DEMYSTIFYING LANFORD’S THEOREM

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Abstract. The issue of identify if/where an irreversible ingredient enters in the proof of the Lanfords theorem has been discussed recently in the literature. Sometimes, in focusing on the details, we can loose the overall picture. In this brief commentary I express few reflection about the general place of Lanfords theorem at the foundation the problem of the origin of irreversibility in particular. I argue that the importance of this theorem is more mathematical/technical than conceptual or explanatory.

1. On explaining irreversibility

The problem of reconciling a reversible micro-dynamics with the thermodynamic behaviour has been ind it continues to be a debated topic in foundation of physics. Recently some articles appeared on this Journal has focus the on the Lanford’s Theorem (henceforth LT) and in particular on what in its proof is responsible for the emergence of irreversibility[9, 1]. I am not going to enter directly in this debate here. What I intend to do is to develop some reflection on the conceptual relevance of LT inside the so called problem of the origin of irreversibility. This requires first a clarification on what I mean here by the problem of irreversibility. For thermodynamic behavior I have in mind the classical image of the gas that, initially confined in a corner of a container, uniformly spreads over the entire available space as soon as the confining partition is removed. Explaining irreversibility in this case means to account, on the base of a reversible micro-dynamics, for the existence and uniqueness of a special macro-state (M_{equilibrium}). This special state acts as an attractor state i.e. the system converges to it and remains there:

\begin{equation}
M \rightarrow M_{equilibrium},
\end{equation}

no matter what the initial state M is [3].

In modern terms, Boltzmann’s solution of the problem so defined rely on the concept of typicality[8, 6]. In a closed system, the volume of the phase-space region corresponding to a macrostate is a measure of how commonly we can expect to observe it in that state. Combinatorial arguments show that Hamiltonian mechanical systems have energy hyper-surfaces dominated by one macrostate, the one that we identify as equilibrium state [2]. M_{equilibrium} dominates in the sense that its measure, calculated as a fraction of the whole phase space, is close to 1. The explanation of the observed irreversibility based on Boltzmann’s original ideas (henceforth for sake of syntheses I will call it Boltzmann statistical program (BSP)) is based on the fundamental ingredients [2]:

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Key words and phrases. Irreversibility, Lanford’s theorem, Boltzmann.
a): many degrees of freedom in a macroscopic system;  
b): separation of observational scales and;  
c): the initial conditions: the system is supposed to start in a atypical (i.e. with low probability) state in the past;  
to this list we need to add the following:  
d): isotropy and independence of velocities of the particles involved.

On the other hand efforts have been made in the understanding in how/where irreversibility emerges in kinetic equations that model the behaviour of systems that begin away from equilibrium. Boltzmann’s transport equation is the focus here:

\[
\frac{\partial}{\partial t} f + v \cdot \Delta_x f = Q(f, f)
\]

This equation describes the evolution of \( f(x, v, t) \) due to diffusion and collisions with respectively the second therm on the left and the therm on the right. It is the well known molecular chaos hypothesis (Stosszahlansatz), i.e. that colliding particles can be considered uncorrelated, is considered the ingredient that accounts for the time asymmetry. Boltzmann was not able to justify rigorously this hypotheses and here is where LT enters into the picture. In the derivation of the Boltzmann equation, LT needs to enter in the details of the model, the geometry of hard-sphere collisions, etc. As a by-product the theorem shows that if a kind of Stosszahlansatz is present at \( t = 0 \), it will be present even later. In LT the fact that a version of Stosszahlansatz works for different times is not an assumption but it is derived in a rigorous way. Rigorous here means that expression like almost or approximately true are substituted with something like”\( \text{Prob}(\text{something}) = 1 \text{ if some counter} \rightarrow \infty \)”.

To do this some highly idealized assumption must be considered like the Grad-Boltzmann limit and a very particular choice of the initial conditions. About the LT it has been said that (p. 86 from [4], my italics):

We already mentioned that the Lanford result that we formulated and proved in this section is unsatisfactory, because its validity time is unsatisfactorily short on physically relevant scales. On the other hand, the conceptual impact of the result was remarkable and persists; we have proved that a rigorous transition from reversible to irreversible dynamics is possible, and this is significant even if the time interval in question is extremely short.

Here I disagree with statements like this that stress the conceptual role of the LT because they give the false impression that either the BSP is not enough solid for explaining the behavior of the gas or that BSP is in need of a rigorous result to be complemented with. This is not the case and I will illustrate my argument with an example.

