A Quantitative Version of Feynman's Static Field Momentum Example

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Abstract

The Feynman demonstration that electromagnetic field momentum is real—even for static fields—can be made more useful by simplifying its geometry. Instead of Feynman's disk with charged balls on its surface, use a hollow non-conducting sphere with uniform surface charge density. The initial field angular momentum and the final mechanical angular momentum can then be calculated in closed form and shown to be equal. The methods used in the calculation are those available to the average upper-division physics student.

This simplified geometry also provides a test for the current idea that electromagnetic field energy can be considered a form of inertial mass. The mass motion in the simplified Feynman example can be modeled as the static, circular, incompressible flow of a fluid, with distributed nonzero vorticity. But such motion requires a centripetal force or pressure that has yet to be identified.

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1 Introduction

The Feynman lectures[1] give an example of the reality of field momentum in a static electromagnetic field. A non-conducting, non-magnetic, horizontal disk is free to rotate about a vertical axis through its center. Several identical, charged balls are equally spaced around its circumference. In its center is an electromagnet. Due to cancellation, the net electric field is small on the surface of the disk. For distances from the axis of the disk larger than its radius, the electric and magnetic fields give a Poynting vector with a significant azimuthal component. The angular momentum density associated with this Poynting vector produces a nonzero total field angular momentum.

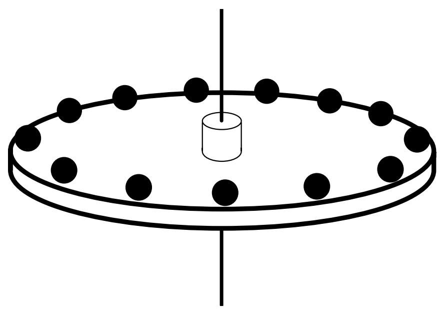

Figure 1: The Feynman example. Charged balls are spaced evenly around the circumference of a disk. A solenoid is placed at the center.

The disk is at rest for times less than zero. If the electromagnet is programmed to turn itself off beginning at time zero, the resulting induced electric field exerts a torque on the charged spheres. After a long time, the magnetic field, and hence the Poynting vector and the field angular momentum, are zero. But the disk is now rotating as a result of the torque exerted on the spheres during the decay of the magnetic field. The initial field angular momentum has become final mechanical angular momentum, which shows that the initial field angular momentum must have been real.

The difficulty with this example is that it depends on qualitative arguments about the field strengths. Obviously, the magnetic field is stronger closer to the electromagnet. One must argue that the weakness of the electric field there more than compensates, so that the Poynting vector is large only in the region outside the radius of the disk where its azimuthal sense is correct. Such qualitative argumentation is not helpful for students who are, after all, not totally convinced that the phenomenon of field angular momentum is real.

Fortunately, it is not difficult to concoct an idealized example like the Feynman one, but with fewer ambiguities. In this simplified example, expressions for both the initial field angular momentum and the final mechanical angular momentum are derived in closed form. The two derived expressions agree *quan-*

[1]Feynman et al (1964), Section 17-4, Section 27-6, and Figure 17-5.

titatively, which increases the persuasive power of Feynman's argument that the initial, static, electromagnetic field momentum must have been real.

2 The Simplified Feynman Example: Static-Field Momentum

Suppose that a non-conducting, non-magnetic spherical shell of outer radius a is suspended by a vertical axis through its center (taken as the z-axis), about which it is free to rotate. The surface at radius a has a uniform surface charge density σ_0. The electric field is thus zero for $r < a$. At the center of the sphere is an electromagnet configured so that its total vector magnetic moment is $\eta_0 \hat{\mathbf{z}}$, parallel to the vertical axis. As in the original Feynman example, the magnetic moment η_0 is a positive constant for $t < 0$, and decreases to zero for $t > 0$. The sphere is at rest for negative time.

