# Follow the Flow: sets, relations, and categories as special cases of functions with no domain

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### Abstract

We introduce, develop, and apply a new approach for dealing with the intuitive notion of function, called Flow Theory. Within our framework all functions are monadic and none of them has any domain. Sets, proper classes, categories, functors, and even relations are special cases of functions. In this sense, functions in Flow are not equivalent to functions in ZFC. Nevertheless, we prove both ZFC and Category Theory are naturally immersed within Flow. Besides, our framework provides major advantages as a language for axiomatization of standard mathematical and physical theories. Russell's paradox is avoided without any equivalent to the Separation Scheme. Hierarchies of sets are obtained without any equivalent to the Power Set Axiom. And a clear principle of duality emerges from Flow, in a way which was not anticipated neither by Category Theory nor by standard set theories. Besides, there seems to be within Flow an identification not only with the common practice of doing mathematics (which is usually quite different from the ways proposed by logicians), but even with the common practice of teaching this formal science.

**Key words:** functions, set theory, category theory.

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# 1   Introduction

Throughout the ages mathematicians have considered their objects, such as numbers, points, etc., as substantial things in themselves.

Since these entities had always defied attempts at an adequate description, it slowly dawned on the mathematicians of the nineteenth century that the question of the meaning of these objects as substantial things does not make sense within mathematics, if at all. The only relevant assertions concerning them do not refer to substantial reality; they state only the interrelations between mathematically "undefined objects" and the rules governing operations with them. What points, lines, numbers "actually" *are* cannot and need not be discussed in mathematical science. What matters and what corresponds to "verifiable" fact is structure and relationship, that two points determine a line, that numbers combine according to certain rules to form other numbers, etc. A clear insight into the necessity of a dissubstantiation of elementary mathematical concepts has been one of the most important and fruitful results of the modern postulational development.

Richard Courant, *What is Mathematics*, 1941.

All usual mathematical approaches for well-known physical theories can be easily associated to either differential equations or systems of differential equations. Newton's second law, Schrödinger's equation, Maxwell's equations, and Einstein field equations are all differential equations which ground classical mechanics, quantum mechanics, classical electromagnetism, and general relativity, respectively. Other similar examples may be found in thermodynamics, gauge theories, the Dirac electron, etc. Solutions for those differential equations (when they exist) are either functions or classes of functions. So, the concept of function plays a major role in theoretical physics. Actually, functions are more relevant than sets, in a very precise sense [4] [5].

In pure mathematics the situation is no different. Continuous functions, linear transformations, homomorphisms, and homeomorphisms, for example, play a fundamental role in topology, linear algebra, group theory, and differential geometry, respectively. And category theory emphasizes such a role in a very clear, elegant, and comprehensive way.

Functions allow us to talk about the dynamics of the world, in the case of physical theories. Regarding mathematics, functions allow us to talk about invariant properties, whether those properties refer to either algebraic operations or order relations.

From a historical point of view, some authors have advocated the idea that functions are supposed to play a strategic role into the foundations of mathematics [17] and even mathematics teaching [9], rather than sets. Notwithstanding, the irony of such discussions lies in a closer look at Georg Cantor's seminal works about the concept of set. Cantor - the celebrated father of set theory - was strongly motivated by Bernard Bolzano's work on infinite multitudes called *Menge* [25]. Those collections were supposed to be conceived in a way such that the arrangement of their components is unimportant. However, Bolzano insisted on an Euclidian view that the whole should be greater than a part, while Cantor proposed a quite different approach. According to the latter, in

order to compare infinite quantities we should consider a *one-to-one correspondence* between collections. That means Cantor's concept of collection (in his famous *Mengenlehre*) was strongly committed to the idea of function. Subsequent formalizations of Cantor's "theory" were developed in a way such that all strategic terms were associated to an intended interpretation of collection. And the result of that effort is a strange phenomenon which we describe in the next paragraphs, based on [21].

Let $S$ be an axiomatic system whose primitive concepts are $c_1$, $c_2$, ..., $c_n$. One of these concepts, say $c_i$, is independent (undefinable) from the remaining if and only if there are two models of $S$ in which $c_1$, ..., $c_{i-1}$, $c_{i+1}$, ..., $c_n$ have the same interpretation, but the interpretations of $c_i$ in such models are different. (Of course, a model of $S$ is a set-theoretic structure in which all axioms of $S$ are true, according to the interpretation of its primitive terms [15].)

As an example, consider a very simple axiomatic system, namely, a *Minimalist Space* $\langle X, f \rangle$, whose axioms are:

**MS1** $X$ is a non-empty set.

**MS2** $f$ is a function whose domain and codomain are both $X$.

By using Padoa's method [1] [18] [24] we can easily prove that $f$ is undefinable, since we can exhibit two models of a minimalist system such that $X$ has the same interpretation in both models but $f$ has two different interpretations within these models. Consider, for this: the Model A, where $X$ is interpreted as the set of real numbers $\Re$ and $f$ is the identity function $f(x) = x$ defined on $X$; and the Model B, where $X$ is interpreted again as the set $\Re$, but $f$ is the function given by $f(x) = 2x$, with the same domain $X$. This means that the interpretation of $X$ does not fix the interpretation of $f$. In other words, $f$ cannot be defined (or fixed) from $X$. On the other hand, $X$ *is* definable, since any two models with two different interpretations for $X$ would unavoidably entail different interpretations for $f$. The reason for this is grounded on the fact that the domain and the codomain of a function $f$ are ingredients for the definition of the function itself, at least within the scope of a standard set theory like Zermelo-Fraenkel's. Different domains imply different functions.

So, at least two questions remain:

1. How to define $X$?

2. What does it mean to say that $X$ is eliminable?

The answers are:

1. $X = dom(f) = cod(f)$ ($X$ is the domain and the codomain of $f$).

2. We do not need to explicitly mention $X$. We could rephrase the definition of a minimalist system by saying that a minimalist system is just a function $f$ whose domain is equal to its codomain.

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In a similar way, it is possible to prove that in usual axiomatic frameworks for physical theories, time and spacetime are concepts that are definable, and so, eliminable. That happens because time and spacetime are usually considered as domains of functions that describe forces, fields, currents, and so on. For example, according to Padoa's principle, the primitive concept time (described as an interval of real numbers) in a physical theory is independent from the remaining primitive concepts (mass, position, force, speed, magnetic field etc.) if, and only if, there are two models of the physical theory such that time has two interpretations and the remaining primitive symbols have the same interpretation. But usually these two interpretations are not possible, since mass, position, force, speed, magnetic field and other physical concepts are in general described as functions whose domains are time. If we change the interpretation of time, we change the interpretation of the other primitive concepts. So, time is not independent and hence can be defined. Since time is definable, it is eliminable. Time is eliminable in the sense that many physical theories can be rewritten without any explicit mention of time. A similar argument can be used to dispense with spacetime. (Details about this approach can be found in [4, 5].)

Results of this kind suggest the idea that functions are indispensable, but the explicit presence of sets as the domains of these functions is questionable. After all, it's not clear that the notion of a set is playing any strategic explanatory role in the context of physical theories, since these sets are definable by means of functions. Sets seem to be carried along as just a surplus structure of the mathematical framework in which these theories are formulated. Moreover, in the context of standard set theories, such as Zermelo-Fraenkel's, to reformulate a physical theory without any explicit mention of either time or spacetime is not an easy task—after all, the latter notions are typically expressed in terms of sets. Such reformulations of physical theories in Zermelo-Fraenkel are also unnatural, given that usually the functions that are invoked in the theories demand an explicit mention of their domains, and in this way, sets are brought back. (For an example of a mathematical description of thermodynamics without any explicit mention of time, see [5].)

Sets can be viewed as the result of a process of collecting objects. An object is collected if it is assigned to a given set. But the fundamental mechanism here is to attribute something to a certain collection. And that notion of attributing something to a given collection resembles a function. From another point of view, we should recall that sets and functions are meant to correspond to an intuitive notion of properties. Usually properties allow to define either classes or sets (like the Separation Schema in ZFC). But another possibility is that properties correspond to functions. Talking about objects that have a given property $P$ corresponds to associate certain objects to a label which represents $P$; and any other remaining objects are supposed to be associated to a different label. The correspondence itself between $P$ and a given label does have a functional, rather than a set-theoretical, appeal. And usually, those labels are called sets. So, why do we need sets? Why cant we deal only with functions? In other words, why cant we label those intended properties with functions instead

of sets?

What would happen if we could avoid any explicit mention of domains of functions? Could we obtain better axiomatic formulations of physical theories? Could we avoid the presence of time and spacetime structures in a natural way? Could we go more directly to the point, i.e., to the functions that usually describe fields and forces, tensors and metrics, speeds and accelerations?

It could be thought that category theory provides a framework to develop this sort of approach. After all, category theory deals primarily with "functions", called morphisms (see [12]). However, even morphisms have domains, which are other morphisms, and so we still wouldn't have the appropriate framework to develop the approach we have in mind. So, even Category Theory is somehow committed to set-theoretic view about what a function is supposed to be.

What we are looking for is a mathematical theory where functions have no domains at all. In this way, we would immediately avoid the introduction of superfluous primitive notions, such as sets or domains, when we use this theory as the mathematical basis for the formulation of physical theories. Sets work as the stage where functions, the actor, play. So, we advocate a way of doing mathematics where the stage itself is unimportant. The relevant agents of mathematics are functions, and functions alone.

In 1925, John von Neumann introduced his axiomatization of set theory [17]. There are two major assumptions in his approach, namely, the use of two kinds of collections, sets and classes, and the use of functions as the intuitive basic notion, instead of sets or classes. More specifically, von Neumann deals with three kinds of terms: I-objects (arguments), II-objects (characteristic functions of classes), and I-II-objects (characteristic functions of sets). The axiomatic system originally proposed was further developed by R. M. Robinson, P. Bernays, and Kurt Gödel, and it came to be known as the von Neumann-Bernays-Gödel (NBG) set theory. However, NBG is not faithful to the idea of the priority of functions instead of collections. In the end, NBG is a standard approach to set theory, where the novelty is the use of classes (mainly proper classes: those classes which are not sets), besides sets.

Intuitively speaking, a function is supposed to be a term which allows us to uniquely associate certain terms to other terms. In standard set theories, for example, a function is a special case of set, namely, a specific set of ordered pairs of sets. That means standard set-theoretic functions do not actually *act* on terms in the sense of transforming them into other terms. In contrast, in category theory morphisms have an intended interpretation which is somehow associated to functions. But even in that case we show morphisms can always be treated as restrictions of an identity function. Besides, in both cases functions are somehow attached to domains and codomains which are sets in set theories and identity morphisms in category theories. In this paper we develop a new approach - Flow Theory - for dealing with the intuitive notion of function. In a precise sense, in Flow Theory functions have no domain at all. Within our approach, a set is a special case of function. Russell's paradox is avoided without any equivalent to the Separation Scheme. We provide a comprehensive

discussion of Flow Theory as a new foundation for mathematics, where functions explicitly play a more fundamental role.

The name Flow is a reference to Heraclitean flux doctrine, according to which things are constantly changing. Accordingly, in Flow theory all terms are "active objects" under the action of other "active objects".

So, this paper is strongly motivated by [17] and [21] and related papers as well ([4] [5]). In [21] it was provided a reformulation of von Neumann's original ideas (termed $\mathcal{N}$ theory) which allowed the authors to reformulate standard physical and mathematical theories with much less primitive concepts in a very natural way. Nevertheless, in $\mathcal{N}$ theory there are two fundamental constants which are not clarified in any way. Those constants, namely, $\underline{0}$ and $\underline{1}$, allow us to define sets as particular cases of functions, in a way which is somehow analogous to the usual sense of characteristic functions in standard set theories.

In this paper Flow theory is introduced as a generalized formulation of concepts derived from $\mathcal{N}$ theory. Constants $\underline{0}$ and $\underline{1}$ are still necessary. Notwithstanding, we are able to define them from our proposed axioms and some related theorems. And that fact entails an algebra defined over functions. Such an algebra shows us that both category theory and ZFC set theory are naturally present within our framework.

Besides the presentation and discussion of Flow axioms, we introduce several applications and foundational issues by comparing Flow with ZFC set theories and Category Theory.

Our punch line may be summarized by something like this: (i) the concept of set (as a collection of objects) is somehow implicitly assumed through ZF axioms; (ii) nevertheless, sets play a secondary role in mathematics and applied mathematics, since the true actors are always functions, while sets work as just a stage (setting) for such actors; (iii) so, why cannot we explicitly assume the notion of function right at the start on the foundations of mathematical theories?

## 2  Flow theory

Flow is a first-order theory with identity, where the formula $x = y$ should be read as "$x$ is equal to $y$". The formula $\neg(x = y)$ is abbreviated as $x \neq y$. Flow has one functional letter $f_1^2(f, x)$, where $f$ and $x$ are terms. If $y = f_1^2(f, x)$, we abbreviate this by $f(x) = y$, and we say $y$ is the *image* of $x$ by $f$. We call $f_1^2$ *evaluation*. All terms of Flow are called *functions*. We use lowercase Latin and Greek letters to denote functions. Uppercase letters are used to denote predicates (which are eventually defined). The axioms of Flow follow in the next subsections. But first we need to make a remark. Any explicit definition in Flow is an abbreviative one, in the sense that for a given formula $F$, the *definiendum* is just a metalinguistic abbreviation for the *definiens* given by $F$.

## 2.1 Functions

**P1 - Weak Extensionality** $\forall f \forall g(((f(g) = f \wedge g(f) = g) \vee (f(g) = g \wedge g(f) = f)) \Rightarrow f = g))$.

This first axiom is tricky. Any function $f$ such that $f(g) = f$ is said to be *rigid with $g$*. And any function $f$ such that $f(g) = g$ is said to be *flexible with $g$*. So, if both $f$ and $g$ are rigid with each other, then we are talking about the very same function ($f = g$). Another possibility to identify a function is by checking if $f$ and $g$ are both flexible with each other. If that is the case, then again $f = g$.

**P2 - Self-Reference** $\forall f(f(f) = f)$.

Our first theorem has a very intuitive meaning.

**Theorem 1** $\forall f \forall g(f = g \Leftrightarrow \forall x(f(x) = g(x)))$.

**Proof:** By using the substitutivity of identity in the formula $f(x) = f(x)$ (which is a theorem in any first-order theory with identity), proof of the $\Rightarrow$ part is quite straightforward. After all, if $f = g$, then $f(x) = g(x)$, for any $x$. In particular, we have $f(f) = g(g) = f(g) = g(f) = f = g$. Concerning the $\Leftarrow$ part, suppose for any $x$ we have $f(x) = g(x)$. In particular, for $x = f$, we have $f(f) = g(f)$. And for $x = g$, we have $f(g) = g(g)$. Nevertheless, according to **P2**, $f(f) = f$ and $g(g) = g$. So, $g(f) = f$ and $f(g) = g$. And from **P1**, that entails $f = g$.

Axiom **P2** says every function is rigid and flexible with itself. That fact deserves a more detailed discussion. Our main purpose here is to avoid any Flow-theoretic version of Russell's paradox. Consider, for example, the next statement: $y$ is a function such that

$$\forall x(y(x) = r \Leftrightarrow x(x) \neq r).$$

In the formula above we are explicitly trying to *define* a function $y$. On the left side of $\Leftrightarrow$ we have the *definiendum* and on the right side we have the *definiens*. If we ignore **P2**, what about $y(y)$? If $y(y) = r$, then we are considering $y(x) = r$ where $x$ is $y$. Hence, according to the formula above we entail $y(y) \neq r$. Analogously, if $y(y) \neq r$, we are considering $x(x) \neq r$ where $x$ is again $y$. And according to the formula above we have $y(y) = r$. Consequently, we have $y(y) = r \Leftrightarrow y(y) \neq r$. That is Russell's paradox! To avoid such an embarrassment (which could explode Flow theory, since we are grounding our axiomatic system within classical logic) all we need to do is to introduce axiom **P2**. According to **P2**, any function $y$ defined by the formula above guarantees that $y$ cannot be equal to $x$. Since for any $x$ we have $x(x) = x$ and the *definiens* above demands that $x(x) \neq r$, that entails $x \neq r$. But the *definiendum* states $y(x) = r$. Hence, $y(x) \neq x = x(x)$. Therefore, Theorem 1 guarantees $y \neq x$, since $x$ and $y$ do not share all their images. Hence, there is no paradox! After all,

the paradox was entailed from the possibility that $x = y$. Axiom **P2** prohibits the definition of a function like $y$. Otherwise, a formula like the one proposed above would be creative, allowing us to derive contradictions. That is a much simpler solution to Russell's paradox than any equivalent to the Separation Scheme in Zermelo-Fraenkel-like set theories. Besides, as we shall see below, Flow theory allows us to talk about sets and proper classes in the usual sense of standard set theories, like ZFC with classes, NBG and their variations.

It is worth to observe that axioms **P1** and **P2** could be rewritten as one single axiom as it follows:

**P1' - Alternative Weak Extensionality** $\forall f \forall g((( f(g) = f \wedge g(f) = g) \vee (f(g) = g \wedge g(f) = f)) \Leftrightarrow f = g)$.

If that was the case, then **P2** would be a consequence from **P1'**. Ultimately, $f = g$ would entail that $f(g) = f$ (from **P1'**). And substitutivity of identity entails $f(f) = f$. On the other hand, we prefer to keep axioms **P1** and **P2** (instead of **P1'**) for pedagogical purposes. From **P1** and **P2**, we can analogously see that **P1'** is a theorem.

One philosophical remark concerning axiom **P2** refers to Richard Courant's quote presented in the Introduction. Functions, by themselves, are irrelevant. What matters is what they do. That point is gradually clearer thanks to the next postulates.

**P3 - Identity** $\exists f \forall x (f(x) = x)$.

This is the first axiom which guarantees the existence of a specific function. Any function $f$ which satisfies **P3** is said to be an *identity function*.

**Theorem 2** *The identity function is unique.*

**Proof:** Suppose both $f$ and $g$ satisfy axiom **P3**. Then, for any $x$ we have $f(x) = x$ and $g(x) = x$. Thus, $f(g) = g$ and $g(f) = f$. Hence, according to **P1**, $f = g$.

In other words, there is one single function $f$ which is flexible to every function. In that case we simply say $f$ is *flexible*. That means "flexible" and "identity" are synonyms.

**P4 - Rigidness** $\exists f \forall x (f(x) = f)$.

In other words, there is at least one function $f$ which is rigid with any function. Observe the symmetry between axioms **P3** and **P4**! Any function $f$ which satisfies this last postulate is simply said to be *rigid*.

**Theorem 3** *The rigid function is unique.*

**Proof:** Suppose both $f$ and $g$ satisfy axiom **P4**. Then, for any $x$ we have $f(x) = f$ and $g(x) = g$. Thus, for any $x$ we have $f(g(x)) = f(g) = f$ and $g(f(x)) = g(f) = g$. Thus, according to **P1**, $f = g$.

Now we are able to justify the extensionality axiom **P1**. Our purpose here is to define constants $\underline{0}$ and $\underline{1}$, in order to accommodate our view about von Neumann's ideas. So, $\underline{1}$ is the identity (flexible) function and $\underline{0}$ is the rigid function, since we proved they are both unique. In other words

$$\forall x(\underline{1}(x) = x \ \wedge \ \underline{0}(x) = \underline{0})$$

If we recall that $f(x) = y$ is an abbreviation for $f_1^2(f, x) = y$, we can read axioms **P3** and **P4** as statements regarding the existence of two "spurs". Axiom **P3** states there is a function $f$ such that for any $x$ we have $f_1^2(f, x) = x$, while **P4** says there is $f$ such that for any $x$ we have $f_1^2(f, x) = f$.

**Theorem 4** $\underline{0}$ *is the only function which is rigid with* $\underline{0}$.

**Proof:** The statement above is equivalent to say that $\forall x(x \neq \underline{0} \Rightarrow x(\underline{0}) \neq x)$. In other words, $\forall x(x(\underline{0}) = x \Rightarrow x = \underline{0})$. But we already know that $\underline{0}(x) = \underline{0}$. Therefore, if we have $x(\underline{0}) = x \wedge \underline{0}(x) = \underline{0}$, according to **P1**, we have $x = \underline{0}$.

**Theorem 5** $\underline{1}$ *is the only function which is flexible with* $\underline{1}$.

**Proof:** The statement above is equivalent to say that $\forall x(x \neq \underline{1} \Rightarrow x(\underline{1}) \neq \underline{1})$. In other words, $\forall x(x(\underline{1}) = \underline{1} \Rightarrow x = \underline{1})$. But we already know that $\underline{1}(x) = x$. Therefore, if we have $x(\underline{1}) = \underline{1} \wedge \underline{1}(x) = x$, according to **P1**, we have $x = \underline{1}$.

The last two theorems do not say what are the images $x(\underline{0})$ or $x(\underline{1})$ (when $x \neq \underline{1}$, in the last case). Nevertheless, such values prove to be rather important for future applications of Flow Theory. But before discussing that, we introduce another axiom.

**P5 - Composition** $\forall f \forall g \exists! h(h \neq \underline{0} \wedge h \neq \underline{1} \wedge \forall x((x \neq f \wedge x \neq g) \Rightarrow (x \neq h \Rightarrow h(x) = f(g(x)))) \wedge (g \neq h \Rightarrow h(g) = \underline{0}) \wedge (f \neq h \Rightarrow h(f) = \underline{0}))$.

**P5** allows us to define unique functions $h$ from other functions $f$ and $g$. That means this last postulate allows us to define a "binary operation" over functions. To be clearer about that, we state the next definition, based on **P5**.

**Definition 1** *For any functions $f$ and $g$ we may define the* composition $h = f \circ g$ *of $f$ with $g$. Function $h$ is the one stated in axiom* **P5**.

Beware! We never calculate $(f \circ g)(f \circ g)$ as $f(g(f \circ g))$, since $(f \circ g)(f \circ g)$ is always $f \circ g$ according to **P2**.

The idea of composition $f \circ g$ is quite simple, although it is not a constructive process. Given functions $f$ and $g$, we can build the composition $f \circ g$ through a three-step process as it follows:

1. First we establish a label $h$ for $f \circ g$.

2. Next we calculate $f(g(x))$ for any $x$ which is different of $f$ and $g$. By doing that we are assuming those $x$ are different of $h$. Thus, if such a choice of $x$ entails $f(g(x)) = y$ for a given $y$, then $h(x) = y$.

