The classical and the quantum

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Abstract. Newtonian and Scrödinger dynamics can be formulated in a physically meaningful way within the same Hilbert space framework. This fact was recently used to discover an unexpected relation between classical and quantum motions that goes beyond the results provided by the Ehrenfest theorem. The Newtonian dynamics was shown to be the Schrödinger dynamics of states constrained to a submanifold of the space of states, identified with the classical phase space of the system. Quantum observables are identified with vector fields on the space of states. The commutators of observables are expressed through the curvature of the space. The resulting embedding of the Newtonian and Shrödinger dynamics into a unified geometric framework is rigid in the sense that the Schrödinger dynamics is a unique extension of the Newtonian one. Furthermore, under the embedding, the normal distribution of measurement results associated with a classical measurement implies the Born rule for the probability of transition of quantum states. In this paper, the implications of the obtained theory to the process of measurement in quantum theory are analyzed. The double-slit, EPR and Schrödinger cat type experiments are reviewed anew. It is shown that, despite reproducing the usual results of quantum theory, the framework is not simply a reformulation of the theory. New experiments to discover the predicted effects are proposed.

1. Questions to be addressed

The current situation in quantum theory is extremely awkward. On one hand, during more than one hundred years of its existence, the theory proved itself to be very successful and accurate in describing the observed microscopic phenomena. On the other, it represents a constellation of paradoxes and unanswered questions that kept several generations of researchers confused and led to exotic interpretations and numerous attempts to revise or complete the theory. The general issue was and still is to understand how the quantum gives rise to the classical world that we live in. Most notable question is how exactly under an observation a typical superposition of states of a particle or a system of particles produces a single observed outcome. This seemingly simple question generates a multitude of the follow-up questions that indicate the level of the current confusion about the subject:

Is quantum state a part of reality, or only the state of our knowledge of the latter? An observation takes a superposition of states to a single outcome, which is the actual value of a physical quantity, say, the position or momentum of a particle. This outcome was not present before observation. Does it mean that we create reality by making an observation? Also, if the state is real and the observed outcome is also real, then how these two realities are related? Why do macroscopic bodies in the universe have a well defined position at all times, independently of whether we observe them or not? When is the body macroscopic and where exactly is the boundary between the quantum and the classical worlds? What happens during an observation?

What process should we call an observation? Does our brain activity have anything to do with generating the outcomes? The Schrödinger dynamics is deterministic. How does the observed probability make its way into the deterministic quantum theory? Does it mean that measuring devices do not satisfy quantum dynamics and must be described differently? How can we derive the rule that determines the probabilities? How is the basis used to find the components of state determined during an observation? If several measuring devices are turned on at the same time, what determines how a particular basis is selected? What should we make of the delayed choice experiments that demonstrate that a photon is neither a particle nor a wave until observed? Is it also an act of creation by observation? Alternatively, is it a process of going back in time and fixing the nature of the photon before observation? Going beyond the non-relativistic framework, to what extent is the seemingly instantaneous transition of states compatible with the special and general relativity?

In a recent series of papers [1]-[6], an important new connection between the classical and quantum dynamics was derived. The starting point was a realization of classical and quantum mechanics on an equal footing within the same Hilbert space framework and identification of observables with vector fields on the sphere of normalized states. This resulted in a physically meaningful interpretation of components of the velocity of state. Newtonian dynamics was shown to be the Schrödinger dynamics of a system whose state is constrained to the classical phase space submanifold in the Hilbert space of states. This also resulted in a formula relating the normal probability distribution and the Born rule and interpretation of quantum collapse in terms of diffusion of state on the projective space of states.

In simple words, the classical space and classical phase space of a system of particles can be identified with a submanifold of the space of states of the corresponding quantum system. When the system is constrained to the submanifold, it behaves classically. Otherwise, it behaves quantum-mechanically. The velocity of state at any point of the classical space submanifold can be decomposed into classical (velocity, acceleration) and non-classical (phase velocity, spreading) components. The curvature of the sphere of states is determined from the canonical commutation relations. An observation creates a diffusion on the sphere of states. During the diffusion the state can reach the classical space submanifold and trigger a detector in it. The probability of reaching a particular point of the classical space submanifold is given by the Born rule.

These results suggest that there is an alternative approach to quantum mechanics that is much more appropriate for understanding and visualizing the theory and for addressing its fundamental problems and paradoxes. In this paper, such an approach will be presented. It will be shown that by accepting the space of states as a new arena for physical events and identifying the classical space and classical phase space with submanifolds of thereof we can fruitfully explore the relationship of classical and quantum dynamics and address the above-listed questions in a coherent and fundamentally simple way.

2. The measurement problem

Classical mechanics is based on the notion of a *material point*. Position of a material point in the classical space is given by its *coordinates* $\mathbf{x} = \{x^i\}, i = 1, 2, 3$. The motion of a material point is described then by the functions of time $x^i(t)$. Velocity $\mathbf{v}$ of the point is given by the derivative $\mathbf{v} = d\mathbf{x}/dt$. Acceleration $\mathbf{w}$ is given by the second derivative $\mathbf{w} = d^2\mathbf{x}/dt^2$. More complicated systems, for example, rigid bodies, are certain systems of material points. Position of a system of n material points requires in general $3n$ coordinates $\mathbf{x}_1, \ ... \ , \mathbf{x}_n$. The space of possible positions of a system is called the *configuration space* of the system. A constrained system requires fewer coordinates and the configuration space may be a non-trivial submanifold of $\mathbb{R}^{3n}$, described by *generalized* coordinates. Dynamics of a system can be derived from the *principle of stationary action*, where the *action functional* S is the integral of the *Lagrangian function* $L(x, v, t)$ of generalized positions and velocities x, v of all the particles in the system with respect to time.

The Lagrangian of a single free material point of mass m in an *inertial coordinate system* can be derived from the *Galileo principle of relativity* and is equal to the kinetic energy $\frac{m\mathbf{v}^2}{2}$ of the particle. The Lagrangian is *additive* so the Lagrangian of a system of free material points is the sum of Lagrangians of each point. The interaction is introduced by adding a function of coordinates (a *potential*). Equations of motion (the Euler-Lagrange equations) are obtained from the condition $\delta S = 0$ on variation of the action and form a system of second order differential equations for the functions $x(t)$. In particular, the Newton's equations of motion of a system of particles with interaction described by a potential $V(\mathbf{x}_1, \mathbf{x}_2, \ \dots\ )$ are the Euler-Lagrange equations for the Lagrangian $L(x, v, t) = \sum_k \frac{m_k \mathbf{v}_k^2}{2} - V(\mathbf{x}_1, \mathbf{x}_2, \ \dots\ )$, where m_k are masses of the particles. Alternatively, the Euler-Lagrange equations can be replaced with a system of the first order equations $\frac{\partial h}{\partial x} = -\frac{dp}{dt}$, $\frac{\partial h}{\partial p} = \frac{dx}{dt}$(*Hamilton equations*) for the generalized coordinates and momenta $p = mv$ of the particles. Here $h = p\frac{dx}{dt} - L$ is the *Hamiltonian function* or *energy* of the system. The generalized variables x, p provide coordinates for the *phase space* and used to define the cotangent bundle and a *symplectic structure*. A point in the phase space represents the *state* of the mechanical system. Another dynamical equation that follows from differentiation of the action functional S and the Euler-Lagrange equations is the *Hamilton-Jacobi equation* $\frac{\partial S}{\partial t} + h(x, \nabla S, t) = 0$. This equation is a point of contact of classical and quantum dynamics.

A microscopic system in quantum mechanics is maximally described by its *wave* or *state* function φ. In the *coordinate representation* the state function at time t is a function $\varphi(x, s)$ of coordinates x of the system and, possibly, additional quantum-mechanical degrees of freedom s such as spin. The state function is an element of a *Hilbert space* of states. The Hilbert space H is a vector space with an inner product $(\varphi, \psi)_H$ that is complete with respect to the norm $||\varphi||_H = \sqrt{(\varphi, \varphi)_H}$. Physical quantities, or *observables* are described by linear self-adjoint operators on H. The system obtains familiar physical characteristics as a result of measurement. A measurement (in the narrow sense) on the system results in an eigenvalue of the corresponding observable. The state of the system after the measurement is the corresponding eigenvector. The initial state of the measured system can be written as a *superposition* (linear combination) of eigenvectors. The probability to find the system in a specific eigenstate of the measured observable is given by the *Born rule* (i.e., by the modulus squared of the coefficient of the corresponding normalized eigenvector in the superposition). Two observables $\widehat{a}, \widehat{b}$ may not commute: the *commutator* $[\widehat{a}, \widehat{b}] = \widehat{a}\widehat{b} - \widehat{b}\widehat{a}$ is not 0, in general. This results in the *uncertainty principle*, which puts a fundamental limit on the possible precision of measurement of two non-commuting observables. The *canonical commutation relations* between *conjugate observables*, such as position and momentum yield the algebraic structure of quantum mechanics. Symmetries of the system are represented by unitary transformations on the space of states H. Accordingly, the theory of representations of groups becomes a foundational part of quantum mechanics. The motion of the system is described by a time-dependent state function that satisfies the Schrödinger equation, which is a linear differential equation $i\hbar\frac{\partial \varphi}{dt} = \widehat{h}\varphi$. The operator $\widehat{h}$ is the *Hamiltonian*, which is an operator version of the Hamiltonian function. For a system of n interacting particles the Hamiltonian is given by $\widehat{h} = -\frac{\hbar^2}{2}\sum_k \frac{\Delta_k}{m_k} + V(\mathbf{x}_1, \ \dots\ , \mathbf{x}_n)$, where Δ_k is the Laplace operator for the k-th particle and V is the operator of multiplication by the potential of the interaction. The linear nature of the Schrödinger equation signifies that the *superposition principle* holds true. Namely, a linear combination of physically possible states of a system represents another physically possible state of the system.

The linear nature of quantum mechanics poses a persistent problem when trying to reconcile the classical and quantum mechanics. The superposition principle is totally foreign to classical physics, which governs the motion of macroscopic bodies. Physical consequences of the principle feed all controversies of the theory and represent the major paradox of modern science. All questions raised in the previous section are rooted in the superposition principle. For instance,

let the wave packets $\varphi_{\mathbf{a}}$, $\varphi_{\mathbf{b}}$ be solutions to the Schrödinger equation, describing a particle near points $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$. According to the superposition principle, the sum $\varphi_{\mathbf{a}} + \varphi_{\mathbf{b}}$ must represent a physically meaningful state of the particle, which is somehow "spread over" both points. The superposition principle is confirmed by the countless experiments with microscopic particles. But how could an electron (let alone a large molecule or a microbe) go through two different holes in a plate, and yet arrive as a single particle to the screen behind the plate? This is at odds with our everyday experience as we never observe a macroscopic body in two places at once. This paradox taken in its full scope and the resulting issue of reconciliation of the classical and the quantum constitutes the *measurement problem*.

The problem can be split into several related smaller parts, of which the following three are essential. First, what is the meaning of the wave function? What does it really describe? The experiment shows its relationship to probability to find the particle in a certain state, which is one of the postulates of quantum theory. Does it go deeper than that? Does it have a physical meaning beyond the probability and information? This is the *problem of reality of state*. Second, when a certain state is obtained as a result of a measurement, what is the process that transforms the original state into the observed state? How do the observed outcomes come into being? This is the *problem of definite outcomes*. Third, there are many physical quantities that can be measured. Under each measurement the initial state transforms to a different observed state (an eigenstate of the measured observable). How could the initial state "know" what set of the final states it needs to transform to? What is the role of the measuring device in this selection? This is the *preferred basis problem*.

3. Newtonian mechanics in the Hilbert space of states

Everyday experience shows that macroscopic bodies possess a well-defined position in space at any moment of time. In quantum mechanics, the state of a spinless particle with a known position $\mathbf{a}$ is given by the Dirac delta function $\delta^3_{\mathbf{a}}(\mathbf{x}) = \delta^3(\mathbf{x}-\mathbf{a})$. The map $\omega : \mathbf{a} \longrightarrow \delta^3_{\mathbf{a}}$ provides a one-to-one correspondence between points $\mathbf{a} \in \mathbb{R}^3$ and state "functions" $\delta^3_{\mathbf{a}}$. This allows us to describe points in $\mathbb{R}^3$ in functional terms and identify the set $\mathbb{R}^3$ with the set M_3 of all delta functions in the space of state functions of the particle.

Dirac delta states are considered an idealization. But so is the notion of a material point in Newtonian mechanics. Both idealizations are the building blocks in their respective theories. As we will see, they are also important for understanding the relationship between Newtonian physics and quantum mechanics. We will see that Newtonian physics in the Euclidean space $\mathbb{R}^3$ is the Schrödinger quantum mechanics of systems whose state is constrained to the submanifold in the Hilbert space of states, formed by the delta-like states of particles.

The space $L_2(\mathbb{R}^3)$ does not contain delta functions. For instance, if f_n is a delta-convergent sequence [7] of continuous, square-integrable functions on $\mathbb{R}^3$, then the sequence $\int f_n^2(\mathbf{x}) d^3\mathbf{x}$ diverges. There are essentially two ways out of this difficulty. One method is to approximate delta functions with the more physical Gaussian functions, which are in $L_2(\mathbb{R}^3)$. Another one is to complete the Hilbert space $L_2(\mathbb{R}^3)$ to obtain a wider space that includes delta functions. The methods are essentially equivalent and will be used interchangeably. To explain, let us write the inner product of functions $\varphi, \psi \in L_2(\mathbb{R}^3)$ as

$$(\varphi, \psi)_{L_2} = \int \delta^3(\mathbf{x}-\mathbf{y})\varphi(\mathbf{x})\overline{\psi}(\mathbf{y}) d^3\mathbf{x} d^3\mathbf{y}, \tag{1}$$

where $\delta^3(\mathbf{x}-\mathbf{y})$ is the kernel of the identity operator. By approximating $\delta^3(\mathbf{x}-\mathbf{y})$ with a Gaussian function, one obtains a new inner product in $L_2(\mathbb{R}^3)$

$$(\varphi, \psi)_{\mathbf{H}} = \int e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{8\sigma^2}} \varphi(\mathbf{x})\overline{\psi}(\mathbf{y}) d^3\mathbf{x} d^3\mathbf{y}. \tag{2}$$

Here σ is a parameter. The Hilbert space $\mathbf{H}$ obtained by completing $L_2(\mathbb{R}^3)$ with respect to this inner product contains delta functions and their derivatives. In particular,

$$\int e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{8\sigma^2}} \delta^3(\mathbf{x}-\mathbf{a})\delta^3(\mathbf{y}-\mathbf{a}) d^3\mathbf{x} d^3\mathbf{y} = 1. \tag{3}$$

Furthermore, the injective map ω is continuous and is, in fact, a homeomorphism onto the image $\omega(\mathbb{R}^3)$ with the topology induced by the metric on $\mathbf{H}$: two delta functions $\delta^3_{\mathbf{a}}, \delta^3_{\mathbf{b}}$ are close in $\mathbf{H}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are close in $\mathbb{R}^3$. Furthermore, ω and its inverse are smooth. It follows that the set M_3 of all delta functions $\delta^3_{\mathbf{a}}(\mathbf{x})$ with $\mathbf{a} \in \mathbb{R}^3$ form a submanifold of the unit sphere in the Hilbert space $\mathbf{H}$, diffeomorphic to $\mathbb{R}^3$. The map $\omega : \mathbf{a} \longrightarrow \delta^3_{\mathbf{a}}$ becomes an embedding of $\mathbb{R}^3$ into $\mathbf{H}$.

The map $\rho_\sigma : \mathbf{H} \longrightarrow L_2(\mathbb{R}^3)$ that relates L_2 and $\mathbf{H}$-representations and identifies the two methods of dealing with delta-states is given by the Gaussian kernel

$$\rho_\sigma(\mathbf{x}, \mathbf{y}) = \left(\frac{1}{2\pi\sigma^2}\right)^{3/4} e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{4\sigma^2}}. \tag{4}$$

In fact, it is easy to see that ρ_σ is one-to-one. Indeed, taking various derivatives of $(\rho_\sigma f)(\mathbf{x})$ one can see that all Fourier coefficients of f in the basis of (multivariable) Hermite functions in $L_2(\mathbb{R}^3)$ vanish. Since these functions form a basis in $L_2(\mathbb{R}^3)$, we conclude that $f = 0$, hence, ρ_σ is one-to-one. Multiplying the operators (integrating the product of kernels) one can see that

$$k(\mathbf{x}, \mathbf{y}) = (\rho^*_\sigma \rho_\sigma)(\mathbf{x}, \mathbf{y}) = e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{8\sigma^2}}, \tag{5}$$

which is consistent with (2) and proves that ρ_σ is an isomorphism of the Hilbert spaces $L_2(\mathbb{R}^3)$ and $\mathbf{H}$.

The isomorphism ρ_σ transforms delta functions $\delta^3_{\mathbf{a}}$ to Gaussian functions $\widetilde{\delta}^3_{\mathbf{a}} = \rho_\sigma(\delta^3_{\mathbf{a}})$, centered at $\mathbf{a}$. The image M^σ_3 of M_3 under ρ_σ is an embedded submanifold of the unit sphere in $L_2(\mathbb{R}^3)$ made of the functions $\widetilde{\delta}^3_{\mathbf{a}}$. The map $\omega_\sigma = \rho_\sigma \circ \omega : \mathbb{R}^3 \longrightarrow M^\sigma_3$ is a diffeomorphism. Here ω is the same as before. Note that the kernel $\delta^3(\mathbf{x}-\mathbf{y})$ of the metric on $L_2(\mathbb{R}^3)$ is analogous to the Kronecker delta δ_{ik}, representing Euclidean metric in orthogonal coordinates. The "skewed" kernel $e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{8\sigma^2}}$ of the metric on $\mathbf{H}$ is then analogous to the Euclidean metric represented in linear coordinates with skewed axes by a constant non-diagonal matrix g_{ik}.

Let $\mathbf{r} = \mathbf{a}(t)$ be a path with values in $\mathbb{R}^3$ and let $\varphi = \delta^3_{\mathbf{a}(t)}$ be the corresponding path in M_3. It is easy to see that the norm $\left\|\frac{d\varphi}{dt}\right\|^2_H$ of the velocity in the space $\mathbf{H}$ is given by

$$\left\|\frac{d\varphi}{dt}\right\|^2_H = \left.\frac{\partial^2 k(\mathbf{x}, \mathbf{y})}{\partial x^i \partial y^k}\right|_{\mathbf{x}=\mathbf{y}=\mathbf{a}} \frac{d\mathbf{a}^i}{dt}\frac{d\mathbf{a}^k}{dt}. \tag{6}$$

Here $k(\mathbf{x}, \mathbf{y}) = e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{8\sigma^2}}$ as in (5), so that

$$\left.\frac{\partial^2 k(\mathbf{x}, \mathbf{y})}{\partial x^i \partial y^k}\right|_{\mathbf{x}=\mathbf{y}=\mathbf{a}} = \frac{1}{4\sigma^2}\delta_{ik}, \tag{7}$$

where δ_{ik} is the Kronecker delta symbol. Assuming now that the distance in $\mathbb{R}^3$ is measured in the units of 2σ, we obtain

$$\left\|\frac{d\varphi}{dt}\right\|_H = \left\|\frac{d\mathbf{a}}{dt}\right\|_{\mathbb{R}^3}. \tag{8}$$

It follows that the map $\omega : \mathbb{R}^3 \longrightarrow \mathbf{H}$ is an isometric embedding. Furthermore, the set M_3 is complete in $\mathbf{H}$ so that there is no vector in $\mathbf{H}$ orthogonal to all of M_3. In fact, if $(f, \delta^3_{\mathbf{a}})_{\mathbf{H}} = 0$, then $\rho_\sigma(f) = 0$ and so $f = 0$, because ρ_σ is an isomorphism.

