probabilistic Comparativism

Abstract: Comparativism is the view that comparative confidences (e.g., being more confident that P than that Q) are more fundamental than degrees of belief (e.g., believing that P with some strength x). In this paper, I outline the basis for a new, non-probabilistic version of comparativism inspired by a suggestion made by Frank Ramsey in ‘Probability and Partial Belief’. I show how, and to what extent, ‘Ramseyan comparativism’ might be used to weaken the (unrealistically strong) probabilistic coherence conditions that comparativism traditionally relies on.

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1 Introduction

Beliefs come in degrees, or so it seems. Assuming they do, one important question concerns the basis of their numerical representation. It is typical to represent the varying strengths with which propositions might be believed using percentages, or real values between 0 and 1, or with intervals thereof. Moreover, it's typical to assume that these numbers encode more than merely ordinal information. For instance, it seems that we can meaningfully talk about intervals of strengths of belief: an agent—let's call her α —might believe one proposition *much more* than she believes another, or she might believe it *just a little more*. Likewise for ratios: if α is 50% confident that the coin she flips will land heads, then most of us would be happy to say that she has *half* as much confidence in that event than she has in the coin landing either heads or tails. And, if she's even a little bit rational, then she'll probably be *at least twice* as confident that it'll land heads on the next toss than that it'll land heads consistently on the next several tosses.

There is, in other words, a widespread prima facie commitment in our understanding of degrees of belief that they can be measured on a ratio scale, or something much like it. Given this, we'll assume for the remainder of this paper that the numbers we use to represent the strengths of our beliefs can, at least in principle, carry *cardinal* (read: at least ratio and therefore also interval) information. Supposing that's correct, it's just the sort of thing that ought to be explained by any adequate account of what degrees of belief *are*. We don't get to posit cardinality for free— α 's doxastic states don't come with little numbers attached to them, and they don't *literally* stand in numerical relationships with one another. Rather, they must have some non-numerical structure that is in some way similar to and hence representable by the real values in the unit interval, and in

particular such that both the ordinal and relevant cardinal properties of and relations between those numbers represent something doxastically meaningful. That much is clear enough—the hard part consists in saying exactly what that structure *is*.

So how is it that we manage to get from the purely non-numerical stuff in our heads through to numerical representations of our doxastic states that encode interesting cardinal information? A few answers to this question have been suggested. One long-standing tradition seeks to explain where the numbers come from and how they get their meaning by considering how beliefs interact with *preferences* (e.g., Ramsey 1931). Others have tried to extract numerical representations out of *comparative expectations*, a special kind of non-propositional comparative attitude (e.g., Suppes and Zanotti 1976). Still other potential approaches have yet to be explored. For instance, if you like the idea that degrees of belief are really just outright beliefs about objective probabilities, then you might think that whatever cardinality they possess is derivative upon the cardinal information possessed by those probabilities—wherever *that* comes from.

I'm inclined to think that each of these possibilities are worth considering seriously; at least, none of them seem to me either obviously correct, or irretrievably hopeless. I have argued elsewhere that the connection with preferences is one promising avenue to explore (Elliott 2019a). But in *this* paper I want to focus on an entirely different kind of approach: *comparativism*.

For the sake of concreteness, I'll take comparativism to be the view that the facts about an agent's degrees of belief supervene on, and indeed hold in virtue of, the facts about what we'll call her *confidence comparisons*. These are purely ordinal comparative doxastic states such as *being more confident that P than that Q*, *being equally confident*

that P as that Q , or *being at least as confident that P as that Q* .¹ With that as their starting point, comparativists tend to see degrees of belief and the numerical representations thereof as a kind of theoretical tool, a way to represent and reason about sufficiently coherent systems of comparative confidence. Or to put that another way: the numbers we use to represent our beliefs ultimately describe a purely ordinal structure imposed over a set of propositions by our confidence comparisons, when those comparisons satisfy some minimum threshold of coherence.

On the face of it, comparativism might seem to struggle with providing any plausible explanation of the possibility of cardinal information. After all, individual confidence comparisons contain no more than purely ordinal information, so how could a system composed of nothing more than such comparisons possess anything more than

¹It won't matter too much for what I have to say exactly how we define 'comparativism', and there of course are many other ways to precisify the general kind of idea that I'm referring to. Most *actual* comparativists have taken a view which is at least in the vicinity of what I below characterise as *probabilistic comparativism*; e.g., (de Finetti, 1931), (Koopman, 1940), (Savage, 1954, Ch. 3), (Fine 1973, 68ff), (Hawthorne, 2016) and (Stefánsson, 2016, 2018); comparativist theories along these lines are also discussed in (Fishburn, 1986) and (Krantz et al. 1971, 200). In some cases, a comparativist might focus on *quarternary* confidence comparisons (e.g., being more confident that P *given* Q than that R *given* S), rather than on binary comparisons like those I've described here. For the sake of brevity, I've limited my discussion to the relatively simple views which consider only binary confidence comparisons. Nevertheless, each of the main points of discussion in §§3–§4 have fairly straightforward analogues for the typical case of the quarternary comparativist.

that?² Nevertheless, comparativists have what is by now a standard explanation of how cardinality can be generated out of nothing more than ordinal confidence comparisons. By drawing on a well-worn analogy with the measurement of mass, length, and other extensive quantities, comparativists have managed to set down conditions (or axioms) under which meaningful cardinal information might be extracted out of a system of confidence comparisons.

That is the current state of play. However, the axioms to which comparativists typically appeal when addressing this kind of challenge are quite strong indeed. Essentially, they impose a comparative variety of probabilistic (and hence logical) coherence on the agents' confidence comparisons. And this is a key limitation with the view in its most typical contemporary form: it lacks an adequate account of how ordinary agents—who do not live up to the very strict standards of probabilistic coherence—might nevertheless have beliefs which carry genuine cardinal information. Consequently, in this paper I want to explore whether, and how, the standard 'probabilistic' axioms might be weakened, while maintaining the same basic strategy for extracting cardinality out of a system of comparative confidences.

Let me say that again, for emphasis: the goal here is to explore whether, and to what extent, the usual probabilistic axioms can be weakened. This is a question of interest to proponents and opponents of comparativism alike, and for those who might be on the fence. I stress however that my results are formal, not evaluative. The present paper is not intended to be a *defence* of comparativism. (It would be woefully inadequate if so!) An evaluation of the overall merits and demerits of the comparativists'

²For a recent complaint along just these lines, see (Meacham and Weisberg 2011, 659).

view is well beyond the scope of this discussion, and I won't try to address the tricky empirical question of whether and to what extent the weakened axioms are satisfied or even approximated by ordinary agents. Still less is this a paper on what our comparative confidences *should* be like, so I will not have anything much to say about how Ramseyan comparativism relates to arguments for probabilism.

I will begin my discussion by reviewing the standard account of how mass can be measured on a ratio scale, and how probabilistic comparativism posits an essentially similar process for the measurement of belief (§§2–3). Following that, I'll discuss in a little more detail the motivations for seeking more general axioms under which cardinality can be extracted out of a system of confidence comparisons (§4). Finally, I will show that the axioms of what I'll call *probabilistic comparativism* can be weakened to a significant extent—though, not without limits. I will do this by developing what I call *Ramseyan comparativism* (§5). Moreover, I will show that the Ramseyan axioms on confidence comparisons are in one important respect *maximally* weak: inasmuch as comparativists want to retain the analogy with the measurement of mass as it's usually understood, the Ramseyan axioms are as weak as they come.

2 The Measurement of Mass

Let a and b be any two concrete objects you like, and compare:

ORDINAL. a is more massive than b

CARDINAL. a is twice as massive as b

CARDINAL obviously contains more information than ORDINAL, and that information has to come from somewhere. Yet masses don't come with little numbers attached to them. Whatever it is that explains the extra information in CARDINAL must ultimately be non-numerical in nature. So how can we get from the non-numerical facts on the ground through to numerical masses that encode interesting cardinal information?

The representational theory of measurement gives us a plausible answer.³ First, note that CARDINAL is true (roughly) if and only if, if you were to take two disjoint objects each as massive as b (call them b_1 and b_2 , b 's *duplicates*) and join them together, then the resulting object would be just as massive as a . Call the operation of joining objects together *concatenation*; we assume that no mass is gained or lost in the act of concatenating. Given this, it's plausible that there's nothing more to the truth of a claim like CARDINAL than what we've just said—that is, ' a is twice as massive as b ' just means something roughly to the effect of ' a is as massive as the concatenation of two duplicates of b .' By reference, then, to purely ordinal comparisons between duplicates and the concatenations thereof, we've been able to give straightforward non-numerical meaning to CARDINAL.

