Cut-off points for the rational believer

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Abstract. I show that the Lottery Paradox is just a (probabilistic) Sorites, and argue that this should modify our way of looking at the Paradox itself. In particular, I focus on what I call "the cut-off point problem" and contend that this problem, well known by students of the Sorites, ought to play a key role in the debate on Kyburg's puzzle.

Very briefly, I show that, in the Lottery Paradox, the premises "ticket n°1 will lose", "ticket n°2 will lose"… "ticket n°1000 will lose" are logically equivalent to soritical premises of the form "buying n tickets does not allow me to win the lottery ⊃ buying $n + 1$ tickets does not allow me to win the lottery" (where ⊃ is the material conditional). As a result, failing to believe, for some ticket, that it will lose comes down to introducing a cut-off point in a chain of soritical premises. I call the view that, for some ticket, we should not believe that it loses the "the cut-off point view".

One important consequence of this reformulation of the Lottery Paradox is that the most popular solution to the puzzle, i.e. denying the *Lockean Thesis*, becomes less attractive. The reason is that keeping *Belief Closure* entails the (rather counterintuitive) cut-off point view. In order to make the counterintuitive character of this view emerge as clearly as possible I consider a heap variant of the original lottery scenario: in this scenario (which is generally used in the context of a different puzzle, viz. the Sorites) the worrying consequences of the cut-off point view become evident.

Finally, I demonstrate that denying *Belief Closure* is not enough. More precisely, it is not enough to solve a puzzle which is closely related to Kyburg's and which puts us before the following dilemma: we should either accept the cut-off point view or reject classical-logic modus ponens. That is, not merely *Belief Closure*, but a fundamental principle of classical logic.

1. The lottery scenario

In the literature on rational belief and rational degrees of belief it is usually claimed that the two following principles cannot be jointly satisfied:

Belief Closure. Rational belief is closed under classical logic.

Lockean Thesis. If and only if, given her evidence, p is very probable (where "very probable" means "equal to or higher than a specified threshold value t"), then the agent should believe p.

Indeed, joint acceptance of *Belief Closure* and the *Lockean Thesis* gives rise to the well-known Lottery Paradox, first proposed by Kyburg (1961).

Consider a fair 1000-ticket lottery with exactly one winner. The probability, for each ticket, that it will win is very low, i.e. it is 0.001. It follows that if $t = 0.999$, then, by the *Lockean Thesis*, one should believe, of each ticket, that it will lose. By multiple applications of *Belief Closure*, one should also believe the conjunction "ticket n°1 will lose & ticket n°2 will lose . . . & ticket n°1000 will lose". However, given that the lottery is fair and has exactly one winner, the negation of "ticket n°1 will lose & ticket n°2 will lose . . . & ticket n°1000 will lose" has a probability of 1; therefore, by the *Lockean Thesis*, one should believe it. So one should believe both "ticket n°1 will lose & ticket n°2 will lose . . . & ticket n°1000 will lose" and its negation. As it is generally accepted that one should not believe two pairwise inconsistent sentences, we conclude that *Belief Closure* and the *Lockean Thesis* are incompatible.

As Leitgeb (2014) puts it, we can classify a huge part of the classical literature on rational belief according to which principle is dropped: for instance, Isaac Levi (1967) accepts *Belief Closure* but rejects the *Lockean Thesis*, while Henry Kyburg (1961) accepts the *Lockean Thesis* and rejects *Belief Closure*.

The most widespread solution to the puzzle, however, consists in denying the *Lockean Thesis*: among the authors who adopt this option are Lehrer (1975; 1990), Kaplan (1981a; 1981b; 1996), Stalnaker (1984), Pollock (1995), Ryan (1996), Evnine (1999), Nelkin (2000), Adler (2002), Douven (2002), Smith (2010; 2016; 2018), and Kelp (2017).

Philosophers who believe, instead, that the Lottery Paradox puts pressure on *Belief Closure* include Klein (1985), Foley (1992), Hawthorne and Bovens (1999), Kyburg and Teng (2001), Christensen (2004), Hawthorne and Makinson (2007), Kolodny

(2007), Easwaran and Fitelson (2015).

One notable exception to this categorization (deniers of the *Lockean Thesis* vs. deniers of *Belief Closure*) is Leitgeb's view. Indeed, Leitgeb (2014; 2015) defends a form of contextualism which, he contends, allows us to keep both the *Lockean Thesis* and *Belief Closure*. I will come back to his proposal in the last section of the paper.

In this paper, I show that the Lottery Paradox is just a (probabilistic) Sorites, and argue that this should modify our way of looking at the Paradox itself. In particular, I focus on what I will call "the cut-off point problem" and contend that this problem, well known by students of the Sorites, ought to play a central role in the debate on Kyburg's puzzle.

Very briefly, I will show that, in the Lottery Paradox, the premises "ticket n°1 will lose", "ticket n°2 will lose"…"ticket n°1000 will lose" are logically equivalent to soritical premises of the form "buying n tickets does not allow me to win the lottery ⊃ buying $n + 1$ tickets does not allow me to win the lottery" (where ⊃ is the material conditional). As a result, failing to believe, for some ticket, that it will lose comes down to introducing a cut-off point in a chain of soritical premises. I call the view that, for some ticket, we should not believe that it loses the "the cut-off point view".