By defining the operations of addition $\oplus$ and multiplication by a scalar $\lambda\odot$ via $\omega(\mathbf{a}) \oplus \omega(\mathbf{b}) = \omega(\mathbf{a} + \mathbf{b})$ and $\lambda \odot \omega(\mathbf{a}) = \omega(\lambda\mathbf{a})$ with ω as before, we obtain M_3 as a vector space isomorphic to the Euclidean space $\mathbb{R}^3$. Since ω is an embedding, these operations are continuous in the topology of $\mathbf{H}$. Of course, the obtained vector structure on M_3 is not the same as the one on the Hilbert space $\mathbf{H}$ and M_3 is not a subspace of $\mathbf{H}$.

With the classical space in place, we can now proceed with a reformulation of Newtonian mechanics in functional terms. The projection of velocity and acceleration of the state $\delta^3_{\mathbf{a}(t)}$ onto the Euclidean space M_3 yields correct Newtonian velocity and acceleration of the classical particle:

$$\left(\frac{d}{dt}\delta^3_{\mathbf{a}}(\mathbf{x}), -\frac{\partial}{\partial x^i}\delta^3_{\mathbf{a}}(\mathbf{x})\right)_{\mathbf{H}} = \frac{da^i}{dt} \tag{9}$$

and

$$\left(\frac{d^2}{dt^2}\delta^3_{\mathbf{a}}(\mathbf{x}), -\frac{\partial}{\partial x^i}\delta^3_{\mathbf{a}}(\mathbf{x})\right)_{\mathbf{H}} = \frac{d^2a^i}{dt^2}. \tag{10}$$

The Newtonian dynamics of the classical particle can be derived from the principle of least action for the action functional S on paths in $\mathbf{H}$, defined by

$$S = \int k(\mathbf{x}, \mathbf{y}) \left[\frac{m}{2}\frac{d\varphi_t(\mathbf{x})}{dt}\frac{d\overline{\varphi}_t(\mathbf{y})}{dt} - V(\mathbf{x})\varphi_t(\mathbf{x})\overline{\varphi}_t(\mathbf{y})\right] d^3\mathbf{x}d^3\mathbf{y}dt. \tag{11}$$

Here m is the mass of the particle, V is the potential and $k(\mathbf{x}, \mathbf{y}) = e^{-\frac{1}{2}(\mathbf{x}-\mathbf{y})^2}$, as in (5) with $2\sigma = 1$, to ensure (8). In fact, under the constraint $\varphi_t(\mathbf{x}) = \delta^3(\mathbf{x} - \mathbf{a}(t))$ the action (11) becomes

$$S = \int \left[\frac{m}{2}\left(\frac{d\mathbf{a}}{dt}\right)^2 - V(\mathbf{a})\right] dt, \tag{12}$$

which is the classical action functional for the particle. An action functional for the time-dependent Schrödinger equation that reduces to the classical action (12) on the properly constrained states will be introduced in section 6.

It follows that a classical particle can be considered a constrained dynamical system with the state φ of the particle and the velocity of the state $\frac{d\varphi}{dt}$ as dynamical variables. A similar realization exists for classical mechanical systems consisting of any number of particles. For example, the map $\omega \otimes \omega : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbf{H} \otimes \mathbf{H}$, $\omega \otimes \omega(\mathbf{a}, \mathbf{b}) = \delta^3_{\mathbf{a}} \otimes \delta^3_{\mathbf{b}}$ identifies the configuration space $\mathbb{R}^3 \times \mathbb{R}^3$ of a two particle system with the embedded submanifold $M_6 = \omega \otimes \omega(\mathbb{R}^3 \times \mathbb{R}^3)$ of the Hilbert space $\mathbf{H} \otimes \mathbf{H}$. Consider a path $(\mathbf{a}(t), \mathbf{b}(t))$ in $\mathbb{R}^3 \times \mathbb{R}^3$ and the corresponding path $\delta^3_{\mathbf{a}(t)} \otimes \delta^3_{\mathbf{b}(t)}$ with values in M_6. For any t, the vectors $\frac{d}{dt}\delta^3_{\mathbf{a}(t)} \otimes \delta^3_{\mathbf{b}(t)}$ and $\delta^3_{\mathbf{a}(t)} \otimes \frac{d}{dt}\delta^3_{\mathbf{b}(t)}$ are tangent to M_6 at the point $\delta^3_{\mathbf{a}(t)} \otimes \delta^3_{\mathbf{b}(t)}$ and orthogonal in $\mathbf{H} \otimes \mathbf{H}$. The space M_6 with the induced metric is isometric to the direct product $\mathbb{R}^3 \times \mathbb{R}^3$ with the natural Euclidean metric. Projection of velocity and acceleration of the state $\varphi(t) = \delta^3_{\mathbf{a}(t)} \otimes \delta^3_{\mathbf{b}(t)}$ onto the basis vectors $\left(-\frac{\partial}{\partial x^i}\delta^3_{\mathbf{a}(t)}\right) \otimes \delta^3_{\mathbf{b}(t)}$ and $\delta^3_{\mathbf{a}(t)} \otimes \left(-\frac{\partial}{\partial x^k}\delta^3_{\mathbf{b}(t)}\right)$ yields the velocity and acceleration of the particles by means of the formulas similar to (9) and (10).

4. Observables as vector fields

Quantum observables can be identified with vector fields on the space of states. Given a self-adjoint operator $\widehat{A}$ on a Hilbert space L_2 of square-integrable functions (it could in particular

be the tensor product space of a many body problem) one can introduce the associated linear vector field A_φ on L_2 by

$$A_\varphi = -i\widehat{A}\varphi. \tag{13}$$

If D is the domain of the operator $\widehat{A}$, then A_φ maps D into the vector space L_2. Because $\widehat{A}$ is self-adjoint, the field A_φ, being restricted to the sphere S^{L_2} of unit normalized states, is tangent to the sphere. The commutator of observables and the commutator (Lie bracket) of the corresponding vector fields are related in a simple way:

$$[A_\varphi, B_\varphi] = [\widehat{A}, \widehat{B}]\varphi. \tag{14}$$

Furthermore, a Hilbert metric on the space of states yields a Riemannian metric on the sphere. For this, consider the realization $L_{2\mathbb{R}}$ of the Hilbert space L_2, i.e., the real vector space of pairs $X = (\mathrm{Re}\psi, \mathrm{Im}\psi)$ with ψ in L_2. If ξ, η are vector fields on S^{L_2}, define a Riemannian metric $G_\varphi : T_{\mathbb{R}\varphi}S^{L_2} \times T_{\mathbb{R}\varphi}S^{L_2} \longrightarrow \mathbb{R}$ on the sphere by

$$G_\varphi(X, Y) = \mathrm{Re}(\xi, \eta). \tag{15}$$

Here $X = (\mathrm{Re}\xi, \mathrm{Im}\xi)$, $Y = (\mathrm{Re}\eta, \mathrm{Im}\eta)$ and (ξ, η) denotes the L_2-inner product of ξ, η.

The Riemannian metric on S^{L_2} yields a Riemannian (Fubini-Study) metric on the projective space CP^{L_2}, which is the base of the fibration $\pi : S^{L_2} \longrightarrow CP^{L_2}$. For this, an arbitrary tangent vector $X \in T_{R\varphi}S^{L_2}$ is decomposed into two components: tangent and orthogonal to the fibre $\{\varphi\}$ through φ (i.e., to the plane C^1 containing the circle $S^1 = \{\varphi\}$). The differential $d\pi$ maps the tangential component to the zero-vector. The orthogonal component of X can be then identified with $d\pi(X)$. If two vectors X, Y are orthogonal to the fibre $\{\varphi\}$, the inner product of $d\pi(X)$ and $d\pi(Y)$ in the Fubini-Study metric is equal to the inner product of X and Y in the metric G_φ:

$$(d\pi(X), d\pi(Y))_{FS} = G_\varphi(X, Y). \tag{16}$$

The resulting metrics can be used to find physically meaningful components of vector fields A_φ associated with observables. Since A_φ is tangent to S^{L_2}, it can be decomposed into components tangent and orthogonal to the fibre $\{\varphi\}$ of the fibre bundle $\pi : S^{L_2} \longrightarrow CP^{L_2}$. These components have a simple physical meaning, justifying the use of the projective space CP^{L_2}. From

$$\overline{A} \equiv (\varphi, \widehat{A}\varphi) = (-i\varphi, -i\widehat{A}\varphi), \tag{17}$$

one can see that the expected value of an observable $\widehat{A}$ in state φ is the projection of the vector $-i\widehat{A}\varphi \in T_\varphi S^{L_2}$ onto the fibre $\{\varphi\}$. Because

$$(\varphi, \widehat{A}^2\varphi) = (\widehat{A}\varphi, \widehat{A}\varphi) = (-i\widehat{A}\varphi, -i\widehat{A}\varphi), \tag{18}$$

the term $(\varphi, \widehat{A}^2\varphi)$ is the norm of the vector $-i\widehat{A}\varphi$ squared. The vector $-i\widehat{A}_\perp\varphi = -i\widehat{A}\varphi - (-i\overline{A}\varphi)$ associated with the operator $\widehat{A} - \overline{A}I$ is orthogonal to the fibre $\{\varphi\}$. Accordingly, the variance

$$\Delta A^2 = (\varphi, (\widehat{A} - \overline{A}I)^2\varphi) = (\varphi, \widehat{A}_\perp^2\varphi) = (-i\widehat{A}_\perp\varphi, -i\widehat{A}_\perp\varphi) \tag{19}$$

is the norm squared of the component $-i\widehat{A}_\perp\varphi$. Recall that the image of this vector under $d\pi$ can be identified with the vector itself. It follows that the norm of $-i\widehat{A}_\perp\varphi$ in the Fubini-Study metric coincides with its norm in the Riemannian metric on S^{L_2} and in the original L_2-metric.

The Schrödinger equation

$$\frac{d\varphi}{dt} = -i\widehat{h}\varphi \tag{20}$$

is an equation for the integral curves of the vector field $-i\widehat{h}\varphi$ on the sphere S^{L_2}. Let's decompose $-i\widehat{h}\varphi$ onto the components parallel and orthogonal to the fibre. The parallel component of $\frac{d\varphi}{dt}$ is numerically

$$\mathrm{Re}(-i\varphi, -i\widehat{h}\varphi) = \overline{E}, \tag{21}$$

i.e., the expected value of the energy. The decomposition of the velocity vector $\frac{d\varphi}{dt}$ into the parallel and orthogonal components is then given by

$$\frac{d\varphi}{dt} = -i\overline{E}\varphi + -i(\widehat{h} - \overline{E})\varphi = -i\overline{E}\varphi - i\widehat{h}_{\perp}\varphi. \tag{22}$$

The orthogonal component of the velocity $\frac{d\varphi}{dt}$ is equal to $-i\widehat{h}_{\perp}\varphi$. From this and equation (19) we conclude that: *The speed of evolution of state in the projective space is equal to the uncertainty of energy.* Equation (22) also demonstrates that the physical state is driven by the operator $\widehat{h}_{\perp}$, associated with the uncertainty in energy rather than the energy itself.

The realization of operators by vector fields yields other interesting results. For instance, the uncertainty relation

$$\Delta A \Delta B \geq \frac{1}{2}\left|\left(\varphi, [\widehat{A}, \widehat{B}]\varphi\right)\right| \tag{23}$$

follows geometrically from the comparison of areas of rectangle $A_{|XY|}$ and parallelogram A_{XY} formed by vectors $X = -i\widehat{A}_{\perp}\varphi$ and $Y = -i\widehat{B}_{\perp}\varphi$:

$$A_{|XY|} \geq A_{XY}. \tag{24}$$

There is also an uncertainty identity, [2]:

$$\Delta A^2 \Delta B^2 = A_{XY}^2 + G_{\varphi}^2(X, Y). \tag{25}$$

The sum on the right hand side of (25) can be written as $||X||^2||Y||^2 \sin^2\theta + ||X||^2||Y||^2 \cos^2\theta$, where θ is the angle between X and Y. In particular, when $\theta = 0$, the uncertainty comes from the inner product term $G_{\varphi}(X, Y)$ in (25) and when $\theta = \pi/2$, the uncertainty is due to the area term. By replacing $\widehat{B}$ with a real linear combination of the operators $\widehat{A}, \widehat{B}$ (i.e., by rotating B_{φ} in the plane through X and Y), we can change θ in any desirable way while preserving the product $\Delta A^2 \Delta B^2$.

5. Commutator of observables and curvature of the sphere of states

The identification of observables with vector fields allows one to relate the commutators of observables with the curvature of the sphere of states. To see this, consider first the space $\mathbb{C}^2$ of electron's spin states. The sphere S^3 of unit-normalized states in $\mathbb{C}^2$ can be identified with the group manifold $SU(2)$. For this, one identifies the space $\mathbb{C}^2$ of complex vectors $\varphi = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ with the space M of 2×2 matrices

$$\widehat{\varphi} = \begin{bmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{bmatrix}. \tag{26}$$

The map $\widehat{\omega} : \varphi \longrightarrow \widehat{\varphi}$ is an isomorphism of (real) vector spaces $\mathbb{C}^2$ and M. The sphere S^3 of unit states in $\mathbb{C}^2$ is identified via $\widehat{\omega}$ with the subset of matrices with unit determinant. The latter subset is the group $SU(2)$ under matrix multiplication.

The differential $d\widehat{\omega}$ of the map $\widehat{\omega}$ identifies the tangent space $T_{e_1}S^3$ to the sphere S^3 at the point $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (that is, the hyperplane $\mathrm{Re} z_1 = 1$) with the Lie algebra $su(2)$ of traceless anti-Hermitian matrices

$$\widehat{A} = \begin{bmatrix} ia_2 & a_3 + ia_4 \\ -a_3 + ia_4 & -ia_2 \end{bmatrix}, \tag{27}$$

$a_2, a_3, a_4 \in R$. Under $d\widehat{\omega}$ the basis $e_2 = \begin{bmatrix} i \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $e_4 = \begin{bmatrix} 0 \\ i \end{bmatrix}$ in the tangent space $T_{e_1}S^3 = R^3$ becomes the basis $\{i\widehat{\sigma}_3, i\widehat{\sigma}_2, i\widehat{\sigma}_1\}$ in the Lie algebra $su(2)$. In particular, the real numbers a_2, a_3, a_4 acquire the meaning of coordinates of points on the tangent space $\mathrm{Re} z_1 = 1$ in the basis $\{e_2, e_3, e_4\}$.

The embedding of S^3 into $\mathbb{C}^2$ induces the usual Riemannian metric on the sphere. A direct verification demonstrates that this metric coincides with the Killing metric on $SU(2)$. The latter metric can be defined on the tangent space $T_eSU(2)$ at the identity e (i.e., on the Lie algebra $su(2)$) by $(\widehat{X}, \widehat{Y})_K = \frac{1}{2} Tr \widehat{X}\widehat{Y}^+$ and then extended to the entire $SU(2)$ by the group action. Here $(\widehat{X}, \widehat{Y})_K$ denotes the Killing inner product of tangent vectors and $\widehat{Y}^+$ on the right is the Hermitian conjugate of $\widehat{Y}$. The constant $1/2$ in the Killing metric together with a proper choice of the unit of measurement ensure the equality of the Riemannian and the Killing metrics. The tangent space $su(2)$ is spanned by the spin operators having the dimension of angular momentum and measured in the units of $\hbar$. Therefore, the Planck system of units will be used. The spin generators $\widehat{s}_1 = \frac{i}{2}\widehat{\sigma}_1, \widehat{s}_2 = \frac{i}{2}\widehat{\sigma}_2, \widehat{s}_3 = \frac{i}{2}\widehat{\sigma}_3$ are orthogonal in the defined metric and have a norm equal to $1/2$ in Planck units.

The integral curves of the left-invariant vector fields $L_{\widehat{X}}(\widehat{\varphi}) = \widehat{\varphi}\widehat{X}$ are geodesics on $SU(2)$. They are given by $\widehat{\varphi}_t = \widehat{\varphi}_0 e^{-i\widehat{X}t}$. In the usual coordinates on $\mathbb{C}^2$, the equation of these geodesics takes the form $\varphi_t = e^{-i\widehat{X}t}\varphi_0$, where $\omega(\varphi_0) = \widehat{\varphi}_0$. The carriers of geodesics are the great circles on the sphere S^3. The commutators of the spin observables are directly related to the sectional curvature of the sphere S^3. This is not surprising as the non-trivial Lie bracket of vector fields whose integral curves are geodesics can only be due to curvature of the underlying space. If $\widehat{X}, \widehat{Y} \in su(2)$ are linearly independent generators and $L_{\widehat{X}}(\widehat{\varphi}), L_{\widehat{Y}}(\widehat{\varphi})$ are the associated left-invariant vector fields, then the sectional curvature $R_\varphi(p)$ of S^3 in the plane p through $L_{\widehat{X}}(\widehat{\varphi}), L_{\widehat{Y}}(\widehat{\varphi})$ is given at any point $\widehat{\varphi}$ by

$$R_\varphi(p) = \frac{1}{4} \frac{\left\| [\widehat{X}, \widehat{Y}] \right\|_K^2}{\left\| \widehat{X} \right\|_K^2 \left\| \widehat{Y} \right\|_K^2 - \left(\widehat{X}, \widehat{Y} \right)_K^2}. \tag{28}$$

In particular, if the generators $\widehat{X}, \widehat{Y}$ are orthonormal in the Killing metric, (28) simplifies to

$$R_\varphi(p) = \frac{1}{4} \left\| [\widehat{X}, \widehat{Y}] \right\|_K^2. \tag{29}$$

Using the formula (28), we obtain the following expression for the sectional curvature $R_\varphi(p)$ in the plane p through orthogonal vectors $L_{\widehat{s}_1}(\widehat{\varphi}), L_{\widehat{s}_2}(\widehat{\varphi})$:

$$R_\varphi(p) = \frac{1}{4} \frac{([\widehat{s}_1, \widehat{s}_2], [\widehat{s}_1, \widehat{s}_2])_K}{(\widehat{s}_1, \widehat{s}_1)_K (\widehat{s}_2, \widehat{s}_2)_K} = 4 (\widehat{s}_3, \widehat{s}_3)_K = 1. \tag{30}$$

This means that the radius of S^3 in Planck units is equal to 1, confirming the isometric nature of the isomorphism $\widehat{\omega}$ considered as a map from the unit sphere S^3 in $\mathbb{C}^2$ onto $SU(2)$ with the

Killing metric. Note that in an arbitrary system of units the sectional curvature would be equal to $1/\hbar^2$ (i.e., radius=$\hbar$). The dimension of sectional curvature is consistent with the fact that the tangent space $su(2)$ is spanned by the spin operators.

