

## Voigt transformations in retrospect: missed opportunities?

*One more essay on the Einstein-Poincaré priority dispute*

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The teaching of modern physics often uses the history of physics as a didactic tool. However, as in this process the history of physics is not something studied but used, there is a danger that the history itself will be distorted in, as Butterfield calls it, a “Whiggish” way, when the present becomes the measure of the past.

It is not surprising that reading today a paper written more than a hundred years ago, we can extract much more of it than was actually thought or dreamed by the author himself. We demonstrate this Whiggish approach on the example of Woldemar Voigt’s 1887 paper. From the modern perspective, it may appear that this paper opens a way to both the special relativity and to its anisotropic Finslerian generalization which came into the focus only recently, in relation with the Cohen and Glashow’s very special relativity proposal.

With a little imagination, one can connect Voigt’s paper to the notorious Einstein-Poincaré priority dispute, which we believe is a Whiggish late time artifact. We use the related historical circumstances to give a broader view on special relativity, than it is usually anticipated.

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### 1 Introduction

Sometimes Woldemar Voigt, a German physicist, is considered as “Relativity’s forgotten figure” [1]. Although his contribution was not quite forgotten and cited “by those who have taken the trouble to read the original literature” [2], it is a matter of fact that this contribution had no influence at all on the development of special relativity. Therefore, in our opinion, it is more appropriate in this case to talk about missed opportunities in the sense of Dyson [3], than about forgotten contribution.

Nearly two decades before the vigorous development of special relativity has started, in 1887 Woldemar Voigt published an article on the Doppler effect in which some fundamental principles underlying the relativity theory were anticipated. Namely, he was the first who used Einstein's second postulate (universal speed of light) and the restricted form of the first postulate (invariance of the wave equation when changing the inertial reference system) to show that the Doppler shift of frequency was incompatible with Newtonian absolute time and required a relative time identical with the Lorentz's local time introduced later.

The modern reader may wonder why this paper published by Voigt remained unnoticed and played no role in the emergence and development of special relativity. The simple answer is that we are reading too much in too little: what we see in Voigt's paper today is affected by our modern scientific knowledge and it doesn't coincide with the readings of Voigt himself and his contemporaries. Because it is in general true that a perception of the world by human beings, and of scientific papers in particular, depends on everybody's background shaped by historical circumstances. As put in words by English poet William Blake, in his idiosyncratic and provocative form, "A fool sees not the same tree as the wise man sees".

Although often "it is considered legitimate, if not necessary, that the historian should 'intervene' in the past with the knowledge that he possesses by virtue of his placement later in time" [4], such an approach is dangerous and produces what has been called "Whig" history [5, 6]. In Whig historiography "what one considers significant in history is precisely what leads to something deemed significant today" [6], so to say, the present becomes the measure of the past. However, "the study of the past with one eye, so to speak, upon the present is the source of all sins and sophistries in history, starting with the simplest of them, the anachronism" [5]. In the present paper our objective is not the research in the history of special relativity. Instead we use the known historical facts and try to properly contextualize the original works and thus, from one side, avoid a Whig approach to the history of special relativity while using it in teaching, and from another one, frame and sharpen modern approach to key problems of special relativity, because one danger of Whiggish approach is that under its influence students can develop an anachronic and archaic perceptions of key features of special relativity.

The relativity of perception of scientific papers is valid, of course, for Poincaré's far more important contributions too and, in our opinion, is

in the base of the pointless Einstein-Poincaré priority dispute.

This priority dispute originated because history and heritage (in the terminology of Grattan-Guinness [7, 8]) were illegitimately intermingled by those whose “normal attention to history is concerned with heritage: that is, how did we get here? Old results are modernized in order to show their current place; but the historical context is ignored and thereby often distorted” [8].

We argue that in reality there is no priority problem in the history of special relativity, and this priority dispute originated at later times from the retrospective reading of Poincaré’s papers and ignoring the historical context. However, we use this artificial priority dispute, inspired by the Wiggish approach to the history of special relativity, to present a broad modern perspective on special relativity which we hope will be helpful. Besides, these modern developments and views show that this priority dispute is no longer relevant as far as basic aspects of special relativity are concerned.

Moreover, we present a detailed derivations of the anisotropic generalization of special relativity by Lalan, Alway and Bogoslovsky and of the corresponding Finslerian metric. This generalization is characterized by one extra parameter  $b$  which has the same conceptual status as the cosmological constant and thus, is expected to be very small. Formally,  $b \rightarrow 0$  limit of the anisotropic special relativity is the very special relativity of Cohen and Glashow which is an active research topic today.

The paper is organized as follows. In the next section some circumstances related to the emergence of special relativity is considered. In particular we are interested in to find out why the role played by Poincaré was not properly acknowledged at that time by his contemporaries. Our hypothesis is that this happened because Poincaré’s approach required a higher level of mathematical education than the majority of physicists had at that time. Minkowski belonged to a few who were in a position to duly appreciate Poincaré’s contribution. However he had a psychological reasons to downplay the importance of Poincaré’s work because otherwise his own contribution might appear as derivative of Poincaré’s ideas. We think that Minkowski’s (wrong) conviction that it was Voigt whom the priority of the discovery of the Lorentz transformations should be attributed played a dramatic role in his perilous decision not to mention Poincaré in his highly influential 1908 Cologne lecture *Space and Time*. Thus we devote the third section to the so-called Voigt transformations, obtained by Voigt in 1887, while studying the Doppler effect, well before

Lorentz, Einstein and Poincaré's pioneering works. Our aim in this section is to convey the spirit of Voigt's reasoning, not to repeat his derivation. Therefore our approach differs in some respect from Voigt's original one and sometimes is even less general. Then we perform intentionally Whiggish analysis of the Voigt's transformations and demonstrate that the modern reader can find a lot of content in it which was impossible for Voigt and his contemporaries. In the fourth section we provide more modern derivation of Voigt's transformations which reveals tacit assumptions behind Voigt's original derivation and later derivations of Lorentz transformations by Poincaré and Einstein. Relaxation of some of these tacit assumptions opens a way for interesting generalizations of special relativity and in the fifth section we consider one such generalization, namely the anisotropic special relativity of Lalan-Alway-Bogoslovsky, and provide some arguments why we believe the dimensionless parameter of this generalization should be very small but nevertheless the nonzero value of this parameter is more natural than the very special relativity recently in the focus of research. In the concluding sixth section we return to the Einstein-Poincaré priority dispute and provide further arguments that this priority dispute is a result of Whiggish approach to the history of special relativity. We also argue that modern developments of special relativity makes this priority dispute pointless for another reason too: many key features of special relativity which by founders of this theory were considered as crucial and revolutionary doesn't seem so from the modern perspective.