And we can easily generalise this idea to explain other rational ratio comparisons. For positive integers n, m , say that a is n/m times as massive as b whenever there's some object c such that

1. a is as massive as the concatenation of n duplicates of c , and
2. b is as massive as the concatenation of m duplicates of c .

³The *locus classicus* for this theory is (Krantz et al., 1971).

Now let x designate c 's mass in whatever units you like—let's say *slugs* (~ 14.6 kg).

Intuitively, a must then have a mass of $n \cdot x$ slugs, and b must have a mass of $m \cdot x$ slugs.

Hence, a is n/m times as massive as b . Indeed, with a little bit more work, we can generalise the idea even further to explain arbitrary *real* ratio comparisons. However, for the sake of simplicity we'll stick with rational ratios throughout this discussion.

Hiding in the background is a crucial empirical assumption: that the operation of concatenation behaves as a kind of non-numerical analogue of *addition*. We rely on exactly this assumption to move from, e.g., ' a is as massive as the concatenation of n duplicates of an object with a mass of x slugs' to ' a has a mass of $n \cdot x$ slugs'—that is, we assume that the mass of a concatenation is just the sum of the masses of the concatenands. (Imagine if, instead, concatenation behaved like quaddition: whenever you concatenate up to 57 duplicates together, things are as usual; but concatenate more and the result is always as massive as 57 duplicates. We could have then used concatenations to define our way up to one object's being 57 times as massive than another, but no further.)

Fortunately, the analogy between concatenation and addition is quite close. Where

$$a \succsim^m b \text{ iff } a \text{ is at least as massive as } b,$$

$$a \sim^m b \text{ iff } a \text{ is exactly as massive as } b,$$

$$a \oplus b = \text{the concatenation of } a \text{ and } b,$$

then it's plausible that \succsim^m is transitive and complete, and \sim^m is its symmetric part.

Furthermore, \oplus behaves with respect to \succsim^m a lot like $+$ behaves with respect to \geq : for all disjoint objects a, b, c ,

1. $a \oplus b \succsim^m b$
2. $a \oplus b \sim^m b \oplus a$
3. $a \oplus (b \oplus c) \sim^m (a \oplus b) \oplus c$
4. $a \succsim^m b$ iff $a \oplus c \succsim^m b \oplus c$

Now compare these with the following properties of $+$ in relation to \geq , where n and m are non-negative real numbers:

1. $n + m \geq m$
2. $n + m = m + n$
3. $n + (m + k) = (n + m) + k$
4. $n \geq m$ iff, for any k , $n + k \geq m + k$

Indeed, if we posit a rich enough space of concrete objects and make one further ‘Archimedean’ assumption—roughly: that no object is infinitely more massive than any other—then we can say something stronger still: if \mathcal{O} is the set of \sim^m -equivalence classes of concrete objects and \mathbb{R}^+ the positive reals, then the relational system $\langle \mathcal{O}, \succsim^m, \oplus \rangle$ has essentially the same structure as $\langle \mathbb{R}^+, \geq, + \rangle$. Thus, we can assign a number to each object in such a way that \succsim^m is represented by \geq , and \oplus is represented by $+$. And with that in hand, we can start to define up ratios of masses, numerical differences in mass, ratios of differences in mass, and so on. In other words, we have all the basic resources needed to explain how numerical representations of mass manage to carry all sorts of interesting cardinal information.

The upshot: numerical masses represent a fully non-numerical system of ordinal mass comparisons which have an ‘additive’ structure over concatenations. We’re justified in treating ratios of masses as meaningful because there exists an operation on objects that is intuitively and formally like ‘adding’ masses together. And we can apply the same basic idea outlined here to account for the measurement of other (extensive) quantities: a is *twice as long as* b iff a is as long as two length-duplicates of b laid end-to-end; a has *twice the volume of* b iff a has the same volume as two volume-duplicates of b joined together; and an event e_1 has *twice the duration of* e_2 iff e_1 can be split into two disjoint events with the same duration as e_2 .

To apply the same idea to the measurement of beliefs, comparativists have therefore historically sought an operation on the relata of confidence comparisons (i.e., propositions) that behaves, with respect to those comparisons, similarly enough to addition to justify treating it as a non-numerical analogue thereof. As Krantz et al. put it, the strategy is ‘to treat the assignment of [subjective] probabilities as a measurement problem of the same fundamental character as the measurement of, e.g., mass or duration’ (1971, p.200). So let’s see how that plays out in practice.

3 Probabilistic Comparativism

In this section I’ll provide an overview of probabilistic comparativism. I’ll begin by laying out some basic notation and assumptions (§3.1), followed by the mathematical underpinnings of the view (§3.2). Finally, I’ll define two specific varieties of probabilistic comparativism—one ‘precise’ (§3.3), and the other ‘imprecise’ (§3.4).

3.1 Notation and assumptions

Let α be an arbitrary thinking subject whose beliefs we are trying to represent. I will assume that the propositions regarding which α has beliefs can be modelled as subsets of some space of logically possible worlds, Ω . By ‘logically possible’, I mean no more than that the worlds are closed under a consequence relation at least as strong as that of classical propositional logic. So, you can assume that Ω includes metaphysically or even epistemically impossible worlds, if that’s what floats your boat—as long as the worlds are classically logically consistent. (I’ll talk more about this assumption in §4.)

Next, let $\mathcal{B} \subseteq \wp(\Omega)$ denote that set of propositions regarding which α has beliefs. Without loss of generality, I’ll assume throughout that \mathcal{B} is a Boolean algebra of sets on Ω . So, \mathcal{B} contains at least Ω and \emptyset , and it’s closed under relative complements and binary intersections/unions. I’ll also assume throughout that \mathcal{B} is finite. Doing this will simplify much of the ensuing discussion and formalities.⁴

I’ll assume that α ’s full system of confidence comparisons can be modelled with a single binary relation \succsim defined over \mathcal{B} , where

$$P \succsim Q \text{ iff } \alpha \text{ believes } P \text{ at least as much as she believes } Q$$

⁴The finitude of \mathcal{B} plays a minor (simplifying) role in relation to THEOREM 1. We can do without it if we instead make use of a more complicated version of DEFINITION 5. The finiteness assumption also plays a role in the existence proof of THEOREM 2. Where \mathcal{B} is uncountable, additional ‘continuity’ assumptions can be placed on the comparative confidence relation which will guarantee the existence of the relevant type of representation. See (Evren and Ok, 2011) for discussion on these types of conditions.

I'll refer to \succsim as α 's *confidence ranking*. Consequently, where \succ and \sim stand for the comparatives *more probable* and *equally probable* respectively, I am in effect assuming that

$$P \sim Q \text{ iff } (P \succsim Q) \& (Q \succsim P)$$

$$P \succ Q \text{ iff } (P \succsim Q) \& \neg(Q \succsim P)$$

Nothing about this last assumption should be treated as obvious or trivial. For example, α might be at least as confident in P as in Q without being more confident in P than in Q , or without being equally confident in P as in Q . Nevertheless, it will simplify the discussion, and nothing of great importance will hang on it.

Finally, where a function Cr assigns real numbers to the propositions in \mathfrak{B} , I'll say that Cr *almost agrees with* \succsim iff, for all $P, Q \in \mathfrak{B}$,

$$P \succsim Q \text{ only if } Cr(P) \geq Cr(Q);$$

and we'll say that Cr *agrees with* \succsim just in case

$$P \succsim Q \text{ iff } Cr(P) \geq Cr(Q).$$

For ease of expression, I'll treat *agreement* (but not *almost agreement*) as symmetric: \succsim agrees with Cr just in case Cr agrees with \succsim .

3.2 Agreeing with probabilities

Any Cr that agrees with confidence comparisons \succsim is ipso facto at least an ordinal-scale representation of \succsim . Our task now is to lay out axioms under which such a function can be said to also carry cardinal information. This is where probabilities come in handy:

DEFINITION 1. $Cr : \mathcal{B} \mapsto \mathbb{R}$ is a *probability function* iff, $\forall P, Q \in \mathcal{B}$,

1. $Cr(\Omega) = 1$,
2. $Cr(P) \geq 0$, and
3. If $P \cap Q = \emptyset$, then $Cr(P \cup Q) = Cr(P) + Cr(Q)$

It follows immediately from the third criterion that *if* some probability function—*any* probability function—agrees with \succsim , then the union of disjoint sets is to \succsim just as \oplus is to \succsim^m , or as $+$ is to \geq . Great! That’s exactly the kind of thing needed for the analogy with the measurement of mass to hold water.