3. Next we evaluate the following possibilities: is $h$ equal to either $f$, $g$ or something else? How can we answer to that question? If $h$ is different of $g$, then $h(g)$ is supposed to be $\underline{0}$. If $h(g) = \underline{0}$ entails a contradiction, then $h$ is simply $g$. And an analogous method is used for assessing if $h$ is $f$. Eventually, $h$ is neither $f$ nor $g$, as we can see in the next theorems.

Remarkable examples of how to calculate compositions can be found in the proofs of Theorems 18 and 19.

**Theorem 6** *Composition $\circ$ is associative.*

**Proof:** Here is a sketch for the proof. Both situations $f \circ (g \circ h) = p$ and $(f \circ g) \circ h = q$ correspond, according to Definition 1, to the formula $y = f(g(h(x)))$, as long we are talking about values $x$ which are different of $f$, $g$, $h$, $p$, and $q$. That means $f \circ (g \circ h) = (f \circ g) \circ h$, if $x \neq f \wedge x \neq g \wedge x \neq h \wedge x \neq p \wedge x \neq q$. Now, all we have to do is to consider four situations which contemplate all possible relations among $f$, $g$, $h$, $p$, and $q$, in order to evaluate the images $p(x)$ and $q(x)$ when $x$ is either one of those remaining terms: (i) $p \neq f \wedge p \neq g \wedge p \neq h$; (ii) $p = f$; (iii) $p = g$; (iv) $p = h$. If we have situation (i), then $p(f) = p(g) = p(h) = \underline{0}$. That means $p$ has all the features of composition $q$. But, according to **P5**, any composition is unique. Thus, $p = q$. Regarding situation (ii), which states $p = f$, we should consider two possibilities: either $p(g \circ h) \neq \underline{0}$ or $p(g \circ h) = \underline{0}$. If $p(g \circ h) \neq \underline{0}$, that means $g \circ h = p$ (according to **P5**). That entails $p \circ p = p$ (an idempotent function); and associativity among idempotent functions is a trivial result, when $g = p$ and $h = p$. On the other hand, if $g \neq p$ and $h \neq p$, then we have $p(g) = p(h) = \underline{0}$, and once again $p$ has all the features of $q$. Hence, from uniqueness of composition, $p = q$. If $g = p$, while $h \neq p$, then from $f \circ (g \circ h) = p$ we have $p \circ (p \circ h) = p$, which entails $(p \circ p) \circ h = p$ for the case $p \circ h = p$, since in that case $p$ is idempotent. And the case $p \circ h \neq p$ was already discarded within the first possibility of situation (ii). And if $h = p$, while $g \neq p$, then $p(g) = \underline{0}$, and once again $p$ has the same features of $q$; hence, $p = q$. Going back to the second possibility of situation (ii), when $p = f$ while $p(g \circ h) = \underline{0}$, the last identity implies $g \circ h \neq p$, since no composition $p$ can ever be $\underline{0}$ (the only function $f$ such that $f(f) = \underline{0}$). So, neither $g$ nor $h$ can be $p$, since $p \circ (g \circ h) = p$. That implies $p \circ g = p$ and $p \circ h = p$. Hence, $(p \circ g) \circ h = p \circ h = p$. The proof of situations (iii) and (iv) is analogous to that one for situation (ii).

This last theorem is somehow interesting, since evaluation $f_1^2$ is not associative and composition is defined from evaluation. Consider, for example, $x(\underline{1}(x))$, for $x$ different of $\underline{0}$ and different of $\underline{1}$, and such that $x(\underline{1}) = \underline{0}$ (functions like this will be available afterwards). Thus, $x(\underline{1}(x)) = x(x) = x$. If evaluation was associative, we would have $x(\underline{1}(x)) = x(\underline{1})(x) = \underline{0}(x) = \underline{0}$. A contradiction! Of course this *rationale* works only if we prove the existence of other functions besides $\underline{0}$ and $\underline{1}$. That happens thanks to the last axiom, as we discuss below.

So, although evaluation $f_1^2$ is not associative, we are still able to define a binary functional letter $\circ$ from $f_1^2$ such that $\circ$ is associative. That happens because $f$ and $f(x)$ are not necessarily the same thing. So, contrary to the usual slogan from category theory [11], evaluation is not a special case of composition.

Actually, it is good news that evaluation is not associative. According to **P2**, we have, for all $t$, $g(t) = (g(g))(t)$, since for any $g$ we have $g(g) = g$. If evaluation was associative, we would have $g(t) = g(g(t))$ for any $t$, and thus, $g = g \circ g$. So, composition would be an idempotent operation. That would be an undesirable result for anyone who intends to develop, e.g., category theory within Flow.

**Theorem 7** *There is a unique function $h$ such that $h \neq \underline{0}$ but $h(x) = \underline{0}$ for any $x \neq h$.*

**Proof:** According to Definition 1 and axiom **P5**, $\underline{0} \circ \underline{0}$ is a unique function $h \neq \underline{0}$ such that for any $x \neq \underline{0}$, we have $h(x) = \underline{0}(\underline{0}(x)) = \underline{0}(\underline{0}) = \underline{0}$. But since $h \neq \underline{0}$, then **P5** guarantees that $h(\underline{0}) = \underline{0}$. Thus, $h(x)$ is $\underline{0}$ for any $x \neq h$, while $h$ itself is different of $\underline{0}$.

Such a function $h$ of last theorem is rather important for future applications. So, we label it with a special symbol, namely, $\phi_0$. That means $\underline{0} \circ \underline{0} = \phi_0$, where $\phi_0 \neq \underline{0}$.

**Theorem 8** *For any $x$ we have $\underline{0} \circ x = \phi_0$*

**Proof:** According to Definition 1 and axiom **P5**, $(\underline{0} \circ x)(t) = \underline{0}(x(t)) = \underline{0}$ for any $t \neq \underline{0}$ and $t \neq x$. But **P5** demands $\underline{0} \circ x$ is different of $\underline{0}$. Hence, $(\underline{0} \circ x)(\underline{0}) = \underline{0}$. Now, regarding $x$, there are three possibilities: (i) $x = \underline{0}$; (ii) $x = \phi_0$; (iii) $x$ is neither $\underline{0}$ nor $\phi_0$. The first case corresponds to Theorem 7, which entails $\underline{0} \circ x = \phi_0$. In the second case, if $\underline{0} \circ x$ is different of $x = \phi_0$, then $(\underline{0} \circ x)(\phi_0) = \underline{0}$. But that would entail $(\underline{0} \circ x)(t) = \underline{0}$ for any $t \neq \underline{0}$, which corresponds exactly to function $\phi_0$ proven in Theorem 7, a contradiction. So, $\underline{0} \circ x$ is indeed $\phi_0$, when $x = \phi_0$. Concerning the last case, since $x \neq \underline{0}$ and $x \neq \phi_0$, then (according to Theorem 1) there is $t \neq \phi_0$ such that $x(t) \neq \underline{0}$ for $x \neq t$. Thus, according to **P5**, $(\underline{0} \circ x)(t) = \underline{0}$ for such value of $t$. But once again we have a function $\underline{0} \circ x$ such that $(\underline{0} \circ x)(t) = \underline{0}$ for any $t \neq \underline{0}$, which corresponds exactly to function $\phi_0$ proven in Theorem 7

Concerning $x \circ \underline{0}$, that value is supposed to be discussed later, due to Theorem 4.

**Theorem 9** $\underline{0} \circ \underline{1} = \underline{1} \circ \underline{0} = \phi_0$

**Proof:** This proof is similar to the previous one in the last theorem.

**Theorem 10** *There is a unique function $h$ such that $h \neq \underline{1}$ and $h(x) = x$ for any $x$ such that $x \neq \underline{1}$.*

**Proof:** According to Definition 1 and axiom **P5**, $\underline{1} \circ \underline{1}$ is a unique function $h \neq \underline{1}$ such that for any $x \neq \underline{1}$, we have $h(x) = \underline{1}(\underline{1}(x)) = \underline{1}(x) = x$. But since $h \neq \underline{1}$, then **P5** guarantees that $h(\underline{1}) = \underline{0}$. Thus, $h(x)$ is $x$ for any $x \neq h$, while $h$ itself is different of $\underline{1}$.

Such a function $h$ of last theorem is rather important for future applications. So, we label it with a special symbol, namely, $\psi$.

The next theorem is rather important for a better understanding about the weak extensionality axiom **P1** (which entails Theorem 1), although its proof does not demand the use of such a postulate.

**Theorem 11** $\forall x((x \neq \underline{0} \wedge x \neq \underline{1}) \Rightarrow (\underline{1} \circ x = x \wedge x \circ \underline{1} = x)).$

**Proof:** Suppose $h = \underline{1} \circ x$, for the sake of abbreviation. That means for any $t$ different of $\underline{1}$ and different of $x$, we have $h(t) = \underline{1}(x(t)) = x(t)$. Suppose now $h$ is different of $x$. According to **P5**, that would entail $h(x) = \underline{0}$. But $x(t) = x(t)$ for any $t$ such that $t \neq x$ and $t \neq \underline{1}$ as well (as long $x \neq \underline{1} \wedge x \neq \underline{0}$, of course). And according to **P5** $h$ is supposed to be unique, which entails $h = x$. The proof of $x \circ \underline{1} = x$ is analogous to the proof of $\underline{1} \circ x = x$.

**Observation 1** *This last result is quite subtle. If it wasn't for the uniqueness requirement of compositions (axiom **P5**), Flow Theory would be consistent with the existence of many functions, like $x$ and $h$, which "do" the same thing. By multiple functions "doing the same thing" we mean different functions $x$ and $h$ which share the same images $x(t)$ and $h(t)$ for any $t$ different of both $x$ and $h$. Since **P2** demands $x(x) = x$ and $h(h) = h$, that would allow $h(x) = \underline{0}$ and $x(h) = \underline{0}$. And according to Theorem 1, that fact would guarantee $x \neq h$. We use this ambiguity for functions "doing the same thing" in Flow to prove Theorems 7 and 10, since $\underline{0} \circ \underline{0} = \phi_0$ and $\underline{1} \circ \underline{1} = \psi$, where $\phi_0 \neq \underline{0}$ and $\psi \neq \underline{1}$. Function $\phi_0$ "does the same things" $\underline{0}$ "does", and $\psi$ "does the same things" $\underline{1}$ does. Notwithstanding, we stop using that opportunity when we are talking about functions which are neither $\underline{0}$ nor $\underline{1}$. Observe, however, the uniqueness of both $\underline{0}$ and $\underline{1}$ is not imposed. Their uniqueness is granted by Theorems 2 and 3. That is why we refer to **P1** as "weak extensionality". A strong extensionality postulate would demand that functions which "do the same thing" are necessarily the same. But that assumption is inconvenient for us, since it does not allow us to guarantee the existence of other functions besides $\underline{0}$ and $\underline{1}$ without considering other primitive concepts besides evaluation $f_1^2$. In order to guarantee a strong concept of extensionality we use, along several axioms of Flow, the quantifier $\exists!$. The pragmatic impact of axioms of Flow Theory is that all of them work together into the direction of a strong concept of extensionality, in the sense that functions $x$ and $h$ who "do the same thing" are the very same, with the sole exceptions of $\underline{0}$ and $\phi_0$, and $\underline{1}$ and $\psi$. On the other hand, in Section 8 we discuss about some possible variations of Flow. And one those variations considers the possibility of replacing all occurrences of $\exists!$ within our axioms by $\exists$. In this sense, we consider the possibility of grounding such a variation of Flow with an*

*intuitionistic logic rather than a classical predicate calculus, as the one used in this article.*

One of the major advantages of our concept of composition is that it allows us to mimic many-variables functions, although all functions in Flow are monadic. That feature allows us to talk even about non-associative binary operations, despite the fact that composition is associative. For details, see Section 7.

**Theorem 12** *Suppose $f$ is idempotent with respect to composition, and there are some $x$ and $y$ such that $f(x) = y$. Then, $f$ is flexible with $y$.*

**Proof:** If $f(x) = y$, then $f(f(x)) = f(y)$. But $f(f(x)) = (f \circ f)(x)$ for $x \neq f \circ f$. Since $f$ is idempotent with respect to composition, then $f \circ f = f$. Thus, $f(x) = f(y)$. Hence, $y = f(y)$.

This last theorem is not important for further developments of Flow Theory. We just proved it to show that our framework is able to mimic well known results regarding the usual way composition is defined within standard set theories. Similar results about idempotent functions can be generated.

**P6 - Expansion** $\exists!\sigma(\forall f(f \neq \sigma \Rightarrow (\sigma(f) \neq \underline{0} \Leftrightarrow \exists g(g \neq f \land \sigma(f) = g \land f(g) = \underline{0} \land \forall x((x \neq g \land x \neq \sigma) \Rightarrow g(x) = f(x)))))).$

This last axiom guarantees the existence and uniqueness of a special function $\sigma$. Differently from other functions in Flow, this one deserves a special notation. From now on we write $\sigma_f$ instead of $\sigma(f)$, where $f$ is any term. If we take a look at the right hand of $\Leftrightarrow$ above, we see that in the case where $\sigma_f$ is different of $\underline{0}$, we have $g(f) = f(f) = f$, since $f$ is different of $g$. Within this context, $\sigma$ may be defined as it follows:

**Definition 2** *$\sigma$ is a function such that for any $f$, $\sigma_f = g \land g \neq \underline{0}$ iff $g \neq f \land f(g) = \underline{0} \land \forall x((x \neq g \land x \neq \sigma) \Rightarrow g(x) = f(x))$.*

The intuitive idea of a term like $\sigma_f$ is that of *successor* of a given function $f$. If the successor of $f$ is a non-$\underline{0}$ term $g$, then $f$ and $g$ share the same images for any $x$ different of $g$ and $\sigma$, although $f$ and $g$ are different. Besides, $f(g) = \underline{0}$. That is why $f$ and $g$ are different, since $f(g) = \underline{0}$ while $g(g) = g$ (remember we are considering the case where $g$ is a non-$\underline{0}$ term). In the case where there is no $g$ which satisfies such demands, then $\sigma_f$ is simply $\underline{0}$, and once again $f$ and $g$ are different (if, of course, we guarantee the existence of any function like $\sigma$, as it is done in axiom **P6**).

Axioms **P1**-**P5** work as "a soil prep to enhance the germination of functions". Axiom **P6**, on the other hand, states the existence of another function $\sigma$. And that fact (together with the next axiom) entails the existence of infinitely many other functions. Besides $\underline{0}$ and $\underline{1}$, there is a unique function $\sigma_{\underline{0}}$ whose images are either $\underline{0}$ or $\sigma_{\underline{0}}$ itself, where $\sigma_{\underline{0}}$ is different of $\underline{0}$. In other words, **P6** is consistent with the existence of a $\sigma_{\underline{0}} \neq \underline{0}$ such that for any $t$ different of $\sigma_{\underline{0}}$, both $\underline{0}$ and $\sigma_{\underline{0}}$ share the same images $\sigma_{\underline{0}}(t)$ and $\underline{0}(t)$. Such a function $\sigma_{\underline{0}}$ is simply $\phi_0$.

**Theorem 13** $\sigma_{\underline{0}} = \underline{0} \circ \underline{0} = \phi_0$.

The proof of this last theorem was already done in the previous paragraph.

Function $\phi_0$ is quite handy here. Actually, $\phi_0$ is ubiquitous within our discussions, since we prove latter $\phi_0$ can be associated to the empty set within ZFC. Since we intend to introduce further axioms regarding the existence of multiple functions (specially those functions which capture the everyday needs of standard mathematics), it is perfectly possible that some compositions $f \circ g$ correspond to certain functions whose images $(f \circ g)(t)$ are always $\underline{0}$, except for $t = f \circ g$, of course. In view of the fact that axiom **P5** demands the composition $f \circ g$ can never be $\underline{0}$, function $\phi_0$ proves to be quite valuable to cope with such situations. In other words, if $(f \circ g)(t)$ is always $\underline{0}$ for any $t$ different of $f \circ g$, then $f \circ g$ is simply $\phi_0$.

**Observation 2** *A word of caution is necessary here. Rigorously speaking, the label "Definition 2" by itself does not necessarily refer to a definition. Consider, for example, there is a function $\phi_0'$ such that $\phi_0' \neq \phi_0$ and $\phi_0' = \sigma_{\underline{0}}'$, where $\sigma'$ has the same properties of $\sigma$ in Definition 2. In that case, we have $\phi_0(\phi_0') = \underline{0}$ and $\phi_0'(\phi_0) = \underline{0}$. That is a result which confirms $\phi_0 \neq \phi_0'$, according to the axiom of weak extensionality. On the other hand, something odd is happening here, since there seems to be* two *successors for the same function $\underline{0}$, despite the fact that $\underline{0}$ is unique. From an intuitive point of view, we cannot actually* see *or* decide *which is which. It does not matter which function is a successor of $\underline{0}$, if there is more than one successor $\sigma$ which satisfies the allegedly definition 2. All that matters is how this successor does work. An analogous remark can be done about the successor of any function $f$ which admits a non-$\underline{0}$ successor (as we intend to pursue in the next paragraphs). Nevertheless, if $\phi_0 = \sigma_{\underline{0}}$ and $\phi_0' = \sigma_{\underline{0}}' = \sigma_{\underline{0}}$, that entails $\phi_0 = \phi_0'$, which conflicts with the assumption that $\phi_0 \neq \phi_0'$. That means, from a rigorous point of view, "Definition 2" may somehow be a creative statement. After all, if "Flow without Definition 2" is consistent, then "Flow with Definition 2" may allow us to entail a contradiction. That means our choice above for stating Definition 2 and axiom **P6** has a pedagogical rationale. That is why we used the quantifier $\exists!$ in **P6**. In the next postulate, we intend to talk about the successor of some other functions, in the sense that the successor of the successor of $\underline{0}$ does exist and so on. But from now on we don't have to worry with the $\exists!$ quantifier, since the uniqueness of $\sigma$ guarantees the uniqueness of $\sigma_f$ for any $f$. Our pedagogical solution to cope with Flow is based on the convenience of how to easily read our axioms.*

**P7 - Infinity** $\exists i((\forall t(i(t) = t \vee i(t) = \underline{0})) \wedge \sigma_i \neq \underline{0} \wedge (i(\sigma_{\underline{0}}) = \sigma_{\underline{0}} \wedge \forall x(i(x) = x \Rightarrow (i(\sigma_x) = \sigma_x \wedge \sigma_x \neq \underline{0}))))$.

**Definition 3** *Any function $i$ which satisfies axiom **P7** is said to be* inductive.

Since the existence of $\phi_0$ is granted by **P6** and **P7** (and independently by **P5**), we can now apply $\sigma$ again to get a function $\phi_1 = \sigma_{\phi_0} = \sigma_{\sigma_{\underline{0}}}$ such that $\phi_1(\phi_1) = \phi_1$, $\phi_1(\phi_0) = \phi_0$, and for the remaining functions $t$ (functions $t$ which

are neither $\phi_0$ nor $\phi_1$) we have $\phi_1(t) = \underline{0}$. Concerning **P7**, this postulate states the existence of another function $i$. It says if a function $x$ admits a non-$\underline{0}$ successor $\sigma_x$ (in a way such that $i(x) = x$), then $i(\sigma_x) = \sigma_x$. Analogously we can get (from **P6**) functions $\phi_2$, $\phi_3$, and so on. Besides, according to **P7**, any inductive function $i$ admits its own non-$\underline{0}$ successor $\sigma_i$.

Subscripts 0, 1, 2, 3, etc., are simply metalinguistic symbols based on an alphabet of ten symbols (the usual decimal numeral system) which follows the lexicographic order. The lexicographic order is denoted here by $\prec$, where $0 \prec 1 \prec 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8 \prec 9$. If $n$ is a subscript, then $n+1$ corresponds to the next subscript, in accordance to the lexicographic order. In that case, we write $n \prec n+1$. $n+m$ is an abbreviation for $(...(...((n+1)+1)+...1)...)$ with $m$ occurrences of $+$ and $m$ occurrences of pairs of parentheses. And again we have $n \prec n+m$. As it is well known for any finite alphabet, $\prec$ is a strict total order. That fact allows us to talk about a minimum value between two subscripts $m$ and $n$. Within that context, $\min\{m,n\}$ is $m$ iff $m \prec n$, it is $n$ iff $n \prec m$, and it is either one of them if $m = n$. Of course, $m = n$ iff $\neg(m \prec n) \wedge \neg(n \prec m)$. If $m \prec n \vee m = n$, we denote this by $m \preceq n$.

Such a vocabulary of ten symbols endowed with $\prec$ is called here (meta) language $\mathcal{L}$.

Thus, **P6** provides us some sort of "recursive definition" for functions $\phi_n$, while **P7** allows us to guarantee the existence of inductive functions:

- $\phi_0$ is such that $\phi_0(x)$ is $\phi_0$ if $x = \phi_0$ and $\underline{0}$ otherwise.

- $\phi_{n+1}$ is such that $\phi_{n+1}(\phi_{n+1}) = \phi_{n+1}$, $\phi_{n+1} \neq \phi_n$, and $\phi_{n+1}(x) = \phi_n(x)$ for any $x$ different from $\phi_{n+1}$.

Observe that $\phi_{n+1}(\phi_n) = \phi_n(\phi_n) = \phi_n$, while $\phi_n(\phi_{n+1}) = \underline{0}$. Moreover, $\phi_{n+2}(\phi_{n+1}) = \phi_{n+1}$, and $\phi_{n+2}(\phi_n) = \phi_{n+1}(\phi_n) = \phi_n$; while $\phi_n(\phi_{n+2}) = \underline{0}$. For a generalization of such results, see Theorems 15, 16, and 17.

Notwithstanding, **P7** says much more, since it states function $i$ itself has its own non-$\underline{0}$ successor $\sigma_i$.

The diagrams below (Figure 1) help us to illustrate how can we represent any function $f$ in a quite straightforward way. Each diagram is formed by a rectangle. The left top corner of any rectangle introduces the label $f$ of the function which is represented by the diagram. The remaining labels refer to functions $x$ such that $f(x) \neq \underline{0}$. For each label $x$ there is a unique corresponding arrow which indicates the image of $x$ by $f$. Since for any function $f$ we have $f(f) = f$, then the function represented at the left top corner of the rectangle does not need to be attached to any arrow. So, our first three examples below refer to functions $\phi_0$, $\phi_1$, and $\phi_2$.