## 2 Voigt — a missing link in the Einstein-Poincaré mysterious connection?

It is unfortunate that Einstein didn't cite anybody in his seminal paper on the foundations of special relativity [9]. On one hand, this circumstance can induce a wrong impression that the creation of special relativity was a miraculous event, a single strike of a young genius, at that time a 26-year-old junior clerk at the Swiss patent office in Bern who worked it out in his spare time within a few months in complete scientific isolation.

Einstein's own account [10] seems to confirm this simplified version of how the special relativity was created. He recalls that after reading the Lorentz's 1895 monograph *Treatise on a Theory of Electrical and Optical Phenomena in Moving Bodies* he spent almost a year in vain trying to resolve a difficult problem of how to reconcile the invariance

of the velocity of light, implied by the validity of Maxwell equations in the reference frame of the moving body, to the addition rule of velocities used in mechanics. Then suddenly during a conversation with his friend Michele Besso a solution came to him: “An analysis of the concept of time was my solution. Time cannot be absolutely defined, and there is an inseparable relation between time and signal velocity. With this new concept, I could resolve all the difficulties completely for the first time. Within five weeks the special theory of relativity was completed” [10].

We have no reason not to believe Einstein. Rindler argues that “contrary claims notwithstanding, apparently no one before Einstein in 1905 ventured the idea that all clocks in motion might go slow” [11]. It seems very natural that this startling new idea psychologically played a role of a trigger for Einstein to develop his version of special relativity. This version turned out very influential and determined the route of special relativity for years ahead.

Nonetheless, historically the genesis of the special relativity theory was a long process involving at least two more key players: Lorentz and Poincaré. Besides, “the construction of the special theory of relativity did not end with Einstein’s papers of 1905. Some features that today’s physicists judge essential were added only later” [12]. Many prominent physicists participated in this process of shaping the special relativity to its modern form, most notably Hermann Minkowski, Arnold Sommerfeld, Alfred Robb, Max Planck and Max von Laue, to name a few.

Einstein’s paper [9] is undoubtedly a strike of a genius. It gives an extraordinary clear presentation of fundamentally new concepts about space and time which shattered the ideas which existed before. However, there is a mystery here. As observed by Dyson, “when the great innovation appears, it will almost certainly be in a muddled, incomplete and confusing form. To the discoverer himself it will be only half-understood; to everybody else it will be a mystery” [13]. How can we reconcile the extreme clarity of Einstein’s presentation with Dyson’s observation?

First of all, Einstein’s great discovery has not appeared out of nowhere. As Einstein himself wrote in 1955 “there is no doubt, that the special theory of relativity, if we regard its development in retrospect, was ripe for discovery in 1905. Lorentz had already observed that for the analysis of Maxwell’s equations the transformations which later were known by his name are essential, and Poincaré had even penetrated deeper into these connections” [14]. Although, as the known historical documents witness [14, 16], Einstein’s version of special relativity is

solely his own long-sought creature, barely influenced by whatever background reading he had at that time, undoubtedly Einstein benefited from his scientific background in this process, at least unconsciously: Einstein received good education for his time. Although “the scientific courses offered to him in Zürich soon seemed insufficient and inadequate, so that he habitually cut his classes. His development as a scientist did not suffer thereby. With a veritable mania for reading, day and night, he went through the works of the great physicists — Kirchhoff, Hertz, Helmholtz, Föpl” [14]. Probably we should add Poincaré to this list as it is known that a collection of essays *a Science et l’Hypothèse* published by Poincaré in 1902 deeply impressed young Einstein and his friends from Olympia Academy and kept them “breathless for weeks on end” [15].

As Einstein cites nobody in his paper [9], we don’t know for sure what contemporary papers, if any, influenced him while writing his masterwork [9] and whether his development of special relativity was eased by his readings. Letters of Einstein, written between 1898 and 1902, confirm [17] his later recollections that he was engaged in the problems of electrodynamics of moving bodies many years before writing his epoch-making article [9]. However, surviving correspondence sheds very little light on what happened in the crucial years between 1902 and 1905. “Whatever reading and writing he may have done at this time, Einstein published nothing on the subject of optics and electrodynamics of moving bodies for 3.5 years” [14, 17].

When Einstein was asked in 1955 whether Poincaré had had any influence on his development of special relativity, he answered [14]: “Concerning myself, I knew only Lorentz’s important work of 1895 — ‘La théorie électromagnétique de Maxwell’ and ‘Versuch einer Theorie der elektrischen und Optischen Erscheinungen in bewegten Körpern’ — but not Lorentz’s later work, nor the consecutive investigations by Poincaré. In this sense my work of 1905 was independent. The new feature of it was the realization of the fact that the bearing of the Lorentz transformation transcended its connection with Maxwell’s equations and was concerned with the nature of space and time in general. A further new result was that the ‘Lorentz invariance’ is a general condition for any theory”.

Einstein is not very accurate here. The universal character of Lorentz invariance, the fact that it is a general requirement for all laws of physics, not merely a property of Maxwell’s equations, was, no doubt, anticipated by Poincaré [12, 18]. Moreover, there is still another mystery. From the modern perspective, it may appear that the Einstein’s synchronization

procedure via light signals and not absolute nature of simultaneity were also Poincaré's inventions. Already in 1898, "Poincaré had presented exactly the same light signaling and clock synchronization thought experiment that would later be found in Einstein's 1905 relativity paper" [19], although Poincaré's presentation is without any mention of the relativity principle and Lorentz's local time.

Two years later in his lecture "Lorentz's theory and the principle of reaction" Poincaré used his light signaling and clock synchronization thought experiment to explain the physical meaning of the Lorentz's local time [19]. It is true however that Poincaré never abandoned the concept of æther and considered the above mentioned relativity of simultaneity as only an apparent, not genuine, physical effect related to the fact that we have synchronized the clocks in the moving frame as if the light velocity were the same in all directions, like what happens in the preferred æther frame. For Poincaré, only clocks so synchronized in the æther frame show the real time, despite the fact that he was fully aware that, because of the relativity principle, it was impossible to experimentally find out that the local time defined in the moving frame was not the real time. In 1902 letter to the Nobel committee to nominate Lorentz for the Nobel prize in Physics, which he indeed was awarded, Poincaré praises very highly Lorentz's "most ingenious invention" of "local time" and writes: "Two phenomena happening in two different places can appear simultaneous even though they are not: everything happens as if the clock in one of these places were late with respect to that of the other, and as if no conceivable experiment could show evidence of this discordance" [19].

