Semi-formally stated open problems on computable sets $X = \{n \in \mathbb{N} : \varphi(n)\}$, where $\varphi(n)$ has the same intuitive meaning for every $n \in \mathbb{N}$ and the finiteness (infiniteness) of $X$ remains conjectured

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Abstract

Let $b = ((24!)!)!$, and let $\mathcal{P}_{n+1}$ denote the set of all primes of the form $n^2 + 1$. Let $\mathcal{M}$ denote the set of all positive multiples of elements of the set $\mathcal{P}_{n+1} \cap (b, \infty)$. The set $X = \{0, \ldots, b\} \cup \mathcal{M}$ satisfies the following conditions: (1) card($X$) is greater than a huge positive integer and it is conjectured that $X$ is infinite, (2) we do not know any algorithm deciding the finiteness of $X$, (3) a known and short algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in X$, (4) a known and short algorithm returns an integer $n$ such that $X$ is infinite if and only if $X$ contains an element greater than $n$. The following problem is open: define a set $X \subseteq \mathbb{N}$ such that $X$ satisfies conditions (1)-(4) and a known and simple formula $\varphi(x)$ satisfies $X = \{n \in \mathbb{N} : \varphi(n)\}$, where $\varphi(n)$ has the same intuitive meaning for every $n \in \mathbb{N}$ (5). The statements $\varphi(n)$ in condition (5) have always the same intuitive meaning, if the predicate $\varphi(x)$ expresses a natural property, the term propounded by the philosopher David Lewis (1941-2001). Let $f(3) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 3$. For an integer $n \geq 3$, let $\Psi_n$ denote the following statement: if a system of equations $S \subseteq \{x_i! = x_{i+1} : 1 \leq i \leq n - 1\} \cup \{x_i \cdot x_j = x_{j+1} : 1 \leq i \leq j \leq n - 1\}$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leq f(n)$. We prove that for every statement $\Psi_n$ the bound $f(n)$ cannot be decreased. The author’s guess is that the statements $\Psi_3, \ldots, \Psi_9$ are true. We prove that the statement $\Psi_9$ implies that the set $X$ of all non-negative integers $n$ whose number of digits belongs to $\mathcal{P}_{n+1}$ satisfies conditions (1)-(5).

Key words and phrases: computable set $X \subseteq \mathbb{N}$ whose finiteness remains conjectured, computable set $X \subseteq \mathbb{N}$ whose infiniteness remains conjectured, David Lewis’s notion of a natural property, intuitive meaning of a mathematical formula, Zenkin’s super-induction.

1 Introduction and basic definitions and lemmas

In this article, we discuss semi-formally stated open problems on computable sets $X = \{n \in \mathbb{N} : \varphi(n)\}$, where $\varphi(n)$ has the same intuitive meaning for every $n \in \mathbb{N}$ and the finiteness (infiniteness) of $X$ remains conjectured.

Definition 1. Let $b = ((24!)!)!$.

Lemma 1. $b \approx 10^{10^{10^{25.16114896940657}}}$. □


Definition 2. We say that an integer $m \geq -1$ is a threshold number of a set $X \subseteq \mathbb{N}$, if $X$ is infinite if and only if $X$ contains an element greater than $m$, cf. [10] and [11].

If a set $X \subseteq \mathbb{N}$ is empty or infinite, then any integer $m \geq -1$ is a threshold number of $X$. If a set $X \subseteq \mathbb{N}$ is non-empty and finite, then all threshold numbers of $X$ form the set $\{\max(X), \max(X) + 1, \max(X) + 2, \ldots\}$. 


Definition 3. We say that a non-negative integer \( m \) is a weak threshold number of a set \( X \subseteq \mathbb{N} \), if \( X \) is infinite if and only if \( \text{card}(X) > m \).

Theorem 1. For every \( X \subseteq \mathbb{N} \), if an integer \( m \geq -1 \) is a threshold number of \( X \), then \( m + 1 \) is a weak threshold number of \( X \).