One important consequence of this reformulation of the Lottery Paradox is that the most popular solution to the puzzle, i.e. denying the *Lockean Thesis*, becomes less attractive. The reason for this is that keeping *Belief Closure* entails the (rather counterintuitive) cut-off point view. In order to make the counterintuitive character of this view emerge as clearly as possible I will consider a heap variant of the original lottery scenario: in this scenario (which is generally used in the context of a different puzzle, viz. the Sorites) the worrying consequences of the cut-off point view become evident.

Finally, I demonstrate that denying *Belief Closure* is not enough. More precisely, it is not enough to solve a puzzle which is closely related to Kyburg's and which puts us before the following dilemma: we should either accept the cut-off point view[1] or reject classical-logic modus ponens. That is, not merely *Belief Closure*, but a fundamental principle of classical logic.

The next section will be devoted to reformulating Kyburg's original puzzle so that its connection with the Sorites becomes obvious. However, before that, one more preliminary remark is needed. In presenting the Lottery Paradox, I used the expression *Belief Closure*. I could have been more specific, though: in Kyburg's puzzle a specific principle is applied, i.e. the closure of rational belief under conjunction introduction. From now on, I will call such a principle "conjunction introduction*", in order to distinguish this epistemic version of the conjunction introduction schema from its non-epistemic, classical-logic counterpart. I will do the same for the other logical principles: e.g. when referring to the closure of rational belief under classical-logic modus ponens and modus tollens, I will use the labels "modus ponens*" and "modus tollens*" respectively.

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[1] As it will become clear in what follows, in this further puzzle the expression "cut-off point" refers to something a bit different from what it denotes in Kyburg's original puzzle. Specifically, in this additional puzzle we have a cut-off point if and only if one in a series of soritical conditionals of the form "I should believe that buying n tickets does not allow me to win the lottery $\supset$ I should believe that buying $n + 1$ tickets does not allow me to win the lottery" is false.

2. The Wide Scope Paradox

Consider again our fair 1000-ticket lottery, with $t = 0.999$. Also assume that tickets are numbered from 1 to 1000. By the *Lockean Thesis*, in this scenario one should believe, for instance, "ticket n°1 wins ∨ ticket n°2 wins…∨ ticket n°999 wins", as its probability is 0.999. That is, one should believe that the set which includes tickets from n°1 to n°999 contains the winning ticket. At the same time, one should not believe "ticket n°1 wins ∨ ticket n°2 wins…∨ ticket n°998 wins", as this sentence only has a probability of 0.998. That is, we should suspend our judgement on whether the winning ticket is to be found between ticket n°1 and ticket n°998 (included).

From now on, instead of saying that one should (or should not) believe the sentence "ticket n°1 wins ∨ ticket n°2 wins… ∨ ticket n°*n* wins" I will say that one should (or should not) believe "buying *n* tickets allows me to win the lottery". That is, I will assume that the probability of "buying *n* tickets allows me to win the lottery" equals the probability of "ticket n°1 wins ∨ ticket n°2 wins…∨ ticket n°*n* wins". Of course, as a matter of fact someone who buys 999 tickets could buy the tickets from n°2 to n°1000 and not those from n°1 to n°999. This simplification will not affect my point, though, and it will make my presentation smoother[2].

Let me now introduce another logical equivalence that will be of much help in what follows. "Ticket n°1 will lose" (that is, given the above, "buying 1 ticket does not allow

[2]Instead of "buying *n* tickets allows me to win the lottery" I could have used a different formulation, i.e. for instance, "someone who buys *n* tickets wins the lottery". The only reason why in what follows I will not adopt this alternative formulation (or a still different one) is that the use, in the former, of the first person makes things easier to formulate.

me to win”) is just equivalent to a negated conjunction, i.e. to the negation of “buying 0 tickets does not allow me to win & buying 1 ticket allows me to win”. That is, “ticket n°1 will lose” (or “buying 1 ticket does not allow me to win”) is equivalent to “buying 0 tickets does not allow me to win ⊃ buying 1 ticket does not allow me to win”. More generally, it can be noted that “ticket n°*n*+1 will lose” = “~(buying *n* tickets does not allow me to win & buying *n* + 1 tickets allows me to win)” = “buying *n* tickets does not allow me to win ⊃ buying *n* + 1 tickets does not allow me to win” = “~(the winning ticket is not among the tickets from 1 to *n* included & it is among the tickets from 1 to *n* + 1 included)”, where ~ is the negation symbol.

What is the point of introducing these equivalences? First, these equivalences can be used to show that starting from a lottery scenario we can generate a (probabilistic) Sorites. I will call this argument “Wide Scope Paradox” (WSP), in order to distinguish it from a related argument that I will label “Narrow Scope Paradox”, and that I present in section 5. I have called it “WSP” because in it the rational belief operator has wide scope over the conditional premises; in the Narrow Scope Paradox instead, as we will see, the belief operator has narrow scope over the antecedent and the consequent of the same conditionals.

