Reductive Explanation and the Construction of Quantum Theories

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‘[...I]n our philosophical reflections upon science, we *should* by now have learned explicitly—that successful scientific theories are to be taken very seriously as clues to the deeper understanding of phenomena: i.e., clues in the search for better and more fundamental theories.’
— H. Stein ([1989], p. 57), ‘Yes, but...’

Abstract

I argue that philosophical issues concerning reductive explanations help constrain the construction of quantum theories with appropriate state spaces. I illustrate this general proposal with two examples of restricting attention to physical states in quantum theories: regular states and symmetry-invariant states.

1 Introduction

It is common to require that a newly constructed physical theory be able to explain the success of the theory it supersedes. In the context of quantization, or the construction of quantum theories, this becomes the requirement that a quantum theory explain the success of its corresponding classical theory in the *classical limit*. But this requirement is vague as it stands—what could count as an explanation of past success? And how could this guide us in the construction of new theories? The purpose of this paper is to make a part of this requirement precise and to show that it puts substantive constraints on theory construction.

I will focus on just one aspect of explaining the success of a past physical theory: the *state space*. The state space of a physical theory often contains states that are not used to represent genuine physical possibilities. I will call these states, which are sometimes said to be 'mere artifacts', *unphysical states*. As I see it, an explanation of the state space involves explaining why only certain parts of the mathematical structure of the theory—the physical states—are required for successful application of the theory. I will not be concerned with the ontological status of the physical and unphysical states, but rather their methodological role in these successful applications. I will assume that there is at least a normative ideal, whether or not it is actually achievable, to construct physical theories that allow only for physical states and rule out unphysical states. Feintzeig ([2018a]) establishes a general procedure for adapting quantum theories to allow for only physical states once a physical state space has been specified, but leaves open how one might go about figuring out which states to deem physical. I will argue that what I call the *requirement of reductive explanations* provides a way of filling this gap and deciding which states to consider as physical.

The central contribution of this paper is thus the proposal of a philosophically significant heuristic method for constructing quantum theories. The proposal is, roughly, that we should aim to construct quantum theories in which the classical limits of physical quantum states are physical classical states. In §2, I will provide some background to the proposal from discussions of physical possibility and reductive explanations in the literature. In §3, I will state the central proposal in a schematic form and provide a general, context-independent argument in an attempt to justify it. However, I will argue that the general formulation of the proposal is significantly limited and one should instead take up the proposal on a case by case basis. This motivates further discussion of two context-specific applications in the following sections. In §4, I discuss recent work by Feintzeig ([2018b]) showing that the proposal leads to the appropriate answer for systems of finitely many free particles, in which the quantum theory can be understood through a regular representation of the Weyl algebra. In §5, I discuss recent work by Browning and Feintzeig ([2019]) showing that the proposal leads to the appropriate answer for systems of particles moving in an external gauge field, in which the quantum theory can be understood through the charge superselection structure of an algebra of observables. Finally, in §6 I will draw some conclusions from what has been said for discussions in philosophy of science concerning heuristics and scientific discovery.

2 Background

To begin, I introduce the conceptual issues that play a role in the main proposal of the paper: the notion of *physical reasonableness* and the *requirement of reductive explanations*.

2.1 Physical states

This paper aims to leverage the following fact: we often do not treat all states allowed by a physical theory as physically possible in the same way.[1] For example, it is common practice in general relativity to rule out as physically unreasonable models of spacetime that have certain kinds of singularities, extensions, or violations of determinism. [2] Similarly, in quantum

[1]I will use the terms 'physical', 'physically possible', and 'physically reasonable' synonymously.

[2]For an overview, see Earman ([1995]) or Manchak ([2013]); for a sampling of recent work, see Manchak ([2009], [2011], [2014], [2016]).

theories a variety of necessary conditions have been proposed for a quantum state to be physically reasonable—for example, we might restrict attention to locally definite states (Arageorgis [1995], p. 168), states accessible from Fock states (Petz [1990], p. 29), DHR states (Haag [1992]; Halvorson [2007]), normal states (Halvorson [2001]; Ruetsche [2011b]; Arageorgis *et al.* [2017]), locally normal states (Dell'Antonio *et al.* [1966]; Hugenholtz and Wieringa [1969]; Takesaki and Winnink [1973]), or regular states (Halvorson [2004]; Feintzeig *et al.* [2019]; Feintzeig and Weatherall [2019]). The details of the last condition will play a role in what follows, but before discussing any particular cases, I want to first try to make sense of this somewhat curious practice.

The practice of restricting what we consider the physically reasonable states is particularly puzzling when viewed from the perspective of what Ruetsche ([2011a], p. 6) calls 'the standard account' of interpreting physical theories. According to the standard account, the content of a physical theory is a specification of the worlds it deems physically possible, as distinguished from the vast background of logical or metaphysical possibilities. If a physical theory, by specifying a collection of states that satisfy its laws, already fixes what it deems physically possible, one might wonder what further we might be doing when restricting to a collection of physical states that form a subset of those allowed by a physical theory. On the standard account, the collection of states allowed by a physical theory *already* specifies the physical possibilities, so what further restriction could be required?

I believe one can understand the practice of restricting attention to a collection of physical states through an alternative to 'the standard account' proposed by Ruetsche ([2003], [2011a]) herself. Ruetsche opposes the standard account by arguing that the notion of physical possibility is *context-dependent*, which implies that the collection of worlds a theory deems possible cannot be specified once and for all.[3] This leaves room for one to restrict attention to a subspace of physical states if the context calls for it. I won't commit to Ruetsche's particular view of physical possibility, but I will argue later that the way we apply heuristics for theory construction based on physical state spaces should be sensitive to contextual details. So it is

[3]Similar themes are familiar from elsewhere in the literature—e.g., Wilson ([2006], [2013]). See also, e.g., Williams ([2018]) for other reasons to reject the standard account.

worth exploring a view like Ruetsche's in some detail to start.

On Ruetsche's view, one can associate many different notions of physical possibility with a given theory. One notion of physical possibility might be the most general kind compatible with the kinematics of the theory, while another notion of physical possibility might be specific to a class of states that we think the actual world lies in. For example, we might think that while certain singular spacetimes are possible in the most general sense, they are not in the more specific sense of possibility governing cosmological models with a well-defined time evolution. Or, while we might think that distinct ground states breaking the electroweak symmetry in the standard model of particle physics are equipossible in a general sense, only one of these states is possible in the more specific sense of being dynamically accessible from the actual world. On Ruetsche's alternative view, then, it makes sense to distinguish a class of physical states from all of the states allowed by a physical theory because the physical states are possible according to a more specific notion of possibility than the notion of possibility for all states of the theory.

In the rest of this paper, I hope to understand the vague phrase 'physically reasonable state' in a way that at least allows it to be made precise in various context-dependent ways. This should not be taken to mean that I endorse Ruetsche's controversial arguments for her conclusion, although I do find the conclusion attractive. Instead, I hope to remain agnostic about what it means in general for a state to be physically reasonable. I will do so by allowing the notion to be specified in a variety of context-dependent ways.

One might wonder whether one can say anything interesting in general about physical states prior to focusing on a specific context. I think one *can*, with the aim of discovering what kinds of tools or strategies one might employ in the context-specific cases, as I will do in the examples of §4 and §5. Ultimately, the central proposal of this paper aims at describing how considerations about physical states provide a heuristic for theory construction. As a heuristic, the best we can hope for is an understanding of how and why it might be applied in possibly different ways in different contexts to yield fruitful, but defeasible results.

In fact, the results of Feintzeig ([2018a]) show us one way in which a choice of physical states might be employed to aid theory construction. Feintzeig shows that if one is given a physical theory formulated in an algebraic framework, and then one specifies a subspace of the state space corresponding to the physical states, there is a general procedure for constructing a new theory that allows for only physical states. I will describe this procedure in some detail because I refer to it throughout this paper.