From left to right, the first diagram refers to $\phi_0$. It says, for any $x$, $\phi_0(x)$ is $\underline{0}$, except for $\phi_0$ itself. The second diagram says $\phi_1(\phi_1) = \phi_1$, and $\phi_1(\phi_0) = \phi_0$. Observe the circular arrow attached to label $\phi_0$ in the second diagram is not a reference to the fact that $\phi_0(\phi_0) = \phi_0$. Circular arrows referring to axiom **P2** are simply omitted. So, the circular arrow associated to $\phi_0$ in the second diagram says solely that $\phi_1(\phi_0) = \phi_0$. Finally, the third diagram says $\phi_2(\phi_2) = \phi_2$,

$\phi_2(\phi_1) = \phi_1$, and $\phi_2(\phi_0) = \phi_0$. The diagram representations for $\underline{0}$ and $\underline{1}$ are, respectively, a blank rectangle and a filled in black rectangle. More sophisticated examples of functions are represented by diagrams in the next Section.
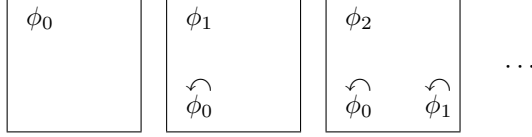


**Figure 1:** From left to right, diagram representations for functions $\phi_0$, $\phi_1$, and $\phi_2$.

Observe those diagrams above may be easily identified with reflexive graphs, from Graph Theory. Since objects and morphisms of a category (in the sense of Category Theory) may be viewed as, respectively, the vertices and edges of a graph, that fact seems to ease our discussion in Section 4 concerning Category Theory. Nevertheless, we show latter that is not the case.

**Theorem 14** *If $i$ is inductive, then for any $n$ of language $\mathcal{L}$ we have $i(\phi_n) = \phi_n$*

The proof is straightforward.
The next theorems are provable by induction.

**Theorem 15** *For any $m$ and $n$ of the vocabulary given above, $\phi_{m+n}(\phi_m) = \phi_m$ and $\phi_{m+n}(\phi_n) = \phi_n$.*

**Theorem 16** *For any $m$ and $n$ of the vocabulary given above, if at least one of them is different of $0$, then $\phi_m(\phi_{m+n}) = \underline{0}$ and $\phi_n(\phi_{m+n}) = \underline{0}$.*

Recall our previous argument for the non-associativity of evaluation holds, since we can now guarantee the existence of other functions besides $\underline{0}$ and $\underline{1}$.

**Definition 4** *$f[t]$ iff $t \neq f \land f(t) \neq \underline{0}$.*

While $f(t)$ is a term for any $f$ and $t$, $f[t]$ is a metalinguistic abbreviation for a formula. We read $f[t]$ as "$f$ *acts on* $t$". And $f$ acts on $t$ iff $t$ is not $f$ itself and $f(t) \neq \underline{0}$. The intuitive idea of this last definition is to allow us to talk about what effectively a function $f$ does. For example, both $\underline{0}$ and $\phi_0$ do nothing at all, since there is no $t$ on which they act. On the other hand, there is a term $t$ on which $\phi_1$ acts, namely, $\phi_0$.

**Theorem 17** *For any $m$ and $n$ of the vocabulary of language $\mathcal{L}$, $\phi_m \circ \phi_n = \phi_n \circ \phi_m = \phi_{\min\{m,n\}}$.*

**Proof:** We present here a sketch for the proof. Without loss of generality, suppose first $m \prec n$. That is equivalent to say there is some $p$ such that $m + p = n$. So, we can use the previous propositions regarding functions $\phi_n$. According to Definition 1, if $x \neq \phi_m \circ \phi_n$, then the images

of $\phi_m \circ \phi_n$ are given by $(\phi_m \circ \phi_n)(x) = \phi_m(\phi_n(x))$. But according to the last two propositions, those are exactly the same images of $\phi_m$. Since those functions of kind $\phi_n$ are generated by axiom **P6**, then $\phi_m \circ \phi_n$ is exactly $\phi_m$. An analogous argument shows that $\phi_n \circ \phi_m = \phi_m$. For the case where $m = n$, the proof is straightforward.

This last proposition proves all functions $\phi_n$ are idempotent with respect to composition. Besides, composition is commutative among functions $\phi_n$, although a given $\phi_n$ does not necessarily commute with any arbitrary function $x$, as we can see in the next two theorems.

**Theorem 18** *For any $n$ from language $\mathcal{L}$, $\sigma \circ \phi_n$ is a function $h$ such that: (i) $h(h) = h$; (ii) $h(\phi_m) = \phi_{m+1}$ for any $m \prec n$ (if there is any); (iii) $h(\sigma) = h(\phi_n) = \underline{0}$; and (iv) $h(x) = \phi_0$ for the remaining values of $x$.*

**Proof:** Item (i) is a direct consequence from axiom **P2**. If $\sigma \circ \phi_n = h$, then $h(x) = \sigma(\phi_n(x))$, for $x \neq h$, $x \neq \sigma$, and $x \neq \phi_n$. If $m \prec n$, then $h(\phi_m) = \sigma(\phi_n(\phi_m)) = \sigma(\phi_m) = \phi_{m+1}$. So, item (ii) is satisfied. That means $h$ is different of $\phi_n$, which entails $h(\phi_n) = \underline{0}$, according to **P5**. On the other hand, if, e.g., $x = \underline{1}$ (which is different of $\sigma$, of $\phi_n$ and of $h$), then $\phi_n(x) = \underline{0}$, which entails $h(x) = \sigma(\phi_n(x)) = \sigma(\underline{0}) = \phi_0$. That means $h$ is not $\sigma$ either. Therefore, item (iii) is satisfied. For the remaining terms (those $x$ which are different of $\phi_m$ for $m \prec n$, different of $\phi_n$, different of $\sigma$ and different of $h$), we have $h(x) = \sigma(\phi_n(x)) = \sigma(\underline{0}) = \phi_0$. Therefore, item (iv) is satisfied.

The proof of last theorem helps us to understand the non-constructive character of the calculation of compositions. In standard differential and integral calculus, for example, the definition of limit of a real function on a given point does not allow us to calculate limits, even when they do exist. Theorems about limits are the usual tools which allow us to calculate limits. A similar situation happens regarding composition in Flow. Axiom **P5** does not provide any methodology for calculating compositions in a constructive fashion. But all theorems about composition can provide useful tools for calculations. Next theorem together with the previous one, e.g., show us that $\sigma \circ \phi_n$ is never equal to $\phi_n \circ \sigma$, for any $n$.

**Theorem 19** *For any $n$ from language $\mathcal{L}$, $\phi_n \circ \sigma$ is a function $h$ such that: (i) $h(h) = h$; (ii) $h(\phi_m) = \phi_{m+1}$ for any $m \prec n$ (if there is any); (iii) $h(\sigma) = h(\phi_n) = \underline{0}$; (iv) $h(\underline{0}) = \phi_0$; and (v) $h(x) = \underline{0}$ for the remaining values of $x$.*

**Proof:** Item (i) is a direct consequence from axiom **P2**. If $\phi_n \circ \sigma = h$, then $h(x) = \phi_n(\sigma(x))$, for $x \neq h$, $x \neq \sigma$, and $x \neq \phi_n$. If $m \prec n$, then $h(\phi_m) = \phi_n(\sigma(\phi_m)) = \phi_n(\phi_{m+1}) = \phi_{m+1}$. So, item (ii) is satisfied. That means $h$ is different of $\phi_n$, which entails $h(\phi_n) = \underline{0}$, according to **P5**. On the other hand, if, e.g., $x = \phi_{n+1}$ (which is different of $\sigma$, of $\phi_n$ and of $h$), then $\sigma(x) = \phi_{n+2}$, which entails $h(x) = \phi_n(\sigma(x)) = \phi_n(\phi_{n+2}) = \underline{0}$.

That means $h$ is not $\sigma$ either. Therefore, item (iii) is satisfied. Besides, $\phi_n(\sigma(\underline{0})) = \phi_n(\phi_0) = \phi_0$ for any $n$, which satisfies item (iv). For the remaining terms $x$, all we have to do is to remember $\phi_n(x) \neq \underline{0}$ only for those $x$ such that $x = \phi_m$, where either $m \prec n$ or $m = n$. But those cases were already analysed. Therefore, $h(x) = \phi_n(\sigma(x)) = \underline{0}$. That concludes item (v).

The next diagram represents function $\phi_n \circ \sigma$. If $\phi_n \circ \sigma = h$, then $h(\underline{0}) = \phi_0$ (first arrow from left to right), $h(\phi_0) = \phi_1$, and so on; until $h(\phi_{n-1}) = \phi_n$, and $h(\phi_n) = \underline{0}$. The remaining values have images $\underline{0}$. That is why do no not represent them in the diagram.
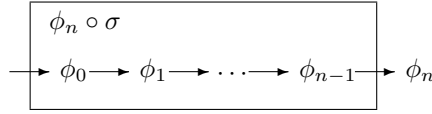


**Theorem 20** $\sigma_{\underline{1}} = \underline{0}$.

**Proof:** Suppose $g = \sigma_{\underline{1}}$. According to definition 2, $\underline{1}(g) = \underline{0}$. That happens only if $g = \underline{0}$. That means the successor of $\underline{1}$ does not share all images of $\underline{1}$. That happens because the successor of $\underline{1}$ is not a non-$\underline{0}$ term.

**Theorem 21** $\sigma \circ \underline{1} = \underline{1} \circ \sigma = \sigma$.

**Proof:** That is a corollary from the fact that for any $x$, we have $x \circ \underline{1} = \underline{1} \circ x = x$, if $x$ is neither $\underline{0}$ nor $\underline{1}$.

**Definition 5** $\mathcal{C}_y(f)$ *iff* $\exists y \forall x (x \neq f \Rightarrow f(x) = y)$.

Formula $\mathcal{C}_y(f)$ is read as "$f$ is a constant function with constant value $y$" or simply "$f$ is a constant function", if there is no risk of confusion. That means a constant function is a term $f$ such that, for a given $y$, $f(x)$ is either $y$ (for any $x \neq f$) or $f$ itself (which is consistent with **P2**).

The next theorem says the composition $\sigma \circ \underline{0}$ is a constant function with constant value $\phi_0$.

**Theorem 22** $\mathcal{C}_{\phi_0}(\sigma \circ \underline{0})$.

**Proof:** For any $t$ we have $(\sigma \circ \underline{0})(t) = \sigma(\underline{0}(t)) = \sigma(\underline{0}) = \sigma_{\underline{0}} = \phi_0$. That means $\sigma \circ \underline{0}$ has images $\phi_0$ for any $t \neq \sigma \circ \underline{0}$ and image $\sigma \circ \underline{0}$ for $t = \sigma \circ \underline{0}$. Thus, $\sigma \circ \underline{0}$ is a constant function with constant value $\phi_0$.

**Theorem 23** $\underline{0}$ *is the only constant function which assumes one single image for any $x$.*

**Proof:** According to **A5**, any constant function $f$ has at most two images, namely, either a constant value $c$ or $f$ itself (see axiom **P2**). So, if $f$ is a constant function and it has one single image for any $x$, then that image is supposed to be $f$ itself, according to **P2**. Well, that is exactly the statement of axiom **P4**. And according to theorem 3, there is one single function like this, namely, $\underline{0}$.

On the other hand, **P6** guarantees the existence of at least one other constant function, namely, $\phi_0$ such that $\phi_0(x) = \underline{0}(x)$ for any $x$ different from $\phi_0$. Nevertheless, $\phi_0$ is not $\underline{0}$, despite the fact that both $\underline{0}$ and $\phi_0$ are constant functions with the same constant value $\underline{0}$.

**Theorem 24** *For any $n$ of language $\mathcal{L}$, there is a constant function $f$ whose constant value is $\phi_n$.*

**Proof:** From the proof of Theorem 22 it is easy to see that $\mathcal{C}_{\phi_1}(\sigma \circ \sigma \circ \underline{0})$, $\mathcal{C}_{\phi_2}(\sigma \circ \sigma \circ \sigma \circ \underline{0})$, and so on.

The last theorems state there are infinitely many constant functions in Flow, namely, those with constant values $\underline{0}$, $\phi_0$, $\phi_1$, $\phi_2$, etc.

It is worth to observe as well, both evaluation $f_1^2$ and composition $\circ$ are not commutative. For example, while $\sigma \circ \underline{0}$ is a constant function with constant value $\phi_0$, $\underline{0} \circ \sigma$ is $\phi_0$. Besides, $\underline{0}(\sigma) = \underline{0}$, but $\sigma(\underline{0}) = \sigma_{\underline{0}} = \phi_0$. Other examples are provided in the next paragraphs.

A final word of caution is necessary here regarding evaluation versus composition. We cannot make confusion between formulas $f(x) = \underline{1}$ and $f \circ x = \underline{1}$. The former is perfectly possible for $f = x = \underline{1}$. The latter is impossible, since no composition results $\underline{1}$.

Next we want to guarantee the existence of *proper restrictions* of a given function. By *proper restriction* of a function $f$ we mean a function $g$ such that: (i) $g(f) = \underline{0}$; and (ii) for the remaining values $x$, we may have either $g(x) = f(x)$ or $g(x) = \underline{0}$ (except, of course, when $x$ is $g$; in that case, $g(g) = g$).

So, if $F(x)$ is a formula where all occurrences of $x$ are free and such that there is no free occurrences of $g$ in $F(x)$, then the following is an axiom.

**P8$_F$ - Restriction** $\forall f(f \neq \underline{0} \Rightarrow \exists! g(g \neq \underline{0} \wedge (g \neq f \Rightarrow g(f) = \underline{0}) \wedge \forall x \forall y((x \neq g \wedge x \neq f) \Rightarrow (g(x) = y \Leftrightarrow ((f(x) = y \wedge F(x)) \vee (y = \underline{0} \wedge \neg F(x)))))))$.

The antecedent for the first conditional $\Rightarrow$ above guarantees the necessary condition for the existence of any *restriction* $g$ of a given function $f$, namely, $f \neq \underline{0}$.

Hence, if $f$ is different of $\underline{0}$, then there is a unique function $g$ such that: (i) $g(f) = \underline{0}$ if $g \neq f$; (ii) $g(g) = g$; (iii) $g$ and $f$ share non-$\underline{0}$ images for some $x$ as long $x$ does satisfy formula $F$; and (iv) when $g$ and $f$ do not share non-$\underline{0}$ images for any $x$, then $g(x) = \underline{0}$. We call $g$ a *restriction* of $f$ under $F(x)$, or simply a restriction of $f$. In a sense, this last axiom is very similar to the Separation Scheme in ZFC. Nevertheless, Separation Scheme's role is not

limited to guarantee the existence of subsets. Thanks to that postulate, ZFC avoids antinomies like Russell's paradox. In the case of Flow Theory, those antinomies are avoided by means of the self-reference axiom **P2**.

To adjust the mathematics of Flow into common practice, the next definition if quite handy.

**Definition 6** *For any function $f$ different of $\underline{0}$, its restriction $g$ is either $f$ itself, or any proper restriction $g$ of $f$, as long $g$ is not $\underline{0}$. Formally, we denote this by*

$$g \subseteq f \ \ iff \ \ (g = f) \vee (g(f) = \underline{0} \wedge \forall x \forall y ((x \neq g \wedge x \neq f) \Rightarrow (g(x) = y \Rightarrow f(x) = y)),$$
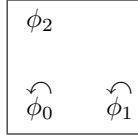
*where both $f$ and $g$ are different of $\underline{0}$.*

Proper restrictions are defined as:

**Definition 7** $g \subset f$ *iff* $g \subseteq f \wedge g \neq f$.

We abbreviate $\neg(g \subseteq f)$ and $\neg(g \subset f)$ as, respectively, $g \nsubseteq f$ and $g \not\subset f$. Accordingly, for all $f$ we have $\underline{0} \nsubseteq f$ and $f \nsubseteq \underline{0}$.

As an example, consider $f = \phi_2$.



**Figure 2:** Diagram of function $\phi_2$.

According to axiom **P8**$_F$, there are *three* proper restrictions $g$ to $\phi_2$. If $F(x)$ is the formula "$x = \phi_0$", then $g = \phi_0$. If $F(x)$ is the formula "$x = \phi_0 \vee x = \phi_1$", then $g = \phi_1$. If $F(x)$ is "$x = x$", then again $g = \phi_1$. If $F(x)$ is "$x \neq x$", then once more $g = \phi_0$. Both $\phi_0$ and $\phi_1$ have their diagrams already represented some paragraphs above. The novelty here, however, happens for the formula $F(x)$ given by "$x = \phi_1$". In that case we have a proper restriction $\gamma$ such that $\gamma \neq \phi_1$, $\gamma(\gamma) = \gamma$, $\gamma(\phi_1) = \phi_1$, and $\gamma(x) = \underline{0}$ for any $x$ different of $\phi_1$ and $\gamma$ itself. Its diagram is as follows.
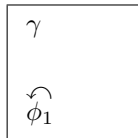


**Figure 3:** Diagram of function $\gamma$, a special restriction of $\phi_2$.

Thus, $\phi_2$ admits four restrictions: $\phi_0$, $\phi_1$, function $\gamma$ in the diagram above, and $\phi_2$.

For practical purposes, it seems useful to adopt a rule of thumb for a better understanding of the concept of restriction. Any function $f$ which satisfies the antecedent of the first conditional in axiom $\mathbf{P8}_F$ is a function which "acts on something". That means there is a $t$ different of $f$ such that $f(t)$ is not $\underline{0}$. For example, $\phi_2$ acts on $\phi_0$ and $\phi_1$. So, all restrictions of $\phi_2$ correspond, intuitively speaking, to all possible combinations of $\phi_0$ and $\phi_1$. Those possible combinations, in that case, are: (1) *nothing at all*, since $\phi_0$ does not act on anyone; (2) $\phi_0$, since $\phi_1$ acts only on $\phi_0$; (3) $\phi_1$, since $\gamma$ acts only on $\phi_1$; and, finally, (4) *everything*, since $\phi_2$ acts both on $\phi_0$ and $\phi_1$.

There are infinitely many other functions (besides $\underline{1}$) which do not have any non-$\underline{0}$ successor, as stated by one of the next theorems. But before that, it is useful to adopt the next convention. The term below

$$ f\big|_{F(x)} $$

denotes a restriction of $f$ by use of axiom $\mathbf{P8}_F$ and formula $F(x)$.

**Theorem 25** *There is a function $g$ such that $\sigma_g = \underline{1}$*

**Proof:** Consider $g = \underline{1}\big|_{x \neq \underline{1}}$. That means $g(x) = \underline{0}$ only if either $x = \underline{0}$ or $x = \underline{1}$. For the remaining values we have $g(x) = x$, which satisfies the definition of successor in the sense that $\sigma_g = \underline{1}$.

**Theorem 26** $\sigma_{\underline{1} \circ \underline{1}} = \underline{1}$.

**Proof:** The formula above is equivalent to say that the successor $\sigma_h$ of function $h$ from Theorem 10 is $\underline{1}$. Well, function $h$ of Theorem 10 is such that $h(x) = x$ for any $x$ different of $\underline{1}$. Besides, $h(\underline{1}) = \underline{0}$. And that is exactly function $g$ of Theorem 25.

Recall the unique function $h$ from Theorem 10 was abbreviated as $\psi$. That means $\psi = \underline{1} \circ \underline{1}$. And now again we see $\psi$ as a function whose successor is $\underline{1}$. In other words, $\sigma_\psi = \underline{1}$.

**Theorem 27** *The successor $\sigma_g$ for any $g = \Phi_n =_{def} \underline{1}\big|_{x \neq \phi_n}$ is $\underline{0}$, if $n$ belongs to language $\mathcal{L}$.*

**Proof:** According to $\mathbf{P8}_F$, $g(\phi_n) = \underline{0}$. That means $g$ is different of $\underline{1}$, since $\underline{1}(\phi_n) = \phi_n$. And Theorem 1 entails that $g \neq \underline{1}$. Consequently, $g(\underline{1}) = \underline{0}$, according to $\mathbf{P8}_F$; and $g(\underline{0}) = \underline{0}$, since $g$ is a restriction of $\underline{1}$ and $\underline{1}(\underline{0}) = \underline{0}$. Now, suppose there is $\sigma_g \neq \underline{0}$. Definition 2 demands that $g(\sigma_g) = \underline{0}$. Besides, $g \neq \sigma_g$ and $g$ and $\sigma_g$ are supposed to share the same images for any $x$ different of $\sigma_g$. But $g(x) = \underline{0}$ iff $x = \underline{0}$ or $x = \underline{1}$ (as already established) or $x = \phi_n$, according to $\mathbf{P8}_F$. For the remaining values of $x$, $g(x) = x \neq \underline{0}$ (axiom $\mathbf{P8}_F$ again). That means there are only three

possible values for $\sigma_g$, namely, $\underline{0}$, $\underline{1}$ or $\phi_n$. And only two of them are different of $\underline{0}$. Now consider $\phi_m$, where $n \prec m$. Function $\phi_m$ is different of either one of those three possible values. So, if anyone of them is $\sigma_g$, it is supposed to share the same images of $g$, for $x = \phi_m$. Notwithstanding, $\phi_n(\phi_m) = \underline{0}$ (according to the recursive definition of functions $\phi_n$), while $g(\phi_m) = \phi_m$. That means $\sigma_g$ cannot be $\phi_n$. Finally, $\underline{1}(\phi_n) = \phi_n$, while $g(\phi_n) = \underline{0}$. That means $\sigma_g$ cannot be $\underline{1}$ either. Thus, the only possible value for $\sigma_g$ is $\underline{0}$, despite the fact that $g$ and $\underline{0}$ do not share the same images for any $x$.

This last theorem does not consider all possible cases of functions $f$ with no successor $\sigma_f \neq \underline{0}$. Similar results may be obtained, e.g., for $\underline{1}\big|_{x \neq \phi_n \vee x \neq \phi_m}$ (where $m \neq n$), $\underline{1}\big|_{x \neq \phi_n \vee x \neq \phi_m \vee x \neq \phi_p}$ (for $m \neq n$, $n \neq p$ and $m \neq p$) etc. Even if we consider $F(x)$ as any finite disjunction of the form $x \neq \phi_{n_1} \vee x \neq \phi_{n_2} \vee \cdots \vee x \neq \phi_{n_m}$ for any $m$, we still cannot guarantee that all possible cases of functions with no successor different of $\underline{0}$ are ran out. But this last theorem is proven to be rather important for our discussion regarding the translation of ZFC axioms into Flow, as we see in the next Section.