Could Poincaré's insights about simultaneity and clock synchronization somehow influenced Einstein? This is not an easy question. We know from Einstein himself that he had read Lorentz's 1895 treatise and "was therefore aware of the local time and the role it served in preserving the form of the Maxwell-Lorentz equations to first order" [12]. We also know that he and his friends from Olympia Academy were fascinated by Poincaré's *La science et l'hypothèse*, which contained the relativity principle and a brief critical discussion of simultaneity. "He may also have known the exact form of the Lorentz transformations, for in 1904 several German theorists commented on them in journals that he regularly read" [12]. For Torretti this is enough to conclude that "there is no doubt that Einstein could have drawn inspiration and support for his first work on relativity from the writings of Poincaré" [20]. However

matters are not as simple, because Einstein denies such an influence and we have no reason not to believe him. Knowing the paradoxical nature of human memory and psychology, we cannot exclude that the true story was hidden for Einstein too.

“It is certain that sometime before 17 May 1906, Einstein did read Poincaré’s 1900 paper — on that day Einstein submitted a paper of his own that explicitly used the contents of Poincaré’s article (though not local time)” [21]. This was the first and last occasion Einstein ever cited Poincaré in the context of relativity (in connection with the  $E = mc^2$  formula).

Similar ideas about clock synchronization and physical definition of the local time can be also found (in German) in Emil Cohn’s November 1904 article “Toward the electrodynamics of moving systems” [21]. Cohn’s views are even closer to Einstein’s than to Poincaré’s, as he rejected the æther, preferring “the vacuum”. We know that sometime before 25 September 1907 Einstein did read the Cohn’s paper — at that day he sent to the editor of a journal his review paper on relativity which contains a footnote-compliment: “The pertinent studies of E. Cohn also enter into consideration, but I did not make use of them here” [21]. However, again we have no evidence that Einstein knew the Cohn’s work prior his founding 1905 paper. How can we then explain a strange fact that “Einstein’s reasoning in 1905 regarding synchronization of clocks using light signals is incredibly similar to Poincaré’s” [22]? Probably Chavan or Solovine (members of the Olympia Academy), who were fluent in French, reported about Poincaré’s 1898 paper (in French) “the measurement of time” at an Olympia Academy session [21]. Even more interesting guess about what could have happened then is suggested in [19].

Norton argues that it is quite possible that thoughts of clocks synchronization by light signals played no essential role in Einstein’s discovery of the relativity of simultaneity: “A plausible scenario is that Einstein was compelled to the Lorentz transformation for space and time as a formal result, but needed some way to make its use of local time physically comprehensible. Thoughts of light signals and clock synchronization would then briefly play their role. It is also entirely possible that these thoughts entered only after Einstein had become convinced of the relativity of simultaneity; that is, they were introduced as an effective means of conveying the result to readers of his 1905 paper and convincing them of it. In both cases, thoughts about the light signals



and clock synchronization most likely played a role only at one brief moment, some five to six weeks prior to the completion of the paper, at the time that Einstein brought his struggle with him to a celebrated meeting with his friend Michele Besso" [23]. Indeed, five to six weeks before completing his relativity paper, Einstein wrote to his friend Conrad Habicht: "The fourth paper is only a rough draft at this point, and is an electrodynamics of moving bodies which employs a modification of the theory of space and time" [19]. It seems that at that time Einstein already discovered the relativity of simultaneity, as he speaks of modification of the theory of space and time, but yet had not found a good way how to present his new discoveries. Then there was a "celebrated meeting" with his friend Michele Besso. At the end of his relativity paper, Einstein writes "In conclusion I wish to say that in working at the problem here dealt with I have had the loyal assistance of my friend and colleague M. Besso, and that I am indebted to him for several valuable suggestions" [9]. What was Besso's role in completing Einstein's relativity paper? Granek suggests "that in the Einstein-Besso 'celebrated meeting' Besso indirectly and unnoticeably conveyed Poincaré's light signaling and clock synchronization thought experiment, without necessarily knowing that this thought experiment originated in Poincaré's works" [19]. Probably Besso was told about this light signaling and clock synchronization thought experiment by another member of Olympia Academy who read Poincaré's papers (as was already mentioned, it is possible that Solovine read Poincaré's 1898 paper). Then he conveyed to Einstein a loose idea about the light signaling and clock synchronization intermingled with other ideas [19]. No wonder that Einstein afterwards refused any connection with Poincaré, because he was convinced that he came to the idea by himself, with the help of his friend Besso, and had not realized that in fact this was Poincaré's idea which came to him in such a bizarre way.

We think this is the best we can do on the question whether Poincaré's or Cohn's papers influenced Einstein's reasoning. "The limits of reconstruction are evident, and would be even if it were written in stone that Einstein had seen one of these papers" [21].

However, the real enigma is not the absence of Poincaré's name in Einstein's 1905 paper. As we have seen above, we can imagine a plausible explanation of this fact. The real enigma is why Poincaré's contribution to relativity was downplayed by his contemporaries.

From the modern perspective, Poincaré's treatment of the Lorentz

group in his last pre-Einstein papers [24, 25] is extraordinary advanced and modern [18, 26]. In some sense, Poincaré was far ahead of his time. Perhaps very few physicists (if any) at that time had enough mathematical background to duly appreciate Poincaré's achievements. Besides, Poincaré's 1906 Rendiconti paper [25] was both "rather long (47 pages) and technically quite complex, especially in the final section" where "many of the key new results of Poincaré on Relativity are contained" [27]. However, Hermann Minkowski was undoubtedly among those who were capable to appreciate Poincaré's writings. And a blatant fact, which leaves us almost speechless, is that Poincaré's name was never mentioned in Minkowski's famous Cologne lecture "Raum und Zeit".

Minkowski's September 21st, 1908, lecture "Space and Time" was a crucial event in the history of relativity <sup>1</sup>. In a sense, the appearance of the Einstein's paper [9] was an unexpected and peculiar event which changed the natural flow of ideas of contemporary physics. Suppose Einstein had not existed. How would our understanding of space-time physics have developed then? In an interesting essay [28] Lévy-Leblond tried to answer this question. His conclusion is that "apart from a few details perhaps, the formalism, that is, the notations and equations, would be very similar to ours today. But the language and, underneath, the words and the ideas we would use could be rather different. Many familiar terms, beginning with 'relativity' itself, might be absent from our vocabulary" [28]. In this respect, Minkowski's Cologne lecture constitutes the shift of focus from reference frames and relativity of lengths and time intervals towards the modern concepts of symmetry and associated space-time geometry. After the Cologne lecture, relativity was gradually but firmly embarked on the royal way leading to the modern understanding of special relativity as the Minkowskian chronogeometry of the 4-dimensional flat space-time.