Consider the following sentences. Remember that we have set $t = 0.999$.

(1) Buying 1000 tickets allows me to win the lottery.

(2) Buying 0 tickets does not allow me to win the lottery.

Also consider 1000 sentences of the form:

Buying *n* tickets does not allow me to win the lottery ⊃ buying *n* + 1 tickets does not allow me to win the lottery.

For convenience, I will call the above conditionals "P-conditionals". The paradox consists in the fact that by multiple applications of modus ponens* we conclude (3), which contradicts (1):

(3) Buying 1000 tickets does not allow me to win the lottery. (!)

Note that we are bound to believe all the premises. Indeed, (1) and (2) both have a probability of 1, and each of the P-conditionals has a probability of 0.999. Why 0.999? Consider, for instance, the conditional "buying 499 tickets does not allow me to win ⊃ buying 500 tickets does not allow me to win": it is equivalent to "buying 499 tickets allows me to win ∨ buying 500 tickets does not allow me to win". The reason why this disjunction has a probability of 0.999 is that its probability is calculated by adding the probability of "buying 499 tickets allows me to win", i.e. 0.499, to the probability of "buying 500 tickets does not allow me to win", i.e. 0.5; the probabilities must be added here because the sentences we are dealing with are mutually inconsistent. Another way of reaching the same result is by focusing on what would make the P-conditionals false: in order to falsify one of them, it is both necessary and sufficient that the winning ticket is exactly ticket n°*n*+1, which has a probability of 0.001.

So one should believe (1), (2) and each of the conditional premises; nonetheless, one should not believe (3), which has a probability of 0, whence the problematic outcome: by modus ponens*, we should believe two pairwise inconsistent sentences, i.e. (1) and its negation.

Now, it is not simply the case that we can generate a (probabilistic) Sorites starting from the lottery scenario. Actually, the premises of the original puzzle by Kyburg and those of WSP are equivalent.

Recall the logical equivalences I have introduced. It can be noted that (1) is equivalent to the sentence that, in the original version of the puzzle, says that the 1000-ticket lottery is fair and has one winner (i.e. to the disjunction "ticket n°1 wins ∨ ticket n°2 wins…∨ ticket n°1000 wins"). (2) corresponds, instead, to a premise which is left implicit in the original argument, i.e. to a premise which is trivially true in the scenario and which says that the lottery is not a 0-ticket lottery which has a winner. Finally, "buying n tickets does not allow me to win the lottery ⊃ buying $n + 1$ tickets does not allow me to win the lottery" is equivalent to "ticket n°n+1 will lose"; that is, the P-conditionals are equivalent to the premises of Kyburg's original argument that say that ticket n°1 will lose, ticket n°2 will lose, etc.

So WSP is just a reformulation of the standard Lottery Paradox[3]. However, one clear difference between Kyburg's original argument and WSP is that in the original

[3]This strengthens a point by Dorothy Edgington (1992; 1997). Indeed, Edgington claims that the Lottery Paradox and the Sorites are structurally similar, so that a common strategy should be applied to solve both. However, unlike mine, her view presupposes the acceptance of a degree-theoretic framework (i.e. the idea that there are such things as degrees of truth). Moreover, and most importantly, my claim is

argument conjunction introduction* is used, whereas in WSP modus ponens* is applied. Now, it turns out that it is possible to reformulate WSP so that conjunction introduction* is used.

Consider the conjunction “buying 0 tickets does not allow me to win & buying 1000 tickets allows me to win”; call it (C). It is equivalent to (D):

(D): (buying 0 tickets does not allow me to win & buying 1 ticket allows me to win) ∨ (buying 1 ticket does not allow me to win & buying 2 tickets allows me to win)… ∨ (buying 999 tickets does not allow me to win & buying 1000 tickets allows me to win).

As we have seen, given t = 0.999, one should believe both (1) and (2). So by conjunction introduction* one should believe (C); but then one should believe (D). However, each disjunct of (D) only has a probability of 0.001, so one should believe its negation. Now, by applying conjunction introduction* to ~(buying 0 tickets does not allow me to win & buying 1 ticket allows me to win), ~(buying 1 ticket does not allow me to win & buying 2 tickets allows me to win)… ~(buying 999 tickets does not allow me to win & buying 1000 tickets allows me to win), one should believe the negation of (D). That is, by conjunction introduction*, one should believe both a sentence and its negation, exactly as in Kyburg’s original version of the puzzle.

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stronger than Edgington’s: the Lottery Paradox is not merely similar to the Sorites Paradox; the Lottery Paradox just *is* a (probabilistic) Sorites.

It could be asked what is the point of introducing this reformulation of the Lottery Paradox, i.e. of introducing WSP. Actually, I think that WSP is interesting in itself, as it shows that the Lottery Paradox just is a (probabilistic) Sorites. However, this is not all there is to WSP. The main reasons why in what follows I will use WSP instead of the original formulation of the Paradox are matters of clarity for my current purposes. Indeed, using WSP makes it more natural to rerun the Lottery Paradox starting from a heap scenario; as a result, the advantages of switching to a heap scenario are more apparent and the main point of the paper will emerge more clearly.