Moreover, we have known for a long time the exact conditions under which a confidence ranking will agree with some probability function on \mathcal{B} . The following five axioms are individually necessary and jointly sufficient (see Scott 1964). For all $P, Q, R \in \mathcal{B}$,

COMPLETENESS. $P \succsim Q$ or $Q \succsim P$

PREORDER. (i) $P \succsim P$, and (ii) if $P \succsim Q$ and $Q \succsim R$, then $P \succsim R$

NON-TRIVIALITY. $\Omega \succ \emptyset$

NON-NEGATIVITY. $P \succ \emptyset$

SCOTT'S AXIOM. Where $\mathbf{1}_P$ denotes the indicator function of P , $(P_i)_{i=1}^n$ and $(Q_i)_{i=1}^n$ are finite sequences of propositions, and $(k_i)_{i=1}^n$ is a finite sequence of natural numbers, then if

1. $\sum_{i=1}^n k_i \cdot \mathbf{1}_{P_i}(\omega) = \sum_{i=1}^n k_i \cdot \mathbf{1}_{Q_i}(\omega)$ for all $\omega \in \Omega$, and
2. $P_i \succsim Q_i$, for $i = 1, \dots, n-1$,

then $Q_n \succsim P_n$

Call the conjunction of the above five axioms the COMPLETE PACKAGE.⁵

Comparativists have frequently suggested that, *when \succsim conforms to the COMPLETE PACKAGE*, beliefs can be measured on a ratio scale with the union of disjoint sets playing the role of concatenation (e.g., Fine 1973, 68ff; Stefánsson 2016, 2018).

It is possible to say something a little more general than this, though, and doing so will be useful in demonstrating a general continuity between probabilistic comparativism and the Ramseyan comparativisms that I'll develop below. First, note that if $\mathcal{C}r$ is a probability function, then if $\mathcal{C}r(P \cap Q) = 0$, then $\mathcal{C}r(P \cup Q) = \mathcal{C}r(P) + \mathcal{C}r(Q)$. That is to say: probability functions are also additive with respect to the union of what we'll call *pseudodisjoint* propositions, where P and Q are pseudodisjoint for α just in case she has no confidence in their intersection. Or, more precisely,

DEFINITION 2. For all $P \in \mathcal{B}$, P is:

⁵In the context of the other axioms, PREORDER is redundant, and SCOTT'S AXIOM is equivalent to the slightly weaker formulation found in (Scott, 1964) (see Harrison-Trainor et al. 2016). I've done it this way to make later discussions easier.

1. *minimal* iff $Q \succsim P$ for all $Q \in \mathcal{B}$,
2. *maximal* iff $P \succsim Q$ for all $Q \in \mathcal{B}$,
3. *middling* iff P is neither minimal nor maximal

DEFINITION 3. $\mathcal{P} \subseteq \mathcal{B}$ is a *set of pseudodisjoint propositions* iff, for any minimal Q and any $\mathcal{P}^* \subseteq \mathcal{P}$ such that $|\mathcal{P}^*| \geq 2$, $\bigcap \mathcal{P}^* \sim Q$; furthermore, propositions P_1, \dots, P_n are *pairwise pseudodisjoint* iff there's a set of pseudodisjoint propositions \mathcal{P} such that $P_1, \dots, P_n \in \mathcal{P}$

Assuming that α has exactly zero confidence in P whenever P is minimal, DEFINITION 3 plausibly characterise in comparativist terms what it is for α to believe that at most one proposition from P_1, \dots, P_n is true.⁶

With all that in hand, we can note that the COMPLETE PACKAGE implies that \succsim is ‘Archimedean’—roughly: no proposition is infinitely more probable than any other—and furthermore, where propositions P, Q, R are pairwise pseudodisjoint,

1. $(P \cup Q) \succsim Q$
2. $(P \cup Q) \sim Q \cup P$
3. $(P \cup (Q \cup R)) \sim ((P \cup Q) \cup R)$
4. $P \succsim Q$ iff $(P \cup R) \succsim (Q \cup R)$

⁶DEFINITION 3 implies that every singleton set $\{P\} \in \mathcal{B}$ is trivially a ‘set of pseudodisjoint propositions’. This is a feature, not a bug. The rather tortured definition will be useful later when we generalise away from probability functions.

Again, this is exactly what comparativists need to draw the analogy with the measurement of mass. So let's turn the foregoing mathematical points into a philosophical theory.

3.3 Precise probabilistic comparativism

Assuming that $\mathcal{C}r$ agrees with α 's confidence ranking, say henceforth that $\mathcal{C}r$ constitutes a *fully adequate model* of α 's beliefs whenever

$$\alpha \text{ believes } P \text{ } n/m \text{ times as much as she believes } Q \text{ iff } \mathcal{C}r(P) = \frac{n}{m} \cdot \mathcal{C}r(Q)$$

I assume that full adequacy is worth striving for—after all, most theorists will be happy to make both of the following kinds of inferences:

1. α believes P to degree x , and Q to degree y

2. $x = n \cdot y$

$\therefore \alpha$ believes P n times as much as she believes Q

and in the other direction,

1. α believes P n times as much as Q

2. α believes P to degree y

$\therefore \alpha$ believes Q to degree $x = n \cdot y$

Only full adequacy licenses inferences in both of these directions, and so I take it that full adequacy stands as an important desideratum for any comparativist theory. With that said, we can also say that $\mathcal{C}r$ is *L-to-R adequate* iff the left-to-right direction of the above biconditional holds, and *R-to-L adequate* iff the right-to-left direction holds. A comparativist may well want to reject full adequacy in favour of mere L-to-R or R-to-L adequacy, provided that the rejection is well-motivated and they are able to explain away any intuitions in support of full adequacy. (I'll say a little more about this in §5.3.)

Next, let *precise probabilistic comparativism* denote any comparativist theory that's committed to the following conditional:

PRECISE PROBABILISTIC COMPARATIVISM. If $\mathcal{C}r$ is the unique probability function that agrees with α 's confidence ranking, then $\mathcal{C}r$ is a fully adequate model of α 's beliefs

Note the stated requirement that the probability function be *unique*. This is needed to avoid contradiction: for *any* non-trivial algebra \mathcal{B} , there will always be some collection of probability functions on \mathcal{B} that agree with one and the same confidence ranking—and since any two probability functions on the same domain will disagree on at least some ratios, any inference from ' $\mathcal{C}r(Q) = n/m \cdot \mathcal{C}r(Q)$ ' to ' α believes P n/m times as much as Q ' will be valid *only* when the $\mathcal{C}r$ is unique in the relevant sense. In short, R-to-L adequacy presupposes uniqueness, which in turn requires further constraints on \succsim .

There are multiple ways to ensure uniqueness. Of particular note is the following, which Stefánsson (2016, 2018; cf. also Suppes 1969, 6–7; Savage 1954) uses to ensure uniqueness in his recent defences of probabilistic comparativism:

CONTINUITY. For all non-minimal P, Q , there are P', Q' such that $P \sim P', Q \sim Q'$, and

P' and Q' are each the union of some subset of a finite set of disjoint propositions $\{R_1, \dots, R_n\}$ such that $R_i \sim R_j$ for $i, j = 1, \dots, n$

The interested reader can see (Krantz et al., 1971, §5.2) and (Fishburn, 1986) for other conditions sufficient to ensure uniqueness.

Now, probabilistic comparativism clearly has resources to put forward an account of how a system of confidence comparisons might end up carrying cardinal information, *in the event that \succsim satisfies the requisite axioms*. In particular, consider the following principle, which in essence is just the comparative probability version of how we defined rational ratio comparisons for mass earlier in §2:⁷

GENERAL RATIO PRINCIPLE. α believes P n/m times as much as Q if

1. For $0 < n \leq m$, there are m non-minimal, equiprobable pairwise pseudodisjoint propositions R_1, \dots, R_m such that $Q \sim (R_1 \cup \dots \cup R_m)$ and $P \sim (R_1 \cup \dots \cup R_n)$; or
2. α believes P n'/m' times as much as R , and believes R n''/m'' times as much as Q , where $n/m = n' \cdot n''/m' \cdot m''$

So, for instance, suppose that $Q \cap Q'$ is minimal. Then, α will take P to be twice as probable as Q inasmuch as $Q \sim Q'$ and $(Q \cup Q') \sim P$. In this case, Q and Q' are acting as ‘duplicates’ of one another, and $Q \cup Q'$ is their ‘concatenation’.

⁷The first clause of the General Ratio Principle is a close relative of Stefánsson’s (2018) ‘Ratio Principle.’ The second (inductive) clause is new—in the context of a condition like CONTINUITY it’s redundant, but see §4 for it put to work.