Initially Einstein was skeptical about Minkowski's innovations and called them "superfluous learnedness" [29]. According to Sommerfeld's recollections, he said on one occasion: "Since the mathematicians have invaded the relativity theory, I do not understand it myself any more" [30]. However, as early as 1912 Einstein fully appreciated the strength of

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<sup>1</sup>Annalen der Physik in 2008 (vol. 17, issues 9 and 10, pp. 613-852) published a collection of articles written to celebrate the 100th anniversary of the Minkowski's 1908 lecture. We cite two articles from this collection (by Damour and by Kastrup) in this manuscript. However, readers can find other articles from this collection also interesting.

Minkowski's 4-dimensional approach. At that year he wrote to Sommerfeld: "I am now occupied exclusively with the gravitational problem, and believe that I can overcome all difficulties with the help of a local mathematician friend [Marcel Grossmann]. But one thing is certain, never before in my life have I troubled myself over anything so much, and that I have gained great respect for mathematics, whose more subtle parts I considered until now, in my ignorance, as pure luxury! Compared with this problem, the original theory of relativity is childish" [30].

The influence of the Cologne lecture was enormous. Its published version "sparked an explosion of publications in relativity theory, with the number of papers on relativity tripling between 1908 (32 papers) and 1910 (95 papers)" [31]. The response to the Minkowski's lecture was overwhelmingly positive on the part of mathematicians, and more mixed on the part of physicists — only in the 1950s their attitude began to converge toward Minkowski's space-time view [31].

Due to the importance of the Cologne lecture and to the impact it had on the historical development of special relativity, a sad omission of Poincaré's name from it played, in our opinion, a crucial role in the subsequent downplaying of Poincaré's contribution to relativity by contemporary physicists for a long time.

Some leading scientists remarked the absence of Poincaré's name in the Cologne lecture and even tried somehow to correct injustice. In particular, "in the notes he added to a 1913 reprint of this lecture, Sommerfeld attempted to right the wrong by making it clear that a Lorentz-covariant law of gravitation and the idea of a 4-vector had both been proposed earlier by Poincaré" [32]. Though Sommerfeld acknowledges Poincaré's contribution only partially, the young Pauli does "a better job" [27] in giving Poincaré a due credit in his famous review article on the theory of relativity [33]. In fact, this happened under Felix Klein's direct influence. In a letter of acknowledgment of the receipt of the manuscript Klein advises Pauli to pay great attention to historical facts and in particular emphasizes Poincaré's priority before Einstein in recognizing the group properties of the Lorentz transformations [34]. However, these efforts were insufficient. As the citation record shows [35], Poincaré's papers [24, 25] were rarely cited during the founding period of relativity 1905-1912 and thus had virtually no effect on the development of special relativity (in this time period Einstein's paper [9] was cited about five times more frequently than Poincaré's papers [24, 25]).

Why did Minkowski not mention Poincaré? Minkowski was certainly

aware of Poincaré's papers. One year before the celebrated lecture in Cologne, in his lecture to the Göttinger Mathematischen Gesellschaft, the text of which was published posthumously in 1915 through the efforts of Sommerfeld, Minkowski frequently and positively cites Poincaré. In fact Poincaré was the third most cited author (cited six times), after Planck (cited eleven times) and Lorentz (cited ten times), in this lecture [27]. For comparison, Einstein was cited only twice. How can we then explain the complete disregard of Poincaré's contribution only one year later? In absence of any historical document resolving this conundrum, only speculations on Minkowski's motivations may be entertained, as was done in [27] and [32]. Maybe the following circumstances played a psychological role.

According to Max Born's recollections, later after the Cologne lecture Minkowski told him that "it came to him as a great shock when Einstein published his paper in which the equivalence of the different local times of observers moving relative to each other was pronounced; for he had reached the same conclusions independently but did not publish them because he wished first to work out the mathematical structure in all its splendor" [30]. We may suppose that the discovery of the 4-dimensional formalism was also a result of this process of working out the mathematical structure behind the Lorentz transformations and was made by Minkowski independently of Poincaré's 1905 papers. "If that reconstruction is correct, he must have been all the more eager, when he later realized that he had been preceded by Poincaré, to find reasons for downplaying Poincaré's work" [27]. So, perhaps, in 1907 Minkowski had only a background reading of Poincaré's papers. Then, as he prepared for the Cologne lecture, he more thoroughly studied Poincaré's papers and realized that if he "had chosen to include some mention of Poincaré's work, his own contribution may have appeared derivative" [32]. However, in our opinion, to make the decision to exclude Poincaré's name from the Cologne lecture Minkowski needed some serious reason to psychologically justify such an unfair omission. Maybe another oddity of the Cologne lecture gives a clue what was this reason.

In the Cologne lecture Minkowski never mentions Lorentz transformations (so named by Poincaré in [24, 25]), instead he refers to transformations of the group  $G_c$ . Even more surprisingly, on December 21, 1907, Minkowski talked to the Göttingen scientific society and subsequently in April 1908 published its contents in *Göttinger Nachrichten* as a technical paper *Die Grundgleichungen für die electromagnetischen Vorgänge*

*in bewegten Körpern* where one can find all the results of the Cologne lecture presented with great details [27, 30]. In this paper Minkowski quotes Poincaré only twice and “in a rather derogatory manner” [27]. In particular, at the beginning, he mentions Poincaré only as an originator of the name “Lorentz transformations”. In fact, probably, this talk of Minkowski was a beginning of his downplaying of the Poincaré’s contribution, as witnessed by the following facts [36]. In it Minkowski gives Lorentz the credit of having found the “purely mathematical fact” of the Lorentz covariance of the Maxwell equations and of having “created the relativity-postulate”. None of these claims is true — it is Poincaré who should be credited for these discoveries as was explicitly stated by Lorentz himself. The priority in clarifying the link between the Lorentz transformations and a new concept of time is fully given to Einstein without mentioning Poincaré’s important insights in this question. Overall impression that the reader could infer from this historically important publication might be that Minkowski is “suggesting that the main (if not the only) contribution by Poincaré is to have given the Lorentz transformations ... their name” [36].

The reason for suppression of the name “Lorentz transformations” in the Cologne lecture is unknown, but very probably [32] it was linked to Minkowski’s discovery that the essential application of the Lorentz symmetry in optics goes back to Woldemar Voigt’s 1887 paper [37].