The background to the procedure for restricting state spaces is the algebraic framework for physical theories. A theory formulated in the algebraic framework specifies a C*-algebra $\mathfrak{A}$ of physical quantities.[4] This framework is flexible enough to encompass both classical and quantum theories. In classical theories, we use commutative C*-algebras corresponding to collections of functions on a system's phase space with pointwise algebraic operations. In quantum theories, we use non-commutative C*-algebras corresponding to collections of operators on a Hilbert space. States in both theories can be understood as positive, normalized linear functionals on the C*-algebra of physical quantities. The state space $\mathcal{S}(\mathfrak{A})$ forms a subset of the dual space $\mathfrak{A}^*$ of bounded linear functionals on $\mathfrak{A}$.

Suppose that one has reason to take only a subset of $\mathcal{S}(\mathfrak{A})$ to represent the physically reasonable states. The subset of physically reasonable states will generate a subspace V of the space $\mathfrak{A}^*$ of linear functionals. Feintzeig ([2018a]) shows that under very general conditions on V (namely, V is weak* closed, and its annihilator $N(V)$ forms a closed two-sided ideal in $\mathfrak{A}$),[5] one can construct another C*-algebra $\mathfrak{A}/N(V)$ that allows for a natural identification of the physical states in V with the entire state space of this new algebra. This construction is indeed employed in mathematical physics when dealing with constrained systems (Grundling and Hurst [1998]; Grundling [2006]).

The construction Feintzeig describes leads to a new kinematical framework of physical quantities in the C*-algebra $\mathfrak{A}/N(V)$ and a new, restricted collection of possible states in the

[4]For mathematical background on C*-algebras, see, e.g., Kadison and Ringrose ([1997]), Sakai ([1971]). For applications of C*-algebraic tools to quantum theory, see, e.g., Emch ([1972]), Haag ([1992]), Bratteli and Robinson ([1987]), Bratteli and Robinson ([1996]). For philosophical introductions to algebraic quantum theory, see, e.g., Arageorgis ([1995]), Halvorson ([2007]), Ruetsche ([2011a]).

[5]Actually, the second example in §5 demonstrates that these conditions on V just stated need not even be satisfied for this construction to be possible.

dual space $(\mathfrak{A}/N(V))^* \cong V$, corresponding to precisely the states that were initially deemed physically reasonable. This construction provides a tool that one can employ in quite general circumstances for the construction of quantum theories. Thus, even without a definitive statement of what counts as a physically reasonable state, we know that *however* one specifies a collection of physical states (subject to a few technical restrictions), one can use this information to aid theory construction.

2.2 Reductive explanations

Now that we've narrowed our goal to discerning a collection of physically reasonable states, the next task is to investigate what kind of information might help us achieve this goal. For this, I turn to constraints of intertheoretic reduction.

Although some classical accounts (e.g., Nagel ([1961], [1998])) present intertheoretic reduction as a relation deriving a higher level theory from a lower level theory, Nickles ([1973]) argues that there is a different notion of reduction at play when previous theories are obtained as limiting cases of new theories. One example of this kind of limiting relation appears in the reduction of general relativity to Newtonian gravitation in the limit where $v/c << 1$. According to Nickles, these limiting reductions 'might be said to explain why the predecessor theory worked as well as it did' (Nickles [1973], p. 185, fn. 4). Although Nickles (p. 201) is skeptical of the significance of limiting relations in which physical constants vary (of which the classical $\hbar \to 0$ limit analyzed in this paper is one example), Fletcher ([2019]) argues to the contrary that the $c \to \infty$ limit of general relativity may be interpreted as the same approximation of low relative velocities just mentioned (see also Ehlers [1998]). On Fletcher's account, limiting relations can be understood to capture a sense in which the reduced theory *approximates* the reducing theory.[6]

In these limiting explanations, the phenomenon to be explained is the success of the predecessor theory. If we desire that newly constructed theories have explanatory power, it is natural to hope that new theories have the ability to explain this success; that is, one might consider it a constraint on any proposed successor theory that it explain the success of its

[6] Cf. Nickles ([1973], p. 194-6); Schaffner ([1967]).

predecessor theory. I call this constraint the *requirement of reductive explanations*.

Some authors have explicitly proposed principles for theory construction that resemble this requirement of reductive explanations. For example, Post ([1971]) proposes this requirement as the heuristic he calls the 'generalized correspondence principle' (p. 228-235). And Hesse ([1961], [1970]) argues for the necessity of correspondences between new and old theories in terms of analogies between models. In the case at hand, the $\hbar \to 0$ limit provides one kind of analogy or correspondence between classical and quantum theories by establishing that the theoretical or mathematical structures are similar with respect to the notion of similarity implicitly encoded in the limit (cf. Fletcher [2016]).

There is ambiguity, however, regarding what it means to understand the requirement of reductive explanations as a heuristic for theory construction. On the one hand, this means that we can and ought to use the principle to guide the construction of new theories. But Hesse ([1961], p. 55-56) argues that the existence of an appropriate analogy or correspondence is *not* sufficient to justify the newly constructed theory. One reason is that any procedure for constructing new theories is fallible; if a new theory makes a prediction that is not borne out by experiment, then this could overrule the heuristics that led to the theory. However, I think we have *some reason* to prefer or pursue theories that satisfy the requirement of reductive explanations, even if these reasons are defeasible. To capture the heuristic value of the requirement of reductive explanations, I'll say that satisfying the requirement makes a candidate theory more *plausible*.[7]

These ideas about reductive explanations, correspondence, and analogies have also been taken up in the literature already specifically in connection with the construction of quantum theories. For example, Hesse ([1952]) uses analogies between quantum mechanics and classical physics to illustrate her ideas about models and contrast her view with operationalism. Further, Post's ideas about correspondence are taken up by Radder ([1991]) in connection with the construction of quantum theories. Since explanations of the success of previous theories might take many different forms in different domains or different contexts,

[7] See Bertolaso and Sterpetti ([2017]) for discussion of plausibility and evidence, and see Fraser ([2019], p. 12) for discussion of plausibility and analogies.

we learn something by analyzing whether and how quantum theories explain the success of their classical predecessors.

One can attempt to explain the success of a classical theory from a quantum theory by employing the $\hbar \to 0$ limit.[8] In order to satisfy the requirement of reductive explanations, this limit would need to give rise to explanations of the success of the mathematical and theoretical structure of the classical theory (more in §3). Radder argues that the $\hbar \to 0$ limit cannot accomplish this task because it gives rise to a merely formal correspondence. In the same spirit, Rosaler ([2015b], [2015a], [2016]) has argued that quantum theories should only be thought of as explaining the empirical success of classical theories in a roughly instrumentalist fashion. If Radder and Rosaler were correct, then one might worry about whether the requirement of reductive explanations could or should be used as a heuristic tool to obtain substantive constraints on the construction of new theories. However, Feintzeig ([2019b]) argues that one can explain some aspects of the theoretical structure of classical theories from quantum theories by employing the tools of strict deformation quantization to analyze the $\hbar \to 0$ limit. Thus, while historically there is little doubt that the $\hbar \to 0$ limit has been employed during the construction of quantum theories, there is some disagreement about its significance.[9]

In this paper, I follow Feintzeig ([2019b]) and pursue the requirement of reductive explanations in quantum theories under the assumption that the $\hbar \to 0$ limit *can* be used to provide explanations of the theoretical or mathematical structure of corresponding classical theories by using strict deformation quantization. Although many of the technical details will not be relevant to the discussion of this paper, it is important to know that a strict deformation quantization provides precisely the mathematical tools needed to analyze the classical limits

[8] I analyze the $\hbar \to 0$ limit, but there are other limits that may also be relevant—for example, $N \to \infty$. Landsman ([2007]) argues that multiple limits might be required to explain the success of classical physics.