The obtained relationship between commutators of spin observables and radius of the sphere of states can be extended to other observables. In particular, the commutator $[\widehat{p},\widehat{x}]$ of position and momentum observables of an arbitrary non-relativistic particle with states in the space $L_2(\mathbb{R})$ yields similarly the sectional curvature of the unit sphere S^{L_2} in $L_2(\mathbb{R})$. In fact, let's compute the sectional curvature of the sphere S^{L_2} in the plane through the tangent vectors $-i\widehat{p}\varphi$, $-i\widehat{x}\varphi$ at a point $\varphi \in S^{L_2}$. It is convenient to represent the action of operators $\widehat{p}$, $\widehat{x}$ in the basis $\varphi_n(x) = \frac{1}{\sqrt[4]{\pi}2^n n!}H_n(x)e^{-\frac{x^2}{2}}$, $n = 0, 1, 2, ...$ of the quantum harmonic oscillator. Here $H_n(x)$ are the Hermite polynomials. Note that the vectors φ_n are in the domain of the operators $\widehat{p}$, $\widehat{x}$, $\widehat{p}\widehat{x}$ and $\widehat{x}\widehat{p}$. The matrices of the operators $\widehat{p}$, $\widehat{x}$ in the basis are given by

$$\widehat{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix} \tag{31}$$

and

$$\widehat{p} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 & 0 & \cdots \\ i & 0 & -i\sqrt{2} & 0 & \cdots \\ 0 & i\sqrt{2} & 0 & -i\sqrt{3} & \cdots \\ 0 & 0 & i\sqrt{3} & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}. \tag{32}$$

Because the operators $\widehat{x}$, $\widehat{p}$ are unbounded, the validity of such a matrix representation requires a discussion. However, for the purpose of computing the sectional curvature it will be sufficient to point out that the matrices (31) and (32) correctly reproduce the action of operators on all vectors with finitely many non-vanishing components in the basis $\{\varphi_n\}$.

Let us find the sectional curvature of the sphere S^{L_2} at the "vacuum" point $\varphi_n|_{n=0} = \varphi_0$. For this, consider the subspace $\mathbb{C}^2 \subset L_2(\mathbb{R})$ formed by the first two vectors of the basis. Note that up to the coefficient $\frac{1}{\sqrt{2}}$, the sub-matrices formed by the first two rows and columns of matrices (31) and (32) coincide with the Pauli matrices $\widehat{\sigma}_x, \widehat{\sigma}_y$ respectively. Let us introduce the bounded operators $\widehat{s}_p, \widehat{s}_x$ on $L_2(\mathbb{R})$ defined by $\widehat{s}_x\varphi = \frac{1}{\sqrt{2}}\widehat{\sigma}_x\varphi$, $\widehat{s}_p\varphi = \frac{1}{\sqrt{2}}\widehat{\sigma}_y\varphi$ for φ in $\mathbb{C}^2$, and by $\widehat{s}_x\varphi = 0$, $\widehat{s}_p\varphi = 0$ for φ in the orthogonal complement of $\mathbb{C}^2$ in $L_2(\mathbb{R})$. Note that the action of operators $\widehat{p}$, $\widehat{x}$ and $[\widehat{p}, \widehat{x}]$ on the point φ_0 is correctly reproduced by the operators $\widehat{s}_p, \widehat{s}_x$:

$$\widehat{x}\varphi_0 = \widehat{s}_x\varphi_0 \tag{33}$$

$$\widehat{p}\varphi_0 = \widehat{s}_p\varphi_0 \tag{34}$$

$$[\widehat{p},\widehat{x}]\,\varphi_0 = [\widehat{s}_p,\widehat{s}_x]\,\varphi_0. \tag{35}$$

Consider the sphere $S^3 = S^{L_2} \cap \mathbb{C}^2$ with the metric induced by the inclusion. As discussed, this metric coincides with the Killing metric on the group $SU(2) = S^3$. The point φ_0 is given in the basis $\{\varphi_0, \varphi_1\}$ in $\mathbb{C}^2$ by the column $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The image $\widehat{\varphi}_0$ of the column φ_0 under the isomorphism (26) is the identity e in the group $SU(2)$. Accordingly, one can compute the norms of the right sides of (33), (34) and (35) in the Killing metric. Such a computation verifies that these norms are equal to the norms of the corresponding left sides in the L_2-metric. For example, the norm of the

right side of (35) in the Killing metric is given by $\left\|\widehat{\varphi}_0 \frac{1}{2}[\widehat{\sigma}_y, \widehat{\sigma}_x]\right\|_K = \|i\widehat{\sigma}_z\|_K = \sqrt{\frac{1}{2}Tr(\widehat{\sigma}_z)^2} = 1$. This coincides with the L_2-norm of the corresponding left side: $\|[\widehat{p}, \widehat{x}]\varphi_0\|_{L_2} = \|\varphi_0\|_{L_2} = 1$.

The sectional curvature of S^{L_2} in the plane through vector fields $-i\widehat{x}\varphi$, $-i\widehat{p}\varphi$ at $\varphi = \varphi_0$ is equal to the sectional curvature $R_{\varphi_0}(p)$ of S^3 in the plane p through the fields $-i\widehat{\sigma}_x\varphi$, $-i\widehat{\sigma}_y\varphi$ at this point. By (28), (33), (34) and (35), this sectional curvature is given in terms of the Lie brakets of these fields, i.e., in terms of the commutator $[\widehat{p}, \widehat{x}]$ evaluated at φ_0 and is equal to 1. Because sphere has a constant sectional curvature, the same result applies to any point. It follows that the commutator of vector fields associated with the operators of position and momentum has the same geometric interpretation as the commutator of vector fields associated with the operators of spin. Namely, the commutators give the sectional curvature of the sphere of states and produce the same value 1 ($\hbar$ in an arbitrary system of units) for the radius of the sphere. This provides one with a purely geometric approach to quantum observables and their commutators in terms of vector fields on the sphere of states and their Lie bracket.

6. Components of the velocity of state under the Schrödinger evolution

We now have all necessary ingredients to put the classical and quantum mechanics on an equal footing and to discover their innermost relationship. From (22), we know that for any state $\varphi \in S^{L_2}$, the velocity of state $\frac{d\varphi}{dt}$ in the Schrödinger equation can be decomposed onto the components parallel and orthogonal to the fibre $\{\varphi\}$ of the bundle $\pi : S^{L_2} \longrightarrow CP^{L_2}$:

$$\frac{d\varphi}{dt} = -i\overline{E}\varphi - i\widehat{h}_\perp\varphi. \tag{36}$$

The norm of the parallel component $-i\overline{E}\varphi$ is the expected value of energy $\overline{E}$. It represents the phase velocity of state. The norm of the orthogonal component $-i\widehat{h}_\perp\varphi$ is equal to the uncertainty of energy ΔE on the state φ. It represents the velocity of motion of the fibre $\{\varphi\}$. In particular, from (36) it follows that under the Schrödinger evolution, the speed of evolution of state in the projective space is equal to the uncertainty in energy.

The orthogonal component $-i\widehat{h}_\perp\varphi$ of the velocity can be further decomposed into physically meaningful components. To see this, let's begin with an equation that follows from the Schrödinger dynamics:

$$\left(\frac{d\varphi}{dt}, -i\widehat{A}\varphi\right) = \left(\varphi, \frac{1}{2}\{\widehat{A}, \widehat{h}\}\varphi\right) - \left(\varphi, \frac{1}{2}[\widehat{A}, \widehat{h}]\varphi\right). \tag{37}$$

The left hand side of (37) is the projection of the velocity of state onto the vector field associated with the observable $\widehat{A}$. The imaginary part of the projection (the term with the commutator $[\widehat{A}, \widehat{h}]$) yields the Ehrenfest theorem for a time-independent observable $\widehat{A}$. The real part of this projection (the term with the anticommutator $\{\widehat{A}, \widehat{h}\}$) is the projection in the sense of Riemannian metric on S^{L_2}. This Riemannian projection can be used to identify further components of the velocity of state.

Suppose that at $t = 0$, a microscopic particle is prepared in the state

$$\varphi_{\mathbf{a},\mathbf{p}}(\mathbf{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{3/4} e^{-\frac{(\mathbf{x}-\mathbf{a})^2}{4\sigma^2}} e^{i\frac{\mathbf{p}(\mathbf{x}-\mathbf{a})}{\hbar}}, \tag{38}$$

where σ is the same as in (4) and $\mathbf{p} = m\mathbf{v}_0$ with $\mathbf{v}_0$ being the initial group-velocity of the packet. Consider the subset $M^\sigma_{3,3}$ of all initial states $\varphi_{\mathbf{a},\mathbf{p}}$ given by (38) in $L_2(\mathbb{R}^3)$. The map $\Omega : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow M^\sigma_{3,3}$,

$$\Omega(\mathbf{a}, \mathbf{p}) = \varphi_{\mathbf{a},\mathbf{p}}(\mathbf{x}), \tag{39}$$

is a homeomorphism from the classical phase space onto $M^{\sigma}_{3,3}$ with the topology induced by the metric on $L_2(\mathbb{R}^3)$. In fact, it is one-to-one and the points $(\mathbf{a},\mathbf{p})$ and $(\mathbf{b},\mathbf{q})$ are close in $\mathbb{R}^3\times\mathbb{R}^3$ if and only if the functions $\varphi_{\mathbf{a},\mathbf{p}}$, $\varphi_{\mathbf{b},\mathbf{q}}$ are close in $L_2(\mathbb{R}^3)$. The map Ω and its inverse are also smooth, so that $M^{\sigma}_{3,3}$ is a 6-dimensional embedded submanifold of $L_2(\mathbb{R}^3)$ diffeomorphic to the classical phase space.

Consider the set of all fibres of the bundle $\pi : S^{L_2}\longrightarrow CP^{L_2}$ through the points of $M^{\sigma}_{3,3}$. The resulting bundle $\pi : M^{\sigma}_{3,3}\times S^1\longrightarrow M^{\sigma}_{3,3}$ identifies $M^{\sigma}_{3,3}$ with a submanifold of CP^{L_2}, denoted by the same symbol. For $\Omega(\mathbf{a},\mathbf{p}) = re^{i\theta}$, where r is the modulus and θ is the argument of $\Omega(\mathbf{a},\mathbf{p})$, the vectors $\frac{\partial r}{\partial a^{\alpha}}e^{i\theta}$ and $i\frac{\partial\theta}{\partial p^{\beta}}re^{i\theta}$ are orthogonal in the Riemannian metric on the sphere S^{L_2}. They are also orthogonal to the fibre $\{\varphi_{\mathbf{a},\mathbf{p}}\}$ in $L_2(\mathbb{R}^3)$ and can be, therefore, identified with vectors tangent to the projective manifold $M^{\sigma}_{3,3}$ at $\{\varphi_{\mathbf{a},\mathbf{p}}\}$. The Riemannian metric induced on $M^{\sigma}_{3,3}$ is the Fubini-Study metric on CP^{L_2}, constrained to $M^{\sigma}_{3,3}$.

For any path $\{\varphi\} = \{\varphi_\tau\}$ with values in $M^{\sigma}_{3,3}\subset CP^{L_2}$, the norm of velocity vector $\frac{d\{\varphi\}}{d\tau}$ in the Fubini-Study metric is given by

$$\left\|\frac{d\{\varphi\}}{d\tau}\right\|^2_{FS} = \frac{1}{4\sigma^2}\left\|\frac{d\mathbf{a}}{d\tau}\right\|^2_{\mathbb{R}^3} + \frac{\sigma^2}{\hbar^2}\left\|\frac{d\mathbf{p}}{d\tau}\right\|^2_{\mathbb{R}^3}. \tag{40}$$

It follows that under a proper choice of units, the map Ω is an isometry that identifies the Euclidean phase space $\mathbb{R}^3\times\mathbb{R}^3$ of the particle with the submanifold $M^{\sigma}_{3,3}\subset CP^{L_2}$ furnished with the induced Fubini-Study metric. The map Ω is an extension of the isometric embedding $\omega_\sigma = \rho_\sigma\circ\omega$ introduced in section 3 from the classical space to the classical phase space.

The obtained embedding of the classical phase space into the space of quantum states is physically meaningful. To see this, let us first project the orthogonal component $-\frac{i}{\hbar}\widehat{h}_{\perp}\varphi$ of the velocity $\frac{d\varphi}{dt}$ onto vectors tangent to the curves of constant values of $\mathbf{p}$ and $\mathbf{a}$ (classical space and momentum space components) in the projective manifold $M^{\sigma}_{3,3}$. Calculation of the projection of the velocity $\frac{d\varphi}{dt}$ onto the unit vector $-\widehat{\frac{\partial r}{\partial a^{\alpha}}e^{i\theta}}$ (i.e., the classical space component of $\frac{d\varphi}{dt}$) for an arbitrary Hamiltonian of the form $\widehat{h} = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{x})$ yields

$$\mathrm{Re}\left(\frac{d\varphi}{dt}, -\widehat{\frac{\partial r}{\partial a^{\alpha}}e^{i\theta}}\right)\Bigg|_{t=0} = \left(\frac{dr}{dt}, -\widehat{\frac{\partial r}{\partial a^{\alpha}}}\right)\Bigg|_{t=0} = \frac{v^{\alpha}_0}{2\sigma}. \tag{41}$$

Calculation of the projection of velocity $\frac{d\varphi}{dt}$ onto the unit vector $i\widehat{\frac{\partial\theta}{\partial p^{\alpha}}\varphi}$ (momentum space component) gives

$$\mathrm{Re}\left(\frac{d\varphi}{dt}, i\widehat{\frac{\partial\theta}{\partial p^{\alpha}}\varphi}\right)\Bigg|_{t=0} = \frac{mw^{\alpha}\sigma}{\hbar}, \tag{42}$$

where

$$mw^{\alpha} = -\left.\frac{\partial V(\mathbf{x})}{\partial x^{\alpha}}\right|_{\mathbf{x}=\mathbf{x}_0} \tag{43}$$

and σ is assumed to be small enough for the linear approximation of $V(\mathbf{x})$ to be valid within intervals of length σ.

The velocity $\frac{d\varphi}{dt}$ also contains component due to the change in σ (spreading), which is orthogonal to the fibre $\{\varphi\}$ and the phase space $M^{\sigma}_{3,3}$, and is equal to

$$\mathrm{Re}\left(\frac{d\varphi}{dt}, i\widehat{\frac{d\varphi}{d\sigma}}\right) = \frac{\sqrt{2}\hbar}{8\sigma^2 m}. \tag{44}$$

Calculation of the norm of $\frac{d\varphi}{dt} = \frac{i}{\hbar}\widehat{h}\varphi$ at $t = 0$ gives

$$\left\| \frac{d\varphi}{dt} \right\|^2 = \frac{\overline{E}^2}{\hbar^2} + \frac{\mathbf{v}_0^2}{4\sigma^2} + \frac{m^2\mathbf{w}^2\sigma^2}{\hbar^2} + \frac{\hbar^2}{32\sigma^4 m^2}, \tag{45}$$

which is the sum of squares of the found components. This completes a decomposition of the velocity of state at any point $\varphi_{\mathbf{a},\mathbf{p}} \in M^{\sigma}_{3,3}$.

For a closed system, the norm of $\frac{d\varphi}{dt} = \frac{i}{\hbar}\widehat{h}\varphi$ is preserved in time. For a system in a stationary state, this amounts to conservation of energy. In fact, in this case $\varphi_t(\mathbf{x}) = \psi(\mathbf{x})e^{-\frac{iEt}{\hbar}}$, which is a motion along the phase circle, and

$$\left\| \frac{d\varphi}{dt} \right\|^2 = \frac{E^2}{\hbar^2}. \tag{46}$$

For a closed system in any initial state, the norm of the phase component (expected energy) and orthogonal component (energy uncertainty) of the velocity $\frac{d\varphi}{dt}$ are both preserved.

In the linear potential approximation, valid in the considered case of small σ (the choice of σ is in our hands; the largest value of σ consistent with observations is related to the boundary between classical and quantum), the first term in (45) is the square of the term

$$\frac{1}{\hbar}\left(U + K + \frac{\hbar^2}{4m\sigma^2}\right), \tag{47}$$

where $U = V(\mathbf{x}_g)$ and $K = \frac{m\mathbf{v}_g^2}{2}$ are potential and kinetic energy of the packet considered as a particle with position $\mathbf{x}_g = \mathbf{x}_0 + \mathbf{v}_0 t + \frac{\mathbf{w}t^2}{2}$ and velocity $\mathbf{v}_g = \mathbf{v}_0 + \mathbf{w}t$. The last term in parentheses in (47) accounts for the difference in energy of the packets with the same U and K, but different values of σ. Up to a constant factor, this term equals the component of velocity due to spreading given by (44). With the unit of length 2σ given by Compton length, this term is equal to the rest energy mc^2 of the particle, making it possible to identify the mass with the speed of motion of state due to spreading.

From (41) and (42), and a simple consistency check showing that the rate of change of the projection in (41) is given by acceleration $\mathbf{w}$, one can see that the phase space components of the velocity of state $\frac{d\varphi}{dt} = -\frac{i}{\hbar}\widehat{h}\varphi$ assume correct classical values at any point $\varphi_{\mathbf{a},\mathbf{p}} \in M^{\sigma}_{3,3}$. This remains true for the time dependent potentials as well. The immediate consequence of this and the linear nature of the Schrödinger equation is that: *Under the Schrödinger evolution with the Hamiltonian $\widehat{h} = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{x}, t)$, the state constrained to $M^{\sigma}_{3,3} \subset CP^{L_2}$ moves like a point in the phase space representing a particle in Newtonian dynamics. More generally, Newtonian dynamics of n particles is the Schrödinger dynamics of n-particle quantum system whose state is constrained to the phase-space submanifold $M^{\sigma}_{3n,3n}$ of the projective space for $L_2(\mathbb{R}^3) \otimes \ ... \ \otimes L_2(\mathbb{R}^3)$, formed by tensor product states $\varphi_1 \otimes \ ... \ \otimes \varphi_n$ with φ_k of the form (38).*

Note again that the velocity and acceleration terms in (45) are orthogonal to the fibre $\{\varphi_{\mathbf{a},\mathbf{p}}\}$ of the fibration $\pi : S^{L_2} \longrightarrow CP^{L_2}$, showing that these Newtonian variables have to do with the motion in the projective space CP^{L_2}. The velocity of spreading is orthogonal to the fibre and to the phase space submanifold $M^{\sigma}_{3,3}$. The implication of this is that the "concentration" of state under the collapse has nothing to do with a motion in the classical space. This will be important when discussing collapse of the wave function under a measurement.

Note that the functional

$$S[\varphi] = \int \overline{\varphi}(\mathbf{x}) \left[i\hbar\frac{\partial}{\partial t} - \widehat{h} \right] \varphi(\mathbf{x}) d^3\mathbf{x}dt \tag{48}$$

with $\widehat{h} = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{x}, t)$ is the action functional for the Schrödinger equation. At the same time, for the states φ constrained to the manifold $M^{\sigma}_{3,3}$ this functional is equal to the classical action. Namely, for φ varying over the states $\varphi_{\mathbf{a},\mathbf{p}}$ of the form (38), the action $S[\varphi]$ is equal to

$$S = \int \left[\mathbf{p}\frac{d\mathbf{a}}{dt} - h(\mathbf{p}, \mathbf{a})\right] dt, \tag{49}$$

where $h(\mathbf{p}, \mathbf{a}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{a}) + C$ is the Hamiltonian function and the constant C is the "rest energy" term in (47). It follows that there exists a single action functional for the classical and quantum dynamics, which was perviously observed in a related context by John Klauder in [8].