3. Setting the threshold at 1

If we reject the *Lockean Thesis* the Lottery Paradox is blocked. However, it is traditionally assumed that accepting $t = 1$ allows us to keep both the *Lockean Thesis* and *Belief Closure* (among the authors who argue that we should accept $t = 1$ are van Fraassen (1995), Arló-Costa (2001), Arló-Costa and Parikh (2005)). One main consequence of this solution is that we are forced to accept the cut-off point view, i.e. as specified above, we are bound to disbelieve[4], for some ticket, that it will lose (in the original scenario, for *each* ticket, we are bound to disbelieve that it will lose). In order to better evaluate the consequences of this view, let us put aside for a moment the standard lottery scenario and use instead a classical example from the literature on vagueness, i.e. the heap example.

Note that replacing the lottery scenario with a different scenario (and more specifically, a heap scenario) is a perfectly legitimate move. Indeed, the Lottery Paradox

[4]Throughout this paper, by “disbelieving *p*” I mean “not believing *p*”, which includes suspending one’s judgement on *p* and believing the negation of *p*.

is not a puzzle about lotteries. On the contrary, the point of the Paradox is a general one, and consists in showing that the *Lockean Thesis* (with t short of 1) and *Belief Closure* are incompatible. Moreover, we are perfectly allowed to assign probabilities to the premises of the Sorites, based on our evidence[5].

For reminder, here is the classical, textbook version of the Sorites Paradox.

Consider (1') and (2').

(1') 1000 grains are a heap[6].

(2') 0 grains are not a heap.

Also consider 1000 sentences of the form:

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[5]Clearly, this way of looking at the Lottery Paradox is at odds with those accounts of the latter which argue that we should deny the *Lockean Thesis* because evidence which is "merely probabilistic" is not enough for rational belief (see, for instance, Nelkin 2000, or Smith 2010, 2016 and 2018). These accounts usually focus on the original version of the Paradox or, when they consider variants of it, they focus on cases in which the relevant evidence is statistical.

However, it can be noted that the formulation I have given of the *Lockean Thesis* leaves it open whether the kind of evidence the agent relies on is "merely probabilistic" or not. That is, in my formulation of the *Lockean Thesis*, which I take to be standard, evidence need not be "merely probabilistic". A consequence of this fact is that, as I say in the body of the paper, we should regard the original version of the Lottery Paradox as simply *illustrating* the conflict between the *Lockean Thesis* and *Belief Closure*. This is an important point, as it seems clear that the conflict does not vanish if we consider evidence which is not "merely probabilistic". In other terms, reducing the Lottery Paradox to a problem concerning statistical evidence alone does not do justice to the challenge it illustrates, which is much more general.

[6]Here "1000" could be replaced with any sufficiently high number. This of course holds for the Lottery Paradox too: instead of a 1000-ticket lottery we could consider a 5000-ticket lottery, or a 1 million-ticket lottery, etc.

n grains are not a heap ⊃ $n + 1$ grains are not a heap

Let us call these sentences "P'-conditionals". (1'), (2') and the P'-conditionals seem true. However, multiple applications of *classical-logic* modus ponens let us infer the following puzzling conclusion:

(3') 1000 grains are not a heap. (!)

I would like to stress that (1')-(3') is about truth, not rational belief: as students of the Sorites classically put it, the argument's premises are intuitively true whereas the conclusion is intuitively false. However, it is easy to transform "the classical Sorites" into a version of WSP: as it will become clear below, one only needs to assign probabilities to the premises of (1')-(3'), set an appropriate threshold for belief, and apply modus ponens* instead of modus ponens.

So how should we assign probabilities to the Sorites' premises? Keep in mind that we are concerned here with the probabilities a rational agent assigns to sentences based on the relevant evidence. Now, given that we know both that 1000 grains are a heap and that 0 grains are not a heap, it seems that we should assign a probability of 1 to (1') and (2') respectively. What about the P'-conditionals? Clearly, we cannot assign to all of them a probability of 1. The reason is simple: consider the conjunction "0 grains are not a heap & 1000 grains are a heap", which I will call (C'). It is equivalent to (D'):

(D') (0 grains are not a heap & 1 grain is a heap) ∨ (1 grain is not a heap & 2 grains are a heap)… ∨ (999 grains are not a heap & 1000 grains are a heap).