[9] A recent reference that discusses the history of the construction of quantum theories with attention to the role and context-dependence of scientific understanding is De Regt ([2017], especially Ch. 4 & 7).

of quantities and states in quantum theories in the C*-algebraic framework.[10]

A *strict deformation quantization* consists in a family $\{\mathfrak{A}_\hbar\}_{\hbar\in[0,1]}$ of C*-algebras and a family of quantization maps $\{\mathcal{Q}_\hbar : \mathcal{P} \to \mathfrak{A}_\hbar\}_{\hbar\in[0,1]}$ on a dense (Poisson) subalgebra $\mathcal{P}$ of $\mathfrak{A}_0$, which is required to be commutative. Here, the algebra $\mathfrak{A}_0$ represents the collection of quantities in a classical theory while the (non-commutative) algebras $\mathfrak{A}_\hbar$ for $\hbar > 0$ represent the collection of quantities in a corresponding quantum theory. Furthermore, for such a structure to be a strict deformation quantization, the quantization maps are required to satisfy axioms that ensure the continuity of algebraic operations as $\hbar \to 0$ and encode the approximate satisfaction of the canonical commutation relations. Within this framework, the classical limit of a family of quantities $\{\mathcal{Q}_\hbar(A)\}_{\hbar\in[0,1]}$ will be understood to be the classical quantity $A \in \mathfrak{A}_0$. Feintzeig ([2019b]) argues that this understanding of the classical limits of quantities is appropriate for explaining the success of classical theories. Moreover, this notion of classical limits of quantities gives rise to a notion of classical limits of states. A family of states $\{\omega_\hbar \in \mathcal{S}(\mathfrak{A}_\hbar)\}_{\hbar\in[0,1]}$, is called a *continuous fields of states* when for each $A \in \mathfrak{A}_0$, the map $\hbar \mapsto \omega_\hbar(\mathcal{Q}_\hbar(A))$ is continuous. The classical limit of a continuous field of states will be understood to be the classical state ω_0.

In what follows, I will take on this framework for analyzing the classical $\hbar \to 0$ limit in quantum theories via strict deformation quantization, and I will assume that the classical limit, when understood in this way, can provide genuine explanations of the mathematical and theoretical structures of classical physics. The remaining task is now to investigate the *extent* of the theoretical structures in classical physics that one can explain, and how requiring such explanations constrains theory construction.

This at last puts us in a position to bring together the two philosophical strands of background that are required to state the main proposal of the paper. One of the theoretical structures in a classical theory is its collection of physically reasonable states, understood as a subset of the entire state space. A reductive explanation of the success of a classical theory from a quantum theory ought to explain why the classical theory contains the physical state

[10]For a mathematical introduction to strict deformation quantization, see Rieffel ([1989], [1993], [1994]) or Landsman ([1993a], [1998], [2007], [2017]).

space that it does. Enforcing the requirement of reductive explanations as a heuristic for theory construction places the constraint that a newly constructed quantum theory give an explanation of the physical state space of its preceding classical theory. The central proposal of the paper, which we state next in the following section, is an analysis of the conditions under which one can give such an explanation of the physical state space of a classical theory from a quantum theory using the $\hbar \to 0$ limit.

3 The proposed 'correspondence principle'

The central proposal of this paper provides necessary (but not sufficient) conditions for a quantum theory to be able to explain the success of its predecessor:

Proposal. *A necessary condition for a quantum theory to explain the success of its classical predecessor is that:*

(i) every physical classical state is the classical limit of some appropriate family of physical quantum states; and

(ii) the classical limit of every appropriate family of physical quantum states is a physical classical state.

I claim only that conditions (i) and (ii) are required for a quantum theory to explain the success of one theoretical structure in the classical theory—its collection of physically reasonable states. Further resources may be required to explain other aspects of the theory.

I will attempt to provide a general argument to motivate each of these conditions in this section; however, I do not believe this general motivation suffices. Instead, I will argue it is best to take up the proposal in specified contexts where further information helps determine what is meant by a physically reasonable state and an appropriate family of states. I think it is worth first considering the motivation for conditions (i) and (ii) in a context-independent manner. But it is even more important to analyze these conditions in the context of specific examples, as I will do in the following sections.

Condition (i) is the more familiar constraint, stating that for a quantum theory to explain the success of its classical predecessor, it must be able to recover every physical classical state as the classical limit of some physical quantum state. To motivate this condition, notice that part of the explanatory burden of the newly constructed quantum theory is to show why the predictions and explanations given using each physical classical state are at least approximately accurate. If (i) were not satisfied, then the quantum theory would not be able to discharge this explanatory burden—there would be physical classical states that could not be recovered even approximately in the $\hbar \to 0$ limit from physical quantum states. The quantum theory would not be able to explain why the predictions and explanations given using those missing physical classical states were successful, and hence, would not be able to explain the success of the classical theory. Therefore, condition (i) is required for a quantum theory to explain the success of its classical predecessor.

The perhaps more controversial condition (ii) states that for a quantum theory to explain the success of its classical predecessor, it must recover *no more* than the physical classical states as the classical limits of physical quantum states. Condition (ii) has a different character; it is not concerned with whether the quantum theory can provide local explanations of the success of individual physical classical states. Instead, condition (ii) is motivated by considering how a quantum theory might give a global explanation of why a certain subcollection of states in the classical theory is deemed physically reasonable. I think that for a quantum theory to give an explanation of the success of its classical predecessor, it must explain why the classical theory employed the mathematical and theoretical structure it did. Part of the theoretical structure of the classical theory is the collection of states it deems physically reasonable. Thus, it is incumbent upon the quantum theory to explain why the classical theory has the physical state space it has. If the quantum theory could recover *un*physical classical states—ones lying outside the physical state space—as the classical limit of physical quantum states, then the quantum theory would seem to allow these states as approximations to the states the quantum theory deems physically possible. In order to explain why the classical theory has the physical state space it has, the quantum theory must only allow the physical classical states to be obtainable as approximations to physical quantum states in the classical limit.

One might object to condition (ii) on the grounds that once one has recovered from a new theory all of the empirical predictions of the old theory, one has captured all one needs to explain. If this were correct, condition (i) would suffice on its own. I believe, however, that condition (ii) is necessary because there are more phenomena the new theory needs to capture than just the predictions of the old theory. It is also an *empirical phenomenon* that researchers using the mathematical and theoretical framework of the old theory have arrived at accurate predictions and useful applications. I believe this phenomenon—that scientists using the theoretical tools of the old theory have been successful—also ought to be explained. The particular tools that scientists used as part of the old theory may be historically contingent, and the explanation may involve appeal to pragmatic features, but nonetheless I think an explanation is needed of why these tools have been successful. One of the theoretical tools of the old theory whose success must be explained is the splitting of the state space into physical and unphysical states. Satisfying condition (ii) amounts to providing an explanation of why scientists have been successful when employing the theoretical tool of a physical subspace of the state space of the old theory.

For this motivation of condition (ii) to be plausible, one must already accept the assumption made in §2.2 that explanations of the success of classical theories involve the explanation of the theoretical structure of those theories. Only on this assumption does it make sense to require that a quantum theory be able to explain the success of the theoretical structure given by the physical state space of a classical theory. If one thought to the contrary that quantum theories could only explain the empirical predictions of their corresponding classical theories, as Rosaler ([2015a]) seems to, then such a requirement would not make sense. However, on the assumption that genuine explanations of the theoretical structure of classical theories are possible and desirable, I believe condition (ii) is well motivated.