7. Uniqueness of extension of Newtonian dynamics to CP^{L_2}

The velocity of state under the Schrödinger evolution with the Hamiltonian $\widehat{h} = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{x})$ was shown to contain for the states in $M^{\sigma}_{3,3}$ the classical velocity and acceleration (formulae (41) and (42)). This was used to establish that Newtonian dynamics of a particle is the Schrödinger dynamics of the system whose state is constrained to the classical phase space $M^{\sigma}_{3,3}$.

On the contrary, there exists a unique extension of the Newtonian dynamics formulated on the classical phase space $M^{\sigma}_{3,3}$ to a unitary dynamics in the Hilbert space $L_2(\mathbb{R}^3)$. More precisely, suppose that for any initial state $\varphi_{\mathbf{a},\mathbf{p}}$ of the form

$$\varphi_{\mathbf{a},\mathbf{p}}(\mathbf{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{3/4} e^{-\frac{(\mathbf{x}-\mathbf{a})^2}{4\sigma^2}} e^{i\frac{\mathbf{p}(\mathbf{x}-\mathbf{a})}{\hbar}} \tag{50}$$

there exists a path $\varphi = \varphi_t$ in $L_2(\mathbb{R}^3)$, passing at $t = 0$ through the point $\varphi_{\mathbf{a},\mathbf{p}}$, and such that (41) and (42) are satisfied. Suppose further that the evolution $\varphi = \varphi_t$ is unitary, so that, by Stone's theorem, $\frac{d\varphi}{dt} = -\frac{i}{\hbar}\widehat{H}\varphi$ for some self-adjoint operator $\widehat{H}$. It is claimed then that the operator $\widehat{H}$ is uniquely defined and is equal to $-\frac{\hbar^2}{2m}\Delta + V(\mathbf{x})$. In other words, the Schrödinger evolution is the only unitary evolution on $L_2(\mathbb{R}^3)$ for which the system constrained to the classical phase space $M^{\sigma}_{3,3}$ satisfies Newtonian equations of motion for the particle.

To prove, let us first verify that (41) and (42) imply the Ehrenfest theorem on states $\varphi \in M^{\sigma}_{3,3}$. As discussed, the Ehrenfest theorem can be written in the following form:

$$2\mathrm{Re}\left(\frac{d\varphi}{dt}, \widehat{x}\varphi\right) = \left(\varphi, \frac{\widehat{p}}{m}\varphi\right) \tag{51}$$

and

$$2\mathrm{Re}\left(\frac{d\varphi}{dt}, \widehat{p}\varphi\right) = \left(\varphi, -\nabla V(\mathbf{x})\varphi\right). \tag{52}$$

From (41) and (50) we have at $t = 0$,

$$\frac{v^{\alpha}}{2\sigma} = \mathrm{Re}\left(\frac{d\varphi}{dt}, -\widehat{\frac{\partial r}{\partial x^{\alpha}}} e^{i\theta}\right) = \frac{1}{\sigma}\mathrm{Re}\left(\frac{d\varphi}{dt}, (x-a)^{\alpha}\varphi\right). \tag{53}$$

Because of the unitary condition, we have $\mathrm{Re}\left(\frac{d\varphi}{dt}, \varphi\right) = 0$ and so (53) yields

$$2\mathrm{Re}\left(\frac{d\varphi}{dt}, x^{\alpha}\varphi\right) = v^{\alpha} = \frac{p^{\alpha}}{m}. \tag{54}$$

Together with $(\varphi, \widehat{p}\varphi) = (\varphi, \mathbf{p}\varphi) = \mathbf{p}$ this gives the first Ehrenfest theorem (51) on states $\varphi \in M^{\sigma}_{3,3}$.

Similarly, from (42), (43) and (50) we have at $t = 0$,

$$\frac{mw^\alpha\sigma}{\hbar} = \mathrm{Re}\left(\frac{d\varphi}{dt}, i\widehat{\frac{\partial\theta}{\partial p^\alpha}}\varphi\right) = \frac{\hbar}{\sigma}\mathrm{Re}\left(\frac{d\varphi}{dt}, \frac{i(x-a)^\alpha}{\hbar}\varphi\right), \tag{55}$$

with

$$mw^\alpha = -\left.\frac{\partial V(\mathbf{x})}{\partial x^\alpha}\right|_{\mathbf{x}=\mathbf{a}}. \tag{56}$$

On the other hand,

$$\widehat{p}\varphi = -i\hbar\nabla\varphi = -i\hbar\left(-\frac{\mathbf{x}-\mathbf{a}}{2\sigma^2} + \frac{i\mathbf{p}}{\hbar}\right)\varphi. \tag{57}$$

Again, from the unitary condition, we have $\mathrm{Re}\left(\frac{d\varphi}{dt}, \varphi\right) = 0$ and so we can rewrite (55) as

$$\frac{mw^\alpha\sigma}{\hbar} = \frac{\sigma}{\hbar}\mathrm{Re}\left(\frac{d\varphi}{dt}, \widehat{p}^\alpha\varphi\right), \tag{58}$$

or,

$$2\mathrm{Re}\left(\frac{d\varphi}{dt}, \widehat{p}^\alpha\varphi\right) = mw^\alpha. \tag{59}$$

From this and (56), we get the second Ehrenfest theorem (52) on states $\varphi \in M^\sigma_{3,3}$. Note that the components (53) and (55) are the real and imaginary parts of the classical space component of the velocity of state. In particular, the classical phase space submanifold inherits a complex structure from CP^{L_2}.

Now, from the derived Ehrenfest theorem and the Stone's theorem for a unitary evolution

$$\frac{d\varphi}{dt} = -\frac{i}{\hbar}\widehat{H}\varphi, \tag{60}$$

we get the following equations for the unknown self-adjoint operator $\widehat{H}$, valid for all functions φ in $M^\sigma_{3,3}$:

$$\left(\varphi, i[\widehat{H}, \widehat{x}]\varphi\right) = \frac{\hbar}{m}\left(\varphi, \widehat{p}\varphi\right) \tag{61}$$

and

$$\left(\varphi, i[\widehat{H}, \widehat{p}]\varphi\right) = \hbar\left(\varphi, -\nabla V(\mathbf{x})\varphi\right). \tag{62}$$

Because $M^\sigma_{3,3}$ is complete in $L_2(\mathbb{R}^3)$, there exists a unique linear extension of the operators $\widehat{x}$, $\widehat{p}$ and $-\nabla V(\mathbf{x})$ from $M^\sigma_{3,3}$ onto (a dense subset of) $L_2(\mathbb{R}^3)$. Likewise, for a given operator $\widehat{H}$, there exists a unique extension of the quadratic forms in the equations (61) and (62) from $M^\sigma_{3,3}$ to (a dense subset of) $L_2(\mathbb{R}^3)$. The resulting equations define $\widehat{H}$ uniquely. That is, there exists a unique operator $\widehat{H}$ for which

$$\left(f, i[\widehat{H}, \widehat{x}]f\right) = \frac{\hbar}{m}\left(f, \widehat{p}f\right) \tag{63}$$

and

$$\left(f, i[\widehat{H}, \widehat{p}]f\right) = \hbar\left(f, -\nabla V(\mathbf{x})f\right) \tag{64}$$

for all functions f in the dense subset D of $L_2(\mathbb{R}^3)$, which is the common domain of all involved operators. In fact, by choosing an orthonormal basis $\{e_j\}$ in D and considering (63), (64) on

functions $f = e_k + e_l$ and $f = e_k + ie_l$ we conclude that all matrix elements of the operators on the left and right of the equations (63) and (64) must be equal. So the equations can be written in the operator form

$$i[\widehat{H}, \widehat{x}] = \frac{\hbar}{m}\widehat{p} \tag{65}$$

and

$$i[\widehat{H}, \widehat{p}] = -\hbar\nabla V(\mathbf{x}). \tag{66}$$

From (65) and (66), it then follows that, up to an irrelevant constant, $\widehat{H} = \frac{\widehat{p}^2}{2m} + V(\mathbf{x})$.

Because (41) and (42) remain true for the potentials that depend on time and the equations used to obtain the result were considered at a fixed moment of time, the derivation remains valid for the time-dependent potentials $V(\mathbf{x}, t)$ as well. Generalization to the case of n interacting distinguishable particles described by tensor product of states (50) is straightforward and leads to the Hamiltonian $\widehat{H} = \sum_k \frac{\widehat{p}_k^2}{2m_k} + V(\mathbf{x}_1, ..., \mathbf{x}_n)$.

By (39), a point $\varphi_{\mathbf{a},\mathbf{p}}$ in the classical phase space $M^\sigma_{3,3}$ defines the initial position and velocity of the particle in $\mathbb{R}^3$. The solution of Newton's equations with this initial condition defines a unique classical path $(\mathbf{a}_t, \mathbf{p}_t)$ of the particle. Let's call the (non-linear) operator $U_c(t, 0) : M^\sigma_{3,3} \longrightarrow M^\sigma_{3,3}$, given by

$$U_c(t, 0)\left(\Omega(\mathbf{a}_0, \mathbf{p}_0)\right) = \Omega(\mathbf{a}_t, \mathbf{p}_t) \tag{67}$$

with Ω given by (39), the *Newtonian evolution operator*. It was shown that there exists a unique unitary evolution operator $U_q(t, 0) : L_2(\mathbb{R}^3) \longrightarrow L_2(\mathbb{R}^3)$, such that $U_q(t, 0)\varphi_{\mathbf{a},\mathbf{p}} = \varphi_t$ satisfies (41) and (42) for all $\varphi_{\mathbf{a},\mathbf{p}} \in M^\sigma_{3,3}$. It turned out to be the usual Schrödinger evolution operator. The domain $L_2(\mathbb{R}^3)$ of this operator is the (closure of the) linear envelop of the domain $M^\sigma_{3,3}$ of the Newtonian evolution operator. The component of the velocity vector field $\frac{dU_q(t,0)\varphi_{\mathbf{a},\mathbf{p}}}{dt}$ tangent to $M^\sigma_{3,3}$ gives the usual Newtonian velocity and acceleration of the particle. The meaning of the additional components of $\frac{d\varphi}{dt}$ was revealed in (45).

The obtained embedding of the classical phase space into the space of states complemented by existence and uniqueness of extension of Newtonian to Schrödinger evolution signifies that Newtonian dynamics found its full-fledged realization within the realm of quantum physics governed by the Schrödinger equation. This realization is valid independently of whether it is taken to mean the actual physical embedding or only as a mathematical representation.

8. The Born rule and the normal probability distribution

The isometric embedding of the classical space M^σ_3 into the space of states $L_2(\mathbb{R}^3)$ results in a relationship between distances in $\mathbb{R}^3$ and in the projective space CP^{L_2}. The distance between two points $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ is $\|\mathbf{a} - \mathbf{b}\|_{\mathbb{R}^3}$. Under the embedding of the classical space into the space of states, the variable $\mathbf{a}$ is represented by the state $\tilde{\delta}^3_{\mathbf{a}}$. The set of states $\tilde{\delta}^3_{\mathbf{a}}$ form a submanifold M^σ_3 in the Hilbert spaces of states $L_2(\mathbb{R}^3)$, which is "twisted" in $L_2(\mathbb{R}^3)$. It belongs to the sphere S^{L_2} and spans all dimensions of $L_2(\mathbb{R}^3)$. The distance between the states $\tilde{\delta}^3_{\mathbf{a}}$, $\tilde{\delta}^3_{\mathbf{b}}$ on the sphere S^{L_2} or in the projective space CP^{L_2} is not equal to $\|\mathbf{a} - \mathbf{b}\|_{\mathbb{R}^3}$. In fact, the former distance measures length of a geodesic between the states while the latter is obtained using the same metric on the space of states, but applied along a geodesic in the twisted manifold M^σ_3. The precise relation between the two distances is given by

$$e^{-\frac{(\mathbf{a}-\mathbf{b})^2}{4\sigma^2}} = \cos^2\theta(\tilde{\delta}^3_{\mathbf{a}}, \tilde{\delta}^3_{\mathbf{b}}), \tag{68}$$

where θ is the Fubini-Study distance between states in CP^{L_2}. The distance θ in the projective space of states CP^{L_2} appears here for a good reason: the fibres of the fibration $\pi : S^{L_2} \longrightarrow CP^{L_2}$

through the points of the classical space M_3^σ are orthogonal to this space. This is why the distance in M_3^σ can be expressed in terms of the distance in CP^{L_2}. Despite the non-trivial geometry contained in (68), the formula itself is easy to derive. The left hand side is the result of integration in $|(\tilde{\delta}^3_{\mathbf{a}}, \tilde{\delta}^3_{\mathbf{b}})|^2$. On the other hand, the same expression is equal to the right side of (68) by definition of the Fubini-Study metric.

The relation (68) has an immediate implication onto the form of probability distributions of random variables over M_3^σ and CP^{L_2}. In particular, consider a random variable ψ over CP^{L_2}. Suppose that the restricted random variable $\widetilde{\delta}^3_{\mathbf{a}}$, equivalently, $\mathbf{a}$, defined over $M_3^\sigma = \mathbb{R}^3$ is distributed normally on $\mathbb{R}^3$. Then the direction-independent probability distribution of ψ satisfies the Born rule for the probability of transition between arbitrary states. The opposite is also true. In other words, we claim that: *The normal distribution law on M_3^σ implies the Born rule on CP^{L_2}. Conversely, the Born rule on the space of states implies the normal distribution law on M_3^σ.*

The fact that the Born rule implies the normal distribution on M_3^σ is straightforward. According to the Born rule, the probability density $f(\mathbf{b})$ to find the particle in a state $\widetilde{\delta}^3_{\mathbf{a}}$ at a point $\mathbf{b}$ is equal to

$$|\widetilde{\delta}^3_{\mathbf{a}}(\mathbf{b})|^2 = |(\widetilde{\delta}^3_{\mathbf{a}}, \delta^3_{\mathbf{b}})|^2 = \left(\frac{1}{2\pi\sigma^2}\right)^{3/2} e^{-\frac{(\mathbf{a}-\mathbf{b})^2}{2\sigma^2}} \equiv f_{\mathbf{a},\sigma}(\mathbf{b}), \tag{69}$$

which is the normal distribution function. It follows that on the elements of M_3^σ, the Born rule *is* the rule of normal distribution.

The Born rule on M_3^σ can be also written in term of the probability $P(\tilde{\delta}^3_{\mathbf{a}}, \tilde{\delta}^3_{\mathbf{b}})$ of transition between the states $\tilde{\delta}^3_{\mathbf{a}}, \tilde{\delta}^3_{\mathbf{b}}$ in M_3^σ:

$$P(\tilde{\delta}^3_{\mathbf{a}}, \tilde{\delta}^3_{\mathbf{b}}) = |(\tilde{\delta}^3_{\mathbf{a}}, \tilde{\delta}^3_{\mathbf{b}})|^2. \tag{70}$$

Assuming $\widetilde{\delta}^3_{\mathbf{b}}$ is sufficiently sharp, the formulas (69) and (70) mean the same thing. In fact,

$$|(\widetilde{\delta}^3_{\mathbf{a}}, \widetilde{\delta}^3_{\mathbf{b}})|^2 = f_{\mathbf{a},\sqrt{2}\sigma}(\mathbf{b})(\Delta x)^3, \tag{71}$$

where $f_{\mathbf{a},\sqrt{2}\sigma}$ is the normal distribution function with variance $\sqrt{2}\sigma$ and $\Delta x = \sqrt{4\pi\sigma^2}$. This relates the probability in (70) to the normal probability density in (69) and identifies $P(\tilde{\delta}^3_{\mathbf{a}}, \tilde{\delta}^3_{\mathbf{b}})$ with the probability of finding the macroscopic particle near the point $\mathbf{b}$.

Conversely, suppose we have a rule for probability of transition between states in CP^{L_2} which gives the normal distribution law for the states in M_3^σ and depends only on the distance between states. Let's show that this must be the Born rule. In fact, the Fubini-Study distance between the states $\tilde{\delta}^3_{\mathbf{a}}$, $\tilde{\delta}^3_{\mathbf{b}}$ takes on all values from 0 to $\pi/2$, which is the largest possible distance between points in CP^{L_2}. By assumption, the probability $P(\varphi, \psi)$ of transition between any states φ and ψ depends only on the Fubini-Study distance $\theta(\pi(\varphi), \pi(\psi))$ between the states. Given arbitrary states $\varphi, \psi \in S^{L_2}$, let then $\tilde{\delta}^3_{\mathbf{a}}$, $\tilde{\delta}^3_{\mathbf{b}}$ be two states in M^3_σ, such that

$$\theta(\pi(\varphi), \pi(\psi)) = \theta(\tilde{\delta}^3_{\mathbf{a}}, \tilde{\delta}^3_{\mathbf{b}}). \tag{72}$$

It then follows that

$$P(\varphi, \psi) = P(\tilde{\delta}^3_{\mathbf{a}}, \tilde{\delta}^3_{\mathbf{b}}) = \cos^2\theta(\tilde{\delta}^3_{\mathbf{a}}, \tilde{\delta}^3_{\mathbf{b}}) = \cos^2\theta(\pi(\varphi), \pi(\psi)), \tag{73}$$

which yields the Born rule for arbitrary states and proves the claim.

9. Measurements on macroscopic and microscopic particles

We are now in a position to compare the process of measurement in the classical and quantum physics. First of all, the classical space and phase space are now submanifolds in the Hilbert space of states. This allows us to use the same language when analyzing both types of measurement. Second, the Newtonian dynamics is now a restriction of the Schrödinger dynamics to the classical phase space submanifold. Conversely, the Schrödinger dynamics is a unique unitary extension of the Newtonian dynamics from the classical phase space to the Hilbert space. This allows us to begin with a model of measurement satisfying Newton laws and extend it to a model consistent with the rules of quantum mechanics. Finally, the normal probability law is the restriction of the Born rule to the classical space submanifold. Conversely, the Born rule is the unique isotropic extension of the normal probability law from the classical space to the space of states. In particular, a classical model of measurement with a normal distribution of the measured quantity should lead us to a model consistent with the Schrödinger dynamics and the Born rule.

Measurements performed on a macroscopic particle satisfy generically the normal distribution law for the measured observable. This is consistent with the central limit theorem and indicates that the specific way in which the observable was measured is not important. To be specific, consider measurements of position of a particle. A common way of finding the position of a macroscopic particle is to expose it to light of sufficiently short wavelength and to observe the scattered photons. Due to the unknown path of the incident photons, multiple scattering events on the particle, random change in position of the particle, etc., the process of observation can be described by the diffusion equation with the observed position of the particle experiencing Brownian motion from an initial point during the time of observation. This results in the normal distribution of observed position of the particle.