3.4 Imprecise probabilistic comparativism

Say that $\mathcal{C}r$ *confirms* the General Ratio Principle (GRP) just in case, whenever that principle implies that P is believed n/m times as much as Q , then $\mathcal{C}r(P) = n/m \cdot \mathcal{C}r(Q)$; otherwise, it *disconfirms* the GRP. It is easy to check that if any probability function *almost* agrees with \succsim , and \emptyset is minimal, then that function will confirm the GRP. This means that it's possible to extend the account of ratio comparisons just given to *incomplete* confidence rankings.

For ordinary agents, the COMPLETENESS axiom is widely considered highly implausible. Consider the following, adapted from (Fishburn, 1986):

P = The global population in 2100 will be greater than 13 billion

Q = The next card drawn from this old and incomplete deck will be a heart

Are you more confident that P than that Q , or less, or just as confident in either? It's not clear that there must be a fact of the matter. Similar examples abound.⁸

There's a natural way of dealing with incompleteness to which comparativists can (and do) appeal. Where \mathcal{F} is any set of real-valued functions on \mathcal{B} , say this time that the

⁸You don't have to be convinced by the example, and here is not the place for a detailed discussion on whether we should expect 'gaps' in \succsim . What matters is just that there might be gaps, and many think that there are. COMPLETENESS may or may not be plausible for perfectly rational agents, but since our focus is on deidealising the usual probabilistic theory that's neither here nor there.

set \mathcal{F} agrees with \succsim just in case for all relevant P, Q ,

$$P \succsim Q \text{ iff } \forall Cr \in \mathcal{F} : Cr(P) \geq Cr(Q)$$

The idea behind a set-of-functions model is to recapture the structure of the confidence ranking by doing something like supervaluating over the functions in \mathcal{F} —only what’s common to every such function is treated as having representative import. If P and Q are *incomparable* in terms of relative confidence, then \mathcal{F} will contain at least one pair of probability functions that disagree on the relative ordering of P and Q —hence, we still manage to ‘numerically’ represent incomplete \succsim rankings.

Alon and Lehrer (2014) have shown that a set of probability functions agrees with \succsim just in case the latter satisfies the COMPLETE PACKAGE *minus* the COMPLETENESS axiom (henceforth: the NON-COMPLETE PACKAGE). Furthermore, while there will often be more than one set of probability functions \mathcal{F} that agrees with \succsim , the union of all such sets will always agree with \succsim . In sum: whenever \succsim satisfies the NON-COMPLETE PACKAGE, there’s guaranteed to be a unique set of probability functions that agrees with \succsim and which is maximal with respect to inclusion.

Consequently, if we extend the definitions of full / L-to-R / R-to-L adequacy in the natural way (i.e., by inserting ‘ $\forall Cr \in \mathcal{F}$ ’ in the appropriate locations), we can characterise *imprecise probabilistic comparativism* by its commitment to:

IMPRECISE PROBABILISTIC COMPARATIVISM. If a non-empty set of probability

functions \mathcal{F} agrees with α ’s confidence ranking and \mathcal{F} is maximal with respect to inclusion, then \mathcal{F} is a fully adequate model of α ’s beliefs

Imprecise probabilistic comparativism implies the precise version. More precisely, if we assume that \mathcal{F} and $\mathcal{C}r$ are essentially the same representation whenever $\mathcal{F} = \{\mathcal{C}r\}$, then the two varieties of comparativism amount to one and the same thing whenever exactly one probability function agrees with \succsim .

Furthermore, every $\mathcal{C}r$ in a set \mathcal{F} that agrees with \succsim will itself *almost* agree with \succsim . So, if we also extend the definition of ‘confirms the GRP’ in the obvious way to sets of functions, it follows that if a set of probability functions \mathcal{F} agrees with \succsim , then \mathcal{F} confirms the GRP. The upshot is that both the precise and imprecise versions of probabilistic comparativism can extract cardinality from comparative confidences in *basically* the same way; the latter is a natural generalisation of the former.

4 Why Generalise?

We’ve seen now that conformity to the NON-COMPLETE PACKAGE is sufficient for the union of pseudodisjoint sets to behave like addition. But it is by no means necessary. It *is* possible to weaken those axioms still further while maintaining the analogy, and I think it is of some importance for comparativism that this can be done. In this section I’ll say why.

The basic reason is that the axioms of the NON-COMPLETE PACKAGE are, in conjunction, quite strong—it is not likely that they’re jointly satisfied by any ordinary agents. Since I think it’s especially troubling, I’ll focus on one issue in particular: in the context of the (individually rather weak) axioms NON-TRIVIALITY and NON-NEGATIVITY, SCOTT’S AXIOM immediately generates a probabilistic version of

the classical problems of logical omniscience. Those three axioms entail that if $P \subseteq Q$ and $P, Q \in \mathcal{B}$, then $Q \succsim P$. Consequently,

LOGICAL OMNISCIENCE. If the worlds in Ω are closed under the consequence relation \Rightarrow , then for all $P, Q \in \mathcal{B}$, if $P \Rightarrow Q$, then $Q \succsim P$

That is, any confidence ranking that is (i) defined over propositions taken from a space of worlds that's closed under \Rightarrow , and (ii) agrees with a (set of) probability function(s), will ipso facto be 'coherent' with respect to \Rightarrow in the manner just described. In §3.1 it was assumed that \Rightarrow is at least as strong as the consequence relation we find in classical propositional logic, and it's implausible that ordinary agents' confidence rankings are everywhere and always coherent with respect to *that* logic. I'll say more about that in a moment. But the point can also be put in a much more general way: we are (probably) not omniscient with respect to any very interesting logics, so unless \Rightarrow is *extremely* weak indeed, the confidence rankings of any ordinary agents will (probably) falsify at least one of NON-TRIVIALITY, NON-NEGATIVITY, or SCOTT'S AXIOM.

How might a comparativist respond to this fact? Four obvious (but also obviously non-exhaustive) options are:

1. Argue that ordinary agents' comparative confidences do conform to the NON-COMPLETE PACKAGE after all, because they are probabilistically coherent after all.
2. Argue that ordinary agents' comparative confidences do conform to the NON-COMPLETE PACKAGE after all, once we define propositions over a richer space of worlds.

3. Argue that because ordinary agents' comparative confidences do not conform to the NON-COMPLETE PACKAGE, they therefore do not ground any cardinal information (or not the same kind of information).
4. Accept that ordinary agents' comparative confidences do not conform to the NON-COMPLETE PACKAGE, and seek weaker axioms under which cardinality can be extracted from comparative confidences.

The fourth seems to me clearly the best option. After all, nothing about comparativism per se ties it irrevocably to specifically *probabilistic* representations of degrees of belief, and if more general conditions exist then it only makes sense for comparativists to find and use them. But if you prefer one of the others, or something else not listed, then so be it—there's no harm in developing ideas in many different directions. I *will*, however, here give some reasons to think that the fourth option should be preferred.

Regarding the first: I will take it for granted in the following discussion that we are not (classically) logically omniscient. “But maybe we are!”—Sure, and I'm not unsympathetic to the idea that we ordinary agents really are probabilistically coherent. But since this is usually met with an incredulous stare let's just move on already.

The second option seems to be the more common way of arguing that the NON-COMPLETE PACKAGE can actually be satisfied by ordinary (and ordinarily irrational) agents. As I've noted, if the entailment relation \Rightarrow is weak enough, then logical omniscience might not look so bad. So what would happen if we remove the assumption that the worlds in Ω are closed under any interesting logic?

In a little more detail, the idea is this. If we help ourselves to a rich enough space of

possible *and* impossible worlds, then it's well-known that we can construct a probability function properly so-called on that enriched space that 'mimics' the behaviour of a non-probabilistic function defined over the smaller space of classical possible worlds.⁹ So what looks like comparative confidences that are inconsistent with NON-TRIVIALITY, NON-NEGATIVITY, and/or SCOTT'S AXIOM *when they're defined for propositions qua sets of possible worlds*, can in fact be *re-represented* using (sets of) probability functions, if we make use of enough impossible worlds. Hence, to apply the probabilistic comparativists' explanation of cardinality to ordinary agents, we *don't* need to weaken the axioms all. We can keep the the NON-COMPLETE PACKAGE as long as we just make sure to use enough impossible worlds.

That *seems* easy enough, but I do not think that this is a viable strategy for the comparativist to adopt. I'll set out the reasons for this very briefly, since most of the relevant issues are discussed at length in (Elliott 2019b). The problem is that once Ω includes enough impossible worlds for the strategy to work (roughly: for any impossibility, there's an impossible world that verifies it), then most subsets of Ω will be *meaningless* and consequently not representative of any proper contents of belief.