Given that she knows that (C'), a rational agent should assign to (C') a probability of 1. But then she should also assign a probability of 1 to (D'); i.e. the sum of the probabilities of the disjuncts of (D') must be 1. Therefore, it cannot be the case that all the P'-conditionals (which are just the negations of the disjuncts of (D')) have a probability of 1. That is, it cannot be the case that all the disjuncts of (D') have a probability of 0. Still, each of them can be assigned a very low probability. If we do so (i.e. if we assign each of them a very low probability), WSP can be formulated starting from a heap scenario. I will call this alternative formulation "Soritical Wide Scope Paradox" (SWSP). It goes as follows: assume that t is very high, but short of 1. By the *Lockean Thesis*, we should believe (1'), (2'), and each of the P'-conditionals. However, again by the *Lockean Thesis*, we should not believe (3'), which has a probability of 0; in fact, we should believe its negation, which is just (1'). That is, we end up having to believe both (1') and its negation. (Exactly as for WSP, we can provide a version of SWSP in which conjunction introduction* is used, instead of modus ponens*. Indeed, we should believe (D') (which has a probability of 1), but if its disjuncts are assigned very low probabilities, we should believe the negation of each of them, so that, by conjunction introduction*, we should believe the negation of (D'). That is, we should believe both (D') and its negation[7].)

One more precision is in order. We are supposing that the disjuncts of (D') are all assigned low probabilities. That is, the probability distribution we are considering is a

[7]For the sake of completeness, note that, both in the case of SWSP and of WSP, modus tollens* could also be used to generate the unacceptable conclusion. Indeed, as we have seen, by the *Lockean Thesis* we should believe (1') ((1) in WSP) and all the conditional premises. However, by multiple applications of modus tollens*, we should believe the negation of (2') (the negation of (2) in WSP), which has a probability of 0.

"uniform" one (one in which all the disjuncts have the same probability), or at least one in which all the disjuncts have a probability greater than 0. However, for the paradox to arise, we are not at all obliged to assign our probabilities this way. On the contrary, we can assume a probability distribution in which some disjuncts have a probability of 0 (in fact, as many as we wish, provided that the probabilities of the disjuncts in D' sum up to 1).

Here we come to a crucial point. Consider again SWSP. Exactly as WSP, we could block it either by dropping the *Lockean Thesis* or by dropping *Belief Closure*. However, as announced in the title of this section, we could be willing to keep both principles by setting $t = 1$. However, if we assume that $t = 1$ we are forced to conclude that, for at least one n, it is not the case that one should believe "n grains are not a heap ⊃ $n + 1$ grains are not a heap". Indeed, for at least one n, the probability of "n grains are not a heap & $n + 1$ grains are a heap" must be greater than 0 (otherwise the probability of (C') would be 0, whereas, by hypothesis, it is 1). (Of course, if there is only one n such that the probability of "n grains are not a heap & $n + 1$ grains are a heap" is greater than 0, then, given that the probability of (C') has to be 1, the probability of that disjunct must be 1.)

Clearly, if our evidence is distributed uniformly over the disjuncts of D' we should neither believe the conditional "0 grains are not a heap ⊃ 1 grain is not a heap", nor any of the other P'-conditionals. Conversely, if we have absolutely no evidence for some of the disjuncts (i.e. if we assign a zero probability to, say, the first twenty disjuncts), the cut-off point will come "later" in the distribution (e.g. we should believe "19 grains are not a heap ⊃ 20 grains are not a heap", but we should not believe "20 grains are not a heap ⊃ 21 grains are not a heap”). Anyway, what matters is that in both cases we are forced to disbelieve at least one of P’-conditionals.

So we have seen that the cut-off point view follows from the acceptance of $t = 1$. Now, it seems clear that in order to solve WSP one must also solve SWSP, which is a simple variant of WSP, in which a heap scenario is used instead of a lottery one.

Many authors already reject the cut-off point view for the original lottery scenario; notably, all those who defend the *Lockean Thesis* with t short of 1. Switching to a heap scenario raises an interesting problem for those who accept the *Lockean Thesis* with $t = 1$, as perhaps some of them will find the outcome that we should disbelieve at least one of the P’-conditionals unpalatable. Of course, “some” does not mean “all of them”. Still, the fact that if we set $t = 1$ we must accept the cut-off point view with respect to SWSP is something we should keep in mind when evaluating a solution to the Lottery Paradox, and this was the point I wanted to make in this section.

4. Rejecting the Lockean Thesis altogether

As we know, a possible way of solving WSP consists in accepting the *Lockean Thesis* with t short of 1 and rejecting *Belief Closure*. Another possible way out of the puzzle is keeping both the *Lockean Thesis* and *Belief Closure*, while setting t at 1. However, we have seen that the latter option forces us to adopt the cut-off point view with respect to (S)WSP.

Let us now turn to the third and last option, which consists in rejecting the *Lockean Thesis* across the board and accepting, instead, a different norm of belief. The alternative norms I will consider are the most popular competitors of the *Lockean Thesis*, i.e. the truth norm and the knowledge norm of belief. In this section, I show that accepting either of these alternative norms still forces us to endorse the cut-off point view with respect to (S)WSP.

The truth norm may be defined as the norm according to which we should believe *p* if and only if *p* is true. During its history, the truth norm has been precisified in various ways; however, the subtleties of the different definitions are not relevant here[8]. As far as WSP is concerned, this norm provides a clear verdict: the argument has one false premise. Indeed, there is a ticket (the winning one) of which we should not believe that it will lose. That is, one of the P-conditionals is false.