Before proceeding, it is worth dealing with an apparent objection to condition (ii).[11] One might worry that there is an obvious counterexample in the case of spontaneous symmetry breaking in many-body spin systems or in the analogous case of the measurement problem. Landsman ([2013], [2017]) discusses models for each of these examples, and at first glance, the classical or thermodynamic limit of the physical quantum ground state may appear to be an unphysical classical state. In the case of a many-body spin system on a one-dimensional

[11] I am grateful to an anonymous referee for pointing out this objection.

lattice, there are two states of broken symmetry in which the spins all align pointing 'up' or all align pointing 'down', but the ground state is a superposition of these two aligned states. The thermodynamic limit of this pure quantum state is, however, the mixture of the two corresponding states with aligned spins. Similarly, in the simple model for measurement with a double-well potential, there are two states in which the wavefunction is centered in the left and right potential wells, respectively, but the ground state is a superposition of the wavefunctions in each well. Further, the classical limit of this pure quantum state is again the mixture of the two classical states located at the minima of each potential well. If one thought these mixtures obtained in the limit were unphysical, then one might have reason to reject condition (ii) of the proposal because they are classical limits of physical states.

I do not believe, however, that this objection succeeds. First, as far as I can tell, this objection relies on the premise that a mixture of two physical states is not a physical state. This assumption seems, at least to me, to be very counterintuitive. Of course, whether one accepts this premise may depend on how exactly one makes precise the notion of a physical state, but I believe that, at least in some contexts, mixed states should be countenanced as physical because of their usefulness in both classical and quantum statistical mechanics. Moreover, in all of the examples and results discussed in this paper, I focus only on the linear space V generated by the physical states, which of course contains all mixtures of the physical states. Second, Landsman and Reuvers ([2013]) and Landsman ([2013], [2017]) establish that almost any small external perturbation will force an 'effective collapse' of these systems into one of the symmetry breaking states on the way to the limit, which results in a pure state at the limit. This further technical result shows that the limit of the physical ground states *can*, when understood with external perturbations, lead to a physical state. This shows the importance of thinking of the classical limit as a limit of *an entire family* of states. The appropriate family of states is influenced by contextual details including the possibility of external perturbations. It is only when one thinks of appropriate families of states with external perturbations that one can see the instability of the mixed ground state in the limit. Thus, when one takes into account further contextual information about the systems of interest to determine the appropriate families of states, one finds that that the central proposal is vindicated. Indeed, this same point is a theme I hope to illustrate and emphasize in the examples of the following sections.

My contention in this paper is that the central proposal stated above can and should be used to constrain the construction of quantum theories, as follows. Suppose we have constructed a quantum theory whose algebra of quantities is given by $\mathfrak{A}$. Let $V \subset \mathfrak{A}^*$ be the subspace generated by the collection of states whose classical limits are physical states according to the corresponding classical theory. These are the states of the quantum theory that should be deemed physical quantum states in order to explain the success of the classical theory, according to the central proposal. So with the aim of constructing a quantum theory whose entire state space consists in physical states, we should replace the original algebra of quantities $\mathfrak{A}$ with $\mathfrak{A}/N(V)$ in order to restrict attention to physical states.

Notice that using the central proposal in this way to constrain theory construction is only possible if we have antecedent knowledge of what states in the corresponding classical theory are physically reasonable. This is acceptable because we typically have a better grasp on the interpretation and significance of states in classical theories than in quantum theories, or new theories we are constructing. But if we do not antecedently know which classical states are physically reasonable, we might discern this by deeming a classical state physical when it is obtained as the classical limit of a physical quantum state. Likewise, we might decide that a classical state is unphysical when it cannot be obtained as the classical limit of a physical quantum state. This is another genuine function of the classical limit—to help us better understand existing classical theories. However, this function of the classical limit in informing interpretations of classical physics is distinct from the role the classical limit can play in theory construction. The use of the central proposal in theory construction projects forward an understanding of which states are physically reasonable in an existing classical theory. Projecting forward in this fashion, as the quote from Stein ([1989]) in the epigraph of this paper suggests, makes sense only when we are more confident about our understanding of the existing classical theory than the quantum theory we are in the process of constructing.

Recognizing this helps us see one way in which using the central proposal to constrain theory construction is defeasible. Our confidence in this heuristic can at best be as high as our confidence that we have picked out the appropriate collection of physically reasonable states in the existing classical theory. To the extent that the classical states we consider physically reasonable might be revised, the procedure for constructing a quantum theory based on the existing classical theory might also be revised. In this sense, the requirement of reductive explanations, when applied to explanations of physical state spaces, should not be understood to produce certainty in the theories we construct.

This form of the requirement of reductive explanations is also subject to the subtleties mentioned in §2—namely, it is a heuristic principle whose satisfaction does not justify a new theory, but rather makes it more plausible. Further, the notion of physical reasonableness in both classical and quantum theories should be understood contextually. Thus, the proposal should be understood to provide a defeasible tool that might be applied in different context-sensitive ways.

It might be surprising, then, that the arguments I have just provided for conditions (i) and (ii) are completely context-independent. I think this shows a severe limitation of those arguments because they do not take into account the context-specific details concerning why certain states are deemed physically reasonable for certain systems. For the proposal to be well-motivated, the $\hbar \to 0$ limit must teach us that unphysical quantum states fail to be physically reasonable for *the same reasons* the corresponding classical states fail to be physically reasonable. The success of the proposal thus depends on context-specific notions of physical possibility. As such, the main proposal and the arguments for it should be understood only schematically, so that they are to be filled in for specific systems of interest.

I will now take up two such specific examples in the remainder of this paper. The examples illustrate the main proposal, and how further contextual information helps us make sense of when certain collections of states are deemed physically reasonable. In the two examples presented in §4 and §5, when the requirement of reductive explanations is implemented in conjunction with the main proposal, it gives rise to the construction of an appropriate quantum theory for the system of interest. These successful results provide some evidence in favor of the main proposal and its heuristic use for theory construction. This evidence should not be understood as underlying an inductive inference because I will only consider two examples, which forms a sample size that is much too small. Instead, I take the examples to show the promise of the main proposal when it is treated with care in specific contexts.

4 Example: regularity

My first example to illustrate the usefulness of the main proposal is the quantization of the theory of a finite collection of free particles via the Weyl algebra (Manuceau *et al.* [1974]; Petz [1990]; Clifton and Halvorson [2001]). In this case, it is standard practice in mathematical physics to restrict attention to *regular* states on the Weyl algebra to represent physically reasonable quantum states. However, this orthodoxy has been challenged, for example, by Halvorson ([2004]), and some have even pursued the construction of physical theories involving non-regular states (Ashtekar [2009]; Corichi *et al.* [2007]).[12] Here, I will argue that recent work of Feintzeig ([2018b]) illustrates an application of the main proposal to rule out non-regular states in quantum theories, motivating the construction of quantum theories that allow for only regular states by using a different algebra than the Weyl algebra.

We aim to construct a quantum theory corresponding to a classical system whose Hamiltonian formulation is given by the phase space $\mathbb{R}^{2n}$, understood as a symplectic vector space with the standard symplectic form σ. The algebra of quantities for the corresponding quantum theory is specified by the Weyl algebra $\mathcal{W}(\mathbb{R}^{2n}, \hbar\sigma)$, defined as the smallest C*-algebra containing the elements $W_\hbar(x)$ for each $x \in \mathbb{R}^{2n}$ with

$$W_\hbar(x)W_\hbar(y) := e^{-\frac{i\hbar}{2}\sigma(x,y)}W_\hbar(x+y)$$
$$W_\hbar(x)^* := W_\hbar(-x)$$

for all $x, y \in \mathbb{R}^{2n}$. It is known that there is a unique C*-algebra picked out by these conditions which corresponds to the completion of the algebra generated by the linearly independent elements $W_\hbar(x)$ in the so-called minimal regular norm (Manuceau *et al.* [1974]). The algebra $\mathfrak{A}_\hbar := \mathcal{W}(\mathbb{R}^{2n}, \hbar\sigma)$ specifies the kinematical framework of a quantum theory for each

[12] See also Beaume *et al.* ([1974]), Fannes *et al.* ([1974]), Acerbi *et al.* ([1993]) for mathematical investigations of non-regular states.

numerical value $\hbar \in [0,1]$ of Planck's constant.

