A COMPLEMENT TO “Fermat’s last theorem proved by induction” in PhilSci Archive

An elementary proof accessible to Fermat

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Abstract

A small perfection to the elementary proof of Fermat's last theorem by induction published in PhilSci Archive is demonstrated. Only the property of identity is necessary in this second version of the proof. "Symmetry" and "transitivity" of the relation of equality are not necessary in it. This allows for simplifying and shorthening the proof. The refusal of a frequent objection to the proof is explicated. The utilized format is suitable for presenting the proof to wider audience

The proof in detail

I The contemporary formulation of *Fermat's last theorem* (FLT)

II All means necessary and sufficient for the proof

III The general idea and scheme of the proof

IV The modified *modus tollens*

V *Fermat's infinite descent* modified for the proof

VI The enumerated series of *modus tollens*

VII The derivative series of implications

VIII The proof of FLT by induction

I The contemporary formulation of Fermat's last theorem (FLT)

$$\forall x, y, z \in \boldsymbol{N}, \nexists n \in (\boldsymbol{N} \geq 3)\colon\ x^n = y^n + z^n$$

II All means necessary and sufficient for the proof

The property “idenity” of the relation of equality "$\forall A: \; A = A$”

Modus tollens "$\forall A, B: (A \to B) \leftrightarrow (\neg B \to \neg A)$, "$A, B$“ are propositions

Axiom of induction "$\forall p: \{p(1) \wedge [p(n) \to p(n+1)]\} \to p$": “p” is a proposition referring to natural numbers, and p(n) is the same propostion referring to the natural number “n”

The proof of FLT for “n=3”: ”$\forall x, y, z \in \boldsymbol{N}: \neg(x^3 = y^3 + z^3)$"

Furthermore:

Arithmetic (i.e. the Peano axioms) necessary for the formulation of FLT. Only a (trivial) arithmetic statement will be involved in the course of the proof: “x^n” is necessary for “x^{n+1}“: $\forall x, n \in \boldsymbol{N}: \; x^{n+1} \to x^n$

The standard propositional logic for the explication of the proof as a syllogism

III The general idea and scheme of the proof

The general idea:

$$[(x^{n+1} = y^{n+1} + z^{n+1}) \rightarrow (x^n = y^n + z^n)] \leftrightarrow$$

$$\leftrightarrow [\neg(x^n = y^n + z^n) \rightarrow \neg(x^{n+1} = y^{n+1} + z^{n+1})]$$

“x", "y", "z", n” are natural numbers

Another notation of the general idea:

$$[(x^{n+1} = x^{n+1}) \rightarrow (x^n = x^n)] \leftrightarrow$$

$$\leftrightarrow [\neg(x^n = y^n + z^n) \rightarrow \neg(x^{n+1} = y^{n+1} + z^{n+1})]$$

III The general idea and scheme of the proof

The general scheme of the proof

"Identity" and "*modus tollens*" imply "F" :$[(x^{n+1} = x^{n+1}) \to (x^n = x^n)] \leftrightarrow$
$\leftrightarrow [\neg(x^n = y^n + z^n) \to \neg(x^{n+1} = y^{n+1} + z^{n+1})]$

"F(n)" is unfolded in a series "SF": F(3), F(4), … , F(n), F(n+1), …

That "SF" is transformed into a series of implications "SI": I(3), I(4), … I(n), …

Here "I" is $[(x^{n+1} = x^{n+1}) \to (x^n = x^n)] \wedge [\neg(x^n = y^n + z^n)] \to$
$\to \neg(x^{n+1} = y^{n+1} + z^{n+1})$

I is proved by induction for any "n"

"SF is equivalent to "FLT": "FLT would be proved

IV The modified *modus tollens*

$\forall x: x = x$. (A) This is equivalent to: $\forall x: x \leftrightarrow x = x$.

(B) $(\forall x: x = y) \leftrightarrow (\forall x: x \leftrightarrow x = y)$

Proof:

A: 1. $x \rightarrow (x = x)$. Indeed: let $\neg\{x \rightarrow x = x) \rightarrow \exists x: x \neq x \rightarrow \neg(\forall x: x = x)$: contradiction.