Proof. For every \( X \subseteq \mathbb{N} \), if \( m \in [-1, \infty) \cap \mathbb{Z} \) and \( \text{card}(X) > m + 1 \), then \( X \cap [m + 1, \infty) \neq \emptyset \). \( \square \)

We do not know any weak threshold number of the set of all primes of the form \( n^2 + 1 \). The same is true for the sets \( \{ n \in \mathbb{N} : 2^{2n} + 1 \text{ is composite} \} \) and \( \{ n \in \mathbb{N} : n! + 1 \text{ is a square} \} \).

Lemma 2. For every positive integers \( x \) and \( y \), \( x! \cdot y! = y! \cdot x! \) if and only if \( (x + 1 = y) \lor (x = y = 1) \).

Lemma 3. (Wilson’s theorem, [2, p. 89]). For every integer \( x \geq 2 \), \( x \) is prime if and only if \( x \) divides \( (x - 1)! + 1 \).

2 Open Problems[1] and [2]

The following two open problems cannot be stated formally as they refer to the intuitive meaning of a mathematical formula and the current mathematical knowledge.

Open Problem 1. Define a set \( X \subseteq \mathbb{N} \) that satisfies the following conditions:
(1) \( \text{card}(X) \) is greater than a huge positive integer and \( X \) is conjectured to be infinite,
(2) we do not know any algorithm deciding the finiteness of \( X \),
(3) a known and short algorithm for every \( n \in \mathbb{N} \) decides whether or not \( n \in X \),
(4•) a known and short algorithm returns an integer \( n \) such that \( X \) is infinite if and only if \( \text{card}(X) > n \),
(5) a known and simple formula \( \varphi(n) \) satisfies \( X = \{ n \in \mathbb{N} : \varphi(n) \} \), where \( \varphi(n) \) has the same intuitive meaning for every \( n \in \mathbb{N} \).

Open Problem 2. Define a set \( X \subseteq \mathbb{N} \) such that \( X \) satisfies conditions (1)-(3), (5), and a known and short algorithm returns an integer \( n \) such that \( X \) is infinite if and only if \( X \) contains an element greater than \( n \) (4).

The statements \( \varphi(n) \) in condition (5) have always the same intuitive meaning, if the predicate \( \varphi(x) \) expresses David Lewis’s natural property. For the meaning of this term, the reader is referred to [1].

Theorem 2. Open Problem \[2] claims more than Open Problem \[7]

Proof. By Theorem \[2] condition (4) implies condition (4•). \( \square \)

3 A partial solution to Open Problem 2

Edmund Landau’s conjecture states that the set \( \mathcal{P}_{n^2+1} \) of all primes of the form \( n^2 + 1 \) is infinite, see [5, pp. 37–38] and [7]. Let \( M \) denote the set of all positive multiples of elements of the set \( \mathcal{P}_{n^2+1} \cap (b, \infty) \).

Theorem 3. The set \( X = \{ 0, \ldots, b \} \cup M \) satisfies conditions (1)-(4).

Proof. Condition (1) holds as \( \text{card}(X) > b \) and the set \( \mathcal{P}_{n^2+1} \) is conjecturally infinite. By Lemma \[1] due to known physics we are not able to confirm by a direct computation that some element of \( \mathcal{P}_{n^2+1} \) is greater than \( b \). Thus condition (2) holds. Condition (3) holds trivially. The set \( M \) is empty or \( M \) is an infinite subset of \( (b, \infty) \). Therefore, \( b \) is a threshold number of \( X \). Thus condition (4) holds. \( \square \)
4 The statements $\Psi_n (n = 3, 4, 5, \ldots)$, which seem to be true for every $n \in \{3, \ldots, 9\}$

Let $f(3) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 3$. For an integer $n \geq 3$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{align*}
\forall i \in [1, \ldots, n-1] \setminus \{2\} \quad x_i! &= x_{i+1} \\
x_1 \cdot x_2 &= x_3 \\
x_2 \cdot x_2 &= x_3
\end{align*}$$

The diagram in Figure 1 illustrates the construction of the system $\mathcal{U}_n$.