The ability to describe the measurement of the position of a macroscopic particle as a diffusion seems to be a general feature of measurements in the macro-world, independent of a particular measurement set-up. The averaging process making the central limit theorem applicable and leading to the normal distribution of the position random variable can be seen, for example, as the result of random hits experienced by the particle from the surrounding particles participating in the measurement. These random hits are equally likely to come from any direction, independent of the initial position of the particle, leading to Brownian motion and the validity of the diffusion equation for the probability density of the position random variable for the particle.

It is claimed now that at any time t, the initial state ψ of a microscopic particle undergoing a similar measurement of position is equally likely to shift in any direction in the tangent space to the appropriate projective space of states. In proving this result, we will use an example of a particle exposed to a stream of photons of sufficiently high frequency and number density. The scattered photons are then observed to determine the position of the particle. The field of photons in the experiment will be treated classically, as a fluctuating potential in a region surrounding the source. Despite the classical treatment of the field and other assumptions made about the potential, a more general proof in section 11 will confirm that the result derived here is general.

Recall first that the space $M^{\sigma}_{3,3}$ is complete in $L_2(\mathbb{R}^3)$. Consider the subset of $M^{\sigma}_{3,3}$ formed by the states

$$\varphi_{\mathbf{mn}}(\mathbf{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{3/4} e^{-\frac{(\mathbf{x}-\alpha\mathbf{n})^2}{4\sigma^2}} e^{i\frac{\beta\mathbf{m}\mathbf{x}}{\hbar}}, \tag{74}$$

where $\alpha = \sqrt{2\pi}\sigma$, $\beta = \frac{h}{\sqrt{2\pi}\sigma}$ and $\mathbf{m}, \mathbf{n}$ take values on the lattice $\mathbb{Z}^3 \times \mathbb{Z}^3$ of points with integer coordinates in $\mathbb{R}^3 \times \mathbb{R}^3$. The set of functions (74) is known to be also complete in $L_2(\mathbb{R}^3)$. Any state in $L_2(\mathbb{R}^3)$ can be then represented by a linear combination of states $\varphi_{\mathbf{mn}}$. (For $\alpha\beta < h$ the system of functions $\varphi_{\mathbf{mn}}$ is called the Gabor or Weil-Heisenberg frame.) In particular, the

initial state ψ of the particle can be represented by a sum

$$\psi = \sum_{\mathbf{m},\mathbf{n}} C_{\mathbf{mn}} \varphi_{\mathbf{mn}}. \tag{75}$$

The set M_3^σ is also complete in $L_2(\mathbb{R}^3)$. Here too there exist countable subsets of M_3^σ that are complete in $L_2(\mathbb{R}^3)$. Moreover, an arbitrary initial state ψ in $L_2(\mathbb{R}^3)$ can be approximated as well as necessary by a finite discrete sum

$$\psi \approx \sum_{\mathbf{n}} C_{\mathbf{n}} \widetilde{\delta}^3_{\mathbf{a}-\gamma\mathbf{n}}, \tag{76}$$

where $\mathbf{a}$ is arbitrary, $\mathbf{n} \in \mathbb{Z}^3$, and the value of $\gamma > 0$ together with the number of terms in the sum depend on ψ and the needed approximation. Taking γ sufficiently small, let us partition the space $\mathbb{R}^3$ into the cubical cells of edge γ centered at the lattice points $\mathbf{a} - \gamma\mathbf{n}$ and consider the indicator function $1_{\mathbf{n}}$ for each cell. Assume first that the potential $\widehat{V}$ acts on the entire space $\mathbb{R}^3$. The potential can be written as a sum $\sum_{\mathbf{n}} 1_{\mathbf{n}} \widehat{V}_{\mathbf{n}}$. The components $\widehat{V}_{\mathbf{n}}$ for different $\mathbf{n}$ can be assumed to be independent, identically distributed random variables. In the case of position measurement by scattering photons off the particle, the components $\widehat{V}_{\mathbf{n}}$ can be associated with a single photon at time t.

For simplicity, let us neglect the kinetic energy term in the Hamiltonian $\widehat{h}$. We will see when the resulting approximation is valid later. Let us denote the solution of the Shrödinger equation with the initial state ψ by $\Psi(t)$ and set $\Psi(t) = e^{-\frac{i\overline{V}t}{\hbar}}\psi(t)$, where $\overline{V} = (\widehat{V}\psi, \psi)$ and $\psi(0) = \psi$. We then have at $t = 0$:

$$\frac{d\psi}{dt} = -\frac{i}{\hbar}\widehat{V}_\perp \psi, \tag{77}$$

where $\widehat{V}_\perp = \widehat{V} - \overline{V}$, as before. This equation gives the velocity of the state $\Psi(t)$ in the projective space CP^{L_2} at $t = 0$. To prove that under the action of $\widehat{V}_\perp$ all directions of velocity of state in $T_{\{\psi\}}CP^{L_2}$ are equally likely, consider the components of the displacement $\delta\psi = \frac{d\psi}{dt}\tau$ of state during a short time interval τ in the basis $-i\widetilde{\delta}^3_{\mathbf{m}} \equiv -i\widetilde{\delta}^3_{\mathbf{a}-\gamma\mathbf{m}}$

$$\left(\delta\psi, -i\widetilde{\delta}^3_{\mathbf{m}}\right) = \frac{\tau}{\hbar}(\widehat{V}_\perp\psi, \widetilde{\delta}^3_{\mathbf{m}}). \tag{78}$$

For any given potential and a given ψ, the form in (78) is a function of the distance between the points ψ and $\widetilde{\delta}^3_{\mathbf{m}}$ in the Fubini-Study metric and, possibly, of the unit vector $\eta \in T_{\{\psi\}}CP^{L_2}$, tangent to the geodesic from $\{\psi\}$ to $\widetilde{\delta}^3_{\mathbf{m}}$ in CP^{L_2}. We have $\widehat{V} = \sum_{\mathbf{n}} 1_{\mathbf{n}}\widehat{V}_{\mathbf{n}}$, where each $\widehat{V}_{\mathbf{n}}$ is a random variable. Accordingly, (78) defines a random variable for each $\mathbf{m}$. On the level surfaces $|\psi| = \lambda$ of $|\psi|$ the value of $|(\psi, \widetilde{\delta}^3_{\mathbf{m}})|$ is the same and the independence of the distribution of the variables in (78) on the direction is particularly clear. More generally, assuming $|(\psi, \widetilde{\delta}^3_{\mathbf{m}})| \neq 0$, let us divide the random variables in (78) by $|(\psi, \widetilde{\delta}^3_{\mathbf{m}})|$. This yields a new set of random variables

$$\frac{\frac{\tau}{\hbar}(\widehat{V}_\perp\psi, \widetilde{\delta}^3_{\mathbf{m}})}{|(\psi, \widetilde{\delta}^3_{\mathbf{m}})|}. \tag{79}$$

Because $|(\psi, \widetilde{\delta}^3_{\mathbf{m}})|$ depends only on the distance between $\{\psi\}$ and $\widetilde{\delta}^3_{\mathbf{m}}$, the probability distributions of the random variables given by (78) and (79) are either both dependent or both independent of η. Provided the potential does not change much within each cell, the expression (79) is equal up to a constant phase factor to

$$\tfrac{\tau}{\hbar}(V_{\mathbf{m}} - \overline{V}). \tag{80}$$

From the decomposition (76), the near-orthogonality of the functions $\widetilde{\delta}^3_{\mathbf{m}}$ and the definition of $\overline{V}$, we have

$$\overline{V} = \sum_{\mathbf{n}} V_{\mathbf{n}} |C_{\mathbf{n}}|^2. \tag{81}$$

Because $\sum |C_{\mathbf{n}}|^2 = 1$, the mean value of the random variable in (80) is zero:

$$E(V_{\mathbf{m}} - \overline{V}) = E(V_{\mathbf{m}}) - E(V_{\mathbf{m}}) \sum_{\mathbf{n}} |C_{\mathbf{n}}|^2 = 0. \tag{82}$$

As discussed, the random variables $V_{\mathbf{m}}$ in different cells, i.e., for different values of $\mathbf{m}$ can be considered independent and identically distributed. It follows that the probability distributions of the random variables $V_{\mathbf{m}} - \overline{V}$ have a zero mean and are identical for all values of $\mathbf{m}$. With the help of the central limit theorem, one can also claim that these distributions are normal. So, disregarding the phase factors, the random vector with components (79) has an isotropic multivariate Gaussian distribution (the covariance matrix is proportional to the identity).

Now, a complex random vector $\delta\psi$ is called *circularly symmetric* if for any constant phase α the distribution of $e^{i\alpha}\delta\psi$ is equal to the distribution of $\delta\psi$. Because $V_\perp$ depends only on the value of $\mathbf{x}$ but not on the constant phase α, we have $e^{i\alpha}\delta\psi = -\frac{i}{\hbar}\widehat{V}_\perp e^{i\alpha}\psi\tau$. It follows that the frequency of a particular value of $\delta\psi$ is the same as the frequency of the value of $e^{i\alpha}\delta\psi$. We conclude that the random variable $\delta\psi$ can be assumed to take values in the tangent space $T_{\{\psi\}}CP^{L_2}$ and is equally likely to point in any direction in $T_{\{\psi\}}CP^{L_2}$ at $t = 0$. To make this result valid at an arbitrary moment of time and independent of the direction at previous times, we may assume that the distribution of potentials is time-independent (stationary) and that potentials at different moments of time are independent random variables.

Note that if $\psi(\mathbf{m}) = 0$, then the division in (79) is not valid. However, one could make $\psi(\mathbf{m})$ small, different than zero and take the limit. In this case, one would see that $\delta\psi$ is as likely to point in the direction tangent to the geodesics from ψ to $\widetilde{\delta}^3_{\mathbf{m}}$ as it is in any other direction in $T_{\{\psi\}}CP^{L_2}$. The fact that the said component goes to zero together with $\psi(\mathbf{m})$ simply means that ψ and $\delta\psi$ are orthogonal to $\widetilde{\delta}^3_{\mathbf{m}}$. In particular, such state can never reach $\widetilde{\delta}^3_{\mathbf{m}}$ under a measurement.

Let us check that the assumptions used in the derivation of the isotropy of the distribution of the displacement random variable $\delta\psi$ are realistic. Suppose for example that the position of an electron is measured by subjecting it to a stream of photons. Assume first that the initial state of the electron belongs to the classical phase space submanifold $M^\sigma_{3,3}$ of the space of states. Suppose also that the wave length of the photons is of the order of $1nm = 10^{-9}m$ (x-rays) or larger. Let us estimate the terms of the decomposition (45) for the velocity of state of the electron. From the Compton scattering formula, we have for the difference in wave length of the incoming and scattered photons

$$\lambda_f - \lambda_i = \frac{h}{mc}(1 - \cos\theta) \sim 10^{-12}m. \tag{83}$$

The transferred energy is then

$$\Delta E = \frac{hc}{\lambda_i} - \frac{hc}{\lambda_f} \sim 10^{-20}J. \tag{84}$$

With the electron initially at rest, we have for the speed v acquired during the interaction

$$\frac{mv^2}{2} \sim 10^{-20}J, \text{ or, } v \sim 10^5 m/s. \tag{85}$$

The accuracy of position measurement is limited by the wave length. Setting $\sigma = \lambda \sim 10^{-9}m$, we have for the classical velocity component of $\frac{d\varphi}{dt}$, given by the second term in (45)

$$\frac{v}{2\sigma} \sim \frac{10^5}{10^{-9}} = 10^{14}s^{-1}. \tag{86}$$

Estimating the time of interaction τ by $\lambda/c \sim 10^{-17}s$, we have for the classical acceleration component, given by the third term in (45):

$$\frac{mw\sigma}{\hbar} = 10^{17}s^{-1}. \tag{87}$$

For the spreading component, given by the last term in (45), we obtain

$$\frac{\hbar}{4\sqrt{2}\sigma^2 m} \sim 10^{13}s^{-1}. \tag{88}$$

In the estimate, the acceleration term is the largest of the three. Also, the resolution parameter σ in the non-relativistic position measurement experiments is typically much larger than the used value of $1nm$. With the increase in σ, the spreading term decreases as $1/\sigma^2$, the velocity term decreases as $1/\sigma$ and the acceleration term increases linearly with σ. In particular, for the scattering of visible light off an electron, we have $\lambda \sim 10^{-5}m$, which gives the acceleration term of the order of $10^{16}s^{-1}$, velocity term $\sim 10^{10}s^{-1}$ and the spreading term $\sim 10^7 s^{-1}$. Furthermore, with an increase in the mass m, the value of the spreading and velocity terms further decrease, while the acceleration term remains the same, showing that it is by far the dominant term under these conditions.

Let us now write an arbitrary initial state ψ as a superposition (75) of states in $M^{\sigma}_{3,3}$. Then the variation $\delta\psi = \frac{d\psi}{dt}\tau$ can be also written as a series in functions from $M^{\sigma}_{3,3}$, so that each term of the series is a constant times a function in $M^{\sigma}_{3,3}$. The initial speed v of each component function in $M^{\sigma}_{3,3}$ is limited by the speed of light c. If v is of the same order as c, then the velocity and acceleration terms in the component function are of the same order. However, given the non-relativistic character of the problem, the major terms in the series correspond to $v \ll c$. The spreading term in each term of the series is the same and is much smaller than the acceleration term. Therefore, assuming as before that the terms of the series are near-orthogonal, we can neglect the velocity and spreading parts in each term, which amounts to keeping only the potential term in the Hamiltonian. In particular, the motion of the state in these conditions amounts to a jiggling of the wave packet without much spreading or displacement.

Let us check now that under reasonably general measurement conditions, the periods of a free evolution of the electron state can be neglected. In other words, interaction with the field is happening continuously in time. From the number density of photons, we can estimate the number of photons in one cubic meter of space by $N \approx 2.02 \times 10^7 T^3$ and the average energy of a photon by $2.7k_BT$, where k_B is the Boltzmann constant and T is temperature. For instance, taking $T \sim 500K$, we obtain $N \sim 10^{15}$. The photon with the average energy at this temperature has the wave length $\lambda \sim 10^{-5}m$. Under these conditions, at any time t there is about one photon per cube of the volume λ^3. So, at any t, each $M^{\sigma}_{3,3}$ component of ψ experiences the potential of a photon passing by. Given these conditions, neglecting the free evolution of the electron state is a reasonable approximation.

Despite being heuristic, these estimates demonstrate that during the type of measurement considered in this section, the potential term is the main term in the Hamiltonian responsible for the dynamics of the particle under the Schrödinger evolution. On the other hand, if the number density of photons is significantly lower, the evolution will consist of a free Schödinger evolution

combined with the periods when the state is driven by the potential term alone. We will return to this issue in section 11, where an alternative approach to the problem will be discussed.

When measuring the position of a macroscopic particle, the observed particle is exposed to a random potential that is responsible for the normal distribution of the position random variable. We now see that the state of a microscopic particle undergoing a similar measurement and exposed to the same potential will experience a random motion on the space of states CP^{L_2}, such that any direction $\delta\psi$ of displacement of the state is equally likely. From the section 8 we know that the normal probability distribution of position on M_3^σ and direction-independence of the distribution of states on CP^{L_2} result in the Born rule for the probability of transition between states. That is, under a random potential produced by the measuring device the state ψ of the measured microscopic particle performs a random walk on the sphere of states and the probability for the state of reaching a neighborhood of any point φ on the sphere is given by the Born rule: $P(\varphi,\psi)=|(\varphi,\psi)|^2$.

Given the lack of Lebesgue measure on an infinite-dimensional Hilbert space, one may wonder how the state would have any chance of reaching a neighborhood of a given point in CP^{L_2}. However, a realistic measuring device occupies a finite volume in the classical space. So the potential created by it can only affect a bounded region Q in $\mathbb{R}^3$. The initial state ψ of the particle can be split onto the state $\psi_Q=\psi|_Q$ with support in Q (restriction of ψ to Q) and the leftover state $\chi=\psi-\psi_Q$. The state χ is not going to change under the potential and will not participate in the measurement (the probability for it of reaching a detector in Q is zero). By (76), the state ψ_Q is a finite linear combination of the states $\widetilde{\delta}^3_{\mathbf{n}}$. Furthermore, under the motion in the random potential $\widehat{V}$ described by equation (77), the state will continue to stay in the finite-dimensional linear envelop L_Q of the states $\widetilde{\delta}^3_{\mathbf{n}}$ centered at the points $\mathbf{a}-\gamma\mathbf{n}$ in Q. In particular, the Lebesgue volume of a ball of a positive radius in L_Q exists and is positive. It follows that the state ψ_Q has a non-vanishing probability of reaching a neighborhood of the state $\widetilde{\delta}^3_{\mathbf{a}}$ and the relative probabilities of reaching neighborhoods of states $\widetilde{\delta}^3_{\mathbf{a}}$ for different points $\mathbf{a}$ are given by the Born rule.

10. The Born rule for measurements of spin

To get another insight into the origin of the Born rule, let us return to the space $\mathbb{C}^2$ of spin states of a spin 1/2 particle. To obtain the corresponding projective space of states CP^1, consider the complex lines $\{\varphi\}$ formed for a given $\varphi=\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}$ by the vectors $\lambda\varphi$, $\lambda\in\mathbb{C}$. Provided $\varphi_1\neq 0$, there is a unique point of intersection of the line with the affine plane of vectors $\begin{bmatrix}1\\ \xi\end{bmatrix}, \xi\in\mathbb{C}$ in $\mathbb{C}^2$. Namely, by setting

$$\lambda\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}=\begin{bmatrix}1\\ \xi\end{bmatrix}, \tag{89}$$

one obtains

$$\xi=\frac{\varphi_2}{\varphi_1}. \tag{90}$$

The map $u=\{\varphi\}\longrightarrow\xi$ provides a coordinate chart on CP^1 that identifies CP^1 without a point (the complex line through $\begin{bmatrix}0\\ 1\end{bmatrix}$) with the set $\mathbb{C}$ of complex numbers. Under the isomorphism $\widehat{\omega}$ in (26), the vectors $\begin{bmatrix}1\\ \xi\end{bmatrix}$ form an affine subspace in the Lie algebra $su(2)$. The algebra $su(2)$ with the Killing metric is the Euclidean space $\mathbb{R}^3$ of vectors $\mathbf{x}=\sum_k x^k i\widehat{\sigma}_k$. The stereographic projection then identifies the unit sphere S^2 at the origin of $\mathbb{R}^3$ with the above plane $\mathbb{C}$ plus a point, i.e., with CP^1 itself. Moreover, the usual metric on S^2 induced by its embedding into $\mathbb{R}^3$

is the Fubini-Study metric on CP^1. The relationship of the coordinate ξ in the plane $\mathbb{C}$ with coordinates $(x^1 = x, x^2 = y, x^3 = z)$ of the corresponding point on the sphere S^2 is given by

$$\xi = \frac{x + iy}{1 - z}. \tag{91}$$

Solving this for x, y and z and using (90), one obtains:

$$\begin{aligned} x &= \varphi_1\overline{\varphi}_2 + \overline{\varphi}_1\varphi_2, & (92)\\ y &= i(\varphi_1\overline{\varphi}_2 - \overline{\varphi}_1\varphi_2), & (93)\\ z &= \varphi_2\overline{\varphi}_2 - \varphi_1\overline{\varphi}_1. & (94) \end{aligned}$$

The resulting map $\pi : S^3 \longrightarrow S^2$ given by $\pi(\varphi_1, \varphi_2) = (x, y, z)$ is a coordinate form of the bundle projection from the sphere of unit states in $\mathbb{C}^2$ onto the space CP^1 of physical states. The map π relates the spaces of representation of the groups $SU(2)$ and $SO(3)$ and maps the spin state of a particle to a vector in $\mathbb{R}^3$.