Moreover, for any meaningful subset P of Ω , none of P 's subsets or supersets will be meaningful, and nor will any subset of $\Omega \setminus P$ be meaningful. In short, having too many

⁹Where Ω is the space of classically possible worlds, $\mathcal{B} \subseteq \wp(\Omega)$, and $\mathcal{C}r : \mathcal{B} \mapsto [0, 1]$, then if Ω^+ is a rich enough extension of Ω into the space of impossible worlds, there's a probability function $\mathcal{C}r^+$ on an algebra of sets $\mathcal{B}^+ \subseteq \wp(\Omega^+)$ such that $\mathcal{C}r^+$ assigns x to the subset of Ω^+ that verifies φ iff $\mathcal{C}r$ assigns x to the subset of Ω that verifies φ . See (Cozic, 2006), (Halpern and Pucella, 2011), and (Elliott 2019b).

impossible worlds in Ω renders useless for the purposes of comparativism any set-theoretic definition of ‘concatenation’ along the lines described in §3. Furthermore, any algebra of propositions defined on a space of possible and impossible worlds that’s rich enough to represent the contents of belief will contain only meaningful propositions just when the relevant space of worlds is closed under a consequence relation that is, for all intents and purposes, at least as strong as classical propositional logic.

(Of course, comparativists don’t *have* to define their concatenations set-theoretically as I have done in §3.2. But the only other place that we will plausibly find the structure required to defined up an appropriate concatenation operation is in the *logical* relations amongst the *contents* of the propositions. That is, we could define concatenations in terms of *disjunctions of inconsistent contents* (or *disjunctions of contents whose conjunctions are minimal*). But defining the concatenation operation in *this* way brings us straight back to where we started *vis-à-vis* to the problem of logical omniscience, and appealing to impossible worlds will be of absolutely no help here.)

So there’s no easy way to pursue either the first or the second route: if you want to tie the possibility of cardinality to the NON-COMPLETE PACKAGE, then you’ll be tying it to very strong conditions of logical omniscience—and consequently you’ll need to face up to the empirical and intuitive evidence that ordinary agents just aren’t that good at classical logic.

Could we instead take the third route, and argue that ordinary agents whose comparative confidences don’t satisfy the NON-COMPLETE PACKAGE cannot have beliefs which carry ratio and interval information? This doesn’t strike me as very plausible. For example, the literature on the conjunction and disjunction fallacies already

strongly suggests that ordinary agents do not have comparative confidences that respect even relatively simple bits of classical logic. So imagine that α has just committed the conjunction fallacy—she thinks it’s more plausible that Linda is a bank teller (B) and active in the feminist movement (F) than that she’s a bank teller. Are we going to say now that there’s no meaningful way to answer the question of *how much more* α believes $B \cap F$ over B ? Of course not. Similarly, *I* am not logically omniscient, and (like most people) I’ve probably fallen foul of various probabilistic fallacies before. My comparative confidences don’t satisfy the NON-COMPLETE PACKAGE. Maybe they don’t even come close to satisfying those axioms. None of this prevents me from believing some things *much more* than other things, or *at least twice as much* as other things.

Our capacity to believe one proposition much more than another, or (at least) twice as much as another thing, etc., is not hostage to any presupposition of logical coherence, still less should it depend on a condition of probabilistic representability. Most philosophers will see no inconsistencies at all in holding both that (a) ordinary agents’ beliefs cannot be faithfully represented by (a set of) probability functions, and (b) for arbitrary P and Q , an ordinary agent might believe P much more than Q , or (at least) twice as much as Q . These claims *should* be uncontroversial—only someone caught firmly in the grips of a deeply unrealistic picture of belief would think to deny it. Or at least I’ll say this: if you want to argue otherwise, then you’ll be facing a difficult uphill battle. Better, I think, to seek more general axioms under which cardinal information can be extracted from a system of comparative confidences.

5 The Ramseyan Alternatives

What I'm calling Ramseyan comparativism is inspired by a brief remark from Frank Ramsey in 'Probability and Partial Belief': "Well, I believe it to an extent $2/3$ ", i.e. (this at least is the most natural interpretation) 'I have the same degree of belief in it as in $P \vee Q$ when I think P, Q, R equally likely and know that exactly one of them is true'.' (Ramsey 1929, 256.) In a recent paper, Weatherson (2016, pp. 223–4) has also suggested that Ramsey's remark points towards a version of comparativism that's weaker than probabilistic comparativism. However, neither Ramsey nor Weatherson take their discussion beyond this initial suggestion, and (as we'll soon see) there's a bit of work that needs to be done in order to flesh the idea out in full.

In the remainder of this paper, I will develop precise Ramseyan comparativism (§§5.1–5.2), and then an imprecise version (§5.3). Following that, I will prove an important result about the axioms under which Ramseyan comparativism supports the analogy with the measurement of mass (§5.4).

5.1 The Main Ideas

First, it'll be useful to introduce another definition (the term ' n -scale' comes from Koopman 1940):

DEFINITION 4. A set \mathcal{P} of n pseudodisjoint propositions is an n -scale of P iff (i) $P \notin \mathcal{P}$,
(ii) $\bigcup \mathcal{P} \sim P$, and (iii) for all Q, Q' in \mathcal{P} , $Q \sim Q'$

We can take this as a comparativist characterisation of what it is for an agent to think

that Q is as likely as a disjunction of equiprobable propositions at most one of which is true. So, e.g., if α thinks Q is as likely as $P \cup P'$, where P and P' are equiprobable and pseudodisjoint, then $\{P, P'\}$ is a 2-scale of Q . We'll also assume that α is *certain* of P 's truth just in case P is maximal, and we'll represent certainty in P with $\mathcal{C}r(P) = 1$. This is something the Ramseyan view shares with probabilistic comparativism, where in order to fix the scales the values of the minimal and maximal propositions need to be stipulated.

In light of DEFINITION 4, Ramsey's idea can be recast as: α believes P to degree n/m when $P \sim (Q_1 \cup \dots \cup Q_n)$, where the Q_1, \dots, Q_n belong to an m -scale $\{Q_1, \dots, Q_n, \dots, Q_m\}$ of some maximal proposition R . A good start—but there's a natural extension that will be helpful to incorporate into what follows.

Consider, to begin with, the following situation. Let \mathcal{B} designate the powerset of $\Omega = \{\omega_1, \omega_2, \omega_3\}$, and let $P_{\langle n \rangle}$ and $P_{\langle nm \rangle}$ designate the possible worlds propositions $\{\omega_n\}$ and $\{\omega_n, \omega_m\}$ respectively. (For example, $P_{\langle 12 \rangle} = \{\omega_1, \omega_2\}$.) Suppose now that \succsim is transitive and reflexive, and (where the square brackets indicate equiprobability):

$$\Omega \succsim \begin{bmatrix} P_{\langle 13 \rangle} \\ P_{\langle 23 \rangle} \end{bmatrix} \succsim \begin{bmatrix} P_{\langle 12 \rangle} \\ P_{\langle 3 \rangle} \end{bmatrix} \succsim \begin{bmatrix} P_{\langle 1 \rangle} \\ P_{\langle 2 \rangle} \end{bmatrix} \succsim \emptyset$$

We can represent \succsim with Figure 1, where the relative sizes of the boxes containing the ω_i correspond to the order of propositions in the confidence ranking:



Figure 1: Indirect R-scalability

Now Ω is maximal, and $\{P_{\langle 12 \rangle}, P_{\langle 3 \rangle}\}$ is a 2-scale of Ω , so Ramsey would say that

$$\mathcal{C}r(P_{\langle 3 \rangle}) = \mathcal{C}r(P_{\langle 12 \rangle}) = 1/2$$

However, $P_{\langle 1 \rangle}$ and $P_{\langle 2 \rangle}$ don't belong to any n -scale of Ω , so Ramsey's idea doesn't yet give us any strength with which they're believed. But since $\{P_{\langle 1 \rangle}, P_{\langle 2 \rangle}\}$ is a 2-scale of $P_{\langle 12 \rangle}$, it's only reasonable to say that

$$\mathcal{C}r(P_{\langle 1 \rangle}) = \mathcal{C}r(P_{\langle 2 \rangle}) = 1/4$$

We can capture the foregoing by means of the following:

DEFINITION 5. For integers n, m such that $m \geq n \geq 0$, $m > 0$, P is

1. $0/m$ -valued if P is minimal and m/m -valued if P is maximal, and
2. n/m -valued if $P \sim (Q_1 \cup \dots \cup Q_{n'})$, where the $Q_1, \dots, Q_{n'}$ belong to an m' -scale of an n''/m'' -valued proposition, and $n' \cdot n''/m' \cdot m'' = n/m$

The new, generalised version of Ramsey's idea now amounts to the claim that α believes P to degree n/m if P is n/m -valued. As such, define a *Ramsey function* as follows:

DEFINITION 6. $\mathcal{C}r : \mathcal{B} \mapsto [0, 1]$ is a *Ramsey function* (relative to \succsim) iff, for all $P \in \mathcal{B}$, if

$$P \text{ is } n/m\text{-valued, then } \mathcal{C}r(P) = n/m$$

The close connection between Ramsey functions and the GRP should at this point be apparent, and it should likewise already be clear that the way Ramsey proposes to measure degrees of belief isn't too different from the strategy the probabilistic comparativists want to adopt. In fact, in the present terminology, the first (non-inductive) clause of the GRP essentially states that for $m \geq n$, P is believed n/m times as much as Q whenever \mathcal{P} is an m -scale of Q , and $\mathcal{P}' \subseteq \mathcal{P}$ is an n -scale of P . In this case, for any Ramsey function $\mathcal{C}r$, $\mathcal{C}r(P) = n/m \cdot \mathcal{C}r(Q)$. With respect to n/m -valued propositions, Ramsey functions always confirm the GRP.