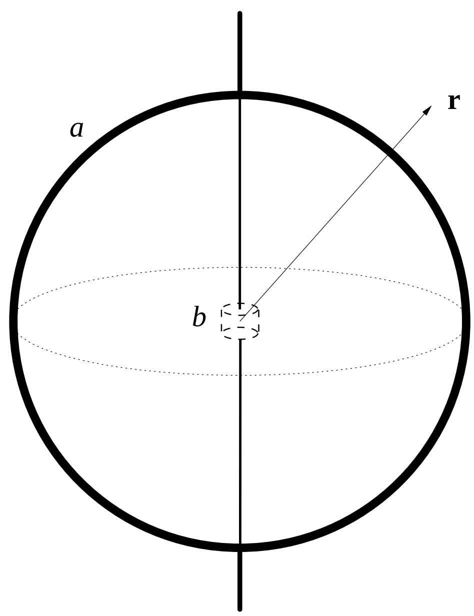

Figure 2: The simplified Feynman example. A non-conducting spherical shell of outer radius a is free to rotate about a vertical shaft. At the center of the sphere, a small electromagnet is attached to the shaft.

All currents in the electromagnet are confined to be less than distance b from the sphere's center. Take b/a to be sufficiently small that, in the region on and outside radius a where the electric field is nonzero, the magnetic field can be approximated adequately by just the dipole term in a multipole expansion.

Then the electric field is zero for $r < a$ and the static fields for $r > a$ are[2]

$$\mathbf{E} = \frac{a^2\sigma_0}{\varepsilon_0 r^2}\,\hat{\mathbf{r}} \qquad\qquad \mathbf{B} = \frac{\mu_0\eta_0}{4\pi r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right) \tag{2.1}$$

The Poynting vector is $\mathbf{S} = \mathbf{E}\times\mathbf{B}/\mu_0$ and the momentum density is $\mathbf{G} = \mathbf{S}/c^2$. The momentum density is thus zero for $r < a$ and

$$\mathbf{G} = \frac{\mu_0 a^2\sigma_0\eta_0\sin\theta}{4\pi r^5}\,\hat{\boldsymbol{\phi}} \tag{2.2}$$

for $r > a$. The angular momentum density for $r > a$ is

$$\mathbf{r}\times\mathbf{G} = -\frac{\mu_0 a^2\sigma_0\eta_0\sin\theta}{4\pi r^4}\,\hat{\boldsymbol{\theta}} \tag{2.3}$$

The total initial static field angular momentum is then

$$\mathbf{L}_0 = \int_a^\infty dr\, r^2 \int_0^\pi d\theta\sin\theta \int_0^{2\pi} d\phi\,\mathbf{r}\times\mathbf{G} = \frac{2\mu_0 a\sigma_0\eta_0}{3}\,\hat{\mathbf{z}} \tag{2.4}$$

3 The Simplified Feynman Example: Torques and their Impulse

At time zero, the current in the electromagnet begins to decrease slowly. Take its magnetic moment for $t \geq 0$ to be $\eta(t) = \eta_0 \mathrm{e}^{-t/\tau}$. Taking $\tau \gg a/c$ so that time delay effects are negligible, one has an instantaneous magnetic vector potential in the vicinity of $r = a$ given, again, by just the dipole term in the multipole expansion

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0\eta(t)\sin\theta}{4\pi r^2}\,\hat{\phi} \tag{3.1}$$

From this vector potential and the surface charge density σ_0, we use $\mathbf{E} = -\nabla\Phi_{\mathrm{elec}} - \partial\mathbf{A}/\partial t$ to derive the instantaneous electric field at radius $r = a$

$$\mathbf{E}(a,\theta,\phi,t) = \frac{\sigma_0}{\varepsilon_0}\,\hat{\mathbf{r}} - \frac{\mu_0\sin\theta}{4\pi a^2}\,\frac{d\eta}{dt}\,\hat{\boldsymbol{\phi}} \tag{3.2}$$