Thanks to a letter sent by Minkowski to David Hilbert in 1889, we know that Minkowski was studying Voigt’s treatment of elasticity [38]. However, it is not known whether Minkowski read Voigt’s 1887 paper on the Doppler principle at that time. Probably he did. At least such a suggestion seems tenable [39]. Even if he read the paper, it seems he had not recognized the potential of Voigt’s transformations. In his letter of 1889 to Hilbert he mentions that “abstract speculation was not sufficient for physical understanding” and that “it was inconceivable to him that anyone would develop mathematical equations only with the hope that someone might later demonstrate their utility” [38].

However, it is quite possible and even more plausible that Minkowski became aware of the Voigt’s 1887 Doppler principle paper only later and his above given quotations concern to another paper by Voigt *Theoretische Studien über die Elasticitätsverhältnisse der Krystalle* published also in 1887 [40]. Interestingly, Walter gives a slightly different translation of Minkowski’s quotation: “how anyone can apply odd calculations in the hope that later, perhaps, someone will be found who can get something

out of it” [40] <sup>2</sup>.

Although Voigt and Lorentz were in correspondence since 1883, Lorentz was unaware of Voigt’s Doppler principle paper and it was not until 1908 that Voigt sent him a reprint of this paper. In a letter of acknowledgment of the receipt of the preprint from July, 1908, Lorentz writes “I would like to thank you very much for sending me your paper on Doppler’s principle together with your enclosed remarks. I really regret that your paper has escaped my notice. I can only explain it by the fact, that many lectures kept me back from reading everything, while I was already glad to be able to work a little bit. Of course I will not miss the first opportunity to mention, that the concerned transformation and the introduction of a local time has been your idea” [39].

Voigt and Minkowski were friends at Göttingen. So it is possible that Voigt informed Minkowski about his correspondence with Lorentz in around July, 1908 and probably that’s how Minkowski became aware of the Voigt’s 1887 Doppler principle paper [39].

In confirmation of this suggestion Minkowski cites (incorrectly) Voigt in the Cologne lecture as the discoverer of the Lorentz symmetry. Namely, he says “Now the impulse and true motivation for assuming the group  $G_c$  came from the fact that the differential equation for the propagation of light waves in empty space possesses the group  $G_c$ ” and adds in the footnote that an essential application of this fact can already be found in Voigt’s 1887 paper [42].

Two days after the Cologne lecture, in a discussion that followed the Alfred Bucherer’s lecture *Measurements of Becquerel rays. The Experimental Confirmation of the Lorentz-Einstein Theory* <sup>3</sup>, Minkowski again tried to promote Voigt as the discoverer of Lorentz transformations. He said: “Historically, I want to add that the transformation, which play the main role in the relativity principle, were first mathematically discussed by Voigt in the year 1887. Already then, Voigt drew some consequences with their aid, in respect to the principle of Doppler” [43].

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<sup>2</sup>The German original can be found in [41] and goes as follows: “Es ist mir ganz unbegreiflich, wie jemand wüste Rechnungen ansetzen kann, in der Hoffnung daß sich später vielleicht jemand findet, der Nutzen daraus zu ziehen im Stande ist”.

<sup>3</sup>Bucherer’s experimental results contributed significantly to the physicists adjustment to the theory of relativity as they refuted earlier results by Walter Kaufmann challenging the empirical adequacy of the Lorentz-Einstein theory. “In the discussion of this lecture, Minkowski expressed joy in seeing the ‘monstrous’ rigid electron hypothesis experimentally defeated in favor of the deformable electron of Lorentz’s theory” [35].

Convincing himself that it was Voigt who should be credited for apprehension of the central role of the Lorentz symmetry, perhaps, it was psychologically more easy for Minkowski to decide to omit Poincaré's name from the Cologne lecture. Very likely, this pernicious decision was further eased by Poincaré's style of writing who "habitually wrote in a self-effacing manner. He named many of his discoveries after other people, and expounded many important and original ideas in writings that were ostensibly just reviewing the works of others, with 'minor amplifications and corrections'. Poincaré's style of writing, especially on topics in physics, always gave the impression that he was just reviewing someone else's work" [44].

Curiously, Voigt's name resurfaces, and in a rather mysterious way, in relation with Einstein's paper [9] too (see [45]). In [9] Einstein doesn't use Lorentz's and Poincaré's nomenclature  $(x, y, z, t) \leftrightarrow (x', y', z', t')$  while writing the final form of Lorentz transformations, but rather the one used in Voigt's paper [37]:  $(x, y, z, t) \leftrightarrow (\xi, \eta, \zeta, \tau)$ . We don't know the reason of this weird coincidence. Did Einstein read Voigt's paper? Or was it reported on the Olympia Academy session? Probably not and all this is just a coincidence. After all  $(\xi_1, \eta_1, \zeta_1)$ ,  $(\xi_2, \eta_2, \zeta_2)$ , etc. were used by Lorentz to denote locations of the molecules in his 1895 treatise which Einstein was well aware of. Poincaré also uses  $(\xi, \eta, \zeta)$ , though to denote velocity components, in [24, 25]. If you still consider such a coincidence as improbable, like the author of [45], one can remember another amazing coincidence of this kind. The following two identities, expressing parametric solutions for representing a forth power as a sum of five forth powers, can be found in Ramanujan's third notebook [46]:

$$(8s^2 + 40st - 24t^2)^4 + (6s^2 - 44st - 18t^2)^4 + (14s^2 - 4st - 42t^2)^4 + (9s^2 + 27t^2)^4 + (4s^2 + 12t^2)^4 = (15s^2 + 45t^2)^4,$$

and

$$(4m^2 - 12n^2)^4 + (3m^2 + 9n^2)^4 + (2m^2 - 12mn - 6n^2)^4 + (4m^2 + 12n^2)^4 + (2m^2 + 12mn - 6n^2)^4 = (5m^2 + 15n^2)^4.$$

What is amazing about this identities is that the first identity was published by Haldeman in 1904 and Ramanujan uses the same notations and precisely the same order of the terms as Haldeman. "One might conclude that Ramanujan saw Haldeman's paper or a secondary source

quoting it. However, this seems highly unlikely as Ramanujan had access to very few journals in India and, moreover, Haldeman's paper was published in a very obscure journal. It is also quite possible that Ramanujan made his discovery before Haldeman did. Thus, in conclusion, the identical notation must be an amazing coincidence" [46].

The second identity is also contained in that Haldeman's paper. However, in this case Ramanujan uses not the same notation as Haldeman. His notation (but not order of the terms) coincides to what was used by A. Martin, another mathematician who published his version of the proof of this identity in the same volume in which Haldeman's paper appeared. "Again, this must be an astonishing coincidence" [46]. Probably, the same can be said about Einstein-Voigt notation coincidence.