2. $(x = x) \rightarrow x$. Indeed: if not, the term “x” of the proposition “$x = x$” would be absent sometimes: contradiction

B: $\forall x: x \leftrightarrow (x = x) \leftrightarrow [x = (x = y)] \leftrightarrow (x = x = y) \leftrightarrow [(x = x) = y] \leftrightarrow$ $(\mathrm{x} = \mathrm{y})$. Consequently, $[\forall x: x \leftrightarrow (x = y)]$

“A"$\wedge$"B" $\rightarrow$ "$(x = x) \leftrightarrow (x = y)$“. *Modus tollens* modified:

$[("x^{n+1} = x^{n+1}" \rightarrow "x^n = x^n") \leftrightarrow (\neg"x^n = x^n" \rightarrow \neg"x^{n+1} = x^{n+1}“)] \leftrightarrow$

$\leftrightarrow [("\boldsymbol{x}^{\boldsymbol{n}+\mathbf{1}} = \boldsymbol{x}^{\boldsymbol{n}+\mathbf{1}}" \rightarrow "\boldsymbol{x}^{\boldsymbol{n}} = \boldsymbol{x}^{\boldsymbol{n}}") \leftrightarrow$
$(\neg"\boldsymbol{x}^{\boldsymbol{n}} = \boldsymbol{y}^{\boldsymbol{n}} + \boldsymbol{z}^{\boldsymbol{n}}" \rightarrow \neg"\boldsymbol{x}^{\boldsymbol{n}+\mathbf{1}} = \boldsymbol{y}^{\boldsymbol{n}+\mathbf{1}} + \boldsymbol{z}^{\boldsymbol{n}+\mathbf{1}}“)]$

V *Fermat's infinite descent* modified for the proof

Fermat's "infinite descent" is modified as an "infinite ascent" for preparing a proof by induction startng from "n=3"

"F" : $\left[\left(x^{n+1} = x^{n+1}\right) \to \left(x^{n} = x^{n}\right)\right] \leftrightarrow$

$$\leftrightarrow \left[\neg\left(x^{n} = y^{n} + z^{n}\right) \to \neg\left(x^{n+1} = y^{n+1} + z^{n+1}\right)\right]$$

"SF": $F_{descent} \leftrightarrow F^{ascent}$

$$\ldots \leftrightarrow \ldots$$

$$F(n+1)_{descent} \leftrightarrow F(n+1)^{ascent}$$

$$F(n)_{descent} \leftrightarrow F(n)^{ascent}$$

$$\ldots \leftrightarrow \ldots$$

$$F(5)_{descent} \leftrightarrow F(5)^{ascent}$$

$$F(4)_{descent} \leftrightarrow F(4)^{ascent}$$

$$F(3)_{descent} \leftrightarrow F(3)^{ascent}$$

START "n=3"

VI The enumerated series of *modus tollens*

"SF" is an enumerated series of *modus tollens*

"F" : $\left[\left(x^{n+1}=x^{n+1}\right) \rightarrow \left(x^{n}=x^{n}\right)\right] \leftrightarrow$

$$\leftrightarrow \left[\neg\left(x^{n}=y^{n}+z^{n}\right) \rightarrow \neg\left(x^{n+1}=y^{n+1}+z^{n+1}\right)\right]$$

"SF": $\boldsymbol{F}_{descent} \leftrightarrow \boldsymbol{F}^{ascent}$

$\ldots \leftrightarrow \ldots$

$\boldsymbol{F}(n+1)_{descent} \leftrightarrow \boldsymbol{F}(n+1)^{ascent}$

$\boldsymbol{F}(n)_{descent} \leftrightarrow \boldsymbol{F}(n)^{ascent}$

$\ldots \leftrightarrow \ldots$

$\boldsymbol{F}(5)_{descent} \leftrightarrow \boldsymbol{F}(5)^{ascent}$

$\boldsymbol{F}(4)_{descent} \leftrightarrow \boldsymbol{F}(4)^{ascent}$

$\boldsymbol{F}(3)_{descent} \leftrightarrow \boldsymbol{F}(3)^{ascent}$

START "n=3"

VII The derivative series of implications

"F"→ "I": $\{[(x^{n+1} = x^{n+1}) \to (x^n = x^n)] \wedge [\neg(x^n = y^n + z^n)]\} \to$