**Fig. 1** Construction of the system $\mathcal{U}_n$

**Lemma 4.** For every integer $n \geq 3$, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $(2, 2, f(3), \ldots, f(n))$.

Let

$$B_n = \{x_i! = x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i \cdot x_j = x_{j+1} : 1 \leq i \leq j \leq n-1\}$$

For an integer $n \geq 3$, let $\Psi_n$ denote the following statement: if a system of equations $S \subseteq B_n$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leq f(n)$. The statement $\Psi_n$ says that for subsystems of $B_n$ with a finite number of solutions, the largest known solution is indeed the largest possible. The author’s guess is that the statements $\Psi_3, \ldots, \Psi_9$ are true.

**Theorem 4.** Every statement $\Psi_n$ is true with an unknown integer bound that depends on $n$.

**Proof.** For every positive integer $n$, the system $B_n$ has a finite number of subsystems. \hfill \Box

**Theorem 5.** For every statement $\Psi_n$, the bound $f(n)$ cannot be decreased.

**Proof.** It follows from Lemma 4 because $\mathcal{U}_n \subseteq B_n$. \hfill \Box

5 The statement $\Psi_9$ solves Open Problem 2

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{align*}
x_2! &= x_3 \\
x_3! &= x_4 \\
x_5! &= x_6 \\
x_8! &= x_9 \\
x_1 \cdot x_1 &= x_2 \\
x_3 \cdot x_5 &= x_6 \\
x_4 \cdot x_8 &= x_9 \\
x_5 \cdot x_7 &= x_8
\end{align*}$$

Lemma 2 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$. 

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Lemma 5. For every integer $x_1 \geq 2$, the system $A$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:

\[
\begin{align*}
  x_2 &= x_1^2, \\
  x_3 &= (x_1^2)! \\
  x_4 &= ((x_1^2)!)! \\
  x_5 &= x_1^2 + 1 \\
  x_6 &= (x_1^2! + 1)! \\
  x_7 &= \frac{x_1^2! + 1}{x_1^2 + 1} \\
  x_8 &= (x_1^2)! + 1 \\
  x_9 &= ((x_1^2)! + 1)! 
\end{align*}
\]

Proof. By Lemma 2, for every integer $x_1 \geq 2$, the system $A$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 5 follows from Lemma 3. □

Lemma 6. There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ which solve the system $A$ and satisfy $x_1 = 1$.

Proof. If a tuple $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ solves the system $A$ and $x_1 = 1$, then $x_1, \ldots, x_9 \leq 2$. Indeed, $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leq 2$. □

Theorem 6. The statement $\Psi_9$ proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set $P_{n=1}$ is infinite.

Proof. Suppose that the antecedent holds. By Lemma 5, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ such that the tuple $(x_1, x_2, x_3, \ldots, x_9)$ solves the system $A$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geq f(7)$. Hence, $(x_1^2)! \geq f(7)! = f(8)$. Consequently,

\[x_9 = ((x_1^2)! + 1)! \geq (f(8) + 1)! > f(8)! = f(9)\]

Since $A \subseteq B_9$, the statement $\Psi_9$ and the inequality $x_9 > f(9)$ imply that the system $A$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 5 and 6, the set $P_{n=1}$ is infinite. □
Let \( \mathcal{F} \) denote the set of all non-negative integers \( k \) whose number of digits belongs to \( P_{n^2+1} \).