### 3 Voigt transformations

Now let us take a closer look at the Voigt's 1887 Doppler principle paper [39, 45, 47, 48, 49]. The first impression is that this paper [37] looks more like a technical note with messy calculations than an ordinary research paper: there is no abstract present nor any explanation of the idea and purpose of the paper, and, like Einstein's [9], it does not contain references [48]. Voigt's objective was to derive the Doppler effect formula from the invariance of the wave equation

$$\square \Phi(x, y, z, t) = \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) \Phi(x, y, z, t) = 0, \quad (1)$$

under the change of variables

$$\begin{aligned} \xi &= m_1(V)x + n_1(V)y + p_1(V)z - \alpha_0(V)t, \\ \eta &= m_2(V)x + n_2(V)y + p_2(V)z - \beta_0(V)t, \\ \zeta &= m_3(V)x + n_3(V)y + p_3(V)z - \gamma_0(V)t, \\ \tau &= t - [a_0(V)x + b_0(V)y + c_0(V)z], \end{aligned} \quad (2)$$

which corresponds to the transition from the æther rest frame to an other frame moving with a constant velocity  $\vec{V}$ . Voigt doesn't explain why such invariance (which implies that the light velocity is the same in both frames) should take place, he simply says that the wave equation in the new frame has the form

$$\left( \frac{1}{c^2} \frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \zeta^2} \right) \Phi(\xi, \eta, \zeta, \tau) = 0, \quad (3)$$



“as it must be” [48]. Neither he provides any rationale behind the coordinate transformation (2).

Comparing (1) and (3), Voigt obtains a system of nine equations for fifteen unknown constants

$$m_1, n_1, p_1, m_2, n_2, p_2, m_3, n_3, p_3, a_0, b_0, c_0, \alpha_0, \beta_0, \gamma_0.$$

Therefore, he needs some assumptions to solve this troublesome system. First of all he notices that in the moving frame  $S'$  the origin ( $x = 0, y = 0, z = 0$ ) of the stationary frame  $S$  moves with the velocity  $-(\alpha_0, \beta_0, \gamma_0)$ . Therefore,  $(\alpha_0, \beta_0, \gamma_0)$  can be identified with  $\vec{V}$  and assuming the standard configuration ( $S'$  moves along the  $x$ -axis with respect to  $S$ ), we get  $\alpha_0 = V, \beta_0 = 0, \gamma_0 = 0$ . The following can then be calculated from (2):

$$\begin{aligned} \frac{\partial}{\partial x} &= m_1 \frac{\partial}{\partial \xi} + m_2 \frac{\partial}{\partial \eta} + m_3 \frac{\partial}{\partial \zeta} - a_0 \frac{\partial}{\partial \tau}, \\ \frac{\partial}{\partial y} &= n_1 \frac{\partial}{\partial \xi} + n_2 \frac{\partial}{\partial \eta} + n_3 \frac{\partial}{\partial \zeta} - b_0 \frac{\partial}{\partial \tau}, \\ \frac{\partial}{\partial z} &= p_1 \frac{\partial}{\partial \xi} + p_2 \frac{\partial}{\partial \eta} + p_3 \frac{\partial}{\partial \zeta} - c_0 \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - V \frac{\partial}{\partial \xi}. \end{aligned} \quad (4)$$

The invariance of the wave equation implies that  $\square' = \gamma^2(V) \square$ , or

$$\frac{1}{c^2} \frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \zeta^2} = \gamma^2(V) \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right), \quad (5)$$

where  $\gamma^2(V)$  is some constant. In combination with (4) this condition gives ten equations which we organize into three groups (starting from this point we somewhat modify Voigt’s original derivation in order to make it more transparent). Equations of the first group correspond to the diagonal elements of the d’Alembert operator and they have the form

$$\begin{aligned} \gamma^2 [1 - (a_0^2 + b_0^2 + c_0^2)c^2] &= 1, \quad \gamma^2 \left[ (m_1^2 + n_1^2 + p_1^2) - \frac{V^2}{c^2} \right] = 1, \\ \gamma^2 [m_2^2 + n_2^2 + p_2^2] &= 1, \quad \gamma^2 [m_3^2 + n_3^2 + p_3^2] = 1. \end{aligned} \quad (6)$$

The second group corresponds to the equations that ensure the vanishing of the mixed derivatives  $\frac{\partial^2}{\partial \tau \partial \xi}$ ,  $\frac{\partial^2}{\partial \tau \partial \eta}$  and  $\frac{\partial^2}{\partial \tau \partial \zeta}$ . These equations are

$$\begin{aligned} m_1 a_0 + n_1 b_0 + p_1 c_0 - \frac{V}{c^2} &= 0, \\ m_2 a_0 + n_2 b_0 + p_2 c_0 &= 0, \quad m_3 a_0 + n_3 b_0 + p_3 c_0 = 0. \end{aligned} \quad (7)$$

The third group of equations ensures the vanishing of the mixed derivatives  $\frac{\partial^2}{\partial \xi \partial \eta}$ ,  $\frac{\partial^2}{\partial \xi \partial \zeta}$  and  $\frac{\partial^2}{\partial \eta \partial \zeta}$ . They have the form

$$\begin{aligned} m_1 m_2 + n_1 n_2 + p_1 p_2 &= 0, \\ m_1 m_3 + n_1 n_3 + p_1 p_3 &= 0, \quad m_2 m_3 + n_2 n_3 + p_2 p_3 = 0. \end{aligned} \quad (8)$$

As we see, we have thirteen unknowns (including  $\gamma^2$ ) and ten equations. So, Voigt concludes that three of them can be chosen arbitrarily. The natural choice is  $m_1 = 1$ ,  $n_1 = 0$ ,  $p_1 = 0$ , which makes the transformation law for  $\xi$  identical to the Galilean transformation. Then equations (8) imply  $m_2 = m_3 = 0$  and

$$n_2 n_3 = -p_2 p_3, \quad (9)$$

while from equations (6) we get

$$\gamma^{-2} = 1 - \frac{V^2}{c^2}, \quad (10)$$

and

$$n_2^2 + p_2^2 = n_3^2 + p_3^2 = \gamma^{-2}. \quad (11)$$

At last, equations (7) will give us in this case

$$a_0 = \frac{V}{c^2}, \quad (12)$$

and

$$n_2 b_0 = -p_2 c_0, \quad n_3 b_0 = -p_3 c_0. \quad (13)$$

Equations (9) and (13) can be combined to produce the following relations

$$(n_2^2 + p_2^2)p_3 c_0 = 0, \quad (n_3^2 + p_3^2)n_2 b_0 = 0, \quad (14)$$

which in light of (11) imply

$$p_3 c_0 = 0, \quad n_2 b_0 = 0. \quad (15)$$

However, in the limit  $V = 0$  we should have  $n_2(0) = 1$  and  $p_3(0) = 1$ . Therefore,  $n_2$  and  $p_3$  can't be identically zero and the only way to satisfy (15) is to take  $b_0 = c_0 = 0$ .