The total instantaneous torque of this electric field on the surface charge density is then

$$\mathbf{N} = a^2\int_0^\pi d\theta\sin\theta\int_0^{2\pi} d\phi\; a\,\hat{\mathbf{r}}\times\sigma_0\,\mathbf{E}(a,\theta,\phi,t) = -\frac{2\mu_0 a\sigma_0}{3}\,\frac{d\eta}{dt}\,\hat{\mathbf{z}} \tag{3.3}$$

The total impulse of this torque is the final, mechanical angular momentum

$$\mathbf{L}_{\mathrm{f}} = \int_0^\infty dt\,\mathbf{N} = \frac{2\mu_0 a\sigma_0\eta_0}{3}\,\hat{\mathbf{z}} \tag{3.4}$$

[2] This paper uses MKSA units in vacuum with the speed of light $(\varepsilon_0\mu_0)^{-1/2}$ denoted by c. Spherical polar coordinates r, θ, ϕ are used with unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$, the z-axis vertical along the shaft, and the origin at the center of the sphere. The x and y-axes are not shown in Figure 2.

4 Comments on Simplified Example

The agreement $\mathbf{L}_0 = \mathbf{L}_\mathrm{f}$ between eqn (2.4) and eqn (3.4) demonstrates that initial field momentum must have been real since it becomes an equal amount of final mechanical momentum. The quantitative agreement is convincing in a way that more qualitative arguments would not be. The methods used: multipole expansion of magnetic induction field and magnetic vector potential, Gauss's law for electric fields, retarded potentials for time-varying currents, the expression for the electric field when $\mathbf{A}$ varies, are all part of a standard junior level Electricity and Magnetism course.[3] That level is also ideal for the introduction of examples such as this one. The simplifying assumptions made are internally consistent. We explicitly assumed $a \gg b$, and $\tau \gg a/c$. An implicit assumption, that the magnetic field produced by the rotating sphere produces negligible residual field angular momentum, can be ensured by using a sphere with very large moment of inertia so that its angular velocity remains small.

5 Addendum: Velocity of the Energy Flow

A standard formula of electrodynamics is $\mathbf{J} = \rho_\mathrm{ch}\mathbf{v}_\mathrm{ch}$, where $\mathbf{J}$ is the charge flux vector ($\mathbf{J}\cdot d\mathbf{a}$ is the charge per second passing through area $d\mathbf{a}$), ρ_ch is the volume density of the flowing charge, and $\mathbf{v}_\mathrm{ch}$ is the flow velocity. An analogous formula can be defined for the flow of electromagnetic energy:

$$\mathbf{S} = \mathcal{E}\,\mathbf{v} \tag{5.1}$$

where the Poynting vector $\mathbf{S} = \mathbf{E}\times\mathbf{B}/\mu_0$ is the energy flux vector, $\mathcal{E} = \left(\varepsilon_0 E^2 + B^2/\mu_0\right)/2$ is the volume energy density, and $\mathbf{v}$ is a velocity of energy flow. Thus, for general fields in vacuum,

$$\mathbf{v} = \frac{\mathbf{S}}{\mathcal{E}} = \frac{2c\,(\mathbf{E}\times c\mathbf{B})}{E^2 + c^2B^2} \tag{5.2}$$

The magnitude of $\mathbf{v}$ is maximum when $\mathbf{E} \perp \mathbf{B}$ and zero when either E or B is zero or $\mathbf{E} \parallel \mathbf{B}$. The identity $(\alpha - \beta)^2 \geq 0$, where $\alpha = E$ and $\beta = cB$, can be used to prove that $0 \leq |\mathbf{v}| \leq c$, the speed of light.[4]