The last theorems teach us the following:

1. If a function $f$ does have a successor $\sigma_f \neq \underline{0}$, that does not necessarily entail that any restriction of $f$ has a non-$\underline{0}$ successor. For example, any function $\Phi_n$ of Theorem 27 is a proper restriction of $\psi$ (Theorem 25). Nevertheless, although there is a successor of $\psi$, which is different of $\underline{0}$, no $\Phi_n$ has a successor different of $\underline{0}$.

2. If a function $f$ has $\underline{0}$ as its successor, that does not entail that every restriction of $f$ has its successor equal to $\underline{0}$. For example, every $\Phi_n$ has successor $\underline{0}$. Nevertheless, $\phi_3$ is a proper restriction of any $\Phi_n$, for $n \neq 3$. And despite the fact that such a $\Phi_n$ has successor $\underline{0}$, $\phi_3$ has its successor different of $\underline{0}$.

3. If a function $f$ has successor $\sigma_f \neq \underline{0}$, that does not entail that $\sigma_f$ has successor different of $\underline{0}$. For example, the successor of $\psi$ is $\underline{1}$. But the successor of $\underline{1}$ is $\underline{0}$. So, there is $\sigma_\psi \neq \underline{0}$, but there is no $\sigma_{\sigma_\psi} \neq \underline{0}$.

Hence, Flow teaches us that restrictions of a function $f$ are not informative enough about the behavior of any $f$. We need something else. And that something else is provided by axiom **P11**, which is displayed some pages below.

So far, most functions $f$ in Flow behave like "children" of $\underline{1}$, in the sense that for any $x$ we have either $f(x) = x$ or $f(x) = \underline{0}$. One exception for this rule is $\sigma$. Notwithstanding, if we want to ground standard mathematics, we need much more than that. So, in order to discuss about that, we need something which resembles the usual notion of ordered pair.

**Definition 8** $f$ *is an* ordered pair $(a, b)$ *iff there are $\alpha$ and $\beta$ such that $\alpha \neq f$, $\beta \neq f$, $\alpha \neq a$, $\beta \neq b$ and*

$$f(x) = \begin{cases} \alpha & if\, x = \alpha \\ \beta & if\, x = \beta \\ \underline{0} & if\, x \neq f \wedge x \neq \alpha \wedge x \neq \beta \end{cases}$$

where $\alpha(a) = a$, $\alpha(x) = \underline{0}$ if $x$ is neither $a$ nor $\alpha$, $\beta(a) = a$, $\beta(b) = b$, $\beta(x) = \underline{0}$ if $x$ is neither $a$ nor $b$ or $\beta$.

Observe we did not demand $\alpha \neq b$. That means we may have two kinds of ordered pairs, namely, those where $\alpha \neq b$ (first kind) and those where $\alpha = b$ (second kind). The diagram for an ordered pair $f = (a, b)$, where $\alpha \neq b$, may be written as follows:
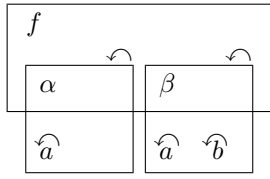


**Figure 4:** Diagram of an ordered pair $f = (a, b)$ of the first kind.

The diagram above says $f$ acts only on $\alpha$ and $\beta$, while $\alpha$ acts only on $a$, and $\beta$ acts only on $a$ and $b$. In the particular case where $a = b$, we have $\alpha = \beta$, and the ordered pair $f$ is denoted by $(a, a)$. Observe that in the diagram above $f(a) = f(b) = \underline{0}$, which means that $f$ never acts neither on $a$ nor on $b$, if $f$ is the ordered pair $(a, b)$ of the first kind. In other words, $f$ is an ordered pair $(a, b)$ iff $f$ acts only on functions $\alpha$ and $\beta$ which act, respectively, only on $a$ and only on $a$ and $b$. To get $(b, a)$, all we have to do is to exchange $\alpha$ by a function $\alpha'$ which acts only on $b$. Hence, our definition for ordered pair is obviously inspired on the standard notion by Kuratowski. In standard set theory an ordered pair $(a, b)$ is a set $\{\{a\}, \{a, b\}\}$ such that neither $a$ nor $b$ belong to $(a, b)$. In Flow, on the other hand, an ordered pair $(a, b)$ of the first kind is a function which does not act neither on $a$ nor on $b$.

Nevertheless, the second kind of ordered pair shows our approach is not equivalent to Kuratowki's. In the case where $\alpha = b$, we have the following diagram.
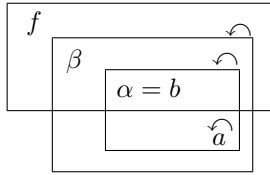


**Figure 5:** Diagram of an ordered pair $f = (a, b)$ of the second kind.

In this non-Kuratowskian kind of ordered pair $f = (a, b)$, $f$ acts on $b$, although it does not act on $a$. In the general case, no ordered pair $f = (a, b)$ ever acts on $a$.

Since any ordered pair $(a, b)$ is a function, we abbreviate $x((a, b)))$ as $x(a, b)$ for a given function $x$.

We intend to use the notion of ordered pair to guarantee the existence of other functions, besides our previous "children" of $\underline{1}$ (which are "children" of $\underline{0}$ as well, since usually most of their images are $\underline{0}$). So, our idea is as follows. Consider, for example, function $\phi_2$, whose restrictions are $\phi_0$, $\phi_1$, $\phi_2$ and $\gamma$, as previously discussed. If we guarantee the existence of a function $f$ which acts only on $\phi_1$ and $\phi_2$ (in a way such that $f(\phi_1) = \phi_1$ and $f(\phi_2) = \phi_2$), then we can easily prove $f$ is the ordered pair $(\phi_0, \phi_1)$. After all, $\phi_1$ acts only on $\phi_0$; and $\phi_2$ acts only on $\phi_0$ and $\phi_1$. On the other hand, if we can guarantee the existence of function $g$ such that $g$ acts only on $\gamma$ and $\phi_2$ (in a way such that $g(\gamma) = \gamma$ and $g(\phi_2) = \phi_2$), we can easily prove that $g$ is the ordered pair $(\phi_1, \phi_0)$. Ultimately, $\gamma$ acts only on $\phi_1$; and $\phi_2$ acts only on $\phi_1$ and $\phi_0$. Observe that $(\phi_0, \phi_1)$ is a non-Kuratowskian ordered pair (second kind), while $(\phi_1, \phi_0)$ is a Kuratowskian ordered pair (first kind).

Once Flow is endowed with ordered pairs $(\phi_0, \phi_1)$ and $(\phi_1, \phi_0)$, all we have to do is to guarantee the existence, e.g., of functions $l$ and $m$ such that $l(\phi_0) = \phi_1$ and $m(\phi_1) = \phi_0$. In that case we are no longer restricted to functions $f$ such that $f(x)$ is either $x$ itself or $\underline{0}$.

Fortunately, the next theorem guarantees we can always define ordered pairs $(a, b)$ for any functions $a$ and $b$ as long we state that none of them is $\underline{1}$. Such a restriction comes from the fact that we use restrictions applied to $\underline{1}$ in order to prove the next result. And any proper restriction of $\underline{1}$ is a function $f$ such that $f(\underline{1}) = \underline{0}$.

**Theorem 28** *If $a$ and $b$ are functions both different of $\underline{0}$ and $\underline{1}$, then there is a function $f$ such that $f = (a, b)$.*

**Proof:** First, we use axiom $\mathbf{P8}_F$ to define the proper restriction of $\underline{1}$ for "$x = a \vee x = b$" as formula $F(x)$. Such a proper restriction can be denoted as $\beta$. So, $\beta$ is a function such that $\beta(a) = a$, $\beta(b) = b$, $\beta(\beta) = \beta$, and $\beta(x) = \underline{0}$ for all remaining values of $x$. Analogously, the proper restriction of $\underline{1}$ for "$x = a$" as formula $F(x)$ in $\mathbf{P8}_F$ gives us the function $\alpha$ such that $\alpha(a) = a$, $\alpha(\alpha) = \alpha$, and $\alpha(x) = \underline{0}$ for the remaining values of $x$. Finally, the proper restriction of $\underline{1}$ for "$x = \alpha \vee x = \beta$ as formula $F(x)$ in $\mathbf{P8}_F$ provides us a function $f$ such that $f(\alpha) = \alpha$, $f(\beta) = \beta$, $f(f) = f$, and $f(x) = \underline{0}$ for all the remaining values of $x$. But function $f$ is exactly that one in definition 8. Hence, $f = (a, b)$.

This last theorem says we do not need $\phi_2$ to produce ordered pairs $(\phi_0, \phi_1)$ and $(\phi_1, \phi_0)$, as we did above. Since axiom $\mathbf{P6}$ guarantees the existence of functions $\phi_n$, we can use $\mathbf{P8}_F$ to obtain any ordered pair $(\phi_m, \phi_n)$.

**Observation 3** *Now, what is the first valuable lesson taught by Flow? From the first five axioms we learn the existence of two functions, namely, $\underline{0}$ and $\underline{1}$. Besides, we learn how to distinct one from the other, thanks to $\mathbf{P1}$. That fact, per se, suggests some notion of duality which is reinforced by the concept of*

*successor: the successor of $\underline{1}$ is $\underline{0}$, and the successor of $\underline{0}$ is $\phi_0$. Thus, more than a principle of duality, we have a principle of complementarity, where the function successor establishes some sort of cycle which connects those two extremes, $\underline{0}$ and $\underline{1}$. In its turn, axiom* **P5** *teaches us how to compose functions. But what is the advantage of composing functions if all we have is a few privileged functions? Compositions involving $\underline{0}$ and $\underline{1}$ do not produce any new functions besides $\phi_0$ and $\psi$. So, axioms* **P6** *and* **P7** *allow us to build infinitely many functions from $\underline{0}$. Those are functions $\phi_n$ $(n = 0, 1, 2, \cdots)$. On the other hand, axiom* **P8**$_F$ *allows us to "deconstruct" $\underline{1}$ to achieve another vast myriad of functions, including ordered pairs. Without* **P7**, **P8**$_F$ *is useless, for the latter demands the existence of a function $f$ which acts on some $t$. And no function can do that in a universe where all we know is the existence of $\underline{0}$, $\underline{1}$, $\phi_0$, and $\psi$. And without* **P8**$_F$, **P7** *is very poor. Hence, $\underline{0}$, under the influence of* **P6** *and* **P7**, *can be seen as a* creation *function. Analogously, $\underline{1}$, under the influence of* **P8**$_F$, *can be seen as an* annihilation *function. Both, creation and annihilation, allow us to shape a whole universe of functions. First we create, then we destroy. That is the main difference between our approach and the usual notions of standard set theories. Standard set theories like ZFC build whole universes of sets from one single source, the empty set. That means the standard approach for deriving sets is by means of a single process of creation. In Flow, however, we build new terms from two fronts: good and evil, light and darkness, creation and annihilation, expansion (* **P6** *) and restriction (* **P8**$_F$ *). That is how Flow Theory flows.*

**Theorem 29** $\forall f(f \neq \underline{0} \Rightarrow (\phi_0 \subseteq f))$.

**Proof:** If $f = \phi_0$, the proof is trivial, since according to Definition 6 every function different of $\underline{0}$ is a restriction of itself. If $f \neq \phi_0$, all we have to do is to use "$x \neq x$" as formula $F(x)$ in axiom **P8**$_F$. In that case $\phi_0$ is a proper restriction of $f$.

**Definition 9** *$z$ is the* power function *of $f$ (or simply the* power *of $f$) iff $z \neq f \wedge \forall x(x \neq z \Rightarrow ((z(x) = x \Leftrightarrow x \subseteq f) \wedge (z(x) = \underline{0} \Leftrightarrow x \nsubseteq f)))$. We denote $z$ as $\mathcal{P}(f)$.*

**Theorem 30** *For any function $h$ different of $\underline{0}$ there is a unique $\mathcal{P}(h)$.*

**Proof:** All we have to do is to apply axiom **P8**$_F$ over function $f = \underline{1}$ and assume "$x \subseteq h$" as formula $F(x)$. Function $g$ guaranteed by **P8**$_F$ is precisely $\mathcal{P}(h)$.

So, for example, $\mathcal{P}(\phi_0) = \phi_1$, $\mathcal{P}(\phi_1) = \phi_2$, and $\mathcal{P}(\phi_2)$ is a function $f$ such that $f \neq \phi_2$, $f(\phi_0) = \phi_0$, $f(\phi_1) = \phi_1$, $f(\phi_2) = \phi_2$, $f(\gamma) = \gamma$, $f(f) = f$, and $f(x) = \underline{0}$ for all the remaining values of $x$.

One interesting side effect of the concept of power function is that $\mathcal{P}(\underline{1})$ is somehow a "smaller" function than $\underline{1}$. What do we mean by that? It means that $\underline{1}$ acts on every single function, with the only exceptions of $\underline{0}$ and $\underline{1}$, since $\underline{1}(\underline{0}) = \underline{0}$ and $\underline{1}(\underline{1}) = \underline{1}$. But $\mathcal{P}(\underline{1})$ is a function $z$ which is different of $\underline{1}$ and such that $z$ acts only on those functions $t$ such that for any $x$ we have either $t(x) = x$

or $t(x) = \underline{0}$. So, Flow is apparently free of any paradox regarding the notion of power.

The next theorem is a first step to prove the existence of relations in Flow. So, contrary to usual set-theoretic notions, relations are special cases of functions.

**Theorem 31** *Let $l$ and $m$ be functions such that they are both different of $\underline{0}$ and $\phi_0$. Then there is a function $g$ such that for any $t \neq g$ we have $g(t) \neq \underline{0}$ iff $t = (a, b)$, where $a$ and $b$ are such that $a \neq l$, $b \neq m$, $l(a) \neq \underline{0}$, and $m(b) \neq \underline{0}$.*

**Proof:** All we have to do is to apply Axiom $\mathbf{P8}_F$ over function $f = \underline{1}$, by assuming as formula $F(x)$ the following one, for a given $l$ and a given $m$:
$$a \neq l \wedge b \neq m \wedge l(a) \neq \underline{0} \wedge m(b) \neq \underline{0} \Leftrightarrow x = (a, b).$$

This unique function $g$ is called the *trivial product* between $l$ and $m$, and it is denoted by $l \otimes m$.

For example, if $l = \phi_3$ and $m = \phi_2$ (both do satisfy the conditions demanded by the theorem above), then $g = \phi_3 \otimes \phi_2$ is the following function.
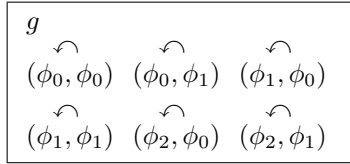


**Figure 6:** The trivial product between $\phi_3$ and $\phi_2$.

The arrows in Figure 6 say that $g(\phi_0, \phi_0) = (\phi_0, \phi_0)$, $g(\phi_0, \phi_1) = (\phi_0, \phi_1)$, $g(\phi_1, \phi_0) = (\phi_1, \phi_0)$, $g(\phi_1, \phi_1) = (\phi_1, \phi_1)$, $g(\phi_2, \phi_0) = (\phi_2, \phi_0)$, and $g(\phi_2, \phi_1) = (\phi_2, \phi_1)$. Besides, $g(g) = g$ and $g(x) = \underline{0}$ for the remaining values of $x$.

As expected, this operation $\otimes$ is not commutative since, e.g., $\phi_3 \otimes \phi_2$ is different of $\phi_2 \otimes \phi_3$. That means we can define relations as it follows:

**Definition 10** *If $g$ is the trivial product between $l$ and $m$, then any $f$ such that $f \subseteq g$ is called a relation with domain $l$ and co-domain $m$.*

If we want to define a relation $f$ with domain $l$ and co-domain $m$, we just need to apply $\mathbf{P8}_F$ over $l \otimes m$ for a given formula $F(x)$. As an example, consider the following definition:

**Definition 11** *Let $l$ and $m$ be functions such that they are both different of $\underline{0}$ and $\phi_0$. Function $g$ is a trivially arbitrary function with domain $l$ and co-domain $m$ iff $f \subseteq l \otimes m$ and for all $a$ such that $l[a]$ there is a unique $b$ such that $m[b]$ and $g(a, b) = (a, b)$. We denote this by $\mathcal{T}_{l \to m}(g)$.*

That means trivially arbitrary functions are special cases of relations.

**Theorem 32** *For any functions $l$ and $m$ which are both different of $\underline{0}$ and $\phi_0$, there is at least one $g$ such that $\mathcal{T}_{l \to m}(g)$.*

**Proof:** All we have to do is to apply Axiom $\mathbf{P8}_F$ over function $f = l \otimes m$, by assuming as formula $F(x)$ the following one: $\forall a((a \neq l \wedge l(a) \neq \underline{0}) \Rightarrow \exists! b(b \neq m \wedge m(b) \neq \underline{0} \wedge x = (a,b)))$.

If we use the particular case illustrated in Figure 6, one example of trivially arbitrary function $f$ with domain $\phi_3$ and co-domain $\phi_2$ is the following:
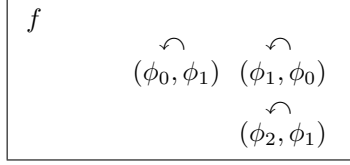


**Figure 7:** Example of a trivially arbitrary function $f$ with domain $\phi_3$ and co-domain $\phi_2$.

Notwithstanding, despite all those results above, all functions in Flow work as some some sort of restriction of $\underline{1}$, in the sense that all our functions $f$ (until now) are such that for any $x$ we have $f(x)$ is either $x$ or $\underline{0}$. To accommodate arbitrary functions, we need the next axiom.

**P9 - Freedom** $\forall l \forall m \forall f (\mathcal{T}_{l \to m}(f) \Rightarrow \exists! g(\forall a \forall b(f(a,b) \neq \underline{0} \Rightarrow g(a) = b)))$.

The intuitive idea of this last axiom is quite simple. If we have a trivially arbitrary function $f$ with domain $l$ and co-domain $m$ which acts on ordered pairs $(a,b)$ in a way such that $f(a,b)$ is always $(a,b)$, then there is a function $g$ such that $g(a) = b$. That means we have now new functions $g$ where $g(a)$ is not necessarily $a$.

If we apply axiom $\mathbf{P9}$, e.g., over function $f$ illustrated in Figure 7, we can get now the following:
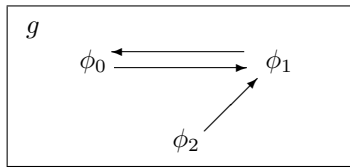


**Figure 8:** Example of a function $g$ obtained from $f$ (of Figure 7) by use of Axiom $\mathbf{P9}$.

The example above refers to a function $g$ such that $g(\phi_0) = \phi_1$, $g(\phi_1) = \phi_0$, $g(\phi_2) = \phi_1$, $g(g) = g$, and $g(r) = \underline{0}$ for the remaining values of $r$.

By using the same ideas, we can define as well, from $i \otimes i$ (where $i$ is an inductive function), a function $\lambda$ such that $\lambda(\phi_n) = \phi_{n+1}$, $\lambda(\lambda) = \lambda$ and $\lambda(r) = \underline{0}$

for the remaining values of $r$. That function $\lambda$ is particularly useful in later discussions.

The next definition is quite useful for dealing with unions, as we intend to do in the next axiom:

**Definition 12** *Let $g$, $h$ and $t$ be functions. Then,*

1. *$\mathcal{X}_\square(g, h \leftarrow t)$ iff $g[t] \wedge h[t] \wedge g(t) = h(t)$,*

2. *$\mathcal{X}_\diamond(g, h \leftarrow t)$ iff $g[t] \wedge h[t] \wedge g(t) \neq h(t)$,*

3. *$\mathcal{X}_\triangle(g, h \leftarrow t)$ iff $t \neq g \wedge t \neq h \wedge (g(t) = \underline{0} \vee h(t) = \underline{0}) \wedge \neg(g(t) = \underline{0} \wedge h(t) = \underline{0})$,*

4. *$\mathcal{X}_\bigcirc(g, h \leftarrow t)$ iff $t \neq g \wedge t \neq h \wedge g(t) = \underline{0} \wedge h(t) = \underline{0}$.*

$\mathcal{X}_\square(g, h \leftarrow t)$ says both $g$ and $h$ act on a given $t$ and share the same value for $g(t)$ and $h(t)$. $\mathcal{X}_\diamond(g, h \leftarrow t)$ says both $g$ and $h$ act on a given $t$; but in that case they do not share the same image for $t$. $\mathcal{X}_\triangle(g, h \leftarrow t)$ says either $g$ or $h$ acts on a given $t$, while the other one gives us an image equal to $\underline{0}$. Finally, $\mathcal{X}_\bigcirc(g, h \leftarrow t)$ simply says both $g$ and $h$ share the same image for a given $t$, and that image is $\underline{0}$.

**P10 - Union** $\forall f((f \neq \underline{0} \wedge \forall x(f[x] \Rightarrow \sigma_x \neq \underline{0})) \Rightarrow$
$\exists! u(\sigma_u \neq \underline{0} \wedge (\forall g \forall h((f[g] \wedge f[h]) \Rightarrow \forall t(t \neq u \Rightarrow$
$(\mathcal{X}_\square(g, h \leftarrow t) \Rightarrow u(t) = g(t)) \wedge$
$((\mathcal{X}_\diamond(g, h \leftarrow t) \vee \mathcal{X}_\bigcirc(g, h \leftarrow t)) \Rightarrow u(t) = \underline{0})) \wedge$
$(\mathcal{X}_\triangle(g, h \leftarrow t) \Rightarrow (u(t) \neq \underline{0} \wedge (u(t) = g(t) \vee u(t) = h(t))))))))$.

We hope the reader does not feel intimidated by the apparent complexity of this last formula. Actually, this postulate is quite intuitive.

Suppose $f$ acts on many functions, like $g$ and $h$. So, we have four possibilities for an arbitrary $t$ (as long neither $g$ nor $h$ is $t$): (i) both $g$ and $h$ act on $t$ and share the same value ($g(t) = h(t)$); (ii) both $g$ and $h$ act on $t$, but do not share the same value ($g(t) \neq h(t)$); (iii) either $g$ or $h$ does not act on $t$, but one of them does act on $t$; (iv) both $g(t)$ and $h(t)$ have value $\underline{0}$. In the first case, $u(t)$ has the value shared by both $g$ and $h$ on $t$. In the second case, $u(t)$ is $\underline{0}$. And the same happens for the fourth case. Finally, in the third case, $u(t)$ has the same value of either $g(t)$ or $h(t)$, as long we are talking about the only one which acts on $t$.

