Reverse Bell's Theorem and Relativity of Pre- and Postselection

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In this paper we prove a Bell's theorem in the setting of postselection ('reverse Bell's theorem'). Specifically, we show under which conditions the Bell inequalities can or cannot be violated with classical postselection, and how this differs from the quantum violations. We then propose a variant of existing experiments that discriminates between quantum violations and classical simulations. The proposed experiment can be adapted to test simultaneously the standard and reverse Bell's theorems. In this case, the distinction between these pre- and postselection effects becomes foliation-dependent.

INTRODUCTION

There are a number of remarkable effects of quantum postselection. Often this is in combination with preselection as in the three-box paradox [1]. Here we treat the case where postselection can give rise to violations of the Bell inequalities as proposed in [2, 3] and realized experimentally in [4] (see also [5]). To distinguish this scenario from the standard Bell inequality violations due to entanglement, we shall refer to it as a violation of the *reverse Bell inequalities.* In this paper, we show what is required to simulate classically a violation of the reverse Bell inequalities and how to saturate the superquantum bound $S = 4$. In so doing, we also identify the conditions under which such simulations are impossible. Thereby we establish when violations of the reverse Bell inequalities provide a test of genuine quantum phenomena, i.e. we prove a *reverse Bell's theorem.* We accordingly propose a modification of the experiment in [4] in order to implement such a test experimentally. The proposed experiment also adds a striking twist to the idea that entanglement is foliation-dependent (the relativity of entanglement), and can be adapted to provide a *simultaneous test* of both the standard and the reverse Bell's theorems.

BELL INEQUALITIES FOR BELL STATES

We begin by summarising a few facts about Bell inequality violations. For a pair of qubits consider the local observables

$$A_i = (P_{\alpha_i} - P_{\alpha_i}^{\perp}) \otimes \mathbb{1},\ B_j = \mathbb{1} \otimes (P_{\beta_j} - P_{\beta_j}^{\perp}) \tag{1}$$

with

$$P_\varphi = \begin{pmatrix} \cos^2\varphi & \cos\varphi\sin\varphi \\ \cos\varphi\sin\varphi & \sin^2\varphi \end{pmatrix}. \tag{2}$$

Next, consider the four CHSH inequalities

$$\begin{aligned} S_1^\psi &= \left| E_{1,1}^\psi + E_{1,2}^\psi + E_{2,1}^\psi - E_{2,2}^\psi \right| \leq 2\,, \\ S_2^\psi &= \left| E_{1,1}^\psi + E_{1,2}^\psi - E_{2,1}^\psi + E_{2,2}^\psi \right| \leq 2\,, \\ S_3^\psi &= \left| E_{1,1}^\psi - E_{1,2}^\psi + E_{2,1}^\psi + E_{2,2}^\psi \right| \leq 2\,, \\ S_4^\psi &= \left| -E_{1,1}^\psi + E_{1,2}^\psi + E_{2,1}^\psi + E_{2,2}^\psi \right| \leq 2\,, \end{aligned} \tag{3}$$

with

$$E_{i,j}^\psi = p^\psi\left(A_i = B_j\right) - p^\psi\left(A_i \neq B_j\right). \tag{4}$$

Maximal violations of these inequalities can be obtained with each of the Bell states

$$\begin{aligned} \left|\Phi^+\right\rangle &= \tfrac{1}{\sqrt{2}}\left(\left|\uparrow\uparrow\right\rangle + \left|\downarrow\downarrow\right\rangle\right), \\ \left|\Psi^+\right\rangle &= \tfrac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right), \\ \left|\Phi^-\right\rangle &= \tfrac{1}{\sqrt{2}}\left(\left|\uparrow\uparrow\right\rangle - \left|\downarrow\downarrow\right\rangle\right), \\ \left|\Psi^-\right\rangle &= \tfrac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right), \end{aligned} \tag{5}$$

which yield the following probabilities:

$$\begin{aligned} p_{i,j}^{\Phi^+} &= \cos^2(\alpha_i - \beta_j)\,, \quad p_{i,j}^{\Psi^+} = \sin^2(\alpha_i + \beta_j)\,, \\ p_{i,j}^{\Phi^-} &= \cos^2(\alpha_i + \beta_j)\,, \quad p_{i,j}^{\Psi^-} = \sin^2(\alpha_i - \beta_j)\,, \end{aligned} \tag{6}$$