For example, a monochromatic plane wave of arbitrary polarization moving in direction $\hat{\mathbf{n}}$ in a vacuum has

$$\mathbf{S} = \varepsilon_0 c\left\{E_\mathrm{a}^2\cos^2(\psi + \delta_\mathrm{a}) + E_\mathrm{b}^2\cos^2(\psi + \delta_\mathrm{b})\right\}\hat{\mathbf{n}} \quad \text{and}$$
$$\mathcal{E} = \varepsilon_0\left\{E_\mathrm{a}^2\cos^2(\psi + \delta_\mathrm{a}) + E_\mathrm{b}^2\cos^2(\psi + \delta_\mathrm{b})\right\} \tag{5.3}$$

where $\psi = k\,\hat{\mathbf{n}}\cdot\mathbf{r} - \omega t$ with angular frequency ω and wave vector $k = \omega/c$. The real, positive amplitudes are E_a, E_b and the phases are δ_a, δ_b, in two perpendicular directions each perpendicular to $\hat{\mathbf{n}}$. Thus, for the monochromatic plane

[3] For example, see Wangsness (1986) and Griffiths (1999).

[4] Sebens (2019), Section 2.1.

wave, the energy flow velocity is the constant vector

$$\mathbf{v} = \frac{\mathbf{S}}{\mathcal{E}} = c\,\hat{\mathbf{n}} \tag{5.4}$$

Note that, regardless of the polarization state, the flow velocity $\mathbf{v}$ is constant, even though both $\mathbf{S}$ and $\mathcal{E}$ may in general vary with both time and position as the plane wave propagates.The extreme cases are linear polarization ($\delta_a = \delta_b$) for which $\mathbf{S}$ and $\mathcal{E}$ at a fixed point pass through zero every π/ω seconds, and circular polarization ($E_a = E_b$ and $\delta_a - \delta_b = \pm\pi/2$) for which $\mathbf{S} = \varepsilon_0 c E_a^2 \hat{\mathbf{n}}$ and $\mathcal{E} = \varepsilon_0 E_a^2$ are constants.

It is interesting to write the energy flow velocity for the $t < 0$ static fields in the simplified Feynman example of Section 2. For that example, for $r > a$,

$$\mathbf{S} = \frac{a^2\sigma_0\eta_0 \sin\theta}{4\pi\varepsilon_0 r^5}\,\hat{\phi} \qquad \text{and}$$

$$\mathcal{E} = \frac{1}{32\pi^2\varepsilon_0 r^4}\left\{\left(4\pi a^2\sigma_0\right)^2 + \left(\frac{\eta_0}{cr}\right)^2\left(3\cos^2\theta + 1\right)\right\} \tag{5.5}$$

The flow velocity is therefore

$$\mathbf{v} = \frac{\mathbf{S}}{\mathcal{E}} = \frac{2c\left(4\pi a^2\sigma_0\right)\left(\frac{\eta_0}{cr}\right)\sin\theta}{\left(4\pi a^2\sigma_0\right)^2 + \left(\frac{\eta_0}{cr}\right)^2\left(3\cos^2\theta + 1\right)}\,\hat{\phi} \tag{5.6}$$

This flow velocity vanishes in the polar directions $\theta = 0$ and $\theta = \pi$ and is maximum in the equatorial direction $\theta = \pi/2$, where, for large r, it falls off as $1/r$ with distance from the center of the sphere,

$$\mathbf{v} = \frac{1}{r}\frac{8\pi a^2\sigma_0\eta_0}{\left(4\pi a^2\sigma_0\right)^2 + \left(\frac{\eta_0}{cr}\right)^2}\,\hat{\phi} \qquad \text{for} \qquad \theta = \pi/2 \tag{5.7}$$

6 Addendum: Relativistic Mass?

Dividing the energy flow equation, eqn (5.1), by c^2 and using the standard equation $\mathbf{G} = \mathbf{S}/c^2$ for the electromagnetic momentum density gives

$$\mathbf{G} = \frac{\mathbf{S}}{c^2} = \frac{\mathcal{E}}{c^2}\,\mathbf{v} \tag{6.1}$$

Using only classical electrodynamics, even before special relativity was developed, it could have been noticed that $\mathcal{E}/c^2$ has the units of a mass density and that the definition $\mathcal{E}/c^2 = \mathcal{M}$ for that density would give the momentum density $\mathbf{G}$ the familiar Newtonian form

$$\mathbf{G} = \mathcal{M}\,\mathbf{v} \tag{6.2}$$

Recent papers[5] refer to $\mathcal{M} = \mathcal{E}/c^2$ as a "relativistic" mass density and investigate the idea that this $\mathcal{M}$ has the properties (inertia, conservation, source of gravity) found in particle masses. The simplified Feynman example in the present paper can serve as a test of this idea.