The classical limit of this quantum theory is specified by the algebra $\mathfrak{A}_0 := \mathcal{W}(\mathbb{R}^{2n}, 0)$, which is known to be *-isomorphic to the collection of *almost periodic functions* on $\mathbb{R}^{2n}$ (Binz *et al.* [2004a]). The C*-algebra of almost periodic functions is the uniform closure of the collection of all finite linear combinations of functions of the form $W_0(x) : \mathbb{R}^{2n} \to \mathbb{C}$ for each $x \in \mathbb{R}^{2n}$ defined by

$$W_0(x)(y) := e^{ix \cdot y}$$

for all $y \in \mathbb{R}^{2n}$, where $\cdot$ signifies the standard inner product on $\mathbb{R}^{2n}$.

That $\mathfrak{A}_0$ should be understood as the classical limit of a quantum theory specified by the Weyl algebra is justified by the fact that the family of algebras $\{\mathfrak{A}_\hbar\}_{\hbar\in[0,1]}$ forms a strict deformation quantization (Binz *et al.* [2004b]; Honegger and Rieckers [2005]; Honegger *et al.* [2008]) with the quantization maps $\{Q_\hbar : \mathfrak{A}_0 \to \mathfrak{A}_\hbar\}_{\hbar\in[0,1]}$ defined as the norm continuous linear extension of[13]

$$Q_\hbar(W_0(x)) := e^{-\frac{\hbar}{4}||x||^2} W_\hbar(x)$$

where $||x||^2 := x \cdot x$. These maps define the restriction of Berezin quantization to the almost periodic functions (Berezin [1975]; Berger and Coburn [1986]). Since this structure forms a strict deformation quantization, we can analyze classical limits of states by specifying continuous fields of states. There is a natural way to specify a continuous field of states given a quantum state $\omega_\hbar \in \mathcal{S}(\mathfrak{A}_\hbar)$. Namely, $\omega_0 := \omega_\hbar \circ Q_\hbar$ defines a classical state on $\mathfrak{A}_0$ since $Q_\hbar$ is positive. And further, the map $\omega_{\hbar'} : \mathfrak{A}_{\hbar'} \to \mathbb{C}$ defined as the continuous extension of $\omega_{\hbar'}(Q_{\hbar'}(A)) = \omega_0(A)$ for all $A \in \mathcal{P}$ specifies a quantum state for any alternative numerical value $\hbar' \neq \hbar$ of Planck's constant. The collection $\{\omega_\hbar\}_{\hbar\in[0,1]}$ then forms a continuous field of states,

[13]There is actually a mistake in Feintzeig ([2018b]) who is missing the exponential factor, which makes the quantization maps positive, and hence norm continuous, as he claims. Positivity allows for the continuous extension of the quantization map to the domain $\mathfrak{A}_0$ and for the analysis of classical limits of states below.

which we will call the *constant field* defined by $\omega_\hbar$, and so the classical state ω_0 can be understood as the classical limit of the quantum state $\omega_\hbar$. Of course, one might define other classical limits of the state $\omega_\hbar$ through different continuous fields of states, and one hopes that the results below generalize to other such classical limits.

In this setup, the algebra $\mathfrak{A}_0$ of the classical theory has a plethora of states that would ordinarily be considered unphysical classical states. The countably additive probability measures on the phase space $\mathbb{R}^{2n}$ form a natural collection of physical states that can be treated using the ordinary methods of classical Hamiltonian mechanics. These states represent either definite positions and momenta specified by a single point in the phase space, or else probabilistic mixtures of such definite phase space points. However, there are other states on $\mathfrak{A}_0$ called 'states at infinity' that cannot be represented by any countably additive probability measure on $\mathbb{R}^{2n}$. I think it is natural to call the countably additive probability measures the physically reasonable states and deem all other states unphysical. For example, one indication the classical states at infinity are unphysical is that it is not clear how to understand their dynamical evolution. Given that these states cannot be represented as countably additive probability measures on phase space, one cannot use them as initial conditions for the standard differential equations of Hamiltonian mechanics.

Given this collection of physical states, we can apply the procedure of §2.1 to reduce the state space of the classical theory. Let $V_C \subseteq \mathfrak{A}_0^*$ consist in the subspace generated by the collection of states that can be represented as countably additive probability measures on $\mathbb{R}^{2n}$. For technical reasons, one needs to first enlarge the algebra $\mathfrak{A}_0$ to the universal enveloping W*-algebra or bidual $\mathfrak{A}_0^{**}$, which can be understood as its bounded completion in the weak* topology (Feintzeig [2019a]). Then the algebra $\mathfrak{A}_0^{**}/N(V_C)$ obtained by restricting attention to the collection of physical states is *-isomorphic to the collection of all bounded measurable functions[14] on $\mathbb{R}^{2n}$, which is a W*-algebra whose normal state space consists in all and only the countably additive probability measures on $\mathbb{R}^{2n}$. The algebra of measurable functions is itself the bidual or bounded completion of the algebra $C_0(\mathbb{R}^{2n})$ of continuous functions on $\mathbb{R}^{2n}$

[14]Specifically, one must consider functions that are measurable with respect to the σ-algebra of universally Radon measurable sets (Fremlin [2003]).

vanishing at infinity. Every state on this algebra can be represented by a countably additive probability measure on $\mathbb{R}^{2n}$. Thus, there is an alternative algebra of classical quantities in which we can understand the standard formulation of the theory allowing for only the physical states, which are the countably additive probability measures.

We can use the collection V_C of physical classical states to constrain the construction of our quantum theory. Let $V_Q \subseteq \mathfrak{A}_\hbar^*$ be the subspace of the dual to the algebra of quantities for the quantum theory generated by the quantum states whose classical limits, understood in the sense of the constant fields of states defined above, lie in V_C. Feintzeig ([2018b]) proves that when we take the classical limit via a constant field of states, the classical limit of a quantum state $\omega_\hbar$ on $\mathfrak{A}_\hbar$ lies in V_C if and only if $\omega_\hbar$ is *regular*. For fixed $\hbar$, a state $\omega_\hbar \in \mathcal{S}(\mathfrak{A}_\hbar)$ is defined to be regular just in case the map

$$t \in \mathbb{R} \mapsto \omega_\hbar(W_\hbar(tx))$$

is continuous for each $x \in \mathbb{R}^{2n}$. Regular states may be familiar from the Stone-von Neumann theorem (see, e.g., Summers [1999]; Ruetsche [2011a]), which states that the GNS representation of any regular state is quasi-equivalent to the Schrödinger representation on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$. Since the classical limit of a state on $\mathfrak{A}_\hbar$ is physical just in case that state is regular, the central proposal suggests that to explain the success of the physical state space of the classical theory, we should understand the physical states on the Weyl algebra to be precisely the regular states.

This constrains theory construction by guiding us to use a different algebra whose irreducible representations are all unitarily equivalent to the Schrödinger representation. As in the classical case, for technical reasons we first need to enlarge the algebra $\mathfrak{A}_\hbar$ to the bidual or bounded completion $\mathfrak{A}_\hbar^{**}$. Then restricting attention to regular quantum states, we find $\mathfrak{A}_\hbar^{**}/N(V_Q)$ is *-isomorphic to the W*-algebra $\mathcal{B}(\mathcal{H})$, which is the bidual or bounded completion of the algebra $\mathcal{K}(\mathcal{H})$ of compact operators. Every normal state on $\mathcal{B}(\mathcal{H})$, or equivalently every state on $\mathcal{K}(\mathcal{H})$, can be represented by a density operator on $\mathcal{H}$. Thus, restricting our attention to regular states leads us to an alternative algebra, which specifies the standard formulation of the quantum theory of a finite number of free particles in terms of density operators in the Schrödinger representation. In other words, the requirement of reductive explanations and the central proposal of this paper lead to the appropriate theory.

I believe that in this example, there is additional contextual information that helps us understand why the collection of regular states, whose classical limits are countably additive probability measures, should be understood as the physical quantum states. As I will now argue, the central proposal draws our attention to analogies between states in classical and quantum theories, so that we can understand non-regular quantum states as unphysical in the same way we understand classical states at infinity as unphysical.