Regarding the knowledge norm, i.e. the norm according to which we should believe that *p* if and only if we know that *p*[9], it also provides a straightforward solution to WSP: we should believe both (1) and (2), as we know that if we buy all the tickets we will win, and that if we do not buy any ticket we will lose. However, we should disbelieve all the P-conditionals. This is because we do not know, of each ticket, that it will lose.

Now consider SWSP. If we accept the truth norm, there are only two ways out of the puzzle: one consists in embracing the cut-off point view, the other in denying *Belief*

[8]Among others, Wedgwood 2002, Boghossian 2003, Shah 2003, Gibbard 2005, Bykvist and Hattiangadi 2007, Engel 2007, and Thomson 2008 contain stimulating remarks on the way the truth norm should be made precise.

[9]The knowledge norm is adopted by a growing number of epistemologists; its most famous defender is Timothy Williamson (see Williamson 2000). Note, though, that in Williamson's work the defence of such a norm is only implicit and must be derived from the author's defence of the knowledge norm of assertion.

Closure[10]. The problem that faces the truth norm's advocate is the following: is there one grain such that when added to a collection of grains which is not a heap turns it into a heap? If the answer is yes, then, by the truth norm, there is one n such that we should not believe the conditional "n grains are not a heap ⊃ n + 1 grains are not a heap". If, instead, she believes that there is not such a n, she must reject *Belief Closure*.

Similar remarks hold for the knowledge norm's defender, even though the problem she faces is slightly different: is there one n such that we know that n grains are not a heap but we do not know that n + 1 grains are not a heap? Depending on her answer, the knowledge norm's advocate will be either endorsing the cut-off point view or denying *Belief Closure*.

However, we have seen that SWSP is an innocent variant of WSP, and that, as a result, we should give a unified answer to the two puzzles. This means that, given that both the truth norm's and the knowledge norm's advocates endorse the cut-off point view with respect to WSP, they should also endorse it with respect to SWSP. In other words, if we accept either the truth norm or the knowledge norm of belief, we are bound to accept the cut-off point conclusion with respect to SWSP.

Of course, rejecting the *Lockean Thesis* does not automatically entail that we should endorse either the truth norm or the knowledge norm. Indeed, among the most classical proposals concerning the Lottery Paradox is that of amending the *Lockean Thesis* by adding a defeat clause (Pollock 1995 is a good example of this kind of approach). Other

[10] Of course, the truth norm's advocate may also solve SWSP by claiming that (2') is false or that (3') is true. The claim according to which (3') is true has been defended in the literature on the Sorites by Peter Unger (1979). However, I will not deal with these very unpopular options here. (And anyway, as it will become clear below, these options are not relevant for my argument's purposes).

(more recent) accounts do not simply add to the *Lockean Thesis* a defeat condition, but propose an outright modification of the threshold constraint (see Lin and Kelly 2012a and 2012b. According to some criteria, Leitgeb's account (2014; 2015) can also be regarded as part of this category (see Staffel forthcoming)). However, these proposals also entail the cut-off point view. So, if I am right in claiming that the solution to WSP should be extended to SWSP, the advocates of these accounts should also endorse the cut-off point view with respect to SWSP[11]. More generally, given that disbelieving (1) or (2) does not seem to be an option, it appears that if we want to preserve *Belief Closure* we are forced to disbelieve at least one of the P-conditionals.

Let us take stock: we can keep *Belief Closure* only if we accept the cut-off point view as far as SWSP is concerned. Indeed, in order to provide a unified solution to WSP and SWSP we only have three options:

(i) accepting the *Lockean Thesis* with t short of 1, which implies rejecting *Belief Closure*.

(ii) accepting the *Lockean Thesis* with $t = 1$, which allows us to keep *Belief Closure*, but forces us to accept the cut-off point view.

(iii) rejecting the *Lockean Thesis* across the board, which also allows us to keep *Belief Closure*, but forces us to endorse the cut-off point view.

[11] Another option consists in denying the *Lockean Thesis* while adopting, at the same time, an eliminativist approach to the notion of full rational belief. That is, it consists in rejecting the whole framework in which the Lottery Paradox is formulated. According to this very radical approach, which I will put aside here, talk about full belief should be entirely replaced by talk about degrees of belief. For a discussion of this option see Foley 1992.

5. The Narrow Scope Paradox

In this last section I will show that, despite the appearances, option (i), i.e. accepting the *Lockean Thesis* with *t* short of 1, does not allow us to avoid the cut-off point conclusion (at least with respect to the argument I am going to present).

Consider (4) and (5).

(4) I should believe that buying 1000 tickets allows me to win the lottery.

(5) I should believe that buying 0 tickets does not allow me to win the lottery.

Also consider 1000 conditionals of the form "I should believe that buying *n* tickets does not allow me to win the lottery ⊃ I should believe that buying *n* + 1 tickets does not allow me to win the lottery".

Repeated applications of *classical-logic* modus ponens lead to (6):

(6) I should believe that buying 1000 tickets does not allow me to win the lottery. (!)