If we introduce two complex numbers  $z_1 = n_2 + ip_2$  and  $z_2 = p_3 + in_3$ , the equations (9) and (11) indicate that  $|z_1| = |z_2| = \gamma^{-1}$  and that  $z_1 z_2$

is a real number. Therefore,  $z_2 = \pm z_1^*$ . Only the plus sign is acceptable, because  $n_2(0) = p_3(0)$ . Thus, the most general solution for  $z_1$  and  $z_2$  has the form

$$\begin{aligned} n_2 &= \gamma^{-1} \cos \phi(V), & p_2 &= \gamma^{-1} \sin \phi(V), \\ n_3 &= -\gamma^{-1} \sin \phi(V), & p_3 &= \gamma^{-1} \cos \phi(V), \end{aligned} \tag{16}$$

where  $\phi(V)$  is any function of the velocity  $V$  such that  $\phi(0) = 0$ . The presence of an angle  $\phi(V)$  reflects the obvious fact that a transformation from the æther frame to the moving frame can always be superimposed by a rotation around the axis parallel to  $\vec{V}$  by an angle  $\phi(V)$ . Excluding this trivial possibility of an additional rotation, we see that the invariance of the wave equation under Voigt's assumptions essentially uniquely leads to Voigt transformations:

$$\begin{aligned} \xi &= x - V t, \\ \eta &= \gamma^{-1} y, \\ \zeta &= \gamma^{-1} z, \\ \tau &= t - \frac{V}{c^2} x. \end{aligned} \tag{17}$$

The inverse transformations have the form

$$\begin{aligned} x &= \gamma^2 (\xi + V \tau), \\ y &= \gamma \eta, \\ z &= \gamma \zeta, \\ t &= \gamma^2 \left( \tau + \frac{V}{c^2} \xi \right). \end{aligned} \tag{18}$$

For a modern reader it is obvious that Voigt was very close to discovering special relativity. His transformations reveal the crucial fact that the invariance of the wave equation is inconsistent with Newtonian absolute time. In fact, Voigt was the first who discovered “that a ‘natural’ clock would alter its rate on motion” [50]. More precisely, to be fair, Voigt missed this opportunity because he apparently did not realize a great conceptual novelty of his transformations. His main objective was the Doppler effect. Using his transformations he concludes that the vibration period experiences time dilation with the (wrong) factor  $\gamma^2$ . However, the conclusion that all moving clocks would actually behave like this, was not ventured and to a great extent “his paper appears to be a mathematical exercise” without giving any conceptual reason for developing this particular set of transformations [50].

Voigt transformations (17) differ from Lorentz transformations only by a scale factor. Many physicists, including Minkowski and Lorentz,

considered this difference insignificant and credited Voigt with the discovery of Lorentz transformations [48]. For example, Lorentz wrote in 1914: “These considerations published by myself in 1904, have stimulated Poincaré to write his article on the dynamics of electron where he has given my name to the just mentioned transformation. I have to note as regards this that a similar transformation has been already given in an article by Voigt published in 1887 and I have not taken all possible benefit from it” [18]. It seems Voigt himself finally shared this opinion. Voigt’s 1887 paper was reprinted in *Physikalische Zeitschrift* on occasion of the tenth anniversary of the principle of relativity in 1915. In the reprinted version Voigt included additional comments and in particular when referring to his transformations he wrote: “This is, except for the factor  $q$  [ $\gamma$ ] which is irrelevant for the application, exactly the Lorentz transformation of the year 1904” [48].

Are Voigt transformations and Lorentz transformations indeed equivalent? The answer depends on the reading of Voigt transformations: modern reader can see quite different contexts for them compared to Voigt’s contemporaries. At first sight, the scale factor “indicates only a uniform magnification of the scales of space and time, or what is the same thing, a change of units. It does not introduce any essential modification” [51]. The same argument is also given in [52].

Indeed, let us assume that to define a time unit the observer in the moving frame uses a standard atomic clock of the æther frame. As this clock moves with respect to the observer, his inferred time unit will be  $\gamma$  times larger than if he would use, as tacitly assumed in usual Lorentz transformations, an atomic clock at rest in his own frame as the standard clock. Modern definition of the length unit relates it through the light velocity  $c$  to the time unit. Therefore, the length unit will be also  $\gamma$  times larger, and to express the Lorentz transformations in these new units of the moving frame we should divide primed coordinates by  $\gamma$ . As a result, we get precisely the Voigt transformations (17).

Therefore, the relativistic reading of Voigt transformations is certainly possible and the resulting theory will be completely equivalent (although perhaps less convenient) to special relativity. Note that the introduction of the preferred æther frame in this case is purely formal: we can choose any inertial reference frame instead of the æther frame and use its standard clocks to define time units in other inertial reference frames.

However, surprisingly enough, at will a non-relativistic Galilean read-

ing of Voigt transformations (17) is also possible. Let us consider a hypothetical Newtonian world with the truly æther frame  $S$  which is the only inertial frame in which light propagates isotropically with velocity  $c$ . Transition to an other inertial frame  $S'$  is given by Galilean transformations

$$\begin{aligned}\xi &= x - V t, \\ \eta &= y, \\ \zeta &= z, \\ \tau &= t.\end{aligned}\tag{19}$$

This form of Galilean transformations assumes Newtonian absolute time. However, if the velocity  $V$  of  $S'$  with respect to the æther frame  $S$  is smaller than light velocity  $c$ , it is possible to perform Poincaré-Einstein synchronization of clocks in  $S'$  and as a result we get another parametrization of the Galilean space-time in  $S'$ . Let us find this parametrization [53].