Since there are challenges to understanding the dynamical evolution of classical states outside V_C, the result that non-regular quantum states lying outside V_Q have these unphysical classical states at infinity as their limits suggests there may be challenges to understanding the dynamical evolution of non-regular quantum states. Indeed, this is borne out by the fact that one cannot straightforwardly apply the dynamics of the Schrödinger equation to non-regular states. There have been suggestions concerning how to extend the Schrödinger dynamics to non-regular states on the Weyl algebra (see, e.g, Fannes and Verbeure [1974]; Narnhofer and Thirring [1993]), but Feintzeig *et al.* ([2019]) argue that any such attempt to deal with dynamics for non-regular states will face interpretive difficulties. Thus, looking at the classical limit helps us see that non-regular quantum states should be considered unphysical for essentially the same reasons that classical states at infinity are considered unphysical. It also helps us understand what we might mean by our notion of physical reasonableness in this context: physical reasonableness in this case involves at least that dynamical evolution makes sense (both mathematically and interpretively).

There is more to say about the boundaries of physical reasonableness in this case. Physical reasonableness cannot be solely about making accurate predictions within the error bounds specified by our measuring devices. What I mean by this is that if one only cares about having states that make accurate predictions, then classical states at infinity are, in a sense, just as good as classical countably additive probability measures. And likewise, regular quantum states are, in a sense, just as good as non-regular quantum states. The reason is that in each case the physical states are dense in the unphysical states in a relevant topology (Feintzeig and

Weatherall [2019]). Every classical state at infinity can be approximated within specified error bounds for a finite number of measurements—that is, with the weak* topology given by $\mathfrak{A}_0^{**}/N(V_C)$—by countably additive probability measures. Similarly, every non-regular quantum state can be approximated within specified error bounds for a finite number of measurements—that is, with the weak* topology given by $\mathfrak{A}_\hbar^{**}/N(V_Q)$—by regular states. This shows a sense in which no finite number of experiments can verify whether a system is in a physical or unphysical state, and so one cannot infer from a finite set of data that the system is in a physical or unphysical state. Thus, I believe we are left with the alternative that the notion of physical reasonableness that distinguishes regular from non-regular states concerns their theoretical role rather than their empirical predictions.

One way to understand what distinguishes physical from unphysical states in this case is that states at infinity and non-regular states are unphysical in the sense that they are idealizations from the physical states (Feintzeig and Weatherall [2019]). While the normal state space of $\mathfrak{A}_0^{**}/N(V_C)$, which is the state space of $C_0(\mathbb{R}^{2n})$, consists in only countably additive probability measures, states at infinity can arise as weak* limits of these states. Similarly, while the normal state space of $\mathfrak{A}_\hbar^{**}/N(V_Q)$, which is the state space of $\mathcal{K}(\mathcal{H})$, consists in only regular states, non-regular states can arise as weak*-limits of these states. These approximations are weak enough that if our system is in a regular state and we use a non-regular state to make predictions about future measurements, we have no guarantee that these predictions will be accurate (see Summers [1999]; Ruetsche [2011a]). This connects to my previous worries about the dynamical evolution of non-regular states; if we use a non-regular state to model the dynamical evolution of a system, we may be led to inaccurate results and predictions. This justifies our interpretation of states at infinity and non-regular states as idealizations—specifically idealizations which are inappropriate for analyzing dynamical evolution. This illustrates how contextual details about a system lead to a better understanding of what we mean by calling some states physically reasonable.

We can also see in this example how the requirement of reductive explanations is a defeasible heuristic principle. Some have proposed (Ashtekar [2009]; Corichi *et al.* [2007]) that non-regular states are important in the construction of quantum theories of gravity.

Everything that I have said so far in this section only applies to systems containing a finite number of particles and so does not strictly have implications for the gravitational field. Still, the arguments of this section suggest that treating non-regular states as physical would be a somewhat radical change of perspective in the sense that one does not have antecedent reason to think that the quantum theories constructed with non-regular states are plausible just from considering the classical theory they are constructed from. I think one would need further information, such as confirming empirical evidence or other theoretical virtues, to make these theories with non-regular states plausible. In fact, Ashtekar ([2009]) has argued that non-regular states are plausible in quantum gravity because they are required to ensure consistency with another theoretical principle in general relativity, the principle of 'background independence'. If this is correct and the theories involving non-regular states are successful, then we may have reason to revise our understanding of non-regular states, and consequently to revise our understanding of the classical states at infinity that we obtain as their classical limits. Thus, the conception of physically reasonable states that I have argued for in this section—and the role it plays in the construction of new theories—is defeasible. It can be overruled by other theoretical or evidential considerations. I think that our current understanding of the classical physics of finite collections of particles gives us good reason to interpret classical states at infinity and non-regular quantum states as unphysical, but I recognize that this understanding is fallible.

In the next section, I will take to heart the lesson that we should aim to construct quantum theories that do not allow for non-regular states, and thus we should not use the Weyl algebra. It is well known how to implement this proposal for systems with finitely many degrees of freedom using the compact operators,[15] which form a C*-algebra whose bidual is the algebra $\mathfrak{A}_\hbar^{**}/N(V_Q)$ recommended above. For this reason, I will use the algebra of compact operators in what follows as a solution to the problem of non-regular states.

[15]Further work has been done to extend similar constructions to systems even with infinitely many degrees of freedom (Grundling [1997]; Grundling and Neeb [2009]).

5 Example: symmetry-invariance

The next example I consider is the quantization of the theory of a particle moving in an external field with gauge symmetry. In this case, the quantum theory has charge superselection structure, which prevents coherent superpositions of different charges. This is encoded in the theory mathematically by the presence of unitarily inequivalent representations of the algebra of quantities for the quantum theory.[16] Unlike some other cases of unitarily inequivalent representations, I think the interpretation of charge superselection is somewhat straightforward: states with different charges are all physically possible (in a quite general sense) and genuinely distinct. So the inequivalent representations are all part of the theory and physically inequivalent in an appropriate sense.

But why do charge superselection rules appear for even a single particle system? The standard lore is that the Stone-von Neumann theorem rules out the existence of inequivalent representations for systems with finitely many degrees of freedom, so one might wonder how superselection rules are possible for a single particle. One way to understand the appearance of these superselection rules is through analysis of how the quantum theory is constructed. The superselection structure is made plausible by an application of the requirement of reductive explanations with my central proposal. In this case, the physical states turn out to be precisely those states that are invariant under gauge symmetry transformations.

We begin by formulating the classical theory for a single particle moving in an external gauge field. This theory contains the geometrical background of a principal fiber bundle with total space P, base space Q, and typical fiber given by a compact Lie group G, which acts smoothly on P on the right by R_g for $g \in G$. The base space Q represents the configuration space of the particle, while the symmetries of the field are encoded in G. Formulating the theory in the total space P allows one to understand how the particle couples to the external field (Sternberg [1977]). One could provide a Hamiltonian formulation of the theory using as phase space the cotangent bundle of the total space T^*P, but one typically instead formulates the theory by applying the Marsden-Weinstein reduction procedure (Marsden and Weinstein

[16]For more on superselection rules in the algebraic approach, see Earman ([2008]), Landsman ([1990b], [1990c]).