This framework in its most general form considers non-interacting particles. Indeed it must be stressed that observed irreversibility does not origin from interactions of the constituents and there is no need of any other particular dynamical property. Assumption d) is seldom stressed in literature but it is fundamental ingredient of the BSP. In particular isotropy is a strong ontological statement about a fact of Nature: in a collection of particles, velocities are uniformly distribute in
Figure 1. If isolated gas evolution is described by its center of mass (CM) irreversibility is introduced.

2. A Simple Model

As an illustrative here I will briefly discuss a different look at the problem of the emergence of irreversibility developed in detail in [5]. The idea is to describe the evolution of an isolated gas as a system of particles and focusing on the dynamics of its center of mass (CM). Describing the dynamics of a gas by its center of mass is an extreme form of dimensional projection that shows easily how irreversibility emerges in classical, deterministic, multi-particle system. With reference to figure 1 if the gas starts in the configuration A with the corresponding CM, spontaneous evolution
will lead to a displacement of $\text{CM}$ as shown. This is a dynamical description where
we observe irreversibility since the reverse evolution $C \rightarrow B \rightarrow A$ is prevented.
The dynamics of $\text{CM}$ can be explained by simple mechanical arguments. When
the partition is removed the gas is confined at $A$. The net external force due to
the momentum transferred to the particles by the walls during collisions is now
different from zero and the $\text{CM}$ moves to $B$ and $C$ and then stops. In $C$ the net
forces acting on the $\text{CM}$ is again zero. The $\text{CM}$ evolution stops in $B$ since the
net external force due to walls is equal to zero. This is easily justified provide
we accept a usual assumption about independence and isotropy of velocities of
particles\(^{2}\). This model is interesting because: a) Irreversibility is introduced in the
\textit{reduced} mechanical description; b) interactions between the particles are irrelevant
because they do not affect the dynamics of $\text{CM}$; c) independence and isotropy
are again fundamental basic assumption; d) the model can be made rigorous in an
appropriate limit regime (number of particles $n \rightarrow \infty$, etc.).

3. \textbf{IRREVERSIBILITY: A DIALECTICAL ACCOUNT}

\textit{Reductionism} is the scientific approach for which higher-level domain phenomena
can be explained by reference to the properties of the lower-level entities that make
them up. From a bird’s-eye view, Boltzmann arguments, culminating in its famous
formula:

\begin{equation}
S = k \log W,
\end{equation}

do not eventually end up in a reductionist project, they represents instead a spectacular
example of \textit{dialectics}: in a process of synthesis, two apparently irreconcilable
domains, \textit{thesis} and \textit{antithesis} (in this case mechanics and thermodynamics) are
resolved in a middle ground theory that is what we now call \textit{statistical mechanics}.
Indeed to solve the problem of irreversibility this is what we need. I summarize it
in the following:

\textbf{Statement 1.} To account for $M_{\text{equilibrium}}$ for the expanding ideal gas, the $\text{BSP}$
does not try to extract irreversibility out of a deterministic, time-reversible Hamiltonian
dynamics in a continuous chain of logical steps. What $\text{BSP}$ accomplish is to show that irreversibility and time-reversible Hamiltonian dynamics can coexist under an appropriate conceptual framework.

$\text{BSP}$ and $\text{LT}$ operate at different levels. $\text{BSP}$ is a master framework that can be
applied to account for the irreversibility observed in a dilute gas. It is so to speak
a top-down logic of explanation: from the general to the particular. $\text{LT}$ works
bottom-up, it aims to derive irreversibility from first principles for a particular
model and under severe assumptions. I claim that the importance of $\text{LT}$ is more
technical-mathematical than conceptual or physical.

On the other end there is a misconception that the final word on the problem of
the origin of irreversibility can be fulfilled ultimately by a pure reductionist project
where irreversibility is extract out of a deterministic, time-reversible Hamiltonian
underlying dynamics, being it classical or quantum. As I explained what is needed

\(^{2}\)A similar argument, with slight modification, can be done even in the case that the gas starts
at the center of the vessel. For details the reader can refer to [5].
is a framework in which we can accommodate both: irreversibility and a deterministic, time-reversible Hamiltonian dynamics and this is what BSP conceptually accomplishes the task.

REFERENCES