Suppose that the z-component of spin of a particle in the initial state φ is measured, causing a random motion of state on S^3 (see the previous section). Suppose for simplicity that the change in the z coordinate of the state at each step of this random motion is $\pm d$ for some positive $d \ll \hbar$, with the positive and negative values being equally likely (isotropy). Then the Born rule for the state follows. In fact, from (94) it follows that

$$|\varphi_1|^2 = \frac{1 + z}{2}, \text{ and } |\varphi_2|^2 = \frac{1 - z}{2}. \tag{95}$$

The gambler's ruin mechanism tells us now that the probability P_2 for the state φ to end at the state $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $(z = -1)$, as a result of the described random motion is equal to

$$P_2 = \frac{\text{number of steps from } z \text{ to } -1}{\text{number of steps from } -1 \text{ to } 1} = \frac{1 - z}{(1 - z) + (1 + z)} = \frac{1 - z}{2} = |\varphi_2|^2. \tag{96}$$

Similarly, the probability P_1 that φ will end at $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is given by $P_1 = |\varphi_1|^2$.

11. The motion of state under measurement

Let us now look into details of the stochastic motion of state under a measurement. Note that in the non-relativistic quantum mechanics, the particle, and therefore its state in a single particle Hilbert space, cannot disappear or get created. The unitary property of evolution means that the state can only move along the unit sphere in the space of states $L_2(\mathbb{R}^3)$. To express this conservation of states in the case of observation of position of the particle, consider the density of states functional $\rho_t[\varphi; \psi]$. Here we begin with an ensemble of particles whose initial state belongs to a neighborhood of the state ψ on the sphere of states $S^{L_Q} \subset L_Q$. The functional $\rho_t[\varphi; \psi]$ measures the number of states that by the time t belong to a neighborhood of a state $\varphi \subset S^{L_Q}$. Under the isometric embedding $\omega : \mathbb{R}^3 \longrightarrow M_3^\sigma \subset L_2(\mathbb{R}^3)$, the states in M_3^σ are identified with positions of particles. So the density of states functional $\rho_t[\varphi; \psi]$ must be an extension of the usual density of particles $\rho_t(\mathbf{a}; \mathbf{b})$ with initial position $\mathbf{b}$ in $\mathbb{R}^3$. In other words, we must have $\rho_t(\mathbf{a}; \mathbf{b}) = \rho_t[\tilde{\delta}^3_{\mathbf{a}}; \tilde{\delta}^3_{\mathbf{b}}]$.

In the case of macroscopic particles, the conservation of the number of particles is expressed in differential form by the continuity equation. For instance, if $\rho_t(\mathbf{a}; \mathbf{b})$ is the density at a point

$\mathbf{a} \in \mathbb{R}^3$ of an ensemble of Brownian particles with initial position near $\mathbf{b}$ and $\mathbf{j}_t(\mathbf{a};\mathbf{b})$ is the current density of the particles at $\mathbf{a}$, then

$$\frac{\partial \rho_t(\mathbf{a};\mathbf{b})}{\partial t} + \nabla \mathbf{j}_t(\mathbf{a};\mathbf{b}) = 0. \tag{97}$$

We will assume that $\rho_t(\mathbf{a};\mathbf{b})$ and $\mathbf{j}_t(\mathbf{a};\mathbf{b})$ are normalized per one particle, i.e., the densities are divided by the number of particles. In this case, the particle density and the probability density can be identified.

The conservation of states of an ensemble of microscopic particles is expressed by the continuity equation that follows from the Schrödinger dynamics. This is the same equation (97) with

$$\rho_t = |\psi|^2, \quad \text{and} \quad \mathbf{j}_t = \frac{i\hbar}{2m}(\psi \nabla \overline{\psi} - \overline{\psi} \nabla \psi). \tag{98}$$

For the states $\psi \in M^{\sigma}_{3,3}$ we obtain

$$\mathbf{j}_t = \frac{\mathbf{p}}{m}|\psi|^2 = \mathbf{v}\rho_t. \tag{99}$$

Because the restriction of Schrödinger evolution to $M^{\sigma}_{3,3}$ is the corresponding Newtonian evolution, the function ρ_t in (99) must be the density of particles, denoted earlier by $\rho_t(\mathbf{a};\mathbf{b})$. Once again, it gives the number of particles that start on a neighborhood of $\mathbf{b}$ and by the time t reach a neighborhood of $\mathbf{a}$. The relation $\rho_t(\mathbf{a};\mathbf{b}) = \rho_t[\tilde{\delta}^3_{\mathbf{a}};\tilde{\delta}^3_{\mathbf{b}}]$ tells us that ρ_t in (98) must be then the density of states $\rho_t[\tilde{\delta}^3_{\mathbf{a}};\psi]$. It gives the number of particles initially in a state near ψ found under the measurement at time t in the state near $\tilde{\delta}^3_{\mathbf{a}}$.

We conclude that the flow of states on the space of states contains the flow of particles and the probability flow on $\mathbb{R}^3$ as particular cases. However, there is much more to it than just an abstract extension of these physical notions. For one, we saw in the previous section how under a certain random potential associated with a position measurement, the initial state ψ was equally likely to be displaced in any direction on the appropriate projective space of states. As a result, the state was undergoing a random motion on the space of states and the probability to find the state at a point φ was shown to be given by the Born rule. In terms of the density of states functional, this result can be described as follows: we are dealing with an ensemble of states initially positioned near the point ψ so that the density of states functional is concentrated at the point ψ. As the time goes by, the states undergo a random motion in accord with the Schrödinger equation with a random potential and the density of states functional "spreads out" in the space of states. As we saw, the density of states at a point φ depends only on the distance from ψ to φ and satisfies the Born rule.

Also, from the Schrödinger equation and the fact that the Schrödinger dynamics constrained to $M^{\sigma}_{3,3}$ is equivalent to the Newtonian one, and using nothing else, we obtained the relationship

$$\rho_t[\tilde{\delta}^3_{\mathbf{a}};\psi] = |\psi_t(\mathbf{a})|^2. \tag{100}$$

This relationship explains the identification of $|\psi_t(\mathbf{a})|^2$ with the probability density, which is one of the postulates in quantum theory. Indeed, the probability density to find the system in a state for an ensemble of states is proportional to the value of the density of states functional on that state, which for the states in M^{σ}_3 is given by (100). So $|\psi_t(\mathbf{a})|^2$ is the probability density to find the particle near $\mathbf{a}$ simply because this quantity is the density of quantum states near the point $\tilde{\delta}^3_{\mathbf{a}}$. If there are more states near $\tilde{\delta}^3_{\mathbf{a}}$, it becomes more likely to find the state under an observation near that point.

Furthermore, the continuity equation (97) in quantum mechanics follows from the Schrödinger equation and is true for *any* potential. Suppose we begin with an arbitrary random potential

V that under the Newtonian dynamics yields the normal distribution of the position random variable. By section 7, there is a unique extension of the Newtonian to Schrödinger dynamics. The formula (100) asserts then the validity of the Born rule for the resulting distribution of states undergoing the Schrödinger evolution with an arbitrary such potential V. This conclusion extends the results of section 9, originally obtained for the potential typically experienced by the particles in a Brownian motion. In addition, a purely geometric derivation of the Born rule in section 8 acquires here its dynamical validation. Note also that the isotropy of the probability distribution that needed to be assumed in the derivation of section 8 now follows directly from the Schrödinger dynamics and its reduction to the Newtonian one.

It is important to clearly distinguish the deterministic and the stochastic Schrödinger evolutions. The motion of state in quantum mechanics normally follows the deterministic Schrödinger equation with a given potential. However, as advocated here, under the conditions typically associated with a measurement, the state evolves by the Schrödinger equation with a random potential. The potential initiates a random motion of the state on the space of states and the resulting change in the density functional. The difference between these two types of evolution is analogous to the difference between the usual Newtonian motion of a macroscopic particle in a given potential and the Brownian motion of the particle under random hits, particularly in modeling a measurement by the diffusion. Of course, in light of the discussed relationship of Newtonian and Schrödinger dynamics, the analogy is not surprising. Note that the typical process of measurement must be sufficiently fast or must satisfy alternative conditions to be able to neglect the deterministic Schrödinger evolution during the measurement. These conditions were discussed in section 9. In the opposite case, the motion of state will consist of the deterministic drift and a random motion about the moving mean. The analogy with the measurement on a macroscopic particle can serve here a guiding principle.

In the integral form, the conservation of states in $L_2(\mathbb{R}^3)$ can be written in the following form:

$$\rho_{t+\tau}[\varphi;\psi] = \int \rho_t[\varphi+\eta;\psi]\gamma[\eta]D\eta, \tag{101}$$

where $\gamma[\eta]$ is the probability functional of the variation η in the state φ and integration goes over all variations η such that $\varphi+\eta \in S^{L_2}$. When the state of the particle is constrained to $M_3^\sigma = \mathbb{R}^3$, this equation must imply the usual diffusion on $\mathbb{R}^3$. The restriction of (101) to M_3^σ means that $\varphi = \tilde{\delta}^3_{\mathbf{a}}$ and $\eta = \tilde{\delta}^3_{\mathbf{a}+\epsilon} - \tilde{\delta}^3_{\mathbf{a}}$, where ϵ is a displacement vector in $\mathbb{R}^3$. As we already know, the function $\rho_t[\tilde{\delta}^3_{\mathbf{a}};\tilde{\delta}^3_{\mathbf{b}}] = \rho_t(\mathbf{a};\mathbf{b})$ is the usual density of particles in space. Let us substitute this into (101), replace $\gamma[\eta]$ with the equal to it probability density function $\gamma(\epsilon) \equiv \gamma[\tilde{\delta}^3_{\mathbf{a}+\epsilon} - \tilde{\delta}^3_{\mathbf{a}}]$ and integrate over the space $\mathbb{R}^3$ of all possible vectors ϵ. As in the Einstein derivation of the Brownian motion, assume that $\gamma(\epsilon)$ is the same for all $\mathbf{a}$ and independent of the direction of ϵ (space symmetry). Therefore, the terms $\int \epsilon^k\gamma(\epsilon)d\epsilon$ and $\int \epsilon^k\epsilon^l\gamma(\epsilon)d\epsilon$ with $k \neq l$ vanish. It follows as in the Einstein derivation that

$$\frac{\partial\rho_t(\mathbf{a};\mathbf{b})}{\partial t} = K\Delta\rho_t(\mathbf{a};\mathbf{b}), \tag{102}$$

where $K = \frac{1}{2\tau}\int \epsilon^2\gamma(\epsilon)d\epsilon$ is a constant.

The diffusion equation (102) describes the dynamics of an ensemble of particles in the classical space M_3^σ. If initially all particles in the ensemble are at the origin, then the density of the particles at a point $\mathbf{a} \in \mathbb{R}^3$ at time t is given by

$$\rho_t(\mathbf{a};0) = \left(\frac{1}{4\pi kt}\right)^{\frac{3}{2}} e^{-\frac{\mathbf{a}^2}{4Kt}}. \tag{103}$$

In particular, for the mean-squared displacement of the Brownian particle we obtain

$$\frac{d\overline{\mathbf{a}^2}}{dt} = 2K. \tag{104}$$

Because the embedding of M^σ into CP^{L_2} is isometric, we have $\mathbf{a}^2 = \theta^2$ for small values of the distance $\|\mathbf{a}\|_{\mathbb{R}^3}$ (this can be also seen from (68)). Also, the density of particles is equal to the density of states functional constrained to M_3^σ. From this and the isotropy of the density of states functional $\rho_t[\tilde{\delta}^3_{\mathbf{a}};\psi]$ it follows that

$$\frac{d\overline{\theta^2}}{dt} = 2K, \tag{105}$$

near $t = 0$. In section 13, the equation (105) will be used to explain why the macroscopic particles are constrained to the classical space M_3^σ.

From section 5, we know that the sphere of states S^{L_2} has a radius equal to one Planck unit. The classical Euclidean space $M_3^\sigma = \mathbb{R}^3$ is isometrically embedded into this microscopic sphere and spirals through its infinite dimensions. Also, for small values of σ even a small change in the position $\mathbf{a} \in \mathbb{R}^3$ yields a state that is near-orthogonal to the original state $\tilde{\delta}^3_{\mathbf{a}} \in M_3^\sigma$. These facts together with (105) may result in a short time interval of spreading of the density of states functional over the sphere. For σ equal to one Planck unit of length this time interval may be of the order of Planck time. This is relevant to the seemingly instantaneous nature of collapse. However, to make a meaningful statement on this basis it would still be necessary to estimate the probability of finding the state in a tubular neighborhood of the manifold $M_3^\sigma \subset CP^{L_2}$.

12. Collapse of quantum state

In the previous section, we considered a diffusion on the space of states, described by the Schrödinger equation with a random potential. Although the probability of transition between states under the diffusion was shown to satisfy the Born rule, the resulting process is very different from what is usually understood by the collapse. The fact that a random potential may lead to a random fluctuation of state is rather trivial and goes against of what one normally tries to achieve when explaining or modeling the collapse. The existing collapse models utilize various ad hoc additions to the Schrödinger equation with the goal of explaining why the state under the resulting stochastic process "concentrates" to an eigenstate of the measured observable (usually, position or energy) [9]-[20]. Instead, it is argued here that under a generic measurement, an ensemble of states with an initial position near ψ "diffuses" isotropically into the space of states by a unitary Schrödinger evolution. Whenever a particular state in the ensemble reaches a neighborhood of an eigenstate of the measured observable, we say that the "collapse" has occurred. In this case, the measuring device can record the value of the measured physical quantity.

According to this scenario, the measuring device has two separate functions. On one hand, it initiates a diffusion by creating a "noise". On the other, it registers a particular location of the diffused state. For instance, the "noise" in the position measuring device could be due to a stream of photons. The device then registers the state reaching a point in M_3^σ. In a similar way, a momentum measuring device registers the states that under the diffusion reach the eigen-manifold of the momentum operator (the image of M_3^σ under the Fourier transformation). Note the similarity in the role of measuring devices in quantum and classical mechanics: in both cases the devices are designed to measure a particular physical quantity and inadvertently create a "noise", which contributes to a distribution of values of the measured quantity.

According to this, the measuring device in quantum mechanics is not responsible for creating a basis into which the state is to be expanded. If several measuring devices are present, they are

not "fighting" for the basis. When the eigen-manifolds of the corresponding observables don't overlap, only one of them can "click" for the measured particle as the state can reach only one of the eigen-manifolds at a time.

What does it all say about measurement of position of macroscopic and microscopic particles? During the period of observation of position of a macroscopic particle, the position is a random variable that satisfies the diffusion equation. Normally, observation happens during a short enough interval of time and the variance of the corresponding probability distribution is sufficiently small. A particular value of position variable during the observation is simply a realization of one of the possible outcomes. The change in observed position of the particle can be equivalently thought of as either a stochastic process $\mathbf{b}_t$ with values in $\mathbb{R}^3$ or a process $\tilde{\delta}^3_{\mathbf{b},t}$ with values in M^σ_3. The advantage of the latter representation is that the position random variable gives both the position of the particle in $M^\sigma_3 = \mathbb{R}^3$ and, under a proper choice of σ, the probability density to find it in a different location $\mathbf{a}$ (in the state $\tilde{\delta}^3_{\mathbf{a}}$), due to uncontrollable interactions with the measuring device under the observation.

Measuring position of a microscopic particle has, in essence, a very similar nature. Under observation, the state ψ is a random variable with values in the space of states CP^{L_2}. To measure position is to observe the state on the submanifold M^σ_3 (or $M^\sigma_{3,3}$) in CP^{L_2}. In this case, the random variable ψ assumes one of the values $\tilde{\delta}^3_{\mathbf{a}}$, with the uniquely defined probability density compatible with the normal density on $\mathbb{R}^3$. This probability density is given by the Born rule. Here too, the random variable ψ gives both the position of the state of the particle in CP^{L_2} and the probability density to find the particle in a different state $\tilde{\delta}^3_{\mathbf{a}}$.

So the difference between the measurements is two-fold. First, under a measurement, the state ψ of a microscopic particle is a random variable over the entire space of states CP^{L_2} and not just over the submanifold M^σ_3. Second, unless ψ is already constrained to M^σ_3 (the case which would mimic the measurement of position of a macroscopic particle), to measure position is to observe the state that "diffused" enough to reach the submanifold M^σ_3. To put it differently, the measuring device is not where the initial state is. Assuming the state has reached M^σ_3, the probability density of reaching a particular point in M^σ_3 is given, as we saw, by the Born rule.

We don't use the term collapse of position random variable when measuring position of a macroscopic particle. Likewise, there seems to be no physics in the term collapse of the state of a microscopic particle. Instead, due to the diffusion of state, there is a probability density to find the state of the particle in various locations on CP^{L_2}. In particular, the state may reach the space manifold $M^\sigma_3 = \mathbb{R}^3$. If that happens and we have detectors spread over the space, then one of them clicks. If the detector at a point $\mathbf{a} \in \mathbb{R}^3$ clicks, that means the state is at the point $\tilde{\delta}^3_{\mathbf{a}} \in CP^{L_2}$ (that is, the state *is* $\tilde{\delta}^3_{\mathbf{a}}$). The number of clicks at different points $\mathbf{a}$ when experiment is repeated is given by the Born rule. The state is not a "cloud" in $\mathbb{R}^3$ that shrinks to a point under observation. Rather, the state is a point in CP^{L_2} which may or may not be on $\mathbb{R}^3 = M^\sigma_3$. When the detector clicks, we know that the state is on M^σ_3.

Note once again that there is no need for any new mechanism of "collapse" in the model. An observation is not about a "concentration" of state and the stochastic process initiated by the observation is in agreement with the conventional Schrödinger equation with a randomly fluctuating potential ("noise"). The origin of the potential depends on the type of measuring device or properties of the environment capable of "measuring" the system. Fluctuation of the potential can be traced back to thermal motion of molecules, atomic vibrations in solids, vibrational and rotational molecular motion, and the surrounding fields.

13. The classical behavior of macroscopic bodies

It was demonstrated that the Schrödinger evolution of state constrained to the classical phase space $M^\sigma_{3,3}$ results in the Newtonian motion of the particle. A similar result holds true for systems of particles. To reconcile the laws of quantum and classical physics, one must also

explain the nature of this constraint. Why would microscopic particles be free to leave the classical space, while macroscopic particles be bound to it?