where $p_{i,j}^\psi = p^\psi(A_i = B_j)$. At most one CHSH inequality can be violated for each Bell state. For sake of definiteness, we fix the spin directions

$$\alpha_1 = 0,\ \alpha_2 = \tfrac{\pi}{4},\ \beta_1 = \tfrac{\pi}{8},\ \beta_2 = -\tfrac{\pi}{8}\,, \tag{7}$$

which give us the violations

$$S_1^{\Phi^+} = S_2^{\Psi^+} = S_2^{\Phi^-} = S_1^{\Psi^-} = 2\sqrt{2}\,. \tag{8}$$

For the other combinations we have $S_i^\psi = 0$.

From (6), one easily shows that all CHSH inequalities are satisfied by equal mixtures of any two Bell states, in particular by the perfectly correlated or anticorrelated mixtures

$$\begin{aligned} \tfrac{1}{2}P^\Phi &= \tfrac{1}{2}\left(\left|\Phi^+\right\rangle\left\langle\Phi^+\right| + \left|\Phi^-\right\rangle\left\langle\Phi^-\right|\right), \\ \tfrac{1}{2}P^\Psi &= \tfrac{1}{2}\left(\left|\Psi^+\right\rangle\left\langle\Psi^+\right| + \left|\Psi^-\right\rangle\left\langle\Psi^-\right|\right). \end{aligned} \tag{9}$$

One also sees that $E^{\Phi^+}_{i,j} = -E^{\Psi^-}_{i,j}$ and $E^{\Phi^-}_{i,j} = -E^{\Psi^+}_{i,j}$. Thus, $|\Phi^+\rangle$ and $|\Psi^-\rangle$ always violate the same CHSH inequalities, and all S_i vanish for equal mixtures of these states. The same is true for $|\Phi^-\rangle$ and $|\Psi^+\rangle$. Finally, an equal mixture of all four Bell states is the maximally mixed state, since the Bell states form an orthonormal basis for two qubits.

Thus, if Vicky prepares an equal mixture of the Bell states (5), and sends the particles to Alice and Bob to perform measurements along the directions (7), each of the four subensembles leads to a maximal violation of either S_1 or S_2, but the total ensemble exhibits no correlations whatsoever.

REVERSE BELL'S THEOREM

Quantum Case

Now we describe the violation of Bell inequalities using postselection. Let Alice and Bob independently prepare qubits. Alice prepares her qubit by measuring either A_1 or A_2 and Bob prepares his qubit by measuring either B_1 or B_2. Their qubits are then sent to Vicky who randomly structures them in pairs in such a way that each pair contains one qubit from Alice and one from Bob. For this ensemble of qubit pairs we have of course that the outcomes of Alice's and Bob's measurements satisfy $S_i = 0$ for all i.

Now Vicky performs on each pair a measurement along the basis (5). Based on the outcome of this measurement Vicky constructs four subensembles of pairs of qubits. For each of these subensembles, the outcomes of Alice's and Bob's measurement do violate one of the CHSH inequalities. This follows simply because of the symmetry of transition probabilities; thus violation of the reverse Bell inequalities is mathematically equivalent to violation of the standard Bell inequalities.

In the standard case, however, we know that under certain locality conditions there is no classical system that can reproduce the quantum predictions. And thus experimental tests rule out classical explanations. We now establish under which conditions the reverse Bell inequalities can or cannot be violated classically.

Classical Simulation

Our task is to subdivide an ensemble that *does not violate the Bell inequalities* into four subsensembles (with the same marginals) that do. Suppose Alice chooses to flip either a U.S. quarter dollar or a Japanese 100 yen piece, and Bob to flip either a 50 euro cents or a British 10 pence [6]. They will get pairs of results with the following distributions (with '=' for two heads or two tails, and '≠' for one head and one tail, and a, b, c, d the proportions in which the four combinations of coins are flipped):

	\$€	\$£	¥€	¥£
$=$	$\frac{a}{2}$	$\frac{b}{2}$	$\frac{c}{2}$	$\frac{d}{2}$
$\neq$	$\frac{a}{2}$	$\frac{b}{2}$	$\frac{c}{2}$	$\frac{d}{2}$