Essentially, a Ramsey function either directly or indirectly scales every middling n/m -valued proposition relative to some maximal proposition, which has a stipulated value. With respect to pairs of propositions that cannot be so scaled, however, a Ramsey function *may* disconfirm the GRP. An especially clear example where this would occur can be seen in Figure 2:

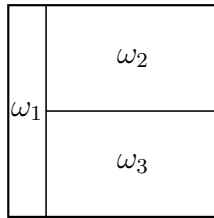


Figure 2: Failure of R-scalability

Where

$$\Omega \succ P_{\langle 23 \rangle} \succ \begin{bmatrix} P_{\langle 12 \rangle} \\ P_{\langle 13 \rangle} \end{bmatrix} \succ \begin{bmatrix} P_{\langle 2 \rangle} \\ P_{\langle 3 \rangle} \end{bmatrix} \succ P_{\langle 1 \rangle} \succ \emptyset$$

In this case, the only non-trivial n -scale is the 2-scale $\{P_{\langle 2 \rangle}, P_{\langle 3 \rangle}\}$ of $P_{\langle 23 \rangle}$. According to the GRP, then, we should be able to say:

$$\mathcal{C}r(P_{\langle 2 \rangle}) = \mathcal{C}r(P_{\langle 3 \rangle}) = 1/2 \cdot \mathcal{C}r(P_{\langle 23 \rangle})$$

However, since $P_{\langle 23 \rangle}$ can't be scaled relative to Ω , Ramsey's suggestion gives us no means of fixing values for $P_{\langle 2 \rangle}, P_{\langle 3 \rangle}$ and $P_{\langle 23 \rangle}$.

Call any proposition that's n/m -valued *R-scalable*. All of the propositions other than Ω and \emptyset in Figure 2 are not R-scalable. Ramsey says nothing about how to measure propositions that aren't R-scalable—though perhaps this is not a very troubling gap in his proposal. One might simply assume that such cases don't exist. Let \mathcal{N} designate the set of R-scalable propositions, then:

R-SCALABILITY. $\mathcal{N} = \mathcal{B}$

R-SCALABILITY is not implied by the COMPLETE PACKAGE. However, *given* that package, it is equivalent to CONTINUITY. (See the appendix for a proof.)¹⁰ In other words, precise probabilistic comparativists don't seem to have anything to fear from an axiom like R-SCALABILITY. (Nevertheless, I'll discuss below how the Ramseyan comparativist can do without it.)

R-SCALABILITY merely guarantees that every proposition in \mathcal{B} is R-scalable. Importantly, this isn't *yet* enough to ground a minimally plausible comparativist theory. There are still two additional problems that can arise in the absence of further

¹⁰The proof rests in part on the assumption that \mathcal{B} is closed under unions. Without that assumption, CONTINUITY will imply R-SCALABILITY but not vice versa.

assumptions about the structure of \succsim :

1. We need to ensure that DEFINITION 6 is *consistent*. Without further assumptions, it's possible that, e.g., $P \sim Q$, where for some R , P belongs to a 2-scale of R and Q belongs to a 3-scale of R . This is clearly unacceptable: α can't believe P to the degrees $1/2$ and $1/3$ simultaneously! If Ramsey functions are to be well-defined, we'll need to ensure that if P is both n/m -valued and n'/m' -valued, then $n/m = n'/m'$.

2. We need to ensure that any Ramsey function relative to \succsim will *agree* with \succsim . Without further assumptions, there's no guarantee that $Cr(P) \geq Cr(Q)$ if *or* only if $P \succsim Q$. For instance, P could be $1/2$ -valued, and Q $1/4$ -valued, yet $Q \succsim P$. This is also undesirable: if the order of the values we assign propositions don't match up to the confidence ranking, then there can be no plausible sense in which those values are a measure of the *strengths* with which those propositions are believed.

In the presence of R-SCALABILITY, we can kill these two birds with a single stone by adding the following rather strong axiom:

R-COHERENCE. If P is n/m -valued and Q is n'/m' -valued, $P \succsim Q$ iff $n/m \geq n'/m'$

R-COHERENCE is sufficient to avoid both worries, as established by the following representation theorem:

THEOREM 1. (i) \succsim satisfies R-COHERENCE iff there exists a Ramsey function Cr with respect to \succsim , and (ii) \succsim also satisfies R-SCALABILITY iff Cr is the unique Ramsey function relative to \succsim that agrees with \succsim .

The proofs for this theorem and the two that follow below can be found in the appendix.

5.2 Precise Ramseyan Comparativism

We will say from now on that one accepts *precise Ramseyan comparativism* just in case they accept the following conditional:

PRECISE RAMSEYAN COMPARATIVISM. If $\mathcal{C}r$ is the only Ramsey function relative to α 's confidence ranking, then $\mathcal{C}r$ is a fully adequate model of α 's beliefs

We can now characterise precisely the respects in which precise Ramseyan comparativism is more lenient than probabilistic comparativism. To start with, it's easy to see that R-COHERENCE is implied already by the COMPLETE PACKAGE. Indeed, if any probability function $\mathcal{C}r$ agrees with \succsim , then $\mathcal{C}r$ is *also* a Ramsey function relative to \succsim . Moreover, where the COMPLETE PACKAGE plus R-SCALABILITY holds, then the unique probability function that agrees with \succsim *just is* the unique Ramsey function that agrees with \succsim . This is important, since (in light of what we said earlier) it means that precise Ramseyan comparativism is a generalisation of any version of precise probabilistic comparativism that makes use of CONTINUITY.

In the other direction, R-SCALABILITY and R-COHERENCE together obviously imply COMPLETENESS and PREORDER. However, they *don't* imply any of NON-TRIVIALITY, NON-NEGATIVITY, or SCOTT'S AXIOM. For a simple (albeit extreme) example where all three of those axioms fail, assume that $\Omega = \{w_1, w_2, w_3, w_4\}$,

\succsim is transitive and reflexive, and:

$$\begin{bmatrix} P_{\langle 4 \rangle} \\ P_{\langle 24 \rangle} \\ P_{\langle 124 \rangle} \\ P_{\langle 234 \rangle} \end{bmatrix} \succsim \begin{bmatrix} \emptyset \\ \Omega \\ P_{\langle 23 \rangle} \\ P_{\langle 34 \rangle} \end{bmatrix} \succsim \begin{bmatrix} P_{\langle 2 \rangle} \\ P_{\langle 14 \rangle} \\ P_{\langle 123 \rangle} \end{bmatrix} \succsim \begin{bmatrix} P_{\langle 1 \rangle} \\ P_{\langle 3 \rangle} \\ P_{\langle 12 \rangle} \\ P_{\langle 13 \rangle} \\ P_{\langle 134 \rangle} \end{bmatrix}$$

It's straightforward (albeit a little tedious) to check that R-SCALABILITY and R-COHERENCE are satisfied in this case. The only non-trivial n -scales (i.e., $n > 1$) that can be defined using this ranking are:

1. The 2-scale $\{P_{\langle 23 \rangle}, P_{\langle 34 \rangle}\}$ of the maximal propositions
2. The 2-scale $\{P_{\langle 123 \rangle}, P_{\langle 14 \rangle}\}$ of \emptyset , Ω , $P_{\langle 23 \rangle}$, and $P_{\langle 34 \rangle}$
3. The several n -scales composed out of minimal propositions, each of some other minimal proposition

Consequently, $\mathcal{C}r(\Omega) = \mathcal{C}r(\emptyset) = 1/2$ because $\{\Omega\}$ and $\{\emptyset\}$ are 1-scales of $P_{\langle 23 \rangle}$ and $P_{\langle 34 \rangle}$, where the latter are $1/2$ -valued; and $\mathcal{C}r(P_{\langle 2 \rangle}) = 1/4$, because $\{P_{\langle 2 \rangle}\}$ is a 1-scale of $P_{\langle 14 \rangle}$ and $P_{\langle 123 \rangle}$, where the latter are $1/4$ -valued. Every other proposition is either maximal or minimal, and assigned either 1 or 0 accordingly. That the example violates NON-TRIVIALITY, NON-NEGATIVITY is obvious; to see that it violates SCOTT'S AXIOM it suffices to consider the two short sequences $P_{\langle 13 \rangle}, P_{\langle 24 \rangle}$ and $P_{\langle 12 \rangle}, P_{\langle 34 \rangle}$.