This last axiom allows us to obtain arbitrary unions of functions, even if they do not share the same images. And the resultant unique union $u$ is a function. So, in a precise sense, this last axiom generalizes the standard notion of union in theories like ZFC, NBG, and others. We denote function $u$ as

$$u = \bigcup_{f[g]} g,$$

where $f$ acts on $g$.

Consider the following example.

Let $f$ be such that $f(g) = g$, $f(h) = h$, $f(f) = f$, and $f(r) = \underline{0}$ for the remaining values of $r$, where $g$ and $h$ are represented below:
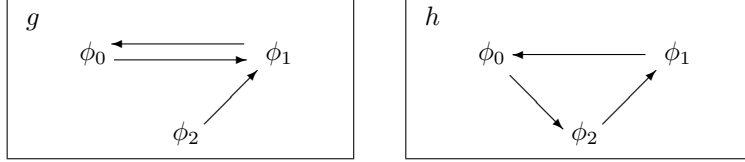


**Figure 9:** Example of functions to be unified.

That means $g(\phi_0) = \phi_1$, $g(\phi_1) = \phi_0$, $g(\phi_2) = \phi_1$, $g(g) = g$, and $g(r) = \underline{0}$ for the remaining values of $r$. Besides, $h(\phi_0) = \phi_2$, $h(\phi_1) = \phi_0$, $h(\phi_2) = \phi_1$, $h(h) = h$, and $h(r) = \underline{0}$ for the remaining values of $r$. Observe that for both cases we have $g(x) \neq \underline{0} \Rightarrow \sigma_x \neq \underline{0}$ and $h(x) \neq \underline{0} \Rightarrow \sigma_x \neq \underline{0}$. That fact entails that both $g$ and $h$ have their respective non-$\underline{0}$ successors. In other words, there is a union $u$ which is associated to $f$. So, we have $\mathcal{X}_\square(g, h \leftarrow \phi_1)$, $\mathcal{X}_\square(g, h \leftarrow \phi_2)$, $\mathcal{X}_\Diamond(g, h \leftarrow \phi_0)$, and $\mathcal{X}_\bigcirc(g, h \leftarrow r)$ for the remaining values of $r$. By applying axiom **P10**, we have that $u = \bigcup_{f[i]} i$ (where $f$ acts on $i$) is simply
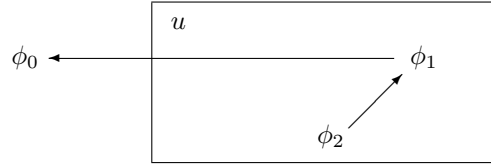


**Figure 10:** Union between functions $g$ and $h$ of Figure 9.

where the arrow which escapes the diagram says that $u(\phi_0) = \underline{0}$, despite the fact that $u(\phi_1) = \phi_0$.

As a second example, consider a function $f'$ such that $f'(g) = h$, $f'(h) = g$, $f'(f') = f'$, and $f'(r) = \underline{0}$ for the remaining values of $r$. Clearly, $f' \neq f$. Nevertheless, we have

$$u = \bigcup_{f'[j]} j = \bigcup_{f[i]} i,$$

where both $f'$ and $f$ act, respectively, on $j$ and $i$. That means different functions may generate the same union $u$, a result which is analogous to what happens, e.g., in ZFC.

If we want the particular case of standard union, all we need to do is to consider the definition given below.

**Definition 13** *Any function $f$ is* strictly unifiable *iff $f$ is not $\underline{0}$ and $\forall g \forall h((f(g) \neq \underline{0} \wedge f(h) \neq \underline{0}) \Rightarrow \forall t((g(t) \neq \underline{0} \wedge h(t) \neq \underline{0}) \Rightarrow g(t) = h(t)))$. We denote this by $\mathcal{U}(f)$.*

So, if $f$ is strictly unifiable, its arbitrary union corresponds, intuitively speaking, to the standard notion of union. That is proved in the next section.

**Theorem 33** *Let $x$ and $y$ be functions such that both $\sigma_x$ and $\sigma_y$ are different of $\underline{0}$. If $x \circ y \neq \underline{0}$ and $y$ does not act on $\sigma_x$, then $\sigma_x \circ \sigma_y \subseteq \sigma_{x \circ y}$.*

**Proof:**

$$(\sigma_x \circ \sigma_y)(t) = \sigma_x(\sigma_y(t)) = \left\{ \begin{array}{ll} \sigma_x \circ \sigma_y & \text{if } t = \sigma_x \circ \sigma_y \\ \sigma_x(y(t)) & \text{if } t \neq \sigma_x \circ \sigma_y \wedge t \neq \sigma_y \\ \sigma_x(\sigma_y) & \text{if } t = \sigma_y \end{array} \right.$$

$$= \left\{ \begin{array}{ll} \sigma_x \circ \sigma_y & \text{if } t = \sigma_x \circ \sigma_y \\ x(y(t)) & \text{if } t \neq \sigma_x \circ \sigma_y \wedge t \neq \sigma_y \wedge y(t) \neq \sigma_x \\ \sigma_x(\sigma_y(t)) & \text{if } t = \sigma_x \\ \sigma_x(\sigma_y) & \text{if } t = \sigma_y \end{array} \right.$$

So, according to the second line of the last brace, unless $y$ acts on $\sigma_x$, we have that $\sigma_x \circ \sigma_y \subseteq \sigma_{x \circ y}$.

**P11 - Coherence** $\forall f((\sigma_f \neq \underline{0} \wedge \forall x(f[x] \Rightarrow \sigma_x \neq \underline{0})) \Rightarrow$
$(\forall g(g \subseteq f \Rightarrow \sigma_g \neq \underline{0}) \wedge \forall g \forall h((h[g] \Rightarrow g \subseteq f) \Rightarrow \sigma_h \neq \underline{0})))$.

This last postulate allows us to establish a frontier between standard objects of Flow Theory and those who are non-standard. For now, standard objects are those directly associated to the concept of a non-$\underline{0}$ successor. If $f$ has a non-$\underline{0}$ successor and it acts only on terms who have non-$\underline{0}$ successor, then any restriction of $f$ has a non-$\underline{0}$ successor and any $h$ which acts on those restrictions has a non-$\underline{0}$ successor. Later on we identify those standard objects to those terms who can be found in ZFC. Its intuitive appeal is quite clear.

**Definition 14** $f \sim g$ *iff* $\forall t((t \neq f \wedge t \neq g) \Rightarrow f(t) = g(t))$.

This last definition has an important role to be discussed at the end of this paper. For now, all the reader needs to know is that its main purpose is to be used in the next postulate.

**P12 - Choice** $\forall f(\forall x \forall y((f(x) \neq \underline{0} \wedge f(y) \neq \underline{0} \wedge x \neq y) \Rightarrow (x \neq \phi_0 \wedge \neg \exists s(x(s) = y(s) \wedge x(s) \neq \underline{0}))) \Rightarrow$
$\exists c \forall r(f(r) \neq \underline{0} \Rightarrow \exists! w(c(w) = r(w) \wedge r(w) \neq \underline{0})) \wedge \forall d(d \sim c \Rightarrow c = d))$.

The term $c$ above is called the *choice function* associated to $f$.
If the reader is missing any axiom regarding *regularity*, see Section 8.

## 2.2 Sets and Proper Classes

In this subsection we introduce concepts which are intuitively associated to some notion of collection. Such collections are organized as classes, proper classes, sets, and ZF-sets.

**Definition 15** $Col(f)$ iff $f \neq \underline{0} \wedge \forall x (f(x) \neq \underline{0} \Rightarrow f(f(x)) \neq \underline{0})$.

In the definition above we read $Col(f)$ as "$f$ is a collection" or "$f$ is a class".

**Theorem 34** $\neg Col(\sigma)$.

**Proof:** We proved in Theorem 26 that $\sigma_{\underline{1} \circ \underline{1}} = \underline{1} \neq \underline{0}$. Nevertheless, $\sigma_{\sigma_{\underline{1} \circ \underline{1}}} = \sigma_{\underline{1}} = \underline{0}$, according to Theorem 20. That means $\sigma_x \neq \underline{0}$ does not entail $\sigma_{\sigma_x} \neq \underline{0}$, for an arbitrary $x$. Therefore, $\sigma$ is not a class.

**Definition 16** $x \in f$ iff $x \neq f \wedge f(x) \neq \underline{0}$.

The negation of the formula $x \in f$ is abbreviated as $x \notin f$. We read $x \in f$ as $x$ belongs to $f$. It is immediate to see that $x \in f$ iff $f[x]$. Observe as well that we do not demand $f$ to be a class. That will allow us, hopefully, to talk about some sort of *fuzzy sets* concept in the case where $f$ is not a class. For example, as we proved above, $\sigma$ is not a class. But since $\sigma_{\underline{0}} = \phi_0$, that means $\underline{0} \in \sigma$. Actually, infinitely many functions belong to $\sigma$, like $\phi_0$, $\phi_1$, etc. Nevertheless, we do not intend to discuss about such *fuzzy* terms in this paper. That is a task for the future. Finally, it is worth to observe that $\underline{0}$ and $\phi_0$ are terms such that no one belongs to any of them; but only $\phi_0$ is a class (by vacuity), since, by definition, $\underline{0}$ cannot be a class. Thus, Flow can be understood as a theory with one single atom (*Urelement*), namely, $\underline{0}$.

**Theorem 35** *The next formulas are all theorems: (i) $Col(\underline{1})$; (ii) $\forall x ((x \neq \underline{1} \wedge x \neq \underline{0}) \Rightarrow x \in \underline{1})$; (iii) $\forall x (x \notin x)$.*

Their proofs are straightforward.

**Definition 17** *A* structure-free class *is a class $f$ such that for any $x$ we have $f(x) \neq \underline{0} \Rightarrow f(x) = x$.*

It is easy to check that every function $\phi_n$ is a structure-free class. The same happens with $\underline{1}$.

**Definition 18** *Any class which is not a structure-free class is said to be a* structured class.

**Definition 19** *$f$ is a* set *iff $f$ is a class and for any $x$, $f[x] \Rightarrow \sigma_x \neq \underline{0}$.*

If $f$ is a set, we denote this by $Set(f)$. Examples of sets are each and every $\phi_n$.

Inspired on **P11**, next we define ZF-sets.

**Definition 20** $Z(f)$ *iff* $\forall x((f[x] \Rightarrow (f(x) = x \wedge \sigma_x \neq \underline{0})) \wedge \forall g(g \subseteq f \Rightarrow \sigma_g \neq \underline{0}))$.

We read $Z(f)$ as "$f$ is a ZF-set". Any ZF-set $f$ is a structure-free class, and if $f$ acts on any $x$, then $x$ has a non-$\underline{0}$ successor. Besides, every restriction of a ZF-set has its own non-$\underline{0}$ successor.

**Theorem 36** $\neg Z(\sigma)$.

**Proof:** Since $\sigma(\phi_0) = \phi_1 \neq \phi_0$, that is enough to prove $\sigma$ is not a ZF-set.

**Theorem 37** *If $f$ is a ZF-set, then $f = \underline{1}\big|_{f[x]}$.*

The proof is straightforward.

**Theorem 38** *If $f$ is a ZF-set, then $\sigma_f \neq \underline{0}$.*

**Proof:** If $f$ is a ZF-set, then for any $x$, $f[x]$ entails $\sigma_x \neq \underline{0}$. But according to axiom **P11**, any restriction of $f$ (under such assumption) has a non-$\underline{0}$ successor. Since $f$ is a restriction of $f$ (for any $f$ different of $\underline{0}$), then $\sigma_f \neq \underline{0}$.

**Theorem 39** *Every $\phi_n$ is a ZF-set.*

**Proof:** $Z(\phi_0)$ is vacuously valid. Now, let $n > 0$. Then any $\phi_n$ acts only on $\phi_m$ and $\phi_n(\phi_m) = \phi_m$, where $0 < m < n$. And each $\phi_m$ has a non-$\underline{0}$ successor, from the definition itself for $\phi_m$. And according to **P11**, that entails that any restriction $g$ of $\phi_n$ has a non-$\underline{0}$ successor. So, $Z(\phi_n)$ for any $n$ from language $\mathcal{L}$.

This last theorem helps us to see how to start building ZF-sets from the axioms of Flow. Next theorem shows us how to build standard hierarchies of ZF-sets.

**Theorem 40** *If $f$ is a ZF-set, then $\mathcal{P}(f)$ is a ZF-set.*

**Proof:** According to Theorem 30, $\mathcal{P}(f) = \underline{1}\big|_{t \subseteq f}$. Let us denote $\mathcal{P}(f)$ by $p$, for the sake of abbreviation. Since $f$ is a ZF-set, that means any restriction $t$ of $f$ has a non-$\underline{0}$ successor $\sigma_t$. In other words, $p[t]$ entails that $p(t) = t$ (since $p$ is a restriction of $\underline{1}$) and $t \subseteq f$. Thus, $\sigma_t \neq \underline{0}$ (since $Z(f)$). But according to **P11**, if $f$ is a ZF-set, then any $h$ such that $h[t] \Rightarrow t \subseteq f$ entails $\sigma_h \neq \underline{0}$. Well, $p$ is exactly like that, since $p[t] \Rightarrow t \subseteq f$. So, there is a non-$\underline{0}$ $\sigma_p$. Consequently, according to **P11**, every restriction $g$ of $p$ has its own non-$\underline{0}$ successor $\sigma_g$. That, finally, corresponds to say that $p$ is a ZF-set. In other words, $\forall t((p[t] \Rightarrow (p(t) = t \wedge \sigma_t \neq \underline{0})) \wedge \forall g(g \subseteq p \Rightarrow \sigma_g \neq \underline{0}))$.

So, we have here a vast universe of ZF-sets who are built from $\phi_0$ and the notion of successor, in a way which allows us to build hierarchies defined through the power function and corresponding restrictions. All of them are ZF-sets.

**Theorem 41** *Any inductive function $i$ is a ZF-set.*

**Proof:** Straightforward from the definitions of inductive function and ZF-set.

All previous results motivate us to define the concept of a proper class.

**Definition 21** *$f$ is a* proper class *iff $f$ is a class and anyone of the next conditions is satisfied: either (i) $\sigma_f = \underline{0}$ or (ii) there is some $x$ such that $f$ acts on $x$ but $x$ has $\sigma_x = \underline{0}$.*

That means no proper class is a ZF-set. If a proper class $f$ is a free-structure class, then we say $f$ is a *free-structure proper class*. Otherwise, we say $f$ is a *structured proper class*.

Examples of proper classes are $\underline{1}$ and $\underline{1}\big|_{x \neq \phi_n}$ (see Theorem 27), for a given $n$ of language $\mathcal{L}$. That happens because neither $\underline{1}$ nor any $\underline{1}\big|_{x \neq \phi_n}$ has a non-$\underline{0}$ successor. Another example of proper class is function $\psi$ from Theorem 25. Although $\psi$ has a non-$\underline{0}$ successor, $\psi$ acts on any $\underline{1}\big|_{x \neq \phi_n}$. So, $\psi$ acts on certain terms $t$ such that there is no non-$\underline{0}$ $\sigma_t$.

**Theorem 42** *There is one single ZF-set $f$ such that for any $x$, we have $x \notin f$.*

**Proof:** $f = \phi_0$. And, according to **P6**, $\phi_0$ is unique. In other words, $\phi_0$ is the empty set, which can be denoted by $\emptyset$.

# 3 ZFC is immersed in Flow

There are two reasons for referring to ZFC at this point. First, presenting the theory provides a framework that will allow us to compare our proposal to a standard and well-known formulation of set theory. Second, having ZFC in place will be useful for our proof that we can still use standard mathematical results when we adopt Flow-theoretic principles. After all, as we'll show shortly, there's a translation from the language of ZFC into a variation of Flow theory such that every translated axiom of ZFC is a theorem in our proposed formal system. However, as we will see, to adopt Flow has the significant advantage of providing a whole new universe to work with.

## 3.1 ZFC Axioms

ZFC is a first-order theory with identity and with one predicate letter $f_1^2$, such that the formula $f_1^2(x, y)$ is abbreviated as $x \in y$, if $x$ and $y$ are terms, and is read as "$x$ belongs to $y$" or "$x$ is an element of $y$". The negation $\neg(x \in y)$ is abbreviated as $x \notin y$.

The axioms of ZFC are the following:

**ZF1 - Extensionality** $\forall x \forall y (\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y)$

**ZF2 - Empty set** $\exists x \forall y (\neg(y \in x))$

**ZF3 - Pair** $\forall x \forall y \exists z \forall t (t \in z \Leftrightarrow t = x \vee t = y)$

The pair $z$ is denoted by $\{x, y\}$ if $x \neq y$. Otherwise, $z = \{x\} = \{y\}$.

**Definition 22** $x \subseteq y =_{def} \forall z (z \in x \Rightarrow z \in y)$

**ZF4 - Power set** $\forall x \exists y \forall z (z \in y \Leftrightarrow z \subseteq x)$

If $F(x)$ is a formula in ZFC, such that there are no free occurrences of the variable $y$, then the next formula is an axiom of ZFC:

**ZF5$_F$ - Separation Scheme** $\forall z \exists y \forall x (x \in y \Leftrightarrow x \in z \wedge F(x))$

The set $y$ is denoted by $\{x \in z / F(x)\}$.

If $\alpha(x, y)$ is a formula where all occurrences of $x$ and $y$ are free, then the following is an axiom scheme of ZFC:

**ZF6$_\alpha$ - Replacement Scheme**

$$\forall x \exists! y \alpha(x, y) \Rightarrow \forall z \exists w \forall t (t \in w \Leftrightarrow \exists s (s \in z \wedge \alpha(s, t)))$$

**ZF7 - Union set** $\forall x \exists y \forall z (z \in y \Leftrightarrow \exists t (z \in t \wedge t \in x))$

The set $y$ from **ZF7** is abbreviated as

$$y = \bigcup_{t \in x} t$$

The intersection among sets is defined by using the Separation Scheme as follows:

$$\bigcap_{t \in x} t =_{def} \left\{ z \in \bigcup_{t \in x} t / \forall t (t \in x \Rightarrow z \in t) \right\}$$

**ZF8 - Infinite** $\exists x (\emptyset \in x \wedge \forall y (y \in x \Rightarrow y \cup \{y\} \in x))$

**ZF9 - Choice** $\forall x (\forall y \forall z ((y \in x \wedge z \in x \wedge y \neq z) \Rightarrow (y \neq \emptyset \wedge y \cap z = \emptyset)) \Rightarrow$
$\exists y \forall z (z \in x \Rightarrow \exists w (y \cap z = \{w\})))$

As is well known, most if not all classical mathematics can be reformulated in ZFC. As a result, ZFC provides a rich framework for the formulation of physical theories—although perhaps not the most economical. As an alternative, we will now consider a different version of set theory, and explore its use in the foundations of physics.

## 3.2 ZFC translation

For the sake of abbreviation, we call Flow Theory $\mathcal{F}$.

Having presented the main features of Flow, we can now prove that standard mathematics, as formulated in Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC), is preserved in a Flow-like axiom system, namely, $\mathcal{F}$. After that, we discuss the meaning of such a result. But first, it is helpful to define the concept of arbitrary intersection.

**Definition 23** *For a given $x \neq \underline{0}$, let $F(r)$ be the formula "$r = z \Leftrightarrow \forall t(x[t] \Rightarrow t[z])$". Then*

$$\bigcap_{x[t]} t =_{def} \left( \bigcup_{x[t]} t \right) \Bigg|_{F(r)} .$$

In the particular case where $x$ acts only on two values $p$ and $q$, such an arbitrary intersection may be rewritten simply as $p \cap q$.

Now, our main result from this section.

**Proposition 1** *There is a translation from the language of ZFC into the language of $\mathcal{F}$ such that every translated axiom of ZFC is a theorem in $\mathcal{F}$.*

To prove this proposition, we need to exhibit a translation from ZFC into $\mathcal{F}$. This translation is given by the table below:

| Translating ZFC into $\mathcal{F}$ | |
|---|---|
| ZFC | $\mathcal{F}$ |
| $\forall$ | $\forall_Z$ |
| $\exists$ | $\exists_Z$ |
| $x \in y$ | $y[x]$ |
| $x \subseteq y$ | $x \subseteq y$ |

where $Z$ is the predicate "to be a ZF-set" from Definition 20.

The proof of Proposition 1 is made through the following lemmas. The first lemma is quite sensitive. A discussion about its proof is delivered afterwards.

**Lemma 1** *The translation of the Axiom of Extensionality in ZFC into Flow is a theorem. That means $\vdash_{\mathcal{F}}$ "Translated $\mathbf{ZF1}''$.*

**Proof:** The translated $\mathbf{ZF1}$ is the formula $\forall_Z x \forall_Z y (\forall_Z z(x[z] \Leftrightarrow y[z]) \Rightarrow x = y)$. If $x$ and $y$ are ZF-sets and $x[z]$ and $y[z]$, that means $x(z) = z$ and $y(z) = z$ (Definition 20). If $\neg x[z]$ or $\neg y[z]$, that means either $x(z) = \underline{0}$ or $y(z) = \underline{0}$; or $z = x$ or $z = y$. So, the translated $\mathbf{ZF1}$ considers the case where both $x$ and $y$ share the same images, except perhaps for $z = x$ or $z = y$. In other words, $x[z] \Leftrightarrow y[z]$ is equivalent to say that for any $z$ we have $x(z) = y(z)$, except perhaps for $z = x$ or $z = y$. Now, suppose $x \neq y$, despite the fact that both $x$ and $y$ share the same images for any $z \neq x$ and any $z \neq y$.

After all, in principle we may have the following situation: $x(y) = \underline{0}$ while $y(y) = y$ (this last identity is due to **P2**). Analogously, we may have $x(x) = x$ while $y(x) = \underline{0}$. In both particular cases ($z = x \lor z = y$), we have $\neg x[y] \land \neg y[y]$, a situation which satisfies the antecedent of $\Rightarrow$ in the translated **ZF1**. Nevertheless, all functions in Flow are built from $\underline{0}$ and $\underline{1}$ through operations like composition, successor, restriction, union, freedom, and choice. And those functions built from $\underline{0}$ and $\underline{1}$ are defined by means of terms where they act. According to Theorem 2, $\underline{1}$ is unique; and according to Theorem 3, $\underline{0}$ is also unique. Besides, any successor $\sigma_f$ for any $f$ is unique. Uniqueness of composition is guaranteed in **P5**. Uniqueness of restriction is guaranteed in **P8**. Uniqueness of union is guaranteed in **P10**. Uniqueness of arbitrary functions (freedom) is guaranteed in **P9**. And the uniqueness of any given choice function $c$ is guaranteed in **P12**, in the sense that once $c$ is obtained and $d \sim c$, for any $d$, then $d = c$. That means there can be no two functions $x$ and $y$ which act on all the same terms $z$ in a way such that $x(z) = y(z)$.