**Lemma 7.** \( \text{card}(\mathcal{F}) \) is greater than a huge positive integer.

**Proof.** The number \((13!)^2 + 1 = 38775788043632640001\) is prime, see [8]. \( \square \)

**Theorem 7.** The statement \( \Psi_9 \) implies that \( X = \mathcal{F} \) satisfies conditions (1)-(5).

**Proof.** Suppose that the antecedent holds. Since the set \( P_{n^2+1} \) is conjecturally infinite, Lemma 7 implies condition (1). Conditions (3) and (5) hold trivially. By Theorem 6, \( 9\ldots9 \) is a threshold number of \( X \). Thus condition (4) holds. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of \( P_{n^2+1} \) is greater than \( f(7) = ((24!)!)! = b \). Thus condition (2) holds. \( \square \)

6 **Open Problems 3 and 4**

**Definition 4.** Let \( (1\diamond) \) denote the following condition: \( \text{card}(X) \) is greater than a huge positive integer and it is conjectured that \( X = \mathbb{N} \).

**Definition 5.** Let \( (2\diamond) \) denote the following condition: we do not know any algorithm deciding the equality \( X = \mathbb{N} \).

The following two open problems cannot be stated formally as they refer to the intuitive meaning of a mathematical formula and the current mathematical knowledge.

**Open Problem 3.** Define a set \( X \subseteq \mathbb{N} \) that satisfies conditions \((1\diamond)-(2\diamond), (2)-(3), (4\bullet), \) and \((5)\).

Open Problem 3 claims more than Open Problem 1 as condition \((1\diamond)\) implies condition \((1)\).

**Open Problem 4.** Define a set \( X \subseteq \mathbb{N} \) that satisfies conditions \((1\diamond)-(2\diamond)\) and \( (2)-(5) \).

Open Problem 4 claims more than Open Problem 2 as condition \((1\diamond)\) implies condition \((1)\).

**Theorem 8.** Open Problem 4 claims more than Open Problem 3.

**Proof.** By Theorem 1, condition \((4)\) implies condition \((4\bullet)\). \( \square \)

7 **A partial solution to Open Problem 4**

Let \( \mathcal{V} \) denote the set of all positive multiples of elements of the set

\[ \{n \in \{b+1, b+2, b+3, \ldots\} : 2^{2^n} + 1 \text{ is composite}\} \]

**Theorem 9.** The set \( X = \{0, \ldots, b\} \cup \mathcal{V} \) satisfies conditions \((1\diamond)-(2\diamond)\) and \((2)-(4) \).

**Proof.** The inequality \( \text{card}(X) > b \) holds trivially. Most mathematicians believe that \( 2^{2^n} + 1 \) is composite for every integer \( n \geq 5 \), see [3, p. 23]. These two facts imply conditions \((1\diamond)\) and \((2\diamond)\). Condition (3) holds trivially. Since the set \( \mathcal{V} \) is empty or \( \mathcal{V} \) is an infinite subset of \((b, \infty)\), the integer \( b \) is a threshold number of \( X \). Thus condition (4) holds. The question of finiteness of the set \( \{n \in \mathbb{N} : 2^{2^n} + 1 \text{ is composite}\} \) remains open, see [4, p. 159]. Hence, the question of emptiness of the set \( \{n \in \{b+1, b+2, b+3, \ldots\} : 2^{2^n} + 1 \text{ is composite}\} \) remains open. Therefore, the question of finiteness of the set \( \mathcal{V} \) remains open. Consequently, the question of finiteness of the set \( X \) remains open and condition (2) holds. \( \square \)

Definition 6. Let \((1^*)\) denote the following condition: \(\text{card}(X)\) is greater than a huge positive integer and it is conjectured that \(X\) is finite.

The following two open problems cannot be stated formally as they refer to the intuitive meaning of a mathematical formula and the current mathematical knowledge.

Open Problem 5. Define a set \(X \subseteq \mathbb{N}\) that satisfies conditions \((1^*)\), \((2)-(3)\), \((4\bullet)\), and \((5)\).

Open Problem 6. Define a set \(X \subseteq \mathbb{N}\) that satisfies conditions \((1^*)\) and \((2)-(5)\).