(10)

Now let Vicky take, say, the top-left subsensemble in (10) and subdivide it at random into four subensembles in the following proportions:

$$\frac{a}{4}p^{\Phi^+}_{1,1} + \frac{a}{4}p^{\Psi^+}_{1,1} + \frac{a}{4}p^{\Phi^-}_{1,1} + \frac{a}{4}p^{\Psi^-}_{1,1} = \frac{a}{2}\,, \tag{11}$$

and similarly with all other boxes in (10), collecting the resulting pairs together in the following four subensembles:

	\$€	\$£	¥€	¥£
$=$	$\frac{a}{4}p^{\Phi^+}_{1,1}$	$\frac{b}{4}p^{\Phi^+}_{1,2}$	$\frac{c}{4}p^{\Phi^+}_{2,1}$	$\frac{d}{4}p^{\Phi^+}_{2,2}$
$\neq$	$\frac{a}{4}q^{\Phi^+}_{1,1}$	$\frac{b}{4}q^{\Phi^+}_{1,2}$	$\frac{c}{4}q^{\Phi^+}_{2,1}$	$\frac{d}{4}q^{\Phi^+}_{2,2}$

(I)

	\$€	\$£	¥€	¥£
$=$	$\frac{a}{4}p^{\Psi^+}_{1,1}$	$\frac{b}{4}p^{\Psi^+}_{1,2}$	$\frac{c}{4}p^{\Psi^+}_{2,1}$	$\frac{d}{4}p^{\Psi^+}_{2,2}$
$\neq$	$\frac{a}{4}q^{\Psi^+}_{1,1}$	$\frac{b}{4}q^{\Psi^+}_{1,2}$	$\frac{c}{4}q^{\Psi^+}_{2,1}$	$\frac{d}{4}q^{\Psi^+}_{2,2}$

(II)

	\$€	\$£	¥€	¥£
$=$	$\frac{a}{4}p^{\Phi^-}_{1,1}$	$\frac{b}{4}p^{\Phi^-}_{1,2}$	$\frac{c}{4}p^{\Phi^-}_{2,1}$	$\frac{d}{4}p^{\Phi^-}_{2,2}$
$\neq$	$\frac{a}{4}q^{\Phi^-}_{1,1}$	$\frac{b}{4}q^{\Phi^-}_{1,2}$	$\frac{c}{4}q^{\Phi^-}_{2,1}$	$\frac{d}{4}q^{\Phi^-}_{2,2}$

(III)

	\$€	\$£	¥€	¥£
$=$	$\frac{a}{4}p^{\Psi^-}_{1,1}$	$\frac{b}{4}p^{\Psi^-}_{1,2}$	$\frac{c}{4}p^{\Psi^-}_{2,1}$	$\frac{d}{4}p^{\Psi^-}_{2,2}$
$\neq$	$\frac{a}{4}q^{\Psi^-}_{1,1}$	$\frac{b}{4}q^{\Psi^-}_{1,2}$	$\frac{c}{4}q^{\Psi^-}_{2,1}$	$\frac{d}{4}q^{\Psi^-}_{2,2}$

(IV)

where $q^{\psi}_{i,j} = 1 - p^{\psi}_{i,j}$. Note that these are just theoretical numbers derived from quantum mechanics to determine the size of purely classical subensembles. But these postselected subensembles of pairs of classical coins reproduce *exactly the same* maximal violations of the Bell inequalities as the postselected quantum ensembles.

In fact, Vicky can do even better, and select instead subensembles of the form:

	\$€	\$£	¥€	¥£
$=$	$\frac{a}{4}$	$\frac{b}{4}$	$\frac{c}{4}$	0
$\neq$	0	0	0	$\frac{d}{4}$

	\$€	\$£	¥€	¥£
$=$	$\frac{a}{4}$	$\frac{b}{4}$	0	$\frac{d}{4}$
$\neq$	0	0	$\frac{c}{4}$	0

(I′–II′)

	\$€	\$£	¥€	¥£
$=$	0	0	$\frac{c}{4}$	0
$\neq$	$\frac{a}{4}$	$\frac{b}{4}$	0	$\frac{d}{4}$

	\$€	\$£	¥€	¥£
$=$	0	0	0	$\frac{d}{4}$
$\neq$	$\frac{a}{4}$	$\frac{b}{4}$	$\frac{c}{4}$	0

(III′–IV′)

These subensembles now violate the same Bell inequalities with $S_i = 4$. Thus, it is possible not only to simulate the quantum violations using classical postselection, but even superquantum violations.