[1974]; Belot [2003]; Butterfield [2007]) to construct the universal phase space $(T^*P)/G$, where G acts smoothly on T^*P by the pullback R_g^*. The idea of symplectic reduction is that P contains redundant degrees of freedom that are eliminated by moving to the universal phase space. Once these redundant degrees of freedom are removed, the universal phase space $(T^*P)/G$ is merely a Poisson space, but not a symplectic space; it foliates into a collection of disjoint symplectic leaves describing different possible charges (See Weinstein [1978]; Landsman [1993b], [1998]).[17]

To put this geometrical theory into an algebraic framework, one employs the algebra $C_0((T^*P)/G)$ of continuous functions vanishing at infinity on the universal phase space. The classical state space corresponding to this algebra of quantities consists in the countably additive probability measures on the universal phase space. This algebraic formulation does not allow for states at infinity and so is compatible with the conclusions of the previous section. However, the space $(T^*P)/G$ has a somewhat complicated structure, and it is useful to formulate the construction of the quantum theory by starting from the space T^*P. Thus, we will take $\mathfrak{A}_0 := C_0(T^*P)$ as the algebra of quantities for our classical theory. If one starts with the space T^*P and uses the algebra $C_0(T^*P)$ then one generally allows for far more states corresponding to all countably additive probability measures on T^*P. It is natural to call a state on $C_0(T^*P)$ physical just in case it is determined from a state on $C_0((T^*P)/G)$ in the universal phase space. The states ω_0 on $C_0(T^*P)$ that are determined by states on $C_0((T^*P)/G)$ are precisely the states that are *symmetry-invariant* in the sense that

$$\omega_0(f \circ R_g^*) = \omega_0(f)$$

for all $f \in C_0(T^*P)$ and $g \in G$. Let V_C be the collection of all symmetry-invariant states on $C_0(T^*P)$; this will form the collection of physical classical states. Applying the construction from §2.1 to restrict attention to physical states produces the algebra $\mathfrak{A}_0/N(V_C)$, which Browning and Feintzeig ([2019]) show is *-isomorphic to $C_0((T^*P)/G)$ and thus allows for only states corresponding to countably additive probability measures on the universal phase

[17]This setup allows for more generality than the previous section because the phase space is a manifold and not necessarily a linear space.

space.[18] Thus, $\mathfrak{A}_0/N(V_C)$ provides the desired kinematical framework for the classical theory with only physical states.

We can construct a quantum theory for this system by applying a generalized Weyl quantization procedure developed by Landsman ([1993b, 1998]). We define the algebra of quantities for our quantum theory to be $\mathfrak{A}_\hbar := \mathcal{K}(L^2(P))$, the compact operators on the Hilbert space $L^2(P)$, for all $\hbar \in (0,1]$. Landsman defines a family of quantization maps $Q_\hbar : \mathcal{P} \to \mathcal{K}(L^2(P))$ on a dense subalgebra $\mathcal{P}$ of $C_0(T^*P)$. Using $\mathcal{K}(L^2(P))$ as the algebra of physical quantities for the quantum theory is also compatible with the conclusions of the previous section as this algebra does not allow for analogues of non-regular states; all states on this algebra are density operators on $L^2(P)$. The family of algebras $\{\mathfrak{A}_\hbar\}_{\hbar\in[0,1]}$ and quantization maps $\{Q_\hbar\}_{\hbar\in[0,1]}$ forms a strict deformation quantization, and so provides a way of taking the classical limit of the quantum theory of a particle in an external gauge field.

We can use the collection V_C of physical classical states to constrain the construction of our quantum theory, just as in the previous example in §4. The key is to identify which quantum states have classical limits in V_C. Towards this end, consider quantum states that satisfy a condition of symmetry-invariance in analogy with the physical states of the classical theory. The symmetry group G has a natural unitary representation $U(g)$ for each $g \in G$ on the Hilbert space $L^2(P)$ of the quantum theory given by

$$(U(g)\psi)(x) = \psi(R_g(x))$$

for all $\psi \in L^2(P)$. We will call a quantum state $\omega_\hbar$ on $\mathfrak{A}_\hbar$ *symmetry-invariant* when

$$\omega_\hbar(U(g)AU(g)^*) = \omega_\hbar(A)$$

for all $A \in \mathfrak{A}_\hbar$ and $g \in G$. Let V_Q be the subspace of $\mathfrak{A}_\hbar^*$ generated by the symmetry-invariant

[18] In this case, the procedure can be applied even though the technical conditions specified by Feintzeig ([2018a]) are not satisfied. $N(V_C)$ is not a two-sided ideal and consequently the canonical quotient projection $\mathfrak{A}_0 \to \mathfrak{A}_0/N(V_C)$ is not a *-homomorphism, but $\mathfrak{A}_0/N(V_C)$ is still a C*-algebra.

quantum states on $\mathfrak{A}_\hbar$.

In considering the classical limits of states of this system, we face a complication that did not arise in the previous example; since $Q_\hbar$ is neither positive nor continuous in the current case, there is no natural way to define the classical limit of a quantum state $\omega_\hbar$ on $\mathfrak{A}_\hbar$ via a constant field of states. So instead of specifying a unique classical limit, we will consider all possible classical limits given by continuous fields of states $\{\omega_\hbar\}_{\hbar\in[0,1]}$ subject to just one condition. If $\omega_\hbar$ is a symmetry-invariant state, then $\{\omega_\hbar\}_{\hbar\in[0,1]}$ is an appropriate family for specifying the classical limit of $\omega_\hbar$ only if $\omega_{\hbar'}$ is symmetry-invariant for each $\hbar' \in (0,1]$. The motivation for this requirement is that we interpret the classical limit along the lines of Feintzeig ([2019b]), who understands changing the numerical value $\hbar$ of Planck's constant in a quantum theory (where $\hbar > 0$) as implementing a change of units and thereby changing the extent to which states approximate one another numerically. On this interpretation, changing the value of $\hbar$ otherwise leaves the physical interpretation of states and quantities the same. Thus, on this interpretation the quantum states $\omega_\hbar$ and $\omega_{\hbar'}$ for $\hbar, \hbar' \in (0,1]$ and $\hbar \neq \hbar'$ should represent the same physical situation in different systems of units. Our constraint then requires that a quantum state $\omega_{\hbar'}$ can represent the same physical situation as a symmetry-invariant state $\omega_\hbar$ only if $\omega_{\hbar'}$ is also symmetry-invariant.

Notice that this constraint only explicitly governs the quantum states *on the way to the limit* because it deals only with states $\omega_\hbar$ for $\hbar > 0$ and leaves open how the classical limits ω_0 of these states behave at $\hbar = 0$. Thus, the constraint just mentioned only fills in some of the context of this application of the main proposal by stating when a family of quantum states is *appropriate* for representing the classical limit. However, with this constraint, Browning and Feintzeig ([2019]) prove that the classical limit of an appropriate family of symmetry-invariant quantum states in V_Q must be a symmetry-invariant classical state in V_C. Moreover, quantum states that fail to be symmetry-invariant will in general have classical limits that fall outside of V_C. The requirement of reductive explanations and the central proposal of this paper suggest that we consider only states in V_Q to be physical quantum states.

We restrict attention to physical states in V_Q in our quantum theory by replacing the algebra of quantities $\mathfrak{A}_\hbar$ with $\mathfrak{A}_\hbar/N(V_Q)$. The results of Browning and Feintzeig ([2019]) and

Landsman ([1993b]) together show that the algebra $\mathfrak{A}_\hbar/N(V_Q)$ is *-isomorphic to $\mathcal{K}(L^2(Q)) \otimes C^*(G)$, where $C^*(G)$ is the group algebra of G. This algebra has unitarily inequivalent representations, which are in one-to-one correspondence with the unitary representations of G (Landsman [1993b], p. 107-110). The algebra $\mathfrak{A}_\hbar/N(V_Q)$ is the one advocated for by Landsman because it captures the superselection structure of the quantum theory. Thus, we should understand the algebra $\mathfrak{A}_\hbar/N(V_Q)$ to afford an appropriate mathematical formulation of the theory, which again illustrates the successful application of the requirement of reductive explanations according to the central proposal of this paper.