The term *relativistic mass* comes from the special relativistic dynamics of a particle with rest (invariant) mass m_0. Such a particle has momentum $\mathbf{p} = m_0\mathbf{v}/\sqrt{1 - v^2/c^2}$ and energy $e = m_0 c^2/\sqrt{1 - v^2/c^2}$ with the covariant relation $e^2 = p^2c^2 + m_0^2c^4$. Then a relativistic mass m_{rel} is defined by

$$m_{\mathrm{rel}} = e/c^2 = \frac{m_0}{\sqrt{1 - v^2/c^2}} \tag{6.3}$$

so that $e = m_{\mathrm{rel}}c^2$ and $\mathbf{p} = m_{\mathrm{rel}}\mathbf{v}$. This last relation has the same form as eqn (6.2).

This definition of relativistic mass as $m_{\mathrm{rel}} = e/c^2$ can also be applied to the case of a particle of zero rest mass such as a photon of energy $e = \hbar\omega$ and momentum $p = \hbar k$. Then $e^2 = p^2c^2 + 0$ and $m_{\mathrm{rel}} = e/c^2 = pc/c^2$, and hence

$$p = m_{\mathrm{rel}}\, c \tag{6.4}$$

When the definition $\mathcal{M} = \mathcal{E}/c^2$ is applied to the monochromatic plane wave of eqn (5.3), the mass density of the wave is

$$\mathcal{M} = \frac{\mathcal{E}}{c^2} = \frac{\varepsilon_0}{c^2}\left\{E_{\mathrm{a}}^2 \cos^2(\psi + \delta_{\mathrm{a}}) + E_{\mathrm{b}}^2 \cos^2(\psi + \delta_{\mathrm{b}})\right\} \tag{6.5}$$

the velocity from eqn (5.4) is the constant vector $\mathbf{v} = c\,\hat{\mathbf{n}}$, and the momentum density from the general formula eqn (6.2) then becomes

$$\mathbf{G} = \mathcal{M} c\, \hat{\mathbf{n}} \tag{6.6}$$

In general, this $\mathcal{M}$ will vary with time and position. However, in the special case of circular polarization, $\mathcal{M} = \varepsilon_0 E_{\mathrm{a}}^2/c^2$ is a constant and thus eqn (6.6) is consistent with the photon relation in eqn (6.4). This is not a surprise, since a monochromatic photon helicity state is the quantum of a monochromatic, plane-wave electromagnetic mode of circular polarization. For plane polarization, however, the electromagnetic mass density $\mathcal{M}$ at a fixed point will reach a maximum and then pass through zero every π/ω seconds.

From eqn (5.5), the definition $\mathcal{M} = \mathcal{E}/c^2$ applied to the simplified Feynman example (with $t < 0$ and $r > a$) gives

$$\mathcal{M} = \frac{\mathcal{E}}{c^2} = \frac{1}{32\pi^2\varepsilon_0 c^2 r^4}\left\{\left(4\pi a^2\sigma_0\right)^2 + \left(\frac{\eta_0}{cr}\right)^2\left(3\cos^2\theta + 1\right)\right\} \tag{6.7}$$

and the velocity $\mathbf{v}$ from eqn (5.6) is

$$\mathbf{v} = \frac{2c\left(4\pi a^2\sigma_0\right)\left(\dfrac{\eta_0}{cr}\right)\sin\theta}{\left(4\pi a^2\sigma_0\right)^2 + \left(\dfrac{\eta_0}{cr}\right)^2\left(3\cos^2\theta + 1\right)}\,\hat{\phi} \tag{6.8}$$

[5] Sebens (2019) and the references cited there.