I will call this puzzle "Narrow Scope Paradox" (NSP). As explained above, the reason for this label is that in NSP the rational belief operator has narrow scope over the antecedent and the consequent of the P-conditionals (whereas in WSP it has wide scope over them).

Now, suppose that we adopt option (i), i.e. that we accept the *Lockean Thesis* with t short of 1: option (i) clearly implies that one of the conditionals in (4)-(6) is false. That is, here too, we have a cut-off point, even though of a different kind than in (S)WSP: what I mean by a cut-off point here is just that one of the conditionals of the form "I should believe that buying n tickets does not allow me to win the lottery ⊃ I should believe that buying $n + 1$ tickets does not allow me to win the lottery" is false (see footnote 1). For convenience, and in spite of the differences with (S)WSP, I will extend the use of the expression "cut-off point view" to the view that one of the conditionals in NSP is false.

Where the cut-off point falls of course depends on the value of t. If we assume, as above, that $t = 0.999$ and that the 1000-ticket lottery is fair, then the false premise will be "I should believe that buying 1 ticket does not allow me to win the lottery ⊃ I should believe that buying 2 tickets does not allow me to win the lottery". Indeed, by the *Lockean Thesis*, one should believe that buying 1 ticket does not allow her to win. However, one should not believe that buying 2 tickets does not allow her to win ("buying 2 tickets does not allow me to win" has a probability of 0.998). In other words, even if the defender of (i) manages to avoid the cut-off point conclusion in the case of WSP, she cannot avoid it in the case of NSP.

This remark can be extended to a heap variant of NSP. As it was already the case with WSP, we can generate NSP starting from a heap scenario instead of a lottery one, i.e. we can generate a Soritical Narrow Scope Paradox (SNSP). One just has to replace (4) with "I should believe that 1000 grains are a heap", (5) with "I should believe that 0 grains are not a heap" and the conditionals in (4)-(6) with sentences of the form "I should believe that n grains are not a heap ⊃ I should believe that $n + 1$ grains are not a heap". Finally, (6) must be replaced with "I should believe that 1000 grains are not a heap".

Here too (i.e. in the case of SNSP too) the advocate of (i) will be obliged to say that there is a cut-off point. Where this cut-off point is will again depend on the value of t and on the specific probability distribution associated with her evidence.

As announced in section 1, the above has some interesting consequences concerning Leitgeb's account of the Lottery Paradox. According to Leitgeb (2014; 2015), the context in which we ask ourselves whether a given ticket n wins and that in which we focus on the fact that some ticket will win (i.e. that the lottery is fair and has one winner) are different and allow us to set different thresholds for rational belief.

More specifically, in a context in which we focus on the fact that some ticket will win, Leitgeb's theory of belief constrains us to set $t = 1$ (and therefore to suspend our judgement on each of the tickets). Instead, a context in which we concentrate on whether a given ticket n will win is one in which we can set $t = 0.999$, and this will not cause the Lottery Paradox to arise, provided that we "partition" (i.e. that we subdivide) the probabilities in our distribution as imposed by the theory. According to Leitgeb, the Lottery Paradox results from fallaciously mixing premises that come from these different contexts.

However, imagine that instead of talking about tickets we were talking about grains: in the context in which $t = 1$ we would have to accept the cut-off point conclusion with respect to SWSP. In the context in which, instead, we ask ourselves whether some specific ticket will win (whether some specific grain turns something that is not a heap into a heap) and we are assuming $t = 0.999$, we would have to say that one of the conditionals in SNSP is false.

It is noteworthy that in (S)NSP the principle that is applied is not *Belief Closure*, but classical-logic modus ponens. Indeed, this is an important fact: until I only considered (S)WSP, the dilemma was between accepting the *Lockean Thesis* with *t* short of 1 on the one hand and accepting both *Belief Closure* and the cut-off point view on the other hand. Thanks to (S)NSP we are now aware that rejecting *Belief Closure* is not enough to avoid the cut-off point view (at least not with respect to this further puzzle): rejecting *classical-logic* modus ponens[12] is necessary. That is, not merely *Belief Closure*, but a fundamental principle of classical logic. The reason is that dropping *Belief Closure* would allow us to block (S)WSP, but not (S)NSP. Instead, giving up classical-logic modus ponens would solve both (S)WSP and (S)NSP: if classical-logic modus ponens is invalid, *Belief Closure* fails; however, the opposite direction of the conditional does

[12] Or rather, classical-logical modus ponens plus at least two other principles, i.e. classical-logic conjunction introduction and classical-logic modus tollens. Indeed, NSP can be generated by using indifferently modus ponens, conjunction introduction and modus tollens. I have explicitly formulated the modus ponens version, but the versions in which conjunction introduction and modus tollens are used are easy to work out. To provide a formulation of NSP in which conjunction introduction is used it suffices to notice that "I should believe that buying 1000 tickets allows me to win the lottery & I should believe that buying 0 tickets does not allow me to win the lottery" is equivalent to "(I should believe that buying 0 tickets does not allow me to win & I should believe that buying 1 ticket allows me to win) ∨ (I should believe that buying 1 ticket does not allow me to win & I should believe that buying 2 tickets allows me to win)… ∨ (I should believe that buying 999 tickets does not allow me to win & I should believe that buying 1000 tickets allows me to win)". If we regard all its disjuncts as false, then, by conjunction introduction, this last disjunction is both true and false (provided, of course, that we assume that "I should believe that buying 1000 tickets allows me to win the lottery & I should believe that buying 0 tickets does not allow me to win the lottery" is true).