After the proof of this first lemma, one natural question seems to be unavoidable. Why didn't we introduce a stronger version for extensionality instead of Axiom **P1**? If we had done something like this, all those strange maneuvers used for proving Lemma 1 could be easily avoided. That is true. Notwithstanding, we intend to suggest here another way of doing mathematics. If we had adopted a stronger version of extensionality, we would have a kind of mathematics which is quite similar to the standard way. So, at the end of this paper we perform a detailed philosophical discussion about this issue. Our main purpose here is to let an open door which can lead us to what we call a *Heraclitean Mathematics*. And such a Heraclitean Mathematics has no room for ZFC.

**Lemma 2** $\vdash_{\mathcal{F}}$ *"Translated* **ZF2**$''$

**Proof:** The translated **ZF2** is the formula $\exists_Z x \forall_Z y (\neg(x[y]))$. That result is a straightforward corollary from Theorem 42. Function $x$ is simply $\phi_0$, which is a ZF-set (and, by the way, unique).

**Lemma 3** $\vdash_{\mathcal{F}}$ *"Translated* **ZF3**$''$

**Proof:** The translated **ZF3** is the formula $\forall_Z x \forall_Z y \exists_Z z \forall_Z t(z[t] \Leftrightarrow (t = x \lor t = y))$. All we have to do is to define $z = \underline{1}\big|_{F(t)}$ for formula $F(t)$ given by "$(t = x \lor t = y)$", where $x$ and $y$ are any two ZF-sets. That can be done thanks to **P8**$_F$. Since both $x$ and $y$ are ZF-sets, according to **P11** there are non-$\underline{0}$ $\sigma_x$ and $\sigma_y$. Hence, $z$ is a ZF-set, since $z[x]$ and $z[y]$ entail $z(x) = x$, $z(y) = y$ (remember $z$ is a restriction of $\underline{1}$), $\sigma_x \neq \underline{0}$, and $\sigma_y \neq \underline{0}$.

**Lemma 4** $\vdash_{\mathcal{F}}$ *"Translated* **ZF4**$''$

**Proof:** The translated **ZF4** is the formula $\forall_Z x \exists_Z y \forall_Z z(y[z] \Leftrightarrow z \subseteq x)$. That corresponds exactly to Theorem 40.

**Lemma 5** $\vdash_{\mathcal{F}}$ *"Translated* **ZF5**″

**Proof:** The translated **ZF5** is the formula $\forall_Z f \exists_Z g \forall_Z x (g[x] \Leftrightarrow f[x] \wedge F(x))$ (we changed the names of variables in order to facilitate the reading of our proof). According to Axiom $\mathbf{P8}_F$, $\forall f(f \neq \underline{0} \Rightarrow \exists g(g \neq \underline{0} \wedge (g \neq f \Rightarrow g(f) = \underline{0}) \wedge \forall x \forall y((x \neq g \wedge x \neq f) \Rightarrow (g(x) = y \Leftrightarrow ((f(x) = y \wedge F(x)) \vee (y = \underline{0} \wedge \neg F(x))))))))$. In other words, $\mathbf{P8}_F$ says that for a given $f$ different of $\underline{0}$ there is a $g$ which shares the same images of $f$ for a given $x$, as long $F(x)$ (where $F$ has the same syntactical restrictions of formula $F$ from translated **ZF5**); otherwise, $g$ has images $\underline{0}$. That entails $g \subseteq f$. And in the case where $g \subset f$, then $g(f) = \underline{0}$. But from Definition 20, it is easy to see that any restriction $g$ of a ZF-set $f$ is also a ZF-set, even in the case where $f$ acts on ZF-sets $x$. So, the translated **ZF5** is simply a straightforward consequence from $\mathbf{P8}_F$.

**Lemma 6** $\vdash_{\mathcal{F}}$ *"Translated* **ZF6**″

**Proof:** The translated **ZF6** is the formula $\forall_Z x \exists!_Z y \alpha(x,y) \Rightarrow \forall_Z z \exists_Z w \forall_Z t(w[t] \Leftrightarrow \exists_Z s(z[s] \wedge \alpha(s,t)))$. That means we are talking about a specific formula $\alpha$ such that for any ZF-set $x$ there is a unique ZF-set $y$ where $\alpha(x,y)$. By applying Axiom $\mathbf{P8}_Z$ over $\underline{1}$ with formula $F(t)$ as "$z[s] \wedge \alpha(s,r) \Leftrightarrow r = t$", for a given ZF-set $z$ and a given formula $\alpha$ like the one demanded by the translated **ZF6**, we get a function $w$. In other words, $w = \underline{1}\big|_{z[s] \wedge \alpha(s,r) \Leftrightarrow r=t}$. So, if $w$ acts on any $t$, then $w(t) = t$ and there is a successor $\sigma_t$ (due to the way formula $\alpha$ is defined and thanks to Theorem 38). Hence, $w$ is a ZF-set.

**Lemma 7** $\vdash_{\mathcal{F}}$ *"Translated* **ZF7**″

**Proof:** The translated **ZF7** is the formula $\forall_Z f \exists_Z u \forall_Z t(u[t] \Leftrightarrow \exists_Z r(r[t] \wedge f[r]))$. Once again we changed the names of the original variables in order to facilitate its reading. According to **P10**,
$\forall f((f \neq \underline{0} \wedge \forall x(f[x] \Rightarrow \sigma_x \neq \underline{0})) \Rightarrow$
$\exists! u(\sigma_u \neq \underline{0} \wedge (\forall g \forall h((f[g] \wedge f[h]) \Rightarrow \forall t(t \neq u \Rightarrow$
$(\mathcal{X}_{\square}(g, h \leftarrow t) \Rightarrow u(t) = g(t)) \wedge$
$((\mathcal{X}_{\lozenge}(g, h \leftarrow t) \vee \mathcal{X}_{\bigcirc}(g, h \leftarrow t)) \Rightarrow u(t) = \underline{0}) \wedge$
$(\mathcal{X}_{\triangle}(g, h \leftarrow t) \Rightarrow (u(t) \neq \underline{0} \wedge (u(t) = g(t) \vee u(t) = h(t))))))))$.
But since we are talking about ZF-sets, the possibility of $\mathcal{X}_{\lozenge}(g, h \leftarrow t)$ is simply discarded. After all, if $f$ acts on both $g$ and $h$, and both $g$ and $h$ act on $t$, then it is impossible that $g(t) \neq h(t)$, since $g(t) = t$ and $h(t) = t$. Now observe that terms $g$ and $h$ from **P10** have the same role of $r$ in translated **ZF7**. Thus, $u(t)$ has the same non-$\underline{0}$ value $t$ of either $g(t)$ or $h(t)$ only in the case where either $g(t) = t$ or $h(t) = t$ (which corresponds to the cases $\mathcal{X}_{\square}(g, h \leftarrow t)$ and $\mathcal{X}_{\triangle}(g, h \leftarrow t)$). That is equivalent to say that $u[t] \Leftrightarrow \exists_Z r(r[t] \wedge f[r])$. But since $f$ is a ZF-set, then it acts on ZF-sets $r$. Since each $r$ is a ZF-set, then each $r$ acts on a ZF-set $t$. That means $u$ acts only on ZF-sets, which makes itself a ZF-set.

**Lemma 8** $\vdash_{\mathcal{F}}$ *"Translated* **ZF8**$''$

**Proof:** The translated **ZF8** is the formula $\exists_Z x(x[\emptyset] \wedge \forall_Z y(x[y] \Rightarrow x[y \cup \{y\}]))$.
Axiom **P7** states that $\exists i((\forall t(i(t) = t \vee i(t) = \underline{0})) \wedge \sigma_i \neq \underline{0} \wedge (i(\sigma_{\underline{0}}) = \sigma_{\underline{0}} \wedge \forall x(i(x) = x \Rightarrow (i(\sigma_x) = \sigma_x \neq \underline{0}))))$. Well, $\emptyset$ is exactly $\phi_0$. So, $i$ acts on $\phi_0$ and $i(\phi_0) = \phi_0$. Besides, $\phi_0$ is a ZF-set. Besides, if $y$ is a ZF-set, then there is $\sigma_y$. And $y \cup \{y\}$ is exactly such $\sigma_y$, where $\{y\} = \underline{1}\big|_{t=y}$. And the union of ZF-sets is a ZF-set, as already proved in the previous lemma. So, if $i$ acts on a ZF-set $t$, then $i$ acts on the ZF-set $\sigma_t$, which makes $i$ itself a ZF-set.

**Lemma 9** $\vdash_{\mathcal{F}}$ *"Translated* **ZF9**$''$

**Proof:** The translated **ZF9** is the formula $\forall_Z x(\forall_Z y \forall_Z z((x[y] \wedge x[z] \wedge y \neq z) \Rightarrow (y \neq \phi_0 \wedge y \cap z = \phi_0)) \Rightarrow \exists_Z y \forall_Z z(x[z] \Rightarrow \exists_Z w(y \cap z = \{w\})))$. On the other hand, **P12** says $\forall f(\forall x \forall y((f(x) \neq \underline{0} \wedge f(y) \neq \underline{0} \wedge x \neq y) \Rightarrow (x \neq \phi_0 \wedge \neg \exists s(x(s) = y(s) \wedge x(s) \neq \underline{0}))) \Rightarrow$
$\exists c \forall r(f(r) \neq \underline{0} \Rightarrow \exists! w(c(w) = r(w) \wedge r(w) \neq \underline{0})))$. Thus, the translated **ZF9** is just a particular case for a ZF-set $x$. Since $c$ acts on ZF-sets, then $c$ is a ZF-set itself.

# 4  Category theory is immersed in Flow

The intuitive notion of a category is quite simple. A category refers to some sort of universe where we can find two kinds of terms, namely, *objects* and *morphisms*. Within a set-theoretic interpretation, objects can be associated to either sets or proper classes, while morphisms can be associated to some sort of general notion of function. Besides, there is a binary operation called *composition*, which is applicable over some pairs of morphisms. Composition, when defined, is associative and it allows the existence of (left and right) neutral elements. Usually Category Theory is referred to as a general theory of functions. Nevertheless, we prove in this section that Category Theory corresponds to a minor fragment of Flow Theory. After all, while composition in Category Theory is not always feasible, within Flow there always exist a composition between any two functions. Those facts lead us to one more important lesson from Flow Theory.

**Observation 4** *We proved in Section 3 that ZFC is immersed within Flow. Nevertheless, we did that by assuming as ZF-sets only special cases of free-structure classes. In this Section we prove Category Theory is immersed within Flow as well. And once again we do that by assuming morphisms (including their domains and co-domains) as special cases of free-structure classes. More than that, we prove next that all standard categories may be dealt with through the exclusive use of free-structure classes. From a philosophical point of view, our results point to an interesting perspective. Despite all the propaganda regarding Category Theory as a general theory of functions, the truth is that all standard*

*categorical results may be reduced to a world of restrictions of $\underline{1}$. So, Category Theory may be reduced to a particular study of functions $f$ whose images for any $x$ are either $x$ itself or $\underline{0}$. The main advantage of Category Theory lurks in its power to establish a connection between different domains, like topology and analysis, algebra and number theory. But that could be achieved within any set theory endowed with proper classes and universes. And once again we are still committed to the standard view that a function is nothing more than a collection of ordered pairs, let it be a morphism, a functor or a natural transformation. One of the epistemological barriers of Category Theory lies in the usual set-theoretic assumption that every morphism is somehow associated to some sort of domain (and a co-domain). And that fact yields to a quite prejudiced perspective about the dynamic nature functions are supposed to have. From a Flow-theoretic point of view, functions have no domain. And from this same perspective, a function $f$ can genuinely act on a given a in a way such that $f(a)$ is not necessarily identical to a. So, after all this discussion about standard mathematics, we explore in the next sections the first steps towards what Flow Theory can really offer to us.*

## 4.1 Category axioms

We follow here a first order language recipe for defining categories as presented by William S. Hatcher in his classical book [7]. Category Theory $\mathcal{K}$ is a first order theory with identity and one ternary predicate letter $K$ of degree three and two monadic function letters $D$ and $C$. The intended interpretation of its terms is that of *morphism*. All terms are represented by lower case Latin letters. Intuitively speaking, we read $K(x, y, z)$ as $z$ is the composition of $x$ with $y$; $D(x)$ as "the domain of $x$"; and $C(x)$ as "the codomain of $x$". The proper axioms of $\mathcal{K}$ are the following.

The domain of the codomain of any morphism $a$ is the codomain of $a$. And the codomain of the domain of $a$ is the domain of $a$:

**K-1** $\forall a(D(C(a)) = C(a) \wedge C(D(a)) = D(a))$.

Composition is unique:

**K-2** $\forall a \forall b \forall c \forall d((K(a, b, c) \wedge K(a, b, d)) \Rightarrow c = d)$.

The composition of $a$ with $b$ is defined if and only if the codomain of $a$ is the domain of $b$:

**K-3** $\forall a \forall b(\exists c(K(a, b, c) \Leftrightarrow C(a) = D(b)))$.

If $c$ is the composition of $a$ with $b$, then the domain of $c$ is the domain of $a$ and the the codomain of $c$ is the codomain of $b$:

**K-4** $\forall a \forall b \forall c(K(a, b, c) \Rightarrow (D(c) = D(a) \wedge C(c) = C(b)))$.

For any $a$, the domain of $a$ is a left identity for $a$ under composition, and the codomain of $a$ is a right identity:

**K-5** $\forall a(K(D(a), a, a) \land K(a, C(a), a))$.

Composition is associative when it is defined:

**K-6** $\forall a \forall b \forall c \forall d \forall e \forall f \forall g((K(a, b, c) \land K(b, d, e) \land K(a, e, f) \land K(c, d, g)) \Rightarrow f = g)$.

## 4.2 Every static category is a category

First we need the concept of *surjective trivially arbitrary function*.

**Definition 24** *Let $r$, $s$, and $g$ be functions such that $\mathcal{T}_{r \to s}(g)$. In other words, $g$ is a trivially arbitrary function with domain $r$ and codomain $s$. We say that $g$ is surjective iff for any $b$ such that $s[b]$, there is $a$ such that $r[a]$ and $g(a, b) = (a, b)$.*

Next we define a *static morphism*.

**Definition 25** *Let $g$, $r$, and $s$ be functions. Then, $\mathcal{M}_{\dagger}(g, r, s)$ iff*

1. *$\forall t(r[t] \Rightarrow r(t) = t) \land \forall t(s[t] \Rightarrow s(t) = t)$,*

2. *$\mathcal{T}_{r \to s}(g)$,*

3. *$g$ is surjective.*

We read the ternary predicate above as "$g$ is a *static morphism* with *domain* $r$ and *codomain* $s$". The first condition says $r$ and $s$ are structure-free classes. Observe that both $r$ and $s$ are restrictions of $\underline{1}$. And any restriction of $\underline{1}$ is a structure-free class. The second one says $g$ is a trivially arbitrary function. In other words, $g$ is a particular case of a structure-free class as well. The third condition guarantees the codomain of a trivially arbitrary function is coincident with it range.

We intend to prove that surjective trivially arbitrary functions work just fine for describing usual categories from standard mathematics. That means the usual concept of category cannot be considered as "a general theory of functions".

**Definition 26** *Let $r$, $s$, and $g$ be functions such that $\mathcal{M}_{\dagger}(g, r, s)$. Then,*

1. *$d_g^{\dagger} = h$ iff $\mathcal{M}_{\dagger}(h, r, r) \land \forall a(r[a] \Leftrightarrow h(a, a) = (a, a))$.*

2. *$c_g^{\dagger} = h$ iff $\mathcal{M}_{\dagger}(h, s, s) \land \forall b(s[b] \Leftrightarrow h(b, b) = (b, b))$.*

*Besides, both $d_g^{\dagger}$ and $c_g^{\dagger}$ have images $\underline{0}$ iff $r$ does not act on $a$ or $s$ does not act on $b$, respectively.*

We read $d_g^\dagger = h$ as "$h$ is *the static domain* of $g$". And $c_g^\dagger = h$ says "$h$ is *the static codomain* of $g$". That means $d_g^\dagger$ is a function which acts on ordered pairs $(a, a)$, as long $g$ acts on $(a, b)$. Analogously, $c_g^\dagger$ acts on ordered pairs $(b, b)$ as long $g$ acts on $(a, b)$. Thus, while $g[(a, b)]$ entails $g(a, b) = (a, b)$, $d_g^\dagger[(a, a)]$ entails $d_g^\dagger(a, a) = (a, a)$, and $c_g^\dagger[(b, b)]$ entails $c_g^\dagger(b, b) = (b, b)$.

**Definition 27** *Let $g$, $r$, $s$, $h$, and $t$ be functions such that $\mathcal{T}_{r \to s}(g)$ and $\mathcal{T}_{s \to t}(h)$. Then $g \circ_\dagger h$ is a function such that,*

1. $\mathcal{T}_{r \to t}(g \circ_\dagger h)$,

2. $\forall a \forall b \forall c((g[(a, b)] \wedge h[(b, c)]) \Rightarrow (g \circ_\dagger h)[(a, c)])$.

3. $\forall a \forall c((g \circ_\dagger h)[(a, c)] \Rightarrow \exists b(g[(a, b)] \wedge h[(b, c)]))$.

We read $g \circ_\dagger h$ as "*the static composition of $g$ with $h$*". The notation $(g \circ_\dagger h)[(a, c)]$ says the composition $g \circ_\dagger h$ acts on $(a, c)$.

**Definition 28** *Let $f$ be a function. Then $\mathcal{C}_\dagger(f)$ iff*

1. $f \neq \underline{0}$,

2. $\forall g(f[g] \Rightarrow f(g) = g)$,

3. $\forall g(f[g] \Rightarrow (\exists r \exists s(\mathcal{M}_\dagger(g, r, s) \wedge \forall h(\mathcal{M}_\dagger(h, r, s) \Rightarrow f[h] \wedge f[d_h^\dagger] \wedge f[c_h \dagger]))))$,

4. $\forall g \forall h((f[g] \wedge f[h] \wedge \exists i(i = g \circ_\dagger h)) \Rightarrow f[i])$.

We read the monadic predicate above as "$f$ is a *static category*". The first two conditions above say any static category is a free-structure class. The third condition says if a static category $f$ acts on any $g$, then $g$ is a static morphism from $r$ to $s$, and $f$ acts on $g$'s static domain and on $g$'s static codomain. Besides, the same happens with every $h$ which is a morphism from $r$ to $s$. Finally, last condition says if $f$ acts on $g$ and $h$, then it acts on the static composition of $g$ with $h$. But that happens obviously if such a static composition exists. In other words, static composition is a quite limited perception about composition, in the sense that static composition in a static category does not necessarily exist, while compositions within Flow always do exist.

**Definition 29** *Let $g$ be a function. Then $\dagger_f(g)$ iff $g$ is a function such that a specific static category $f$ acts on $g$. If there is no risk of confusion, we may rewrite $\dagger_f(g)$ simply as $\dagger(g)$.*

Before we prove static categories do satisfy all axioms of $\mathcal{K}$ (if a proper translation is provided), it might be useful to introduce here a rather simple example (although non-trivial) of a static category. Let $f$ be given as it follows:



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where $g$, $h$, and $i$ are given as:

| $g$ |
|---|
| $(\overset{\frown}{\phi_0, \phi_0})$ $(\overset{\frown}{\phi_1, \phi_1})$ |

| $h$ |
|---|
| $(\overset{\frown}{\phi_0, \phi_1})$ $(\overset{\frown}{\phi_1, \phi_2})$ |

| $i$ |
|---|
| $(\overset{\frown}{\phi_1, \phi_1})$ $(\overset{\frown}{\phi_2, \phi_2})$ |

In that case, $f$ is a static category. Besides, $d_g^\dagger = c_g^\dagger = d_h^\dagger = g$, $d_i^\dagger = c_i^\dagger = c_h^\dagger = i$, and $g \circ_\dagger h = h \circ_\dagger i = h$, while neither $h \circ_\dagger g$ nor $i \circ_\dagger h$ do exist. The ellipsis above just indicates there are other functions with static domain (static codomain) $g$ and static codomain (static domain) $h$.

The translation provided in the next subsection allows us to prove that $f$ given above is a category in the sense given by William Hatcher [7].

## 4.3 Categories translation

Here we prove the main result of this Section.

**Proposition 2** *There are translations from the language of Category Theory $\mathcal{K}$ into the language of $\mathcal{F}$ such that every translated axiom of $\mathcal{K}$ is a theorem in $\mathcal{F}$ in each translation.*

To prove this proposition scheme we need to exhibit a translation from $\mathcal{K}$ into $\mathcal{F}$, for every possible static category $f$. Such a translation is given by the table below:

| Translating $\mathcal{K}$ into $\mathcal{F}$ | |
|---|---|
| $\mathcal{K}$ | $\mathcal{F}$ |
| $\forall$ | $\forall_\dagger$ |
| $\exists$ | $\exists_\dagger$ |
| $D(g)$ | $d_g^\dagger$ |
| $C(g)$ | $c_g^\dagger$ |
| $K(g,h,i)$ | $i = g \circ_\dagger h$ |

where predicate $\dagger$ refers to the specific static category $f$. In other words, $\forall_\dagger x(P)$ means $\forall x(f[x] \Rightarrow P)$, and $\exists_\dagger x(P)$ means $\exists x(f[x] \wedge P)$, where $P$ is a formula from Flow.

The proof of last proposition scheme is made through the following lemmas. We keep the same labels used for terms in $\mathcal{K}$ axioms when it is convenient for us. Otherwise, we change them.

**Lemma 10** $\vdash_{\mathcal{F}}$ *"Translated* **K-1***"*.

**Proof:** The translated **K-1**, for the static category $f$, is $\forall_\dagger g(d_{c_g^\dagger}^\dagger = c_g^\dagger \wedge c_{d_g^\dagger}^\dagger = d_g^\dagger)$. Notwithstanding, $c_g^\dagger$ is a function such that $c_g^\dagger[(b,b)]$ iff $c_g^\dagger(b,b) = (b,b)$ and $s[b]$ for a given $s$; and $c_g^\dagger(t) = \underline{0}$ iff $t$ is different of $c_g^\dagger$ or different of any $b$ where that given $s$ acts, according to Definition 26. But that is precisely the static domain of $c_g^\dagger$, according again to Definition 26 and Theorem 1 An analogous argument can be used for proving that $c_{d_g^\dagger}^\dagger = d_g^\dagger$.