Suppose for simplicity that the macroscopic particle under consideration is a crystalline solid. The position of one cell in the solid defines the position of the entire solid. If one of the cells was observed at a certain point a, the state of the solid immediately after the observation (in one dimension with a being the left most cell) is the product

$$\varphi = \tilde{\delta}_a \otimes \tilde{\delta}_{a+\Delta} \otimes \ ... \ \otimes \tilde{\delta}_{a+n\Delta}, \tag{106}$$

where Δ is the lattice length parameter. The general quantum-mechanical state of the solid is then a superposition of states (106) for different values of a in space:

$$\varphi = \sum_a C_a \tilde{\delta}_a \otimes \tilde{\delta}_{a+\Delta} \otimes \ ... \ \otimes \tilde{\delta}_{a+n\Delta}. \tag{107}$$

Why would non-trivial superpositions of this sort be absent in nature?

The classical phase space $M^{\sigma}_{3n,3n}$ of a n-particle system consists of all tensor products $\varphi_1 \otimes ... \otimes \varphi_n$ with the state φ_k of each particle given by (38). As discussed, the Schrödinger dynamics of n-particle system constrained to $M^{\sigma}_{3n,3n}$ is the Newtonian dynamics of the system. Note also that the isomorphism $\omega_n : \mathbb{R}^3 \times ... \times \mathbb{R}^3 \longrightarrow M^{\sigma}_{3n}$, $\omega_n(\mathbf{a}_1, ..., \mathbf{a}_n) = \tilde{\delta}^3_{\mathbf{a}_1} \otimes ... \otimes \tilde{\delta}^3_{\mathbf{a}_n}$ allows us to interpret n-particle states in M^{σ}_{3n} as points in the classical configuration space $\mathbb{R}^{3n}$ or positions of n particles in the single classical space $\mathbb{R}^3$. A similar map identifies the submanifold $M^{\sigma}_{3n,3n}$ with the classical phase space of n particles. These maps together with the established relationships of the classical and quantum dynamics allow us to think of M^{σ}_{3n} and $M^{\sigma}_{3n,3n}$ as the physical classical space and phase space with n particles.

To understand the dynamics of macroscopic bodies under measurement, consider the Brownian motion of a crystalline solid. The motion of any solid can be represented by the motion of its center of mass under the total force acting on the body and a rotational motion about the center of mass. The motion of the center of mass is the motion of a material point under the random force term, which is the sum of forces acting from the surrounding particles on each cell. Suppose for simplicity that the solid is one-dimensional and consists of n-cells. Let $\rho[\varphi; \psi]$ be the density of states functional on the space CP^{L_2}, where L_2 is the subspace in $L_2(\mathbb{R}) \otimes ... \otimes L_2(\mathbb{R})$, formed by the entangled states (107). The conservation of states for the system reads as before

$$\rho_{t+\tau}[\varphi] = \int \rho_t[\varphi + \eta; \psi]\gamma[\eta]D\eta, \tag{108}$$

where the meaning of terms is clear from (101). Define $\tilde{\delta}^{\otimes}_a = \tilde{\delta}_{a+\Delta_1} \otimes \tilde{\delta}_{a+\Delta_2} \otimes ... \otimes \tilde{\delta}_{a+\Delta_n} \in M^{\sigma}_{3n}$ and consider the functions

$$\rho_t(a; b) = \rho_t[\tilde{\delta}^{\otimes}_a; \tilde{\delta}^{\otimes}_b], \tag{109}$$

and

$$\rho_t(a; \psi) = \rho_t[\tilde{\delta}^{\otimes}_a; \psi], \tag{110}$$

where a, b denote the center of mass and Δ_k describe the positions of each cell relative to the center of mass. Applying the results of sections 9 and 11, we conclude that the state of the solid will experience a random motion on CP^{L_2} and that any direction of displacement of the state in $T_{\{\psi\}}CP^{L_2}$ at any time t is equally likely. In particular, if ψ is constrained to M^{σ}_{3n}, then (108) yields, as in the case of a single particle, the usual diffusion equation for the material point positioned at the center of mass of the solid.

It is a well established and experimentally confirmed fact that macroscopic bodies experience an unavoidable interaction with the surroundings. Their "cells" are pushed in all possible

directions by the surrounding particles. For instance, a typical Brownian particle of radius between $10^{-9}m$ and $10^{-7}m$ experiences about 10^{12} random collisions per second with surrounding atoms in a liquid. The number of collisions of a solid of radius $10^{-3}m$ in the same environment is then about 10^{19} per second. Collisions with photons and other surrounding particles must be also added. Even empty space has on average about 450 photons per cm^3 of space.

Now, suppose the state of a macroscopic body (in one dimension) is initially given by $\psi = \tilde{\delta}_{b+\Delta_1} \otimes ... \otimes \tilde{\delta}_{b+\Delta_n}$. Recall that this means that the initial distribution of position random variable is Gaussian with the center of mass at b. Under interaction with the surroundings the state ψ undergoes a random motion on the space of states CP^{L_2}. Consider the spatial (i.e., restricted to M^{σ}_{3n}) component of the motion near $t = 0$. As we know, the mean position of the center of mass will remain equal to b. Also, macroscopic bodies are distinguished by a large number of "cells", or a large dimension. As a result, the diffusion coefficient K in (102) is negligible so that the diffusion in space is trivial. But we know that the probability density of states under diffusion is direction-independent: if the state does not diffuse in the space M^{σ}_{3n}, then it cannot diffuse in the space of states either! In particular, if $K = 0$ in (104), then $K = 0$ in (105), which means that the density of states $\rho_t(\theta)$ must be constant in time. Therefore, in the absence of additional potentials acting on the macroscopic body it will maintain its original state ψ.

The situation is surprisingly similar to that of a pollen grain and a ship initially at rest in still water. While under the kicks from the molecules of water, the pollen grain experiences a Brownian motion, the ship in still water will not move at all. Because of the established relation of Newtonian and Schrödinger dynamics, this is more than an analogy. In fact, when the state is constrained to the classical phase space submanifold, the "pushes" experienced by the state become the classical kicks in the space that could lead to the Brownian motion of the body.

Let us estimate the value of the diffusion coefficient for a macroscopic body. As known after the works of Stokes and Einstein, the diffusion coefficient for a spherical particle is well described by the expression

$$K = \frac{k_B T}{6\pi\eta r}, \tag{111}$$

where r us the radius of the particle and η is the dynamic viscosity. In particular, for a macroscopic particle of radius $r \sim 1mm$ in the air, $\eta \sim 10^{-5} N \cdot s/m^2$, at room temperature, we get $K \sim 10^{-12} m^2/s$. The variance of position of the particle is given by $\overline{x^2} = 2Kt$. In particular, it would take about $10^6 s$ or more than 10 days for the standard deviation of $1mm$ in the distribution of the displacement of the particle to occur.

Now, the actual time of observation of position of particles in experiments is much shorter. For instance, if we scatter visible light off the particle to determine its position, the time interval of observation could be as short as $10^{-13}s$, which for a $1mm$ of radius particle in the air would amount to the displacement of the order of $10^{-21}m$. This quantity is much less than the accuracy of measurement, limited by the wavelength $\lambda \sim 10^{-5}m$, and cannot be observed in the measurement. Of course, if the diffusion process were to continue, we would start seeing some deviations from the original position of the particle. However, given the significant amount of time that this requires, these deviations would be negligible, when compared to the changes due to interactions of the particle with the surroundings. At the same time, the mean displacement will be always zero, making our ability to discover the motion even more challenging. Note that the Fubini-Study distance between Gaussian states that are $10^{-21}m$ apart in M^{σ}_3 with $\sigma \sim 10^{-5}m$ can be calculated via (68) and is about $10^{-16} rad$. So the state is hardly moving away from its original position and cannot realistically reach points in the space of states that are away from that position. In particular, it becomes impossible to find the state positioned initially in the configuration space M^{σ}_{3n} at a different point of that space.

Suppose now an external potential V is applied to the macroscopic system. According to (45), this will "push" the state that belongs to the classical phase space submanifold in the direction tangent to the submanifold. Therefore, the external potential applied to a macroscopic body will not affect the motion of state in the directions orthogonal to the classical phase space submanifold. That means that the state will remain constrained to the submanifold. On the other hand, as we know from the same section, the constrained state will evolve in accord with Newtonian dynamics in the total potential $V + V_S$, where V_S is the potential created by the surroundings. However, since at any time t the total force $-\nabla V_S$ exerted on the macroscopic body by the particles of the surroundings can typically be neglected, the body will evolve according to Newtonian equations with the force term $-\nabla V$. To be sure, the particles of the surroundings are responsible for the friction. In the Hamiltonian description of interaction of the body with the surroundings (as in the Ullersma model [21]), the friction comes from a contribution to the total potential in the Hamiltonian. However, whenever the friction can be neglected, the dynamics of the solid is determined by the force $-\nabla V$.

So, the origin of the classical behavior of macroscopic bodies in the theory is two-fold. First of all, the initial state of a macroscopic body is on $M^{\sigma}_{3,3}$. That is, a macro body is created at a point of the submanifold $M^{\sigma}_{3,3}$. Second, because of the interaction of the particle with the surroundings (radiation, molecules of air, water and other media), the state undergoes a diffusion process rather than a free Schrödinger evolution. Also, because of the macroscopic character of the body, the diffusion coefficient is extremely small. The probability distribution of the variation of the state of the body has a zero mean and is nearly constant in time. We don't see a quantum evolution of the state, but rather a negligible diffusion. This diffusion does not influence measurement of position of the body as that measurement happens on a much shorter time scale.

From this analysis, it becomes clear that the transition of macroscopic to microscopic happens for the macroscopic bodies for which the Brownian motion in the surrounding media is observable. If a macroscopic body is sufficiently small so that the Brownian motion of the body in the media can be observed in an experiment, then the superposition of states of different positions of the body becomes observable as well. In fact, it was demonstrated that under the conditions typical for the Brownian motion, the state of the system has equal probability of any direction of displacement in the space of states. In particular, the state may become a superposition of distinguishable states of a given position in $\mathbb{R}^3$. Interference effects on such states are then observable.

14. The role of decoherence

So far, the state of the measured system was considered independently of the measuring device and the environment. This is possible as long as the influence of the environment can be modeled, at least approximately, by a potential. However, in many cases the state of the measured particle, whose position is measured, and the surroundings cannot be described independently. The state of the total system is then a linear combination of the terms

$$\tilde{\delta}^3_{\mathbf{a}} \otimes E_{\mathbf{a}}, \tag{112}$$

where $E_{\mathbf{a}}$ represents the state of the surroundings when the particle is in the state $\tilde{\delta}^3_{\mathbf{a}}$. At the same time, the result of measurement is always a single term like (112). How could it be?

Given the observed relationship between the Newtonian and Shrödinger dynamics, let us begin with the following classical mechanical example of an entangled state. Consider a pair of macroscopic particles, for simplicity in one dimension, connected by a weightless rigid rod that keeps the particles at a fixed distance d from each other. Suppose the position of one of the particles is measured. As before, the diffusion mechanism can be used to describe the resulting normal distribution of the position random variable. If position of the particle is found to be

a, then position of the second is guaranteed to be $a + d$. If position of the second particle is measured and found to be b instead, then position of the first is guaranteed to be $b - d$.

In the language of quantum states, the state of the pair at any time is $\tilde{\delta}_a \otimes \tilde{\delta}_{a+d}$ for some $a \in \mathbb{R}$. This state belongs to a one-dimensional submanifold N of the two-dimensional manifold $M_1^\sigma \otimes M_1^\sigma$, which itself is a submanifold of the Hilbert space $L_2(\mathbb{R}) \otimes L_2(\mathbb{R})$. Here M_1^σ is the submanifold of $L_2(\mathbb{R})$ made of the normalized Gaussian functions $\tilde{\delta}_a$ of width σ. The difference between this state and the state of a pair of microscopic particles entangled in a "similar" manner is that now the state belongs to the linear envelop L_N of N, which is a subspace of $L_2(\mathbb{R}) \otimes L_2(\mathbb{R})$. So, a normalized entangled state lives on the unit sphere $S^{L_N} \subset L_N$ and not only on the submanifold N of thereof. A system of two particles connected by a rod is a classical mechanical analogue of the entangled state of a pair of particles and is, at least mathematically, a special case of the entangled state. The measurement of position of one particle in the pair connected by a rod is mathematically a particular case of the collapse of the entangled state of the pair. Furthermore, nothing prevents us now from applying the same construction to an arbitrarily constrained system of more than two macroscopic particles.

The similarity of this picture to the one that relates the classical space and the Hilbert space of states of a single particle is quite obvious. And indeed, the results obtained previously for a single particle can be reproduced here verbatim. As in section 9, we conclude that the distribution of the displacements of the state ψ under a generic measurement is isotropic. It follows that the Born rule must indeed be satisfied for the system in an entangled state. That is, under the measurement the state ψ undergoes a random motion on the space of states and can potentially reach the submanifold N in CP^{L_N}. The probability of reaching a neighborhood of a particular point $\tilde{\delta}_a \otimes \tilde{\delta}_{a+d}$ in CP^{L_N} is given by the Born rule. To find the pair in the state $\tilde{\delta}_a \otimes \tilde{\delta}_{a+d}$ is the same as to find the first particle at a (and, therefore, the second particle at $a + d$).

It is clear that the number of "parts" in a quantum system makes no difference for the proposed mechanism to work. The entanglement between the particle and the surrounding simply enlarges the Hilbert space of possible states of the system. The measurement of only one part of the entangled system results in a random motion of the total state on the space of states. The measurement of position of a particle is successful if under this motion the state reaches the submanifold of the product states $\tilde{\delta}^3_{\mathbf{a}} \otimes E_{\mathbf{a}}$. The relative probabilities of finding the state near a specific point of the submanifold is given by the Born rule.

The issue of where to place the "cut" in the surroundings, to avoid considering the entire universe in an entangled state with the measured particle is analogous now to the same issue in Newtonian mechanics. There too, in principle, the entire universe influences the motion of a measured particle. However, a good approximation can be obtained by assuming that the particle represents a closed system, by reducing the role of the environment to a potential, or by considering a system of finitely many particles, or else, by yet another mechanism that effectively reduces the degrees of freedom of the system and makes the problem solvable.

So far, decoherence was not present in the discussion. Formally, decoherence is a mathematical expression of the fact that a quantum system interacting with the environment behaves like a probabilistic mixture and needs to be described by the probability and not by the state itself. The dynamical part of the process that leads to decoherence consists in the entanglement between the state of the particle and the environment. This part is due to the usual unitary evolution of the total system and it precedes decoherence. The decoherence itself consists in a fast decrease in the interference effects between terms of the total state of the system, due to the near-orthogonality of the states of the environment $E_{\mathbf{a}}(t)$, as the system evolves in time. The theory is centered around, and does not usually go beyond the issue of entanglement and the resulting loss of coherence. It does not describe the way in which specific measurement results are obtained and does not derive the Born rule. At the same time, decoherence theory

uses an array of very useful models that provide physical content for the theory. These models testify to the universal character of the loss of coherence and transition to classical probability resulting from interaction with the environment.

Moreover, interaction with the incident particles in the model of spatial decoherence by scattering is what also triggers the diffusion of state, under the discussion here. In this context, decoherence may be considered a superficial expression of the underlying physical process of diffusion of state. In fact, the diffusion of state under a measurement signifies that the system is described by the density of states functional. When applied to measurement of position, the functional yields a probabilistic distribution of the position random variable, which is the end-result of the (spatial) decoherence. Note that the diffusion is a dynamical process, unlike the "pure entanglement form" of decoherence itself. Despite being a "fake" or "microscopically unitary" decoherence [22], the diffusion of state seems to provide a universal dynamical mechanism for the loss of coherence *and* collapse of state.

15. The double-slit experiment

The derivation of Newtonian from Schrödinger dynamics, the relationship of the Born rule to the normal probability distribution, an explanation of the classical behavior of macroscopic bodies and a clear picture of collapse all suggest that the isomorphism between the classical space $\mathbb{R}^3$ and the manifold M_3^σ must be considered a physical and not just a mathematical identification. Let us accept this hypothesis and use it to analyze quantum-mechanical experiments and to address the paradoxes of quantum mechanics. As discussed, the superposition principle in quantum mechanics represents the main obstacle to reconciliation of the quantum and the classical. Let us therefore begin with the simplest manifestation of the superposition principle: the double-slit experiment.

Different forms of the experiment are well known and don't need to be reviewed here. We are going to discuss the simplest set-up of the experiment, involving an electron gun, a plate with a pair of parallel slits, and a scintillating screen or a photographic plate to observe the interference pattern. Our first task is to identify the Hilbert space of the system and the submanifold of the corresponding classical system. We will deal with a single electron. Also, since the origin of the electron will not be important, the electron gun will be left out of the picture. For now we will also leave out the screen registering the outgoing particles and the surroundings.

The Hilbert space of the system is the tensor product of spaces $L_2(\mathbb{R}^3)$, one for each particle in the system. However, the state of the macroscopic plate with the slits has the form (106) in section 13. That is, the plate is given by a point ψ_P on the submanifold $M_{3n}^\sigma = M_3^\sigma \otimes ... \otimes M_3^\sigma$ in CP^{L_2}. Here L_2 is the tensor product of Hilbert spaces $L_2(\mathbb{R}^3)$ for all particles of the plate. As discussed in the section 13, the isomorphism $\omega_n : \mathbb{R}^3 \times ... \times \mathbb{R}^3 \longrightarrow M_{3n}^\sigma$, $\omega_n(\mathbf{a}_1, ..., \mathbf{a}_n) = \tilde{\delta}^3_{\mathbf{a}_1} \otimes ... \otimes \tilde{\delta}^3_{\mathbf{a}_n}$ allows us to view the states in M_{3n}^σ as points in the classical configuration space $\mathbb{R}^{3n}$ or positions of n particles in the single classical space $\mathbb{R}^3$. That is how our usual view of the plate becomes possible and how the state ψ_P gets identified with a set of material points that represent the particles of the plate in $\mathbb{R}^3$. As discussed in section 14, the interaction between the plate and the environment prevents an entanglement between the states of the macroscopic plate and the electron. Moreover, since the plate is practically unaffected by the electron, its state during the experiment remains ψ_P so that the state of the total system belongs to the manifold $L_2(\mathbb{R}^3) \otimes \psi_P$.

We can now proceed with the analysis of the experiment. First, the wave packet of the electron propagates toward the plate. If the electron is sufficiently fast, the spreading of the packet on the approach to the plate can be neglected. During this time interval, the propagation of the initial packet ψ is happening essentially by a displacement $\psi_t(\mathbf{x}) = \psi(\mathbf{a} - \mathbf{v}t)$. The electron state moves along (parallel to) the classical space submanifold M_3^σ in $L_2(\mathbb{R}^3)$. The state $\Psi_t = \psi_t \otimes \psi_P$ of the total system moves along the submanifold $M_3^\sigma \otimes \psi_P$ in $L_2(\mathbb{R}^3) \otimes \psi_P$, diffeomorphic to

$M_3^\sigma = \mathbb{R}^3$. The motion can be thought of in the classical terms; we have a material point propagating towards the plate. If desirable, we can add photons to this picture, to ensure that we can "see" the plate. In fact, if refraction in the media is neglected the photon wave packet always propagates along the classical space M_3^σ without spreading. The isomorphism ω_n can then be used to interpret the entire process in terms of the electron, the plate and the photon, all in the same classical space $\mathbb{R}^3$.