Proof. By Theorem 1, condition \((4)\) implies condition \((4\bullet)\). □

9 A partial solution to Open Problem [6]

A weak form of Szpiro’s conjecture implies that there are only finitely many solutions to the equation \(x! + 1 = y^2\), see [6].

Lemma 8. ([9, p. 297]). It is conjectured that \(x! + 1\) is a square only for \(x \in \{4, 5, 7\}\).

Let \(W\) denote the set of all integers \(x\) greater than \(b\) such that \(x! + 1\) is a square.

Theorem 11. The set

\[X = \{0, \ldots, b\} \cup \{k \cdot x : (k \in \mathbb{N} \setminus \{0\}) \land (x \in W)\}\]

satisfies conditions \((1^*)\) and \((2)-(4)\).

Proof. Condition \((1^*)\) holds as \(\text{card}(X) > b\) and the set \(W\) is conjecturally empty by Lemma 8. Condition \((3)\) holds trivially. We do not know any algorithm that decides the emptiness of \(W\) and the set

\[Y = \{k \cdot x : (k \in \mathbb{N} \setminus \{0\}) \land (x \in W)\}\]

is empty or infinite. Thus condition \((2)\) holds. Since the set \(Y\) is empty or \(Y\) is an infinite subset of \([b, \infty)\), the integer \(b\) is a threshold number of \(X\). Thus condition \((4)\) holds. □

10 The statement \(\Psi_6\) solves Open Problem [6]

Let \(C\) denote the following system of equations:

\[
\begin{align*}
  x_1! &= x_2 \\
  x_2! &= x_3 \\
  x_5! &= x_6 \\
  x_4 \cdot x_4 &= x_5 \\
  x_3 \cdot x_5 &= x_6
\end{align*}
\]

Lemma 2 and the diagram in Figure 3 explain the construction of the system \(C\).
Lemma 9. For every $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$, the system $C$ is solvable in positive integers $x_2, x_3, x_5, x_6$ if and only if $x_1! + 1 = x_4^2$. In this case, the integers $x_2, x_3, x_5, x_6$ are uniquely determined by the following equalities:

$$
\begin{align*}
  x_2 &= x_1! \\
  x_3 &= (x_1!)! \\
  x_5 &= x_1! + 1 \\
  x_6 &= (x_1! + 1)!
\end{align*}
$$

Proof. It follows from Lemma 2.

Theorem 12. If the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then the statement $\Psi_6$ guarantees that each such solution $(x_1, x_4)$ belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$.

Proof. Suppose that the antecedent holds. Let positive integers $x_1$ and $x_4$ satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 9 the system $C$ is solvable in positive integers $x_2, x_3, x_5, x_6$. Since $C \subseteq B_6$, the statement $\Psi_6$ implies that $x_6 = (x_1! + 1)! \leq f(6) = f(5)!$. Hence, $x_1! + 1 \leq f(5) = f(4)!$. Consequently, $x_1 < f(4) = 24$. If $x_1 \in \{1, \ldots, 23\}$, then $x_1! + 1$ is a square only for $x_1 \in \{4, 5, 7\}$.

Theorem 13. Let $X$ denote the set of all non-negative integers $n$ which have $(((k!)!)!)$ digits for some $k \in \{m \in \mathbb{N} : m! + 1$ is a square$\}$. We claim that the statement $\Psi_6$ implies that $X$ satisfies conditions $(1^*)$ and $(2)-(5)$.

Proof. Let $d = (((7!)!)!)!$. Since $7! + 1 = 71^2$, we obtain that $[1_{d \text{ digits}}, \ldots, 9_{d \text{ digits}}] \subseteq X$. By this and Lemma 8 condition $(1^*)$ holds. Conditions $(2)-(3)$ and $(5)$ hold trivially. By Theorem 12 the statement $\Psi_6$ implies that $9_{d \text{ digits}}$ is a threshold number of $X$. Thus condition $(4)$ holds.

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