In this context, what it means for classical states in V_C or quantum states in V_Q to be physical is rather different from what it meant in the previous example of §4. What makes a classical state unphysical when it fails to be symmetry-invariant (and hence lies outside of V_C) is that its expectation values depend on degrees of freedom that themselves do not represent genuine physical properties. For example, a classical state that fails to be symmetry-invariant for a particle moving in an external electromagnetic field with $G = U(1)$ is one that depends on the values of the electromagnetic potential over and above the values of the electromagnetic field. States that fail to be symmetry-invariant are not consistent with our understanding of the forces classical gauge fields produce on particles. A quantum state that fails to be symmetry-invariant is unphysical in the same sense that it depends on degrees of freedom that do not represent genuine physical properties.

One might worry that this understanding would rule out the dependence of quantum states on the classical electromagnetic potential. This would be undesirable because it is known that we can observe the dependence of quantum states on the classical electromagnetic potential in the Aharanov-Bohm effect (Aharonov and Bohm [1959]).[19] Certainly, it would be a problem for my approach if it led to a theory inconsistent with empirical observations.

However, restricting attention to symmetry-invariant states in the sense of restricting attention to V_Q does not immediately appear to rule out this possibility. Landsman ([1990a], [1990b], [1990c]) takes up the Aharanov-Bohm effect in a model containing the topological

[19]For philosophical discussions of the Aharanov-Bohm effect, see Nounou ([2003]), Earman ([2019]), Shech ([2017]).

idealization that the solenoid in the Aharanov-Bohm effect is infinitely long and its interior is inaccessible to the electron. He shows the kinematical algebra one obtains by implementing this constraint in the configuration space of the electron results in an algebra of the form $\mathfrak{A}_\hbar/N(V_Q)$ with superselection sectors. Moreover, he shows that this provides a natural setting for understanding the dynamics of the Aharanov-Bohm effect because when the free dynamics is implemented in each superselection sector, the Hamiltonian generating time translations contains so-called 'topological terms' that give rise to interference depending on the strength of the electromagnetic potential. Thus, restricting attention to physical states in V_Q might actually be understood to give rise to novel predictions (through changes in the dynamics) for experiments like the Aharanov-Bohm setup.

More work is required here to understand the relationship between the Aharanov-Bohm effect and symmetry-invariant states; one would, for example, like to construct a model that does not contain the topological idealization of an impenetrable solenoid and instead represents the experiment with a non-trivial electromagnetic field in the interior of the solenoid. It should be possible to analyze such a model with the quantization procedure discussed above. It is also worth noting that historically the prediction of the Aharanov-Bohm effect was, of course, not generated by thinking about reductive explanations and restricting the collection of physically reasonable states. Still, I believe it is worth further investigation to see the extent to which the foundation of the Aharanov-Bohm effect may be recovered from the considerations of this paper.

Notice that the sense in which quantum states lying outside V_Q are unphysical is different than the sense in which non-regular quantum states were understood to be unphysical in §4. Quantum states that fail to be symmetry-invariant should not be understood as idealizations, but rather as states involving a dependence on artifactual or redundant degrees of freedom. This demonstrates that what we mean when we call states physical or unphysical depends on the context of the system under consideration. This does not imply that the requirement of reductive explanations and the central proposal are idle, but shows that they must be applied with care depending on the details of the situation at hand.

Moreover, this example demonstrates another way in which the procedure recommended here for constructing quantum theories is defeasible. In this example, the construction procedure leads to the use of a distinct dynamics in the quantum theory for setups like the Aharanov-Bohm effect, which leads to novel predictions. If those predictions were not confirmed by experiments, we would have reason to revise the kinematical framework of the quantum theory. Thus, the requirement of reductive explanations serves only as a heuristic to make theories plausible, but does not produce certainty. Still, when the requirement of reductive explanations is understood as a heuristic and applied in a context-sensitive way, it can lead to a deeper understanding of the foundation for the quantum theories we construct.

6 Conclusion: heuristics and discovery

I hope to have demonstrated that work on heuristics, like the requirement of reductive explanations, can be of philosophical interest and of great importance to real scientific examples. But little recent work in philosophy of science discusses heuristics and scientific discovery. (For notable exceptions, see Crowther and Linnemann [2019]; Crowther [2018a], [2018b].) In fact, there is some history in the philosophical community of skepticism about whether heuristics and discovery are appropriate topics for philosophical investigation. The distinction made by Reichenbach ([1938]) between the contexts of discovery and justification led many positivists and empiricists to focus on justification as the only concept of epistemic relevance. This trend is exemplified by the work of Popper ([1959]), whose view implies that the epistemic attitude we should take toward theories is in some sense independent of the methods used to generate them. Thus, it is worth closing with some remarks to situate my conclusions within the recent history of philosophical discussions of heuristics.

Some aimed to revive issues concerning discovery and heuristics in the 1980’s (e.g., Nickles [1980]), but lingering doubts about the possibility of a philosophy of discovery seem to have overwhelmed these discussions by focusing on the question of whether theory generation is epistemically relevant (e.g., Laudan [1980]). Nickles ([1985]) responds by arguing for a connection between theory generation and justification, in part through considerations of efficiency in the generation of knowledge; he argues that philosophers should be interested in a logical relation he calls ‘discoverability’, rather than the contingent way in which a theory was actually discovered. Similarly, Zahar ([1983]) focuses on defending the concept of rational heuristics from the challenges posed by Popper, arguing that discovery is not merely subject to psychological or descriptive analysis. These defenders of the philosophical relevance of theory construction distance themselves from the idea of a 'logic of discovery' leading us to new theories once and for all, but still attempt to give a general defense of the epistemic relevance of methods for construction as forming something other than a logic.

In some sense, I am friendly to the ideas of Zahar and Nickles (among others) in their attempts to revive philosophy of scientific discovery. I agree that heuristics can have normative force, and I agree that the value of methods for theory construction can be understood separately from historical contingencies associated with how theories were actually generated. For example, my own investigation has taken place solely on the basis of mathematical analysis with no consideration of history. However, I also differ in my approach from these writers because I think that the philosophy of scientific discovery is best served by analyzing the status of heuristics in particular cases, and I am skeptical that general considerations can provide the kind of insight that is needed. Rather than attempting to give a general defense of the epistemic relevance of methods for theory construction, I have focused on how particular methods can be employed in particular contexts in the construction of quantum theories. My reason is that the normative force of heuristic principles depends on details of the context one is interested in, and I worry that one may miss philosophically relevant considerations if one attempts to discuss theory construction at too high a level of generality.

My approach aligns even more closely with that of Post ([1971]) and Hesse ([1961], [1970]), who consider specifically the heuristic role of correspondences between new and old theories. I have argued that one can make at least one aspect of a 'correspondence principle' precise for the construction of new quantum theories as the requirement that a quantum theory give a reductive explanation of the success of its classical predecessor. I have argued that in order to achieve such an explanation, one must explain the structure of the physical state space in the classical theory. I have shown that this requirement places nontrivial constraints on the construction of new quantum theories, and in at least two examples leads to the appropriate construction of the quantum theory. I think that the heuristic role of the requirement of reductive explanations is epistemically relevant in the sense that we have good reason to pursue theories that can explain the success of their predecessors. However, I prefer to think of these reasons slightly differently than the evidential reasons for accepting hypotheses that come from empirical confirmation. I have argued along the lines of Hesse that satisfaction of the requirement of reductive explanations makes newly constructed theories more plausible, but that this plausibility can be overruled by other considerations including new empirical results. The defeasibility of the epistemic and normative force of heuristics may not be so different from empirical evidence, but I think it is worth emphasizing.

I think there are many open questions concerning the role of heuristics in theory construction that are worth investigating. For example, in what ways can the requirement of reductive explanations extend beyond considerations of physical state spaces? Does the requirement of reductive explanations make a difference in other examples besides the models I have considered? And can we understand the normative force behind other heuristics besides those involving reductive explanations? These questions strike me as important in part because they provide guides toward ways in which philosophical work might aid scientific progress. This is especially true in quantum physics, where there is much ongoing and open-ended work on constructing quantum field theories and quantum theories of gravity. I hope more philosophers will take up the task of investigating the methods used in theory construction because they have the potential to make a real impact in these areas.

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