During the second stage of the experiment, the electron goes "through" the slits in the plate. Although it is useful to "visualize" the plate by the state ψ_P, or by the corresponding set of points in $\mathbb{R}^3$, the effect of the plate on the electron can be described by a potential $\widehat{V}$, which is infinite on the plate and zero at the slits. The potential acts non-trivially on the subset of $L_2(\mathbb{R}^3)$ of all state functions whose support has non-empty intersection with the plate. The Schrödinger evolution of the electron is still described by a path ψ_t in the Hilbert space. However, at this time, the shape of the function ψ_t is different. After interaction with the potential the state function is a superposition $c_1\psi_1 + c_2\psi_2$, where the packets ψ_1, ψ_2 represent the state of the electron passing though one of the slits with the second slit closed. The resulting superposition continues propagating in the same direction, forming a path ψ_t.

What happens at this step is very important. Let us describe the motion of the state in terms of the Schrödinger evolution on the space of states $H = L_2(\mathbb{R}^3) \otimes L_2$ of the electron-plate system. The state $\Psi_t = \psi_t \otimes \psi_P$ of the system propagates along the classical space submanifold $M_3^\sigma \otimes \psi_P = \mathbb{R}^3$ on the sphere S^H in H, approaching the plate in the induced metric on the sphere (or the Fubini-Study metric on the projective space CP^H). On interaction with the potential representing the plate, the state ψ_t evolves into a superposition $c_1\psi_{1t} + c_2\psi_{2t}$. In terms of the geometry on the space of states, the path ψ_t is no longer valued in the classical space submanifold M_3^σ in $L_2(\mathbb{R}^3)$. In fact, the classical space submanifold is formed by the Gaussian states. Those states have a single "hump", while ψ_t behind the plate is a "double-humped" state function. As the state interacts with the plate, the distance from the state to the the classical space $\mathbb{R}^3$ increases.

Using the identification of the state ψ_P with the set P in M_3^σ of states of all particles of the plate, we can also view the entire process of interaction with the plate in a single space $L_2(\mathbb{R}^3)$. The set P is the image of the plate in $\mathbb{R}^3$ under the isomorphism ω_σ of section 3. The classical part of the experiment can be formulated within the submanifold M_3^σ alone and consists of the electron state (a point on, or near M_3^σ) approaching the plate P. Under interaction with the plate, the state of the electron becomes $\psi_t = c_1\psi_{1t} + c_2\psi_{2t}$. Even when the states ψ_{1t} and ψ_{2t} are Gaussian states, representing the points in the classical space M_3^σ, the superposition is not a Gaussian state. Therefore, the superposition at time t is *not* a point in the classical space. As the result of interaction with the plate, the path ψ_t moves away from the classical space M_3^σ and, therefore, passes *over* the plate with the slits (which is a subset P of M_3^σ, in this representation).

The origin of the paradox of the double-slit experiment is now clear. When trying to view the dynamics of the electron in the experiment within the classical space $M_3^\sigma = \mathbb{R}^3$, we are facing the dilemma: which slit did the electron go through? When formulated in these terms, the only correct answer seems to be that it went "through both" or to admit that position is not defined. This violently clashes with everything we know about the world around us and contradicts Newtonian mechanics. It forces us to think of the electron in terms of some kind of "electron cloud" that can "assemble" back to the particle (collapse) when measured. Alternatively, that the answer to Einstein's question - "is the moon there, when nobody looks?", - must be negative, at least for the electrons.

Under the Schrödinger dynamics, the evolution of the electron is a path ψ_t in the Hilbert space. It is a path in the usual sense; a continuous and single-valued function of time with values in $L_2(\mathbb{R}^3)$. When the state is constrained to M_3^σ, ψ_t is the usual path of a macroscopic particle in Newtonian dynamics. When the electron interacts with the plate, the path continues into the

Hilbert space. Because the path can be written now as a sum $\psi_t = c_1\psi_{1t} + c_2\psi_{2t}$, we tend to think that both parts, ψ_{1t} and ψ_{2t} are real, so that the path of the electron splits into the paths that go through slits 1 and 2. This is paradoxical and contradictory. In fact, if the same wave function is written as a superposition of eigenstates of a different observable, then, by the same logic, the new components must be real as well. Since there are many observables, the notion of reality becomes ill-defined. The way out is to accept that the adequate way to describe the reality is by the vector ψ_t and not by its components ψ_{1t} and ψ_{2t}, that depend on the choice of a basis. When the state function belongs to the submanifold M_3^σ, the electron behaves classically. In general, however, the state of the electron is not confined to M_3^σ and satisfies the Schrödinger equation.

The issue of reality of the components ψ_{1t} and ψ_{2t} is similar to the following question in classical physics. When a physical vector (say, a velocity vector) is written in terms of its components in a certain basis, should we count the components as real? The answer is obvious: the physical vector itself is real because it is basis independent. However, the components of the vector are just shadows of the real thing as they change with the change of basis, similar to the way a shadow changes when the source of light is moved around. Our problem with the superposition principle is rooted in the desire to attach to the classical components like ψ_{1t} and ψ_{2t} the status of a "real thing". The paradox of the superposition is resolved by accepting the total state ψ_t as an adequate description of reality, while considering ψ_{1t} and ψ_{2t} for what they really are: representation dependent components of the vector ψ_t. To answer Einstein's question: The moon and the electron *are* there, when nobody looks. Their existence is described by the state, at any time and not just when the object is measured. Whenever the state belongs to the classical space $\mathbb{R}^3 = M_3^\sigma$, it describes the usual classical existence in the Newtonian sense. But unlike the classical position, the state also catches the quantum origin of nature.

Suppose that position of the electron is measured by the screen behind the plate. As discussed in sections 11 and 12, a measurement of position produces a diffusion on the projective space of states. If the initial state of the electron was ψ, the density of states functional at the point $\tilde{\delta}^3_{\mathbf{a}}$ was shown to be $|\psi(\mathbf{a})|^2$. Because the state is a superposition of two states that describe the electron passing through one of the slits, the density of states functional contains the cross term. This term in the density results in an alternating probability of reaching different parts of the screen, producing a typical interference picture on the screen.

What happens when we place a source of light between the plate with the slits and the screen? In this case, the diffusion of the electron state begins earlier. After passing through the plate, the electron state is "two-humped". In particular, this initial state of the electron is positioned away from M_3^σ. Suppose that on interaction with the photons of the source of light, the electron is observed near one of the slits. That means, in particular, that the diffused electron state is on the classical space submanifold M_3^σ. So the state function of the electron observed near one of the slits must be "single-humped". The electron in such a Gaussian-like state is later observed on the screen. Clearly, no interference picture would appear on the screen.

What about a delayed-choice version of the experiment when we decide to determine which slit the electron went through *after* the electron has passed the plate with the slits? For instance, we could turn the light on after the electron went through the slits. The paradox is that the electron seems to "decide" retroactively to behave as a particle or a wave, and, accordingly, to go through one slit, or both, depending on our decision to turn the light on. However, the previous analysis is not altered by this change in the experiment. Whether or not the light source is present, the state of the electron after the slits is "two-humped". In particular, inserting a screen between the plate and the light source will show the interference pattern. When the light source is turned on and the electron is observed near one of the slits, the "two-humped" state is transformed to a "single-humped, Gaussian-like state. As a result, the screen behind the light source will not show interference picture.

As before, we see that the paradox is due to our assumption that the electron must be on the classical space manifold M_3^σ at any time. In this case, the observed interference pattern signifies that the electron somehow "spreads out" over both slits and behaves like a wave. On the other hand, if the light source is on, then the electron visibly goes through one of the slits only and behaves like a particle. The paradox is resolved by accepting that evolution of the electron is described by a path φ_t in the space of states CP^{L_2}. When the electron interacts with the plate, the path abandons the classical space submanifold M_3^σ in CP^{L_2}, the state function is "two-humped" and the interference picture is observable. When the source of light is turned on and the electron is observed by one of the slits, the state function is "single-humped" and the interference is not present. The moment when the light source is turned on is irrelevant. The nature of the electron does not change. In particular, the electron does not go back in time to "adjust" its nature depending on our decision to turn the light source on. The electron does not spread over the slits. Moreover, the electron *does not go through* the slits. If anything, it goes *over* the slits into the large dimensions of the space of states and *comes back* whenever its position is measured. This resolves the paradox of the double-slit experiment.

16. EPR experiment

The state of a pair of microscopic particles is an element of the tensor product Hilbert space $H = L_2(\mathbb{R}^3) \otimes L_2(\mathbb{R}^3)$. When positions of both particles are known, the state belongs to the submanifold $M_3^\sigma \otimes M_3^\sigma$ in CP^H. In section 14, a classical-mechanical version of an entangled state was discussed. It consists of two macroscopic particles connected by a weightless rigid rod, considered for simplicity in one dimension. If position of one particle in such a system is measured to be a, then position of the second particle is automatically known to be $a+d$, where d is the length of the rod. The state of the pair is then $\tilde{\delta}_a \otimes \tilde{\delta}_{a+d}$. A pair of microscopic particles in a superposition of such states is an example of an EPR pair. If momentum of the first particle in a pair is found to be p, then the momentum of the second will be $-p$.

There are essentially two paradoxes associated with EPR-pairs. The first one consists of the non-local character of "communication" between the particles of the pair. Namely, how could a measurement performed on one particle instantaneously affect the other particle, no matter how far away? The other paradox is related to our ability to influence the reality of position or momentum of the second particle by choosing to measure either position or momentum of the first. This calls into question the notion of physical reality as well as completeness of quantum theory.

Similarly to the single particle case, the evolution of the pair is a path in the space of states CP^H. Whenever the path takes values in the submanifold $M_3^\sigma \otimes M_3^\sigma$, the position of both particles is known. Moreover, if the state is constrained to $M_3^\sigma \otimes M_3^\sigma$, then the Schrödinger dynamics of the pair is equivalent to the Newtonian one. As before, the constructed isomorphisms ω_n allow us to identify the state of the pair in $M_3^\sigma \otimes M_3^\sigma$ with a point in the configuration space $\mathbb{R}^3 \times \mathbb{R}^3$ of the system of two point-particles or positions of both particles in the classical space $\mathbb{R}^3$.

Suppose the state of the pair is a point on CP^H away from the submanifold $M_3^\sigma \otimes M_3^\sigma$. Suppose that position of one of the particles is measured. As discussed in section 14, the state of the system will undergo a diffusion in CP^H and the probability for the state of reaching a particular point in $M_3^\sigma \otimes M_3^\sigma$ is given by the Born rule. Note that position of only one of the particles needs to be measured for the state to be able to reach the manifold $M_3^\sigma \otimes M_3^\sigma$. Under the measurement, the state of the pair will undergo a random motion while following a continuous path ψ_t from the initial state to a point in $M_3^\sigma \otimes M_3^\sigma$.

It is important that the distance d between the points a and $a+d$ has nothing to do with the motion of the state ψ to an observed position state $\tilde{\delta}_a \otimes \tilde{\delta}_{a+d}$. The observed properties of one particle are not communicated to the other one by any signal or a field in space. Moreover,

there are no particles in the sense of objects on $M_3^\sigma \otimes M_3^\sigma$, or on $M_3^\sigma = \mathbb{R}^3$. Rather, there is a state ψ_t representing the pair. When the state is constrained to $M_3^\sigma \otimes M_3^\sigma$, the particles are described by the classical Newtonian dynamics. So we can think of them in purely classical terms, as indeed, material points. However, the state in CP^H, not constrained to the classical space or phase space submanifolds describes the pair as a quantum object that embraces and supersedes the material point of Newtonian mechanics.

We see that the paradoxical "spooky action at a distance" is not present anymore. The state of the pair takes over the individual reality of the particles. There is no instantaneous collapse that somehow makes the pair "real". The pair is always real and exists in a form described by the state. For the state constrained to the classical phase space $M_6^\sigma \otimes M_6^\sigma$, we recover the usual classical-mechanical description of a pair of material points in $\mathbb{R}^3$. But in all cases, the state is an appropriate entity to describe physical characteristics of the observed world. The paradox of "creation" of reality of position or momentum of one particle by measuring the corresponding quantity of the second clears up as well. These physical characteristics only make sense for the state constrained to the manifold $M_6^\sigma \otimes M_6^\sigma$ and alike. In that particular case, their relation to the motion of state was derived in section 6. Otherwise, these physical characteristics are only "shadows" of the deeper physics described by the state. The space of states is the new physical arena that extends the classical space. The state offers a more complete way of identifying characteristics of physical bodies. It generalizes the notion of position, momentum and other observed quantities and reproduces these quantities when constrained to an appropriate classical submanifold.

17. Schrödinger's cat paradox

The issue here is that the existence of entangled states of microscopic systems results in a contradiction when applied to macroscopic objects. In particular, in the famous Schrödinger thought experiment we get superpositions of states of a cat being alive and dead. As discussed in section 13, a macroscopic system is subjected to interaction with the environment, or, to put it differently, is "measured by the environment". From the same section we know that the state of the "measured" macroscopic system undergoes a trivial diffusion and so it does not change at all. The macroscopic object is therefore constrained to the classical space M_{3n}^σ. Assume that an entangled state of a cat and a decaying atom is somehow created. To measure the system is to measure one component of the system. For instance, when we check if the cat is alive, we also know that the atom has not decayed. The measured system is then in the product state. However, because the cat is always "measured" the state of the total atom-cat system is maintained in the product form. So, under the normal conditions the state of a microscopic and a macroscopic system cannot be entangled. There cannot be Schrödinger cats running around.

Note that the inconsistent view of reality by different observers in the Wigner's friend type of experiment, discovered by Frauchiger and Renner [23], is only present when an entanglement of microscopic and macroscopic objects is possible. As discussed, such an entanglement is not possible. At the same time, there is much more to be investigated now that the physical arena became the space of states. In particular, the notion of reality is altered for the objects not constrained to the classical space submanifold. We need to understand what it means in detail. Further, if the space is now a submanifold in the space of states, then what is an appropriate extension of the space-time manifold? How does it fit into the scheme? Would the results of relativity theory need to be changed? This requires further investigation.

18. Summary and experimental verification

The dynamics of a classical n-particle mechanical system on the classical space $\mathbb{R}^3$ was identified with the Schrödinger dynamics with the states constrained to the classical phase space submanifold $M_{3n,3n}^\sigma$ in the space of states. Conversely, we saw that there is a unique

unitary time evolution on the space of states of a quantum system that yields Newtonian dynamics when constrained to the classical phase space. This resulted in a tight, previously unnoticed relationship between classical and quantum physics. Under this relationship, the classical Euclidean space $\mathbb{R}^3$ is isometrically embedded into the space of states CP^{L_2} with the Fubini-Study metric and is identified with the submanifold M_3^σ of CP^{L_2}. The Newtonian dynamics reigns on M_3^σ, while the Schrödinger dynamics is its unique extension to the space of states CP^{L_2}. The normal probability distribution on M_3^σ has a unique extension to CP^{L_2} and becomes the Born rule for the probability of transition between states. Vector fields on M_3^σ have a unique extension to linear vector fields on the space of states. Quantum observables are identified with the associated linear vector fields. Commutators of observables are Lie brackets of the vector fields and are related to the curvature of the space of states. The physical quantities of velocity, acceleration and mass in Newtonian dynamics are now components of the velocity of quantum state.

The process of measurement in quantum mechanics is now an extension of the measurement in classical physics that itself produces a normal distribution of the measured observable and can be described by a diffusion equation. The state under a measurement is equally likely to fluctuate in any direction on the space of states. This fact is responsible for the validity of the Born rule for the probability of transition of the initial state to a particular final destination. The state is not a cloud in the classical space that somehow "shrinks" under a measurement. Rather, the state undergoes a random motion with a chance of reaching certain areas of the space of states in the process. The evolution remains unitary and satisfies the Schrödinger equation with a random potential without contradicting the known "no-go" results. The "collapse" of the state becomes an unnecessary and redundant concept.

This approach to measurement is applicable to quantum systems consisting of an arbitrary number of particles. When the system is macroscopic, the diffusion of state trivializes and the state remains unchanged in time. As a result, macroscopic particles are constrained to the classical space submanifold of the space of states. On the other hand, microscopic particles can leave the submanifold and exist in a superposition of position eigenstates. The double-slit and numerous other quantum-mechanical experiments demonstrate this property. When position of a microscopic particle is measured and the result is obtained, the state returns to the classical space submanifold. A particular point in M_3^σ where the state was found determines the value of the position variable.

The entangled states of two or more particles are represented by the points in the space of states of the system that is not on the classical space or phase space submanifold. Similar to the case of a single particle, under a measurement the point representing the state of the pair in the space of states undergoes a random motion. To make a measurement on an EPR-pair, it suffices to measure just one of the particles. The measurements yield an isotropic distribution of the displacement of state in the space of states, implying the validity of the Born rule. A measurement on one of the particles in an entangled pair restricts the outcomes of the corresponding measurement on the second particle. However, the measurement does not imply a "communication" between the particles. Instead, the point in the space of states representing the state of the pair moves in a continuous way and reaches an eigenstate of the measured observable. The probability of reaching a particular eigenstate is given by the Born rule.

The obtained realization of the Newtonian mechanics in functional terms and the derived relationship of the classical and quantum theories is not just a reformulation of the theory. The results of the classical and quantum mechanics are indeed reproduced in the theory. However, the embedding resulted in a tighter relationship between the theories. This relationship can be experimentally tested. A meaningful relationship between Newtonian and Schrödinger dynamics can be seen in several places. First, there is a formula (36) that yields the known result that the speed of evolution of state is equal to the uncertainty in energy, derived in a clear geometrical

way. Further, the decomposition (45) relates Newtonian velocity and acceleration, and, for the appropriate value of σ, also the mass of a particle to the corresponding components of the velocity of quantum state. However, these results are consistent with the Schrödinger dynamics itself and the Ehrenfest theorem that follows from it and cannot serve a validation of the constructed embedding.

What helps to come up with an experiment is the "rigidity" of the embedding: the extension of the Newtonian dynamics and Newtonian models to the space of states is *unique*. This allows us to approach the process of measurement in quantum theory in a new way, as an extension of the random motion associated with a classical measurement. An important consequence of this is the notion of a density of state functional and its derived isotropy property that can be tested. Indeed, if several observables are measured on a particular state of a system at the same time, we should be able to test the isotropy of the distribution of frequencies of the measured eigenvalues. That is, the state should be seen "collapsing" equally frequently to the eigenstates of different observables, positioned at the same Fubini-Study distance from the initial state. The observation of different components of spin of a particle at the same time would probably be the easiest way to set up such an experiment.

Another experiment could test the classical to quantum boundary. This boundary is predicted by the theory to be determined by the largest particles for which the Brownian motion in an appropriate media is observable. In fact, as long as the Brownian motion for the particle is observable, the state of the particle is capable of diffusing into the space of states. In particular, superpositions of the position eigenstates become possible and can be observed.

The obtained results lead one to the conclusion that macroscopic and microscopic bodies may not be so different. The only important distinction is that microscopic systems live in the space of states while their macroscopic counterparts live in the classical space submanifold of thereof. Because our life happens in the macro-world and we deal primarily with macroscopic bodies, it is hard for us to understand the infinite-dimensional quantum world around us. As soon as the classical-space-centered point of view is extended to its Hilbert-space-centered counterpart, the new, clearer view of the classical-quantum relationship emerges.

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