**Lemma 11** $\vdash_{\mathcal{F}}$ *"Translated* **K-2***".*

**Proof:** The translated **K-2**, for the static category $f$, is $\forall_\dagger a \forall_\dagger b \forall_\dagger c \forall_\dagger d((c = a \circ_\dagger b \wedge d = a \circ_\dagger b) \Rightarrow c = d)$. According to Definition 27, both $c$ and $d$ share the same images, for a given $a$ and a given $b$. So, from Theorem 1, $c = d$.

**Lemma 12** $\vdash_{\mathcal{F}}$ *"Translated* **K-3***".*

**Proof:** The translated **K-3**, for the static category $f$, is $\forall_\dagger g \forall_\dagger h (\exists_\dagger i (i = g \circ_\dagger h \Leftrightarrow c_g^\dagger = d_h^\dagger))$. According to Definition 27, $i[(a,c)]$ iff $g[(a,b)]$ and $h[(b,c)]$. But according to Definitions 25 and 26, $g[(a,b)]$ entails $c_g^\dagger[(b,b)]$, and $h[(b,c)]$ entails $d_h^\dagger[(b,b)]$. Thus, $i[(a,c)]$ iff $c_g^\dagger[(b,b)]$ and $d_h^\dagger[(b,b)]$, which is equivalent to say that $c_g^\dagger = d_h^\dagger$, according to Theorem 1.

**Lemma 13** $\vdash_{\mathcal{F}}$ *"Translated* **K-4***".*

**Proof:** The translated **K-4**, for the static category $f$, is $\forall_\dagger g \forall_\dagger h \forall_\dagger i (i = g \circ_\dagger h \Rightarrow (d_i^\dagger = d_g^\dagger \wedge c_i^\dagger = c_h^\dagger))$. According to Definition 27, $i[(a,c)]$ iff $g[(a,b)]$ and $h[(b,c)]$. But according to Definitions 25 and 26, $g[(a,b)]$ entails $d_g^\dagger[(a,a)]$, and $h[(b,c)]$ entails $c_h^\dagger[(c,c)]$. Thus, once again Definition 27 shows that $i[(a,c)]$ entails $d_i^\dagger[(a,a)]$ and $c_i^\dagger[(c,c)]$, which is equivalent to say that $d_i^\dagger = d_g^\dagger$ and $c_i^\dagger = c_h^\dagger$, according to Theorem 1.

**Lemma 14** $\vdash_{\mathcal{F}}$ *"Translated* **K-5***".*

**Proof:** The translated **K-5**, for the static category $f$, is $\forall_\dagger g (d_g^\dagger \circ_\dagger g = g \wedge g \circ_\dagger c_g^\dagger = g)$. According to Definitions 26 and 25, $g[(a,b)]$ iff $d_g^\dagger[(a,a)]$ and $c_g^\dagger[(b,b)]$. And according to Definition 27, $d_g^\dagger \circ_\dagger g$ acts on $(a,a)$, while $g \circ_\dagger c_g^\dagger$ acts on $(b,b)$. That is equivalent to say that $d_g^\dagger \circ_\dagger g = g$ and $g \circ_\dagger c_g^\dagger = g$, according to Theorem 1.

**Lemma 15** $\vdash_{\mathcal{F}}$ *"Translated* **K-6***".*

**Proof:** The translated **K-6**, for the static category $f$, is
$\forall_\dagger a \forall_\dagger b \forall_\dagger c \forall_\dagger d \forall_\dagger e \forall_\dagger f \forall_\dagger g ((a \circ_\dagger b = c \wedge b \circ_\dagger d = e \wedge a \circ_\dagger e = f \wedge c \circ_\dagger d = g) \Rightarrow f = g)$. According to Definition 27 and according to the antecedent of conditional $\Rightarrow$ above, we have the following: for certain values $\alpha$, $\beta$, $\gamma$, and $\delta$, $a[(\alpha,\beta)]$, $b[(\beta,\gamma)]$, and $d[(\gamma,\delta)]$. Besides, $c[(\alpha,\gamma)]$ and $e[(\beta,\delta)]$. Thus, $f$ acts on $(\alpha,\delta)$ iff $g$ acts on $(\alpha,\delta)$. That is equivalent to say $f = g$, according to Theorem 1.

Hence, as promised, any category in the general sense provided by Hatcher is reducible to a structure-free class $f$ which acts only on structure-free classes. That is somehow identifiable with the current view that any small category is isomorphic to a subcategory of **Set** (Category of sets, in standard mathematics).

Nevertheless, our result shows that no category (either small or not) demands any notion which goes beyond the intuitive concept of a (structure-free) class. That is one of the main reasons why we try to explore this new approach called Flow Theory. We did not check if Flow is reducible to Category Theory. But that is a task we intend to undertake.

## 4.4 Functors and natural transformations

**Definition 30** *Let $a$ and $b$ be static categories, such that † and ‡ refer, respectively, to $a$ and $b$. A* static covariant functor *from $a$ to $b$ is a function $\theta$ such that:*

**(i)** $\forall t((\theta[t] \Leftrightarrow a[t]) \wedge b[\theta(t)])$

**(ii)** $\forall t(b[t] \Rightarrow \exists r(a[r] \wedge \theta(r) = t)$

**(iii)** $\forall t(\theta(d_t^{\dagger}) = d_{\theta(t)}^{\ddagger})$

**(iv)** $\forall t(\theta(c_t^{\dagger}) = c_{\theta(t)}^{\ddagger})$

**(v)** $\forall t \forall u(\theta(t \circ_{\dagger} u) = \theta(t) \circ_{\ddagger} \theta(u))$

The first condition says a static covariant functor from $a$ to $b$ acts only on those terms where the static category $a$ acts. Besides, the static category $b$ acts on the images of $\theta$. Second item says every static covariant functor is surjective. In other words, we have the following: $\forall t(\exists p \exists q(\mathcal{M}_{\dagger}(t, p, q)) \Rightarrow \exists r \exists s(\mathcal{M}_{\ddagger}(\theta(t), r, s))))$. Conditions (iii) and (iv) say a static covariant functor $\theta$ from $a$ to $b$ associates objects from $a$ to objects in $b$. And the last item says any static covariant functor is supposed to preserve static composition.

Next we define the corresponding dual of static covariant functors, namely, static contravariant functors.

**Definition 31** *Let $a$ and $b$ be static categories, such that † and ‡ refer, respectively, to $a$ and $b$. A* static contravariant functor *from $a$ to $b$ is a function $\theta$ such that:*

**(i)** $\forall t((\theta[t] \Leftrightarrow a[t]) \wedge b[\theta(t)])$

**(ii)** $\forall t(b[t] \Rightarrow \exists r(a[r] \wedge \theta(r) = t)$

**(iii)** $\forall t(\theta(d_t^{\dagger}) = c_{\theta(t)}^{\ddagger})$

**(iv)** $\forall t(\theta(c_t^{\dagger}) = d_{\theta(t)}^{\ddagger})$

**(v)** $\forall t \forall u(\theta(t \circ_{\dagger} u) = \theta(u) \circ_{\ddagger} \theta(t))$

**Definition 32** *If $a$ is a static category, then $t$ is a* static object *of $a$ iff $a[t]$ and $t$ is either the static domain or the static codomain of a given static morphism $m$ such that $a[m]$. We denote this by $Obj_a(t)$.*

**Definition 33** *Let $a$ and $b$ be static categories, such that † and ‡ refer, respectively, to $a$ and $b$. Let $\theta$ and $\vartheta$ be static covariant functors from $a$ to $b$. A static natural transformation from $\theta$ to $\vartheta$ is a function $\eta$ such that:*

**(i)** *For any $x$, if $Obj_a(x)$, then $\eta(x)$ is a static morphism, denoted by $\eta_x$, with static domain $\theta(x)$ and static codomain $\vartheta(x)$ and*

**(ii)** *For every static morphism $f$ with static domain $x$ and static codomain $y$ in $a$ we have the following:*

$$\eta_y \circ_\ddagger \theta(f) = \vartheta(f) \circ_\ddagger \eta_x$$

The corresponding concept of a static natural transformation from a contravariant static functor $\theta$ to a static contravariant functor $\vartheta$ is straightforward.

## 4.5  Set and other standard categories

Before we define examples of standard categories within Flow, it seems useful to show an insightful theorem.

**Theorem 43** $\underline{1}\big|_{Z(x)}$ *has successor $\underline{0}$.*

**Proof:** Let us denote $\underline{1}\big|_{Z(x)}$ by $c$. In other words, $c[x] \Leftrightarrow Z(x)$. That means any $x$ where $c$ acts has a non-$\underline{0}$ successor. Now suppose $c$ has a non-$\underline{0}$ successor. That would entail, from **P11**, that $c$ is a ZF-set. So, $c$ acts on $c$. But no function acts on itself. So, there is no $\sigma_c$ different of $\underline{0}$.

Now, we show how to describe a well known category from standard mathematics.

**Definition 34 Set**, *the static category of ZF-sets, is a function $f$ such that $f = \underline{1}|_{F(x)}$, where $F(x)$ is a formula given as follows:*

$$\exists r \exists s (Z(r) \wedge Z(s) \wedge \mathcal{M}_\dagger(x, r, s))$$

That means the static category **Set** of ZF-sets is a function $f$ which acts on static (surjective) morphisms $x$ whose domains and codomains are ZF-sets. Thus, even if $f$ does not act on certain ZF-sets $g$ such that $g(a)$ is either $a$ or $\underline{0}$, it still acts on static morphisms $x$ such that $x(a, a) = (a, a)$. So, such morphisms $x$ can be easily identified with ZF-sets like $g$. Those terms like $g$ work as the *objects* of **Set**.

## 4.6  The Cantor-Schröder-Bernstein theorem

Despite the fact that Category Theory emphasizes the role of functions (called morphisms) in mathematics, that theory does not allow us to prove the Cantor-Schröder-Bernstein Theorem. That happens because Category Theory algebra is related to composition. In Flow theory, however, our algebra of functions is primarily based on functions valuations.

**Theorem 44 (Cantor-Schröder-Bernstein)** *Let $\mathcal{M}(f, r, s)$ and $\mathcal{M}(g, s, r)$ such that both $f$ and $g$ are injective. Then, there is a function $h$ such that $\mathcal{M}(h, r, s)$ and $h$ is bijective.*

**Proof:** All we have to do is to follow Kolmogorov-Fomin style in their book *Introductory Real Analysis* [10].

## 5   Intuitive Flow theory

For practical purposes, all that matters is how to operate within a Flow-theoretic approach in everyday mathematics. So, in this Section we provide the main features of a naive Flow theory.

The first basic features are as follows:

**Principle I** Every function $f$ has an image $f(t)$ for any function $t$. And that image is either $f$ itself, $\underline{0}$ or another value $g$. And $\underline{0}$ is a privileged function such that $\underline{0}(t)$ is always $\underline{0}$ itself.

**Principle II** For any function $f$ we have $f(f) = f$.

**Principle III** Any functions $f$ and $g$ which share the same images are the same.

**Principle IV** If $t \neq f$ and $f(t)$ is different of $\underline{0}$, then we say $f$ acts on $t$. And we denote this by $f[t]$. So, no function acts on itself.

**Principle V** For any functions $f$ and $g$ there is the associative composition $f \circ g$, such that $(f \circ g)(t) = f(g(t))$, except when $t = f \circ g$. In that case, $(f \circ g)(f \circ g) = f \circ g$.

**Principle VI** The successor $\sigma_f$ of a function $f$ is supposed to be a function $g$ which shares the same images of $f$ for any $t \neq g$, but in a way such that $f \neq g$. If that is not possible, then the successor $\sigma_f$ of $f$ is simply $\underline{0}$, except for the case when $f$ is $\sigma$ itself. In that case, $\sigma_\sigma = \sigma$. In standard mathematics (based on ZFC), however, the successor $\sigma_f$ always corresponds to the first case mentioned above.

**Principle VII** The restriction $g$ of a given function $f$ is such that for any $t$, if $g[t]$, then $f[t]$ and $g(t) = f(t)$.

**Principle VIII** The power $\mathcal{P}(f)$ of any function $f$ is the function which acts on every restriction of $f$.

**Principle IX** There are functions $f$ which are ordered pairs $(a, b)$. That means such functions $f$ act on functions $\alpha$ and $\beta$ which, in turn, act on $a$, and on $a$ and $b$, respectively. Analogously, there are ordered $n$-tuples $(a_1, a_2, \cdots, a_n)$.

**Principle X** The union of functions $g$ and $h$ is a function $f$ which shares the same images shared between $g$ and $h$. If $g$ and $h$ do not share any image for a specific $t$, then $f(t) = \underline{0}$. An analogous result holds for arbitrary unions.

**Principle XI** The intersection of functions may be defined from union in an analogous way how it is done within ZFC.

From those principles above we are able to define usual concepts, since ZFC is immersed within Flow.

**$n$-variable functions** A two-variable function $f$ is a function which acts only on ordered pairs $(a, b)$. An analogous concept can be defined for $n$-variable functions. If $\star$ is a two variable function, then we can abbreviate $\star(a, b)$ as $a \star b$.

**Finiteness** Let $f$ be a $n$-variable function which acts on ordered $n$-tuples $(a_1, \cdots, a_n)$. We say $f$ is *finite* on entry $i$ (where $1 \preceq i \preceq n$) iff there is a finite number of possible values $a_i$ such that $f[(a_1, \cdots, a_i, \cdots, a_n)]$.

**Closed $n$-variable function** A two-variable function $f$ is closed iff $f[(a, b)]$ and $f[(c, d)]$ entails that $f[(f(a, b), f(c, d))]$. An analogous concept can be defined for $n$-variable functions.

**Intersecting $n$-variable functions** Two $n$-variable functions $f$ and $g$ are intersecting iff either $f[(a_1, \cdots, a_n)]$ entails $g[(a_1, \cdots, a_n)]$, or $g[(a_1, \cdots, a_n)]$ entails $f[(a_1, \cdots, a_n)]$.

# 6 Axiomatization as a flow-theoretic predicate

In this section we briefly propose and discuss a static Flow-theoretic version for the axiomatization program proposed by Patrick Suppes [23] [2] [3]. Roughly speaking, Suppes Program is associated to his famous slogan "to axiomatize a theory is to define a set-theoretic predicate". Our proposed slogan can be read like this: "Any theory is a function".

We start with a "static version" for Group Theory. That means we are working only with structure-free classes.

## 6.1 Group theory

**Definition 35** *A static binary operation* is a function $f$ such that $\forall t(f[t] \Rightarrow \exists a \exists b(t = (a, b) \land f(t) = t))$.

So, if $*$ is a static binary operation, we may denote $*(a, b)$ simply as $a * b$.

**Definition 36** *A static binary operation $f$ is* closed *iff* $\forall a \forall b \forall c \forall d((f[(a, b)] \land f[(c, d)]) \Rightarrow f[(f(a, b), f(c, d))]$.

**Definition 37** *A static binary operation $f$ has* neutral element *iff* $\exists e \forall a \forall b(f(f(a,e),b) = f(f(e,a),b) = f(a,b))$. *Term $e$ is the neutral element of $f$.*

**Definition 38** *A static binary operation $f$ is* universally invertible *iff $f$ has neutral element $e$ and $\forall a \exists a^{-1} \forall b(f(f(a,a^{-1}),b) = f(f(a^{-1},a),b) = f(e,b))$. Term $a^{-1}$ is called the static inverse of $a$ in $f$.*

**Definition 39** *A static binary operation $f$ is associative iff $\forall a \forall b \forall c(g(a,g(b,c)) = g(g(a,b),c)))$.*

So, one way to define a static group is like this:

**Definition 40** *A* static group *is a function $g$ such that:*

1. *$\forall t(g[t] \Rightarrow \exists a \exists b(t = (a,b) \land g(t) = t))$.*

2. *$\forall a \forall b \forall c \forall d((g[(a,b)] \land g[(c,d)]) \Rightarrow g[(g(a,b),g(c,d))]$.*

3. *$\forall a \forall b \forall c(g(a,g(b,c)) = g(g(a,b),c)))$.*

4. *$\exists e \forall a \forall b(g(g(a,e),b) = g(g(e,a),b) = g(a,b))$.*

5. *$\forall a \exists a^{-1} \forall b(g(g(a,a^{-1}),b) = g(g(a^{-1},a),b) = g(e,b))$.*

Another way is like this:

**Definition 41** *A* static group *is a binary static operation $*$ which is closed, associative, and universally invertible.*

Next we prove that any ZFC-theoretic group is associated to some Flow-theoretic group, but the converse is not valid.

## 6.2   Other mathematical theories

**Definition 42** *A* static field *is an ordered pair $\langle +, \cdot \rangle$ such that $+$ and $\cdot$ are intersecting two-variable functions which do satisfy the usual axioms of a field.*

## 6.3   Classical particle mechanics

The next example of application of Flow refers to an axiomatic framework for a very simple form of non-relativistic classical particle mechanics. The system below is essentially based on the axiomatization of classical particle mechanics due to P. Suppes [22], which, in turn, is a variant of the formulation by J. C. C. McKinsey, A. C. Sugar and P. Suppes [13]. We call this McKinsey-Sugar-Suppes system of classical particle mechanics "MSS system".

MSS system, grounded on ZFC language, has six primitive notions: $P$, $T$, $m$, **s**, **f**, and **g**. $P$ and $T$ are sets; $m$ is a real-valued unary function defined on $P$; **s** and **g** are vector-valued functions defined on the Cartesian product $P \times T$, and **f** is a vector-valued function defined on the Cartesian product

$P \times P \times T$. Intuitively, $P$ corresponds to the set of particles and $T$ is to be physically interpreted as a set of real numbers measuring elapsed times (in terms of some unit of time, and measured from some origin of time). In turn, $m(p)$ is to be interpreted as the numerical value of the mass of $p \in P$; whereas $\mathbf{s}_p(t)$, with $t \in T$, is a 3-dimensional vector which is to be physically interpreted as the position of $p$ at instant $t$. Moreover, $\mathbf{f}(p, q, t)$, with $p, q \in P$, corresponds to the internal force that the particle $q$ exerts over $p$ at instant $t$. Finally, the function $\mathbf{g}(p, t)$ is to be understood as the external force acting on the particle $p$ at instant $t$.

We can now give the axioms for the MSS system:

**Definition 43** $\langle P, T, \mathbf{s}, m, \mathbf{f}, \mathbf{g} \rangle$ *is a MSS system if and only if the following axioms are satisfied:*

**M1** *$P$ is a non-empty, finite set.*

**M2** *$T$ is an interval of real numbers.*

**M3** *If $p \in P$ and $t \in T$, then $\mathbf{s}_p(t)$ is a 3-dimensional vector $(\mathbf{s}_p(t) \in \Re^3)$ such that $\frac{d^2\mathbf{s}_p(t)}{dt^2}$ exists.*

**M4** *If $p \in P$, then $m(p)$ is a positive real number.*

**M5** *If $p, q \in P$ and $t \in T$, then $\mathbf{f}(p, q, t) = -\mathbf{f}(q, p, t)$.*

**M6** *If $p, q \in P$ and $t \in T$, then $[\mathbf{s}_p(t), \mathbf{f}(p, q, t)] = -[\mathbf{s}_q(t), \mathbf{f}(q, p, t)]$.*

**M7** *If $p, q \in P$ and $t \in T$, then $m(p)\frac{d^2\mathbf{s}_p(t)}{dt^2} = \sum_{q \in P} \mathbf{f}(p, q, t) + \mathbf{g}(p, t)$.*

Some remarks regarding the axioms are in order here: (a) The brackets in Axiom **M6** denote the external product. (b) Axiom **M5** corresponds to a weak version of Newton's Third Law: to every force there is always a counter-force. (c) Axioms **M6** and **M5** correspond to the strong version of Newton's Third Law. Axiom **M6** establishes that the direction of force and counter-force is the direction of the line defined by the coordinates of particles $p$ and $q$. (d) Axiom **M7** corresponds to Newton's Second Law.

Now, in the study of a MSS system, it's sometimes useful to consider only certain parts of the system—perhaps only a subsystem needs to be considered. But is the subsystem of a MSS system still a MSS system? In [13] this question is positively answered in full details. But the point here is not that kind of question. We are interested on the use of Padoa's Principle and its consequences. Now we have all the resources in place to start asking questions regarding the independence of primitive notions in a MSS system. Using Padoa's method, it's not difficult to prove the following theorem:

**Theorem 45** *Mass and internal force are each independent of the remaining primitive notions of a MSS system.*

After presenting the MSS system, and in light of the last theorem, Suppes raised a significant issue regarding the definability of the notions of force in the system. As he points out [22]:

> Some authors have proposed that we convert the second law [of Newton], that is, **M7**, into a definition of the total force acting on a particle. [...] It prohibits within the axiomatic framework any analysis of the internal and external forces acting on a particle. That is, if all notions of force are eliminated as primitive and **M7** is used as a definition, then the notions of internal and external force are not definable within the given axiomatic framework.

It was natural then to extend Suppes' point even further, considering the notions of time and spacetime. And in [4, 5], the authors have proved that time is definable—and, thus, dispensable—in some very natural axiomatic frameworks for classical particle mechanics and even thermodynamics. Furthermore, they have established, in the first paper, that spacetime is also eliminable in general relativity, classical electromagnetism, Hamiltonian mechanics, classical gauge theories, and in the theory of Dirac's electron. Having an axiomatic framework in place allows one to obtain results of this type.

In particular, returning to the MSS system, here is one of the theorems proved in the papers quoted above:

**Theorem 46** *Time is eliminable in a MSS system.*

The proof is quite simple. According to Padoa's principle, the primitive concept $T$ in a MSS system is independent from the remaining primitive concepts (mass, position, internal force, and external force) iff there are two models of MSS system such that $T$ has two interpretations and the remaining primitive symbols have the same interpretation. But these two interpretations are not possible, since position **s**, internal force **f**, and external force **g** are functions whose domains depend on $T$. If we change the interpretation of $T$, then we change the interpretation of three other primitive concepts, namely, **s**, **f**, and **g**. So, time is not independent and hence can be defined. Since time is definable, it is eliminable.