$\underline{\to [\neg(x^{n+1} = y^{n+1} + z^{n+1})]}$

"SI": $I_{descent} \qquad \to I^{ascent}$

$\dots \qquad \to \dots$

$I(n+1)_{descent} \qquad \to I(n+1)^{ascent}$

$I(n)_{descent} \qquad \to I(n)^{ascent}$

$\dots \qquad \to \dots$

$I(5)_{descent} \qquad \to I(5)^{ascent}$

$I(4)_{descent} \qquad \to I(4)^{ascent}$

$I(3)_{descent} \qquad \to I(3)^{ascent}$

START "n=3"

“FLT decomposed by the derivative series of implications” = “FLT proved by indiction”

Let “FLT(n)” means FLT for a certain ”n” [$\neg(\boldsymbol{x^n = y^n + z^n})$]. Then “I”:

$$\{[(\boldsymbol{x^{n+1} = x^{n+1}}) \rightarrow (\boldsymbol{x^n = x^n})] \wedge [\boldsymbol{FLT(n)}]\} \rightarrow \boldsymbol{FLT(n+1)}$$

Thus, if one proves both (1) “x^n is necessary for x^{n+1}” and (2) “FLT(3)”,

“FLT” would be proved by induction as "$\forall p$: $\{p(1) \wedge [p(n) \rightarrow p(n+1)]\} \rightarrow p$“

implies "$\forall p(n \geq)$: $\{p(3) \wedge [p(n) \rightarrow p(n+1)]\} \rightarrow p(n \geq 3)$":

(1) Indeed: “$x^{n+1} = x^n . x$”, thus “x^n” is necessary for “x^{n+1}”

(2) “FLT(3)” was claimed by many mathematicians such as Euler, Gauss, and many others. Kummer’s proof (1847) will be cited for being absolutely rigorous

(3) Consequently, Fermat’s last theorem is proved!

The answer of a frequent objection

The objection: the "modified *modus tollens*" needs "$x^n = y^n + z^n$" to be proved. Fermat's infinite descent modified as in the claimed proof uses the substitution "$\neg(x^n = y^n + z^{n)}$". So, this contradiction, involved in the proof, makes it false.

The answer: "$x^n = y^n + z^n$" is a necessary condition for the "modified *modus tollens*". Thus, the latter implies the former. "$\neg(x^n = y^n + z^{n)}$" is a substitution in the "modified *modus tollens*". Thus, the latter implies the former

Consequently, the "modified *modus tollens*" implies both "$x^n = y^n + z^n$" and "$\neg(x^n = y^n + z^n)$", but <u>separately</u>, i.e by disjunction rather than by conjuction. This is not a contradiction as:

$$[(a \to b) \vee (a \to \neg b)] \leftrightarrow \text{"True"} \ [\forall \text{x}: (\text{"True"} \ \to x) \ \leftrightarrow x]$$

This means only that the proof involves a tautology redundant to the syllogism

This is quite different from the alleged "$[a \to (b \wedge \neg b)] \to$ "$False$", which is absent in the proof

The present proof and its first version in PhilSci Arhive are slightly different

The next two slides can describe the difference. They are according to the first version in PhilSci and should be compared with the corresponding slides above

II All means necessary and sufficient for the proof

The relation of equality defined by: (1) identity "$\forall A:\ A = A$" ; (2) symmetry "$\forall A, B:\ (A = B) \leftrightarrow (B = A)$"; and (3) transitivity "$\forall A, B, C:\ (A = B) \wedge (B = C) \leftrightarrow (A = B = C)$"

Modus tollens "$\forall A, B: (A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A)$, "$A, B$" are propositions

Axiom of induction "$\forall p: \{p(1) \wedge [p(n) \rightarrow p(n+1)]\} \rightarrow p$": "p" is a proposition referring to natural numbers, and p(n) is the same propostion referring to the natural number "n"

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The standard propositional logic for the explication of the proof as a syllogism

III The general idea and scheme of the proof

The general idea:

$$[(a = x^{n+1} = y^{n+1} + z^{n+1}) \rightarrow (b = x^n = y^n + z^n)] \leftrightarrow$$

$$\leftrightarrow [\neg(b = x^n = y^n + z^n) \rightarrow \neg(a = x^{n+1} = y^{n+1} + z^{n+1})]$$

“a, b, x, y, z, n” are natural numbers

A brief notation of the general idea:

$$[(a = x^{n+1}) \rightarrow (b = x^n)] \leftrightarrow$$

$$\leftrightarrow [\neg(x^n = y^n + z^n) \rightarrow \neg(x^{n+1} = y^{n+1} + z^{n+1})]$$