Suppose a light signal is sent from the origin  $O'$  of the inertial reference frame  $S'$  at absolute time  $\tau_1$ , it arrives to some point  $P'(\xi, \eta, \zeta)$  at absolute time  $\tau_2$ , is instantaneously reflected by a mirror at  $P'$  and returns to  $O'$  at absolute time  $\tau_3$ . According to Poincaré-Einstein synchronization convention, to synchronize the clock at  $P'$  with the clock at  $O'$ , one should set its reading to  $t'_2 = (\tau_1 + \tau_3)/2$  at an instant of signal reflection. Taking

$$\tau_3 = \tau, \quad \tau_1 = \tau - \frac{r'}{c'_1} - \frac{r'}{c'_2},\tag{20}$$

where  $r' = \sqrt{\xi^2 + \eta^2 + \zeta^2}$ , and  $c'_1, c'_2$  are light velocities in the frame  $S'$  when it moves from  $O'$  to  $P'$  and back respectively, we get

$$t'_2 = \tau - \frac{1}{2} \left( \frac{r'}{c'_1} + \frac{r'}{c'_2} \right).\tag{21}$$

Therefore, at the time  $\tau_3 = \tau$  at  $O'$ , the Poincaré-Einstein synchronized clock at  $P'$  will show

$$t' = t'_2 + \frac{r'}{c'_2} = \tau - \frac{1}{2} \left( \frac{r'}{c'_1} - \frac{r'}{c'_2} \right) = \tau - \frac{r'}{2} \frac{c'_2 - c'_1}{c'_1 c'_2}.\tag{22}$$

On the other hand, according to the Galilean velocity addition formula,

$$\begin{aligned}c^2 &= (c'_1 + \vec{V})^2 = c'^2_1 + 2c'_1 V \cos \theta' + V^2, \\ c^2 &= (c'_2 + \vec{V})^2 = c'^2_2 - 2c'_2 V \cos \theta' + V^2,\end{aligned}\tag{23}$$

where  $\theta'$  is the angle between the radius-vector  $\vec{r}'$  and the  $x'$ -axis. The positive solutions of (23) are

$$\begin{aligned} c'_1 &= -V \cos \theta' + \sqrt{V^2 \cos^2 \theta' + c^2 - V^2}, \\ c'_2 &= V \cos \theta' + \sqrt{V^2 \cos^2 \theta' + c^2 - V^2}. \end{aligned} \quad (24)$$

Substituting these velocities into (22), we get (instead of  $(\xi, \eta, \zeta)$ , we have returned to the more traditional  $(x', y', z')$  notation)

$$t' = \tau - \frac{r' V \cos \theta'}{c^2 - V^2} = \tau - \gamma^2 \frac{V}{c^2} x'. \quad (25)$$

Therefore, if the Poincaré-Einstein synchronization convention is adopted, instead of Galilean transformations we will get Zahar transformations [53]:

$$\begin{aligned} x' &= x - V t, \\ y' &= y, \\ z' &= z, \\ t' &= t - \gamma^2 \frac{V}{c^2} x' = \gamma^2 \left( t - \frac{V}{c^2} x \right). \end{aligned} \quad (26)$$

Let us note that, contrary to a prevailing belief, the non-relativistic limit of Lorentz transformations, when  $\beta = V/c \ll 1$ , is not the Galilean transformations (19) but the Zahar transformations (26) in which  $\beta^2$  terms are neglected [54, 55, 56], so that  $\gamma^2 \approx 1$ . The Galilean limit additionally requires

$$\beta \ll \frac{ct}{x}, \quad \frac{x}{ct} \sim \beta, \quad (27)$$

which is not necessarily true if the spatial separation  $x$  between two events is comparable or larger than temporal separation  $ct$ . It seems that this simple fact is not widely known as witnessed by its rediscovery in [57, 58]. The reason for this mismatch of non relativistic limits is the use of different synchronization conventions in Lorentz transformations and Galilean transformations: for sufficiently distant events in Minkowski world one cannot ignore not absolute nature of distant simultaneity and only when Poincaré-Einstein synchronization convention is adopted in Galilean world too the resulting Zahar transformations share the common non relativistic limit with the Lorentz transformations [57].

Now let us assume that inhabitants of the frame  $S'$  decided to use an atomic clock of the frame  $S$  as the time standard (and let us assume that clocks when in motion behave as prescribed by Newtonian physics of Galilean space-time). Then (26) shows that the time unit so defined in  $S'$  will be  $\gamma^2$  times larger than the time unit inferred from the stationary atomic clock in  $S'$ , as assumed in (26). Length units, as usually, are determined from the time unit and light velocity. However we should be careful here because the light velocity is not isotropic in  $S'$  in Galilean world even under the Poincaré-Einstein synchronization convention, if the “natural” units, as in (26), are used. The velocity addition rule that follows from (26) has the form

$$V'_x = \frac{V_x - V}{\gamma^2 \left(1 - \frac{V}{c^2} V_x\right)}, \quad V'_y = \frac{V_y}{\gamma^2 \left(1 - \frac{V}{c^2} V_x\right)}, \quad V'_z = \frac{V_z}{\gamma^2 \left(1 - \frac{V}{c^2} V_x\right)}. \quad (28)$$

Suppose light signal moves in  $S'$  along the  $x$ -axis with velocity  $c_{\parallel} = V'_x$ . Then (28) shows that, in the frame  $S$ ,  $V_y = V_z = 0$ . Hence  $V_x = c$  and

$$c_{\parallel} = \frac{c - V}{\gamma^2 \left(1 - \frac{V}{c}\right)} = \frac{c}{\gamma^2}. \quad (29)$$

If the light impulse in  $S'$  moves along the  $y$ -axis with velocity  $c_{\perp} = V'_y$ , then (28) shows that  $V_x = V$ ,  $V_z = 0$ , and hence  $V_y = \sqrt{c^2 - V^2}$ . In this case (28) will give

$$c_{\perp} = \frac{\sqrt{c^2 - V^2}}{\gamma^2 \left(1 - \frac{V^2}{c^2}\right)} = \frac{c}{\gamma}. \quad (30)$$

Therefore, when the time unit is increased  $\gamma^2$  times, the length unit in the  $x$ -direction (parallel to the  $S'$  velocity) will not change at all and in the transverse  $y$ - and  $z$ -directions will be increased  $\gamma$  times. To write Zahar transformations (26) in these new units, we must therefore divide  $t'$  by  $\gamma^2$ ,  $y'$  and  $z'$  by  $\gamma$ , and leave  $x'$  unchanged. As a result we again obtain Voigt transformations (17).

As we see, Voigt transformations by themselves are empirically empty without detailing how coordinates involved in these transformations are related to the behavior of real clocks and rulers. Voigt transformations require an additional input about the nature of real clocks not to be just a mathematical exercise but an instrument to answer questions about the physics of the real world. Let us demonstrate this by considering a natural question whether the Poincaré-Einstein clock synchronization is equivalent to another synchronization method by slow clock transport.

Let us consider an inertial reference frame  $S'$  moving with respect to aether frame  $S$  (which is singled out by Voigt transformations) with velocity  $V$ . Clocks in  $S'$  are synchronized by Poincaré-Einstein procedure. Let one of the so synchronized clocks be transported slowly