For polar directions ($\theta = 0$ or $\theta = \pi$), the velocity vanishes, $\mathbf{v} = 0$, and the mass density takes its maximum value. For equatorial directions ($\theta = \pi/2$), the mass density takes its minimum value

$$\mathcal{M} = \frac{1}{32\pi^2 \varepsilon_0 c^2 r^4} \left\{ \left(4\pi a^2 \sigma_0\right)^2 + \left(\frac{\eta_0}{cr}\right)^2 \right\} \qquad \text{for} \quad \theta = \pi/2 \tag{6.9}$$

and the velocity has its maximum value

$$\mathbf{v} = \frac{2c\left(4\pi a^2 \sigma_0\right)\left(\frac{\eta_0}{cr}\right)}{\left(4\pi a^2 \sigma_0\right)^2 + \left(\frac{\eta_0}{cr}\right)^2} \hat{\boldsymbol{\phi}} \qquad \text{for} \quad \theta = \pi/2 \tag{6.10}$$

The movement of relativistic mass $\mathcal{M}$ in the simplified Feynman example can be modeled as a fluid flowing incompressibly[6] in static, circular streamlines about the z-axis. This flow also has a nonzero distributed vorticity $\boldsymbol{\Omega} = \boldsymbol{\nabla} \times \mathbf{v}$. For example, in the large-r limit it is

$$\boldsymbol{\Omega} = \boldsymbol{\nabla} \times \mathbf{v} \approx \frac{\eta_0 \cos\theta}{\pi a^2 \sigma_0 r^2} \hat{\mathbf{r}} \qquad \text{for} \quad \frac{(\eta_0/cr)}{(4\pi a^2 \sigma_0)} \ll 1 \tag{6.11}$$

This vorticity is independent of time for $t < 0$, which suggests that the fluid must also be inviscid.

If $\mathcal{M}$ and $\mathbf{v}$ are to represent a static flow of actual inertial mass, then the crucial problem is to identify some centripetal force or pressure to maintain the centripetal acceleration of that circular flow.

In the simplified Feynman example, the rate of change of momentum density following the flow (material derivative) is

$$\frac{D\,\mathbf{G}}{Dt} = \mathcal{M}\frac{D\,\mathbf{v}}{Dt} = -\frac{\mu_0 c^2 \left(4\pi a^2 \sigma_0\right)^2 \left(\frac{\eta_0}{cr}\right)^2 \sin\theta}{8\pi r^4 \left[\left(4\pi a^2 \sigma_0\right)^2 + \left(\frac{\eta_0}{cr}\right)^2 \left(3\cos^2\theta + 1\right)\right]} \hat{\boldsymbol{\rho}} \tag{6.12}$$

which is nonzero except at the poles.[7]

If momentum density $\mathbf{G}$ is produced by motion of an inertial mass, then some pressure or force is required to produce this change of momentum. But, in electrodynamics static fields do not act on other fields. Static fields act only on charges, and there are no charges in the region $r > a$ where the mass flow is found. Unless a centripetal force or pressure acting on mass density $\mathcal{M}$ itself can be identified, one must question the treatment of $\mathcal{M}$ as an inertial relativistic mass density.

[6] The eqn (6.7) and eqn (6.8) imply that the material derivative (derivative following the flow, defined as $D/Dt = \partial/\partial t + \mathbf{v} \cdot \boldsymbol{\nabla}$) of the mass density is $D\mathcal{M}/Dt = \partial\mathcal{M}/\partial t + \mathbf{v} \cdot \boldsymbol{\nabla}\mathcal{M} = 0$, which is the condition for an incompressible flow.

[7] Here $\hat{\boldsymbol{\rho}}$ is the radial unit vector of cylindrical polar coordinates.

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