In [5], the authors have shown that time is dispensable in thermodynamics as well, at least in a particular (although very natural) axiomatic framework for the theory. Moreover, in the same paper, they have shown how to define time and how to restate thermodynamics without any explicit reference to time. In the case of the MSS system, time can be defined by means of the domain of the functions **s**, **f**, and **g**. A similar procedure is used in [5].

## 6.4   Reformulating classical particle mechanics

**Definition 44** *Let s and g be two two-variable functions, m a one-variable function, and f a three-variable function. Besides, let v be a one-variable function which acts only on vectors of a three-dimensional real vector space endowed*

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*with usual scalar product ×. A non-relativistic classical particle system is an ordered 4-uple $\langle m, s, f, g \rangle$ such that:*

1. *m is finite on its only entry. Besides, if m acts on p, then m(p) is a real number greater than zero.*

2. *s[(p, t)] iff m[p] and t belongs to an interval i of real numbers. Besides, v[s(p, t)].*

3. *f[(p, q, t)] iff m acts on both p and q, and s acts on both (p, t) and (q, t). Besides, v[f(p, q, t)].*

4. *g[(p, t)] iff s[(p, t)]. Besides, v[g(p, t)].*

5. *s[(p, t)] ⇒ ∃$\frac{d^2 s(p,t)}{dt^2}$.*

6. *f[(p, q, t)] ⇒ f(p, q, t) = −f(q, p, t).*

7. *f[(p, q, t)] ⇒ s(p, t) × f(p, q, t) = −s(q, t) × f(q, p, t).*

8. *f[(p, q, t)] ⇒ m(p)$\frac{d^2 s(p,t)}{dt^2}$ = $\sum_{m[q]} f(p, q, t) + g(p, t)$.*

So, what is a particle? According to our view, only mass, position and forces are primitive concepts. Within this context, a particle is any function $p$ where $m$ acts.

# 7 The full potential of Flow

In this Section we suggest a way of doing mathematics without limiting ourselves to structure-free classes, as already done in the previous sections.

## 7.1 Composition

First of all, it is rather important to realize the Flow-theoretic concept of composition (see Definition 1) is not equivalent to the usual notion of composition within standard mathematics. By "standard mathematics" we mean the study of those trivially arbitrary functions we used here to reconstruct both ZFC and Category Theory. Trivially arbitrary functions and static morphisms (including static functors and static natural transformations) are simply restrictions of $\underline{1}$. In other words, those special cases of functions $f$ act on certain ordered pairs $(a, b)$ in a way such that $f(a, b) = (a, b)$. Within this context, a static composition works, intuitively speaking, like this: if $f$ acts on $(a, b)$ and $g$ acts on $(b, c)$, then $f \circ_\dagger g$ acts on $(a, c)$. Notwithstanding, that does not correspond to the concept of composition introduced in Definition 1. What do we mean by that?

In ZFC, many functions can be bijective. That means many functions $f$ admit the existence of an inverse $f^{-1}$. And such an inverse is defined through the use of composition (ZFC-composition). In Flow Theory, however, $\underline{1}$ is the only restriction of $\underline{1}$ which is an injective function, in the sense that for any

other restriction $f$ different of $\underline{1}$ we have at least two functions $x$ and $y$ such that $f(x) = \underline{0}$ and $g(y) = \underline{0}$. In other words, any $ZF$-set is a non-injective function, in the sense above. That means the standard strategy to define inverse (by means of composition) is not applicable in Flow. Besides, even the successor function $\sigma$ is not injective in this sense. That means it is really hard do find any other injective function besides $\underline{1}$ in Flow.

One natural way of coping with this limitation is by means of a definition of "local" invertibility, as we introduce in the next definition.

**Definition 45** $f$ *is* locally invertible *iff there is $g$ such that $\forall t((g[t] \Rightarrow f(g(t)) = t) \wedge (f[t] \Rightarrow g(f(t)) = t))$. In that case, we call $g$ the* local inverse *of $f$, and denote $g$ as $f^{-1}$.*

**Theorem 47** *If $f$ is a restriction of $\underline{1}$, then it is locally invertible, and its local inverse is $f$ itself.*

**Proof:** If $f$ is a restriction of $\underline{1}$, according to axiom $\mathbf{P8}_F$, then for any $t$ we have $f[t] \Rightarrow f(t) = t$. That entails $f[t] \Rightarrow f(f(t)) = t$.

The shortcoming of this last definition is that any ZF-set is locally invertible, since every ZF-set is a restriction of $\underline{1}$, according to Definition 20. So, even the notion of local invertibility does not correspond to the way how invertibility is addressed in ZFC.

As a non-trivial example (a function which is not a restriction of $\underline{1}$), consider a function $f$ such that $f(\phi_1) = \phi_2$, $f(\phi_2) = \phi_4$, $f(\phi_3) = \phi_6$, $f(f) = f$, and $f(r) = \underline{0}$, where $r$ stands for the remaining possible values. The existence of $f$ is guaranteed by axiom $\mathbf{P9}$. According to our definition above, a local inverse of $f$ is $f^{-1}$ such that $f^{-1}(\phi_6) = \phi_3$, $f^{-1}(\phi_4) = \phi_2$, $f^{-1}(\phi_2) = \phi_1$, $f^{-1}(f^{-1}) = f^{-1}$, and $f^{-1}(r) = \underline{0}$, where $r$ stands for the remaining possible values.

Notwithstanding, there is an intriguing consequence for locally invertible functions. The existence of a local inverse function does not entail its uniqueness. Consider, e.g., the next function:

$$f: \quad \longrightarrow \phi_5 \longrightarrow \phi_6$$
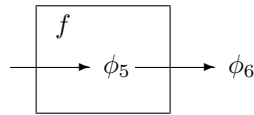
According to the diagram above, $f$ is a function which acts only on $\underline{0}$ and $\phi_5$, in a way such that $f(\underline{0}) = \phi_5$, and $f(\phi_5) = \phi_6$, while $f(\phi_6) = \underline{0}$. If we follow Definition 45, we find both functions $g$ and $h$ below are local inverses of $f$.

$$g: \quad \longrightarrow \phi_6 \longrightarrow \phi_5 \qquad\qquad h: \quad \phi_6 \longrightarrow \phi_5$$
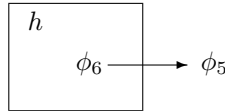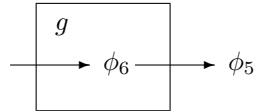
While $g$ acts on $\underline{0}$, that does not happen with $h$.

It is possible the reader finds this result somehow suspicious. After all, since composition is associative, we could expect the uniqueness of the local inverse $f^{-1}$ of a given function $f$. Nevertheless, it is worth to recall our notion of local inverse refers only to values where a given function acts. That means the concept of local inverse does not encompass all possible values of the universe of Flow.

**Definition 46** $f$ *is* locally injective *iff* $\forall r \forall s ((f[r] \wedge f[s] \wedge r \neq s) \Rightarrow f(r) \neq f(s))$.

**Theorem 48** *If $f$ is locally injective, then it is locally invertible.*

**Proof:** If, for a given $t$, $f[t]$ entails $f(t) = u$, all we have to do is to define a function $f^{-1}$ such that $f^{-1}(u) = t$. That is possible because we are assuming $f$ is locally injective. That means $f^{-1}$ is indeed a function. Thus, if $f[t]$, then $f^{-1}(f(t)) = f^{-1}(u) = t$. On the other hand, $f(f^{-1}(u)) = f(t) = u$.

Observe in the proof above we are not assuming neither that $f^{-1}$ necessarily acts on $u$ nor that $u$ is necessarily different of $\underline{0}$.

**Theorem 49** *If $f$ is locally injective and, for any $t$, $f[t] \Rightarrow f[f(t)]$, then its local inverse $f^{-1}$ is unique.*

**Proof:** Here is a sketch for the proof. As previously discussed in the last theorem, if, for a given $t$, $f[t]$ entails $f(t) = u$, all we have to do is to define a function $f^{-1}$ such that $f^{-1}(u) = t$. Now suppose $g \neq f^{-1}$ is a local inverse of $f$. The only way to guarantee that $g \neq f^{-1}$, is by assuming either $f^{-1}(\underline{0}) = u$ or $g(\underline{0}) = u$ for a specific $u$ where $f(t) = u$ for a specific $t$, since $f$ is locally injective. But that would entail $f(u) = \underline{0}$, since both $f^{-1}$ and $g$ are local inverses of $f$. Notwithstanding, that would entail $f(u) = f(f(t)) = \underline{0}$, which contradicts the assumption that $f[f(t)]$ for any $t$ where $f[t]$.

A more detailed discussion about the Flow-theoretic concept of composition is presented in the next subsection.

## 7.2 $n$-ary functions

Consider a function $*$ which acts only on functions $\phi_n$, for any $n$ of language $\mathcal{L}$. For the sake of abbreviation, we may denote $*(r)$ simply as $*_r$. That means $*_r$ is different of $\underline{0}$ (and of $*$ itself, of course) iff $r = \phi_n$ for some $n$. Now let us assume, for any $r$ such that $*[r]$, we have that $*_r$ acts on every $s$ such that $*[s]$, in a way such that $*_r(s)$ is equal to $\phi_m$, for some specific $m$ of language $\mathcal{L}$. That means $*[*_r(s)]$ for any $r$ and any $s$ such that $*[r]$ and $*[s]$.

If we denote $*_r(s)$ as $r * s$ (once again for the sake of abbreviation), we can easily see that $*$ behaves, for practical purposes, as a binary operation, despite

the fact we are not explicitly working with ordered pairs $(r, s)$. Intuitively speaking, $*$ behaves like a "family" (indexed by $r$) of monadic functions $*_r$.

For example, consider a function $+$ such that $+(\phi_n)$ is a function $+_n$ such that $+_n(\phi_m) = \phi_{m+n}$. Intuitively speaking, that corresponds to the summation between any two natural numbers. The idea is something like this: the binary summation $+$ is a "family" of monadic summations; there is the summation $+_0$, the summation $+_1$, the summation $+_2$, and so on. In elementary school, a similar idea has been used for a long time, namely, multiplication tables. Actually, multiplication tables date at least from four thousand years ago [19]. There is the multiplication table of 7, and the multiplication table of 9. Within Flow, we work with the multiplication table of any natural number. And we do not restrict ourselves only to multiplication tables, but rather to any operation's table. Thus, the statement $\phi_n + \phi_m$ can be assumed as an abbreviation for $(+ \circ +_n)(\phi_m)$, which is $+(+_n(\phi_m))$. Notwithstanding, observe we are not talking about natural numbers within a ZFC framework, since neither $+$ nor $+_n$ (for any $n$ of language $\mathcal{L}$, such that $n \neq 0$) is a ZF-set. We are not talking about restrictions of $\underline{1}$ anymore.

A lot of questions can be raised from our approach. And one of them refers to the usual ways of teaching mathematics. Why multiplication tables are so common along history? Are multiplication tables a simple pedagogical strategy? Or can we believe on the possibility that multiplication tables refer to a quite natural way of grounding mathematics itself? Once upon a time, the axiomatic method was nothing more than a methodology of teaching. Now, since 19th century it is well known that the axiomatic method is a genuine branch of mathematics. Can we say something analogous about multiplication tables?

Now, going back to our arbitrary $*$, suppose we want to say $*$ is commutative. That is easily accomplished by stating the following formula:

$$\forall r \forall s (r * s = s * r),$$

which is equivalent to

$$\forall r \forall s (*_r(s) = *_s(r)).$$

Nevertheless, the most interesting question is this: how to say that $*$ is associative?

If we write such a statement by means of our proposed abbreviation, we have obviously this:

$$\forall r \forall s \forall t ((r * s) * t = r * (s * t)).$$

Nevertheless, this last formula corresponds to say:

$$\forall r \forall s \forall t (*_{*_r(s)}(t) = *_r(*_s(t))).$$

In other words, the statement that $*$ is associative corresponds to say:

$$\forall r \forall s (*_{*_r(s)} = *_r \circ *_s),$$

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which is equivalent to say

$$\forall r \forall s ((* \circ *_r)(s) = *_r \circ *_s).$$

Hence, we proved here that associativity of a "binary" operation $*$ is somehow associated to composition of monadic functions. Observe also that our discussion here can be easily extended to operations which act on other terms, besides functions $\phi_n$.

In standard set theories, operations are simply functions which, in turn, are sets. Within Flow, "binary" operations can be coped as special cases of monadic functions (since all functions in Flow are monadic). And that fact helps us to get a new perspective about how operations work in mathematics. Besides, it is worth to observe that the "domain" of $*$ is $*$ itself, since the domain of any function $f$ can be defined as a function which acts on all terms where $f$ acts. Well, that means we can always say the domain of $f$ is $f$. That is the meaning of our claim in the Abstract that functions in Flow have no domain.

## 7.3 Mathematics teaching

Besides times tables, there are other issues regarding Flow theory and its possible repercussion into the common practice of mathematics.

There are many issues regarding at least three ways of doing mathematics: (i) the common practice of the working mathematician (including those who work with applied mathematics, like physicists, engineers, economists, psychologists, statisticians, and so on); (ii) the common practice of logicians; and (iii) the common way how mathematics is taught at school.

For example, within ZFC, a function is simply a set of ordered pairs. Nevertheless, no working mathematician says the ordered pair $(1, 2)$ *belongs* (in the set-theoretic sense of the predicate letter $\in$) to a specific function $f$ whose domain is the set of real numbers and which doubles all real numbers. Working mathematicians simply say "$f(1) = 2$". And the intended interpretation of this last statement is simply that $f$ "transforms" 1 into 2. Well, that intended interpretation is somehow safe within our proposal, as long we do not limit ourselves to situations like those regarding the translation of ZFC into Flow.

However, there are other issues a little more sensitive. For example, usually the quotient $r/s$ between real numbers $r$ and $s$ is referred to as an *operation* over real numbers, despite the fact that from the logical point of view that is not the case. That happens because it is unusual to define $r/0$. The usual way of dealing with this situation is by claiming $r/0$ is not definable. Nevertheless, that is a false claim. For details, see, for example, [22]. From a Flow-theoretic point of view, the quotient $r/0$ can be regarded simply as $\psi$ (a function already discussed above, which cannot be regarded as a real number). In other words, $r/0$ is just a term which is not a real number, nothing else. So, unlike multiplication over real numbers, quotient is not a closed operation over real numbers. That is all.

Other examples may be found in standard text books of differential and integral calculus. For example, it is quite common to say that if $f$ a real function

defined over the set of real numbers, then $\lim_{x \to a} f(x) = L$ iff $\forall \varepsilon > 0 \exists \delta > 0 (0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon)$. Clearly we are not using the standard language of predicate calculus here. But that is not the point. The point is that such a statement is usually referred to as a definition for limit. But that is not the case, since every definition is supposed to be eliminable [22], in the sense that the *definiendum* is supposed to be replaceable by the *definiens*. Nevertheless, that is an impossible demand (according to logicians) for the case where there is no limit $L$. If for every $\varepsilon > 0$ there is no $\delta$ which satisfies the formula used in the alleged *definiens*, then how can we replace the *definiendum* by its corresponding *definiens*? From a Flow-theoretic point of view there is clear answer to that question.

Within standard calculus, there are three distinct cases of non-existent limits: (i) functions which grow indefinitely with positive values; (ii) functions which grow indefinitely with negative values; (iii) and the remaining cases. When a mathematician says $\lim_{x \to a} f(x) = \infty$, that is clearly disturbing, from a logical point of view. After all, $\infty$ is not a real number. Worse than that, $\infty$ is not even a term within ZFC. An analogous situation happens when a working mathematician says $\lim_{x \to a} f(x) = -\infty$. From a Flow-theoretic point of view, that problem is easily solved. We can say, for example, that $\lim_{x \to a} f(x) = \infty$ means $\lim_{x \to a} f(x) = \underline{1}$, and that $\lim_{x \to a} f(x) = -\infty$ means $\lim_{x \to a} f(x) = \underline{0}$, while for other cases of non-existent real limits $L$ we may say $\lim_{x \to a} f(x) = l$, where $l$ is an arbitrary term which is neither a real number, nor $\underline{0}$ or $\underline{1}$, like, e.g., function $\psi$ (that function whose successor is $\underline{1}$). So, limit may be regarded as an operation, like quotient. Nevertheless, limit is not closed over the set of real numbers. That means Flow theory allows us to introduce a justification for common practices among working mathematicians and teachers and authors in general.

# 8   Variations of Flow

We briefly discuss here some possible variations of Flow to be investigated in the future.

## 8.1   Closure

One possible variation of Flow has to do with the inclusion of one more axiom:

**Closure**  $\forall x (x(\underline{0}) = \underline{0} \wedge (x \neq \underline{1} \Rightarrow x(\underline{1}) = \underline{0}))$.

The main advantage of this strategy is that it simplifies many calculations within Flow. Besides, if we add **Closure**, we have a well behaved algebra for composition, specially regarding operations involving $\underline{0}$ and $\underline{1}$. For example, **Closure** guarantees that for any $x$, $\underline{0} \circ x = \underline{0}$; $x \circ \underline{1} = x$; $\underline{1} \circ x = x$; and $x \circ \underline{0} = \underline{0}$. This last formula is not a theorem in Flow, unless we add **Closure**.

On the other hand, an inconvenient side effect of "Flow + **Closure**" is that Theorems 22 and 24 are no longer valid. Those results refer to constant

functions. And both theorems use the fact that $\sigma(\underline{0})$ is $\phi_0$. In other words, "Flow + **Closure**" is inconsistent, since axiom **P6** guarantees the existence of a unique function $\sigma$ such that $\sigma(\underline{0})$ is different of $\underline{0}$. Moreover, **P6** guarantees the existence of a vast universe of functions $x$ such that $x(\underline{0})$ is different of $\underline{0}$. For example, $\sigma(\sigma(\underline{0}))$ is $\phi_1$, $\sigma(\sigma(\sigma(\underline{0})))$ is $\phi_2$, and so on. So, if $x$ is either $\sigma$, $\sigma \circ \sigma$, or $\sigma \circ \sigma \circ \sigma$, then the proposed new axiom **Closure** is not satisfied.

To avoid such an obvious inconsistency, one possible solution is to erase **P6** from the list of axioms of Flow and to introduce a new primitive concept into Flow's language, namely, a monadic functional letter $f_1^1$ (abbreviated, e.g., as $\Sigma$) which plays a similar role of $\sigma$. Besides, all remaining axioms where $\sigma$ occurs are supposed to be rewritten in a way such that $\sigma$ is replaced by $\Sigma$.

We prefer here to avoid new primitive concepts, besides evaluation $f_1^2$. That is why we omitted formula **Closure** as an axiom. Nevertheless, the fact is that such a variation deserves to be explored.

## 8.2 Regularity

One of our aims for the future is a thorough discussion about the metamathematics of Flow Theory. Thus, we should consider the possibility of one extra *Axiom of Regularity* which avoids the possibility of functions $f$ and $g$ such that for both we have $f[g]$ and $g[f]$. That would entail the impossibility of $f \in g$ and $g \in f$. Hopefully, with this extra axiom we will be able to rank all functions in the sense of defining a cumulative hierarchy of terms. So, our additional postulate could be, e.g., the following one:

**Regularity** $\forall f \forall g ((f \neq g \wedge f(g) \neq \underline{0}) \Rightarrow g(f) = \underline{0})$.

In principle that would allow us to talk about Flow-theoretic versions of induction, recursion, rank, inaccessible cardinals, and other usual concepts from standard set theories [8]. But that is a task to be pursued in future works.

## 8.3 Clones and equiconsistency

Half of our axioms use the quantifier $\exists!$ for several functions. Such functions refer to composition (**P5**), successor (**P6**), restriction (**P8**$_F$), terms which are no restriction of $\underline{1}$ (**P9**), union (**P10**), and choice (**P12**). In the specific case of the Axiom of Choice **P12**, our concern with uniqueness is almost ludicrous, since we simply impose that any $d$ such that $d \sim c$, where $c$ is a choice function, is necessarily identical to $c$. So, why do we worry so much with uniqueness?

As previously discussed in Observation 1, if it wasn't for the uniqueness requirement of axioms like **P5** and others, Flow would be consistent with the existence of many functions which "do" the same thing. By multiple functions "doing the same thing" we mean different functions $x$ and $y$ which share the same images $x(t)$ and $y(t)$ for any $t$ different of both $x$ and $y$. That is why we introduced Definition 14, which says that $f \sim g$ iff $\forall t((t \neq f \wedge t \neq g) \Rightarrow f(t) = g(t))$.

The fact is that our weak extensionality **P1** does not guarantee equivalent functions $f$ and $g$ (i.e., $f \sim g$) are necessarily identical. And since we wanted to prove ZFC is immersed within Flow, it seemed reasonable to impose uniqueness for composition, successor, restriction, union, and choice. That was our almost unavoidable way for dealing with the strong extensionality imposed by ZFC axioms.

Nevertheless, another possible variation of Flow is supposed to replace all occurrences of $\exists!$ in axioms **P5**, **P6**, **P8**$_F$, **P9**, **P10**, and **P12** by quantifier $\exists$. Besides, the last part of **P12** ($\forall d(d \sim c \Rightarrow c = d)$) is supposed to be simply deleted. So, what would be the consequences of such a variation? Well, we do not know yet. In order to cope with such a possibility, we feel to be necessary to apply model theoretic techniques into Flow Theory. First of all, can we prove any equiconsistency theorem for any version at all of Flow? If we can prove Flow theory, in its present form, is equiconsistent to, e.g., ZFC, that would be a nice result. But even if that is possible, what about this (in principle possible) variation of Flow where there is no *ad hoc* assumption regarding the uniqueness of composition and other functions? Can we guarantee that such a variation is consistent with classical logic? If there is some ambiguity in defining functions which "do the same thing" but are not necessarily identical, can we guarantee that any given closed formula is necessarily either true or false? If that is not the case, what would be a proper logic for such a variation of Flow? Are we talking about the possibility of an intuitionistic logic whose models rely on a Heyting algebra? Well, if that is the case, then the Axiom of Choice is supposed to be removed, since intuitionistic logic is inconsistent with such a statement [14]. Or can we consider the possibility of a paraconsistent logic [6]?

# 9 Final remarks

As a reference to Heraclitus's flux doctrine, we are inclined to call all terms of Flow *fluents*, rather than functions. That is also an auspicious homage to the *Method of Fluxions* by Isaac Newton [16]. Newton referred to functions as fluents. And their derivatives were termed *fluxions*. Whether Newton was inspired by Heraclitus, that is historically uncertain ([20], page 38). Nevertheless, we find such a coincidence quite inspiring.

# 10 Acknowledgements

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