The interesting 'work' here is of course being done entirely by R-COHERENCE. This axiom imposes a limited kind of additive structure on \succsim , specifically with respect to

confidence rankings between propositions constructed out of members of the same n -scale of any n'/m' -valued proposition. Roughly: *within* an n -scale, \succsim behaves “pseudo-probabilistically”—but not every proposition is constructible out of the members of an appropriate n -scale, and *across* n -scales \succsim can behave quite irrationally indeed.

5.3 Imprecise Ramseyan comparativism

If we wanted to drop R-SCALABILITY out of the picture, we could do so by adopting a set-of-functions representation of \succsim . For that, we will need to add back in the PREORDER axiom. This is obviously necessary for *any* real-valued function *or* set thereof to agree with \succsim , and it is not implied by R-COHERENCE alone.

THEOREM 2. *\succsim satisfies PREORDER and R-COHERENCE iff there is a nonempty set \mathcal{F} of Ramsey functions relative to \succsim such that \mathcal{F} agrees with \succsim , and in such cases there will also be a unique such \mathcal{F} that agrees with \succsim that’s maximal with respect to inclusion.*

Given this, let’s characterise the imprecise variety of Ramseyan comparativism by its commitment to:

IMPRECISE RAMSEYAN COMPARATIVISM. If \mathcal{F} is a non-empty set of Ramsey functions with respect to α ’s confidence ranking, which is maximal with respect to inclusion and agrees with \succsim , then \mathcal{F} is an R-to-L adequate model of α ’s beliefs

Note that imprecise Ramseyan comparativism only claims R-to-L adequacy. This is because (as we’ve seen) PREORDER and R-COHERENCE are not sufficient for a (set of)

Ramsey function(s) to confirm the GRP in full. This is a limitation with the imprecise Ramseyan comparativist’s theory, but perhaps not a devastating one. In effect, R-to-L adequacy says that we won’t go wrong whenever we read cardinal information off of the numbers, though there *may* be some interesting cardinal properties to one’s degrees of belief that aren’t appropriately captured by their cardinal representation. Although it’s not perfect, I suspect that many comparativists would be satisfied by this result—nobody said that our numerical representations had to be perfect after all.

Imprecise Ramseyan comparativism also agrees exactly with (precise and imprecise) probabilistic comparativism whenever the COMPLETE PACKAGE plus R-SCALABILITY are satisfied. We’ve already shown that this is so for precise Ramseyan comparativism, but if this is not obvious in the case of imprecise Ramseyan comparativism then consider: if we assume the COMPLETE PACKAGE plus R-SCALABILITY, then the probability function $\mathcal{C}r$ that agrees with \succsim is the Ramsey function that agrees with \succsim ; from imprecise Ramseyan comparativism, $\mathcal{C}r$ is R-to-L adequate, so $\mathcal{C}r$ determines a unique ratio comparison for every pair of non-minimal propositions; and finally, α cannot believe P n/m times as much as Q and n'/m' as much as Q , for $n/m \neq n'/m'$.

5.4 The importance of R-Coherence

Importantly, we can show that PREORDER and R-COHERENCE are individually *necessary* for coherence with the GRP.

As far as PREORDER is concerned, this is obvious for the reasons already mentioned. The more interesting result concerns R-COHERENCE. Given some very minimal scaling assumptions, violations of that axiom imply that any $\mathcal{C}r$ that agrees

with \succsim cannot confirm the GRP:

THEOREM 3. *If (i) $\mathcal{C}r$ agrees with \succsim , (ii) there are P, Q such that $P \succ Q$, and (iii) $\mathcal{C}r(R) = 0$ whenever R is minimal, then $\mathcal{C}r$ confirms the GRP only if R-COHERENCE is satisfied.*

Corollary: under the same assumptions, *mutatis mutandis*, any set of real-valued functions \mathcal{F} will confirm the GRP only if R-COHERENCE is satisfied.

In other words, assuming just that \succsim has *some* non-trivial structure, and that minimal propositions can be assigned value 0, that a function (or set of functions) confirms the GRP implies that any comparative ranking it agrees with will satisfy PREORDER and R-COHERENCE. Thus we have found two *minimal* axioms necessary for the union of pseudodisjoint sets to behave like addition with respect to \succsim .

6 Conclusion

Let's take stock. The standard comparativist strategy for explaining cardinality is based on a purported analogy with the measurement of certain extensive quantities like length or mass. So, for instance, to say that P is n times more likely than Q , we just need to be able to say that P is as likely as the union of n 'duplicates' of Q , where the 'duplicates' are propositions that are equiprobable and pairwise (pseudo)disjoint. The two Ramseyan varieties of comparativism I've outlined offer an account of when this kind of 'adding' is meaningful that generalises the axioms assumed by the more common probabilistic comparativism, thus applying to a wide range of confidence rankings that aren't probabilistically representable.

In particular, we've shown that comparativists can *in principle* do without any appeal to NON-TRIVIALITY, NON-NEGATIVITY, and SCOTT'S AXIOM, and can avoid the problems that those axioms bring in their wake. This is an interesting result by itself, since it establishes that comparativists can preserve their favourite explanation of cardinality without necessarily committing to the stronger conditions required for probabilistic representability. Moreover, we have been able to show that the union of (pseudo)disjoint sets behaves like addition *only* if the comparative confidence ranking satisfies PREORDER and R-COHERENCE. Inasmuch as comparativists want to retain the analogy with the measurement and mass *as it's usually understood*—i.e., in terms of the union of either disjoint or pseudodisjoint propositions—then Ramseyan comparativism is as general as it gets.

It remains to be seen whether it's *correct* to say that an agent α considers P to be n times more likely than Q if and only if P is as likely for her as the union of n pseudodisjoint duplicates of Q . But we now know the minimal conditions required for the analogy with mass to hold, so we can ask: (a) are PREORDER and R-COHERENCE plausibly satisfied by actual agents—or at least, by the kinds of agents whom we are happy to say have degrees of belief which carry cardinal information? And, (b) if so, does the GRP *in those cases* accurately predict our considered judgements about the degrees of belief of such agents? These are questions that I've not considered in this paper, but they will need careful consideration in future discussions on the viability of the comparativist view.

Appendix: Proofs

A. *Proof that, given the COMPLETE PACKAGE, CONTINUITY is equivalent to R-SCALABILITY:*

Assume the COMPLETE PACKAGE throughout.

For the left-to-right, assume CONTINUITY. This entails that for every middling proposition P , $P \sim (Q_1 \cup \dots \cup Q_n)$, where the Q_1, \dots, Q_n belong to some m -scale of Ω , which gives us R-SCALABILITY.

For the right-to-left, assume R-SCALABILITY, and (for *reductio*) that there exists a non-minimal atom A in the algebra \mathcal{B} such that for every other atom A' , $A' \succsim A$, with ‘ \succsim ’ replaced by ‘ \succ ’ in at least one instance. (Equivalently: assume there are non-minimal atoms not equally ranked by \succsim).

Since $\Omega \setminus A$ is middling, it’s R-scalable *only if* $(\Omega \setminus A) \sim (Q_1 \cup \dots \cup Q_n)$, for some Q_1, \dots, Q_n in an m -scale of some R-scalable proposition S such that $S \succ (\Omega \setminus A)$.¹¹ However, let $\mathcal{C}r$ be any probability function that agrees with \succsim ; then

$$\mathcal{C}r(Q_1) + \dots + \mathcal{C}r(Q_n) = \mathcal{C}r(\Omega) - \mathcal{C}r(A)$$

Furthermore, the Q_i must be more probable than A , since as there exist atoms more probable than A the union of any and all propositions that are as probable as A will be

¹¹We can safely ignore the case where $S \sim (\Omega \setminus A)$, since then S will be R-scalable only if $\Omega \setminus A$ is.

strictly less probable than $\Omega \setminus A$. So, $\mathcal{C}r(Q_i) > \mathcal{C}r(A)$, and thus

$$\mathcal{C}r(Q_1) + \cdots + \mathcal{C}r(Q_n) + \cdots + \mathcal{C}r(Q_m) > \mathcal{C}r(\Omega)$$

But there's no $S \succ \Omega$, so $\Omega \setminus A$ is not R-scalable, contradicting our assumption.