Discriminating the Quantum from the Classical Case

In the classical simulation we made use of the fact that Vicky has full information about the eight subensembles. In particular, for each coin pair, it is known which coins were flipped by Alice and Bob. This information is not required in the quantum case. There Vicky only uses the outcome of the Bell measurement to construct the subensembles.

If, in the classical case, we restrict what Vicky can know about Alice and Bob's coin flips, suddenly the possibility of violating the Bell inequalities disappears. Indeed, with no information about which coins have been flipped, Vicky will at most be able to postselect so as to fix arbitrarily the value of any *single* $E_{i,j}$. But for any such subensemble,

$$E_{1,1} = E_{1,2} = E_{2,1} = E_{2,2}\,. \tag{12}$$

Thus

$$S_1 = S_2 = S_3 = S_4 = 2|E_{i,j}| \leq 2\,, \tag{13}$$

and the reverse Bell inequalities provide a limit to the correlations that can be simulated through classical postselection.

With this restriction we *can* discriminate between the classical and quantum case. What is being used is that quantum mechanical equal mixtures of spin-up and spin-down in different directions are indistinguishable. Assume that Alice and Bob do not prepare classically distinguishable mixtures of pairs of coins, but quantum mechanical mixed states. In this case, if Vicky merely performs a measurement of the classical correlations (via a nonmaximal measurement of the projections P^Φ, P^Ψ in (9)), the reverse Bell inequalities cannot be violated. But they *will* be violated if Vicky postselects based on the results of a complete measurement of the four Bell states (5). Thus, we have proved a *reverse Bell's theorem.*

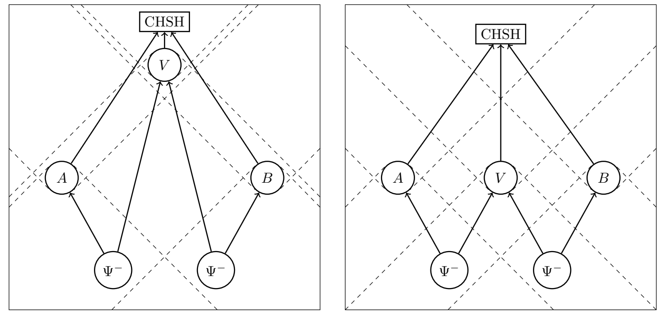

FIG. 1. Spacetime diagram for delayed-choice entanglement swapping (left), and for our proposed experiment (right).

EXPERIMENTAL TEST

A violation of reverse Bell inequalities has already been established experimentally by Ma *et al.* [4]. The protocol used there is somewhat different from the one we described earlier. It starts again with Alice and Bob performing measurements on qubits. Each qubit is now part of a pair prepared in the state $|\Psi^-\rangle$. The two other qubits in each pair are sent to Vicky.

Alice and Bob perform their local experiments on their qubits. Initially, their results will be completely uncorrelated. However, at an arbitrary point in the future, Vicky can decide to perform a measurement in the Bell basis (5). The outcomes of this measurement can then be used to postselect subensembles for which the measurement results of Alice and Bob become correlated and violate a CHSH inequality. This is delayed-choice entanglement swapping [3]. When Vicky's measurement is timelike separated from both Alice's and Bob's measurements as in Figure 1 (left), the explanation of the CHSH violation is unambiguously due to postselection.

Although the experiment by Ma *et al.* [4] was accordingly set up to ensure timelike separation between Vicky's and Alice and Bob's measurements, it is precisely this feature that provides a loophole for a classical explanation of the results. Although Vicky does not explicitly use the total information about Alice's and Bob's measurements required for a classical simulation, this information could in principle be available along with Vicky's qubit pair, and thus could causally influence the outcomes of Vicky's measurements. In order to ensure that information about Alice's and Bob's settings is unavailable, while it is not necessary that Alice and Bob are at spacelike separation from each other, we need to make sure that their measurements are at spacelike separation from

Vicky's, as in Figure 1 (right). We thus propose this as a modification of the Ma *et al.* experiment. With the EPR and Bell measurements at spacelike separation from each other, classical postselection at Vicky's site cannot reproduce a violation of the reverse Bell inequalities. Thus we have an *experimental test* of the reverse Bell's theorem.