Concerning the modus tollens version, the contradiction is generated by assuming both (4) and all the conditionals in (4)-(6), and by applying modus tollens as many times as needed.

not hold. In other words, if we want to solve (S)NSP, we should either endorse the cut-off point view or give up classical-logic modus ponens.

I will not take a stand here on which of these two very radical alternatives is the best. Of course, this new dilemma could be regarded as favouring the cut-off point view, i.e. as a clear indication of the fact that, puzzling as they may be, cut-off points are unavoidable. However, the validity of classical-logic modus ponens has been challenged in the past. Dialetheists, for instance, argue that the derivation of q from $p \supset q$ and p can fail, although in very special circumstances, when both p and $\sim p$ are true (see, most notably, Priest 1979 and Beall 2009). For their part, relevant logicians have questioned the validity of disjunctive syllogism (which is just modus ponens for the material conditional modulo double negation principles; see Anderson and Belnap 1975)[13].

Anyway, I will not tackle this issue here. In this paper I wanted to show that keeping *Belief Closure* becomes a less appealing option when one sees what happens if instead of a lottery scenario a different material is used, notably, a heap scenario. However, it is also worth noting that rejecting *Belief Closure* is not enough: as we have just seen, if we want to avoid the cut-off point conclusion with respect to (S)NSP we should embrace an even more radical solution, i.e. denying classical-logic modus ponens.

The above also teaches us something important concerning the most popular norms of belief on the market. Indeed, it can be noted that they all entail the cut-off point view (either only with respect to (S)NSP or with respect to both (S)WSP and (S)NSP):

[13] I should also mention here the advocates of the so-called "degree-theoretic view of vagueness"; indeed, many "degree-theorists" reject modus ponens when degrees of truth are involved in the inference.

whether we assume the *Lockean Thesis* (with $t = 1$ or with t short of 1), the truth norm or the knowledge norm, we end up with a cut-off point "somewhere". More precisely, if we assume the *Lockean Thesis* with t short of 1, we end up with a cut-off point (only) in (S)NSP. If, instead, we assume either the *Lockean Thesis* with $t = 1$, the truth norm or the knowledge norm, we end up with a cut-off point both in (S)WSP and in (S)NSP. Indeed, accepting any of these three norms makes it the case that for some ticket we should not believe that it loses/that for some grain we should *not* believe that adding it to something which is not a heap does *not* turn it into a heap. So some conditional in (S)WSP is unacceptable. But it is also the case that (S)NSP has one false conditional: for some n, we should believe "buying n tickets does not allow me to win" ("n grains are not a heap"), while we should not believe "buying $n + 1$ tickets does not allow me to win" ("$n + 1$ grains are not a heap").

Consequently, if we want to avoid cut-off points altogether, it is not enough to give up classical-logic modus ponens: denial of modus ponens would block both (S)WSP and (S)NSP, but were we to keep any of these three norms of belief, the cut-off points would still be there. As a result, if we want to avoid cut-off points, we should reject all three norms; i.e. we should deny modus ponens (to block the paradoxes) *and* we should go through a quite radical rethinking of the way we conceive norms of belief.

Of course, this result too could be regarded as favouring the cut-off point view, i.e. as proof of the fact that cut-off points cannot be avoided. On the contrary, I think that the cut-off points' opponents could take up the challenge. Notably, it seems to me that the challenge can be broken into three "smaller" ones: the cut-off points' enemies should (i) propose a suitable non-classical framework in which (S)NSP can be dealt with; (ii) come up with weaker (but still sensible) coherence constraints on rational belief (weaker than *Belief Closure*)[14]; iii) propose a norm of belief which can be naturally associated with such constraints, and which does not entail cut-off points. These certainly are hard challenges, but hard is not impossible.

More generally, I believe that it is premature for both sides (the cut-off points' advocates and their opponents) to claim success: more work has to be done in order to understand the structure of (S)WSP and (S)NSP, as well as their mutual relations. Hopefully, from such work decisive arguments will result against or in favour of cut-off points. For the time being, I take it to be the main lesson of this paper that the "cut-off point problem" (i.e. the question whether our solution to the Lottery Paradox and its variants should allow for cut-off points) ought to play a key role in the debate on the Lottery Paradox. In the literature on the Sorites, this question has always been central. The present article is a plea for writers on rational belief and rational degrees of belief to focus on this issue, which has been neglected so far.

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[14] An interesting attempt to provide a compelling alternative to *Belief Closure* as a coherence requirement for rational belief can be found in Easwaran and Fitelson 2015.

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