R-SCALABILITY therefore implies that $A \sim A'$ for any two non-minimal atoms A and A' ; from this, CONTINUITY straightforwardly follows. \square

B. *Proof of THEOREM 1:*

Part (i): For the left-to-right, assume R-COHERENCE. If P is n/m -valued and n'/m' -valued, then $n/m = n'/m'$. So there exists a function $\mathcal{C}r$ that assigns to each $P \in \mathcal{N}$ a unique rational value in $[0, 1]$, and $\mathcal{C}r$ will be a Ramsey-function relative to \succsim on \mathcal{N} . This function can then be extended to the whole of \mathcal{B} in the event that $\mathcal{B} - \mathcal{N} \neq \emptyset$ in any way you like. The right-to-left is obvious.

Part (ii): For the left-to-right, assume R-COHERENCE and R-SCALABILITY. For any $P, Q \in \mathcal{N}$ ($= \mathcal{B}$), suppose first that $P \succsim Q$. Where P is n/m -valued and Q is n'/m' -valued, $n/m \geq n'/m'$; so for any Ramsey-function $\mathcal{C}r$ relative to \succsim , $\mathcal{C}r(P) \geq \mathcal{C}r(Q)$. Next, suppose $\mathcal{C}r(P) \geq \mathcal{C}r(Q)$; since $\mathcal{C}r$ is a Ramsey function, P is n/m -valued and Q is n'/m' -valued, for $n/m \geq n'/m'$; by R-COHERENCE, therefore $P \succsim Q$. So from R-COHERENCE and R-SCALABILITY, there is a Ramsey-function $\mathcal{C}r$ relative to \succsim that agrees with \succsim . It is obvious from the definitions that the restriction of $\mathcal{C}r$ to \mathcal{N} will always be the unique Ramsey function relative to \succsim on \mathcal{N} , and in this case $\mathcal{N} = \mathcal{B}$.

For the right-to-left, the existence of the Ramsey-function $\mathcal{C}r$ already entails R-COHERENCE by part (i). That its uniqueness condition also entails R-SCALABILITY is

obvious given the finitude of \mathcal{B} . □

C. *Proof of THEOREM 2:*

The right-to-left of the existence part is obvious given part (i) of THEOREM 1.

For the left-to-right of the existence part, assume henceforth PREORDER and R-COHERENCE. We focus on the case where $\mathcal{N} \subset \mathcal{B}$, as R-SCALABILITY trivialises the proof.

From PREORDER, at least one nonempty set $\mathcal{F} = \{f_i : \mathcal{B} \mapsto \mathbb{R} \mid i = 1, \dots, n\}$ exists that agrees with \succsim . (See Evren and Ok 2011, p. 556, Proposition 1.) Suppose that \mathcal{F} is maximal with respect to inclusion. We then just need that there's some nonempty $\mathcal{F}^* \subseteq \mathcal{F}$ such that \mathcal{F}^* also agrees with \succsim and $\forall f \in \mathcal{F}^*$, f has an order-preserving transformation f' that's a Ramsey function w.r.t. \succsim . (We'll say that f' is an order-preserving transformation of f just in case $f(P) \geq f(Q)$ iff $f'(P) \geq f'(Q)$.) The set of all such transformations f' will then agree with \succsim .

There are three cases to consider: (i) \mathcal{N} is empty; (ii) \mathcal{N} contains only the minimal and/or maximal elements of \mathcal{B} ; (iii) \mathcal{N} contains some middling propositions. The first two are straightforward and omitted. For the third, note that if \mathcal{F} agrees with \succsim and $P \succ Q$, then

1. $f(P) \geq f(Q)$ for all $f \in \mathcal{F}$
2. $f(P) > f(Q)$ for some, *but not all*, $f \in \mathcal{F}$

For $P, Q \in \mathcal{N}$, R-COHERENCE requires however that for any Ramsey function $\mathcal{C}r$, if $P \succ Q$, then $\mathcal{C}r(P) > \mathcal{C}r(Q)$; consequently, it's not true that if \mathcal{F} agrees with \succsim , then

every $f \in \mathcal{F}$ has an order-preserving transformation that's also a Ramsey function with respect to \succsim . But define \mathcal{F}^* as follows:

$$\mathcal{F}^* = \{f \in \mathcal{F} \mid \text{if } P, Q \in \mathcal{N} \text{ and } P \succ Q, \text{ then } f(P) > f(Q)\}$$

\mathcal{F}^* will be non-empty, and will agree with \succsim . Let $\mathcal{F}_{\mathcal{N}}$ denote the set of restrictions of every $f \in \mathcal{F}^*$ to \mathcal{N} ; given this, the unique Ramsey function (denoted $\mathcal{C}r_{\mathcal{N}}$) on \mathcal{N} is going to be an order-preserving transformation of every $f \in \mathcal{F}_{\mathcal{N}}$. So we just have to show that each $f \in \mathcal{F}^*$ has an order-preserving transformation bounded by 0 and 1 that's an extension of $\mathcal{C}r^{\mathcal{N}}$ from \mathcal{N} to the whole of \mathcal{B} . Since \mathcal{B} is finite this is straightforward.

The proof of the uniqueness condition is obvious: if \mathcal{F} and \mathcal{F}' both agree with \succsim , then $\mathcal{F} \cup \mathcal{F}'$ will too. □

D. *Proof of THEOREM 3:*

Suppose just that $\mathcal{C}r$ agrees with \succsim and that \succsim violates R-COHERENCE. So, there exist P, Q such that P is n/m -valued, Q is n'/m' -valued, and not:

$$(P \succsim Q) \leftrightarrow (n/m \geq n'/m')$$

There are three cases: (1) neither P nor Q is minimal; (2) both P and Q are minimal; or (3) exactly one of P or Q is minimal.

Start with case (1). Focus on P , and let max designate some maximal proposition. (If P is n/m -valued and non-minimal, then max exists.) P is either (i) as probable as the union of n members of an m -scale of max , or (ii) as probable as the union of n'' members

of an m'' -scale of ... the union of n''' members of an m''' -scale of max . If (i), $\mathcal{C}r$ confirms the GRP only if

$$\mathcal{C}r(P) = {}^{n/m} \cdot \mathcal{C}r(max)$$

If (ii), only if

$$\mathcal{C}r(P) = ({}^{n'' \cdot \dots \cdot n'''} / ({}^{m'' \cdot \dots \cdot m'''})) \cdot \mathcal{C}r(max) = {}^{n/m} \cdot \mathcal{C}r(max)$$

The same applies to Q , *mutatis mutandis*, so $\mathcal{C}r$ confirms the GRP only if

$$\mathcal{C}r(Q) = {}^{n'/m'} \cdot \mathcal{C}r(max)$$

Assume for *reductio* that $\mathcal{C}r$ confirms the GRP, and suppose ${}^{n/m} \geq {}^{n'/m'}$. Hence, $\mathcal{C}r(P) \geq \mathcal{C}r(Q)$, and therefore $P \succeq Q$. In the other direction, suppose $P \succeq Q$; so $\mathcal{C}r(P) \geq \mathcal{C}r(Q)$, and ${}^{n/m} \geq {}^{n'/m'}$. So,

$$(P \succeq Q) \leftrightarrow ({}^{n/m} \geq {}^{n'/m'}),$$

which violates our assumptions.

Now case (2). Assume for this case that there are $P, Q \in \mathfrak{B}$ such that $P \succ Q$, and that if P is minimal, then $\mathcal{C}r(P) = 0$. If P, Q are both minimal then $P \sim Q$, and if $\mathcal{C}r$ agrees with \succeq then $\mathcal{C}r(P) = \mathcal{C}r(Q) > \mathcal{C}r(R)$, for any R such that $R \not\sim P$ (and hence $R \succ P$). Since P, Q are ${}^0/m$ -valued by definition, R-COHERENCE is violated only if P or Q is also ${}^n/m$ -valued, for $n > 0$. Suppose this of P ; then by the earlier reasoning, $\mathcal{C}r$ confirms the GRP only if $\mathcal{C}r(P) = {}^{n/m} \cdot \mathcal{C}r(max)$. Since ${}^{n/m} > 0$ and $\mathcal{C}r(max) > 0$, this is

false; so $\mathcal{C}r$ disconfirms the GRP.

Case (3) is then straightforward, and the proof of the corollary (for sets of functions) follows the same structure. Both proofs are omitted. □

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