RELATIVITY OF PRE- AND POSTSELECTION

In quantum theory, spacelike separated measurements commute. This is the basis for what Shimony has called the 'peaceful coexistence' of quantum theory and relativity [7, 8]. But there is of course a tension with the idea that quantum state collapse occurs instantaneously across space. A proposal to resolve this tension is to embrace the idea that quantum states are defined on spacelike hyperplanes or hypersurfaces and encode the probabilities for results of measurements to the future of the given hyperplane or hypersurface conditional on results of measurements to its past [9–13]. Consequently, entanglement of distant particles becomes a foliation-dependent notion: while the probabilities for Alice's and Bob's results are invariant, whether a qubit pair is entangled when Alice performs a measurement depends on the time order between their measurements. To capture this phenomenon, Myrvold [12, 13] has coined the term 'relativity of entanglement'.

The experiment we propose adds a further dramatic touch to this idea. Namely, the same experiment can be alternatively described in two different ways: either as Vicky performing a series of Bell measurements on maximally mixed pairs prepared by Alice and Bob, or as Alice and Bob performing a series of EPR measurements on maximally entangled pairs prepared by Vicky. In other words, depending on the choice of foliation, Vicky's measurement acts as a preselection or a postselection. We now have *relativity of pre- and postselection* [14].

This suggests considering the case in which all three measurements are at spacelike separation from each other (as indeed in Figure 1 (right); note that in the Ma *et al.* experiment, Alice and Bob are already spacelike separated). By choosing an appropriate foliation, the same three measurements can be given *any arbitrary* time order. Thus in this scenario, not only does the choice of foliation affect whether Alice performs a measurement on an entangled qubit or not, it also affects with which other qubit it is entangled [15].

In fact, the experiment can now be seen both as a modification of the delayed-choice entanglement swapping by Ma *et al.*, and as a modification of the loophole-free Bell-EPR experiment by Hensen *et al.* [16], where Vicky's measurement is part of the preparation procedure of Alice's and Bob's qubits [17]. In this version, the experiment becomes a (loophole-free) *simultaneous test* of both the standard and the reverse Bell's theorem.

CONCLUSION

The quantum mechanical predictions are invariant under change of foliation, because measurements at spacelike separation commute. Because of the relativity of pre- and postselection, instead, the difference between Bell inequality violations due to entanglement and due to postselection is no longer invariant. What in the case of timelike separation appear as physically different effects, in the case of spacelike separation turn out to be *one and the same physical effect.*

When in 1905 Einstein related two seemingly very different effects in the introduction to his 'On the electrodynamics of moving bodies' [18], it led to the unification of electric and magnetic fields as one single physical object. Perhaps the relativity of pre-and postselection in violations of the Bell inequalities is trying to tell us that the very notion of quantum state is in need of equally deep revision. This is indeed what Abner Shimony (1928–2015), to whose memory we wish to dedicate this paper, thought about the relativity of entanglement. As he eloquently put it [8]:

> [T]he two accounts of *processes* from initial to final sets of events are in disaccord. But it is important to note that the process is a theoretical construction. [...] The thesis of peaceful coexistence presupposes a conceptually coherent reconciliation of the descriptions from the standpoints of [the frames] Σ and Σ'. Even more desirable, in the spirit of the geometrical formulation of space-time theory, would be a coordinate-free account.

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[14] Since any Bell measurement consists of entangling a pair of particles with an ancilla and performing a measurement on the latter, already Cohen [2] points out that the difference between pre- and postselection can be reduced to the timing of the measurment on the ancilla. But he cashes this out differently: postselection for him reflects the 'counterfactual entanglement' that would have existed had we preselected instead.

[15] It is also possible to choose the foliation such that the creation of Alice's qubit occurs after Bob's measurement. The scenario in which these events are instead timelike separated was realized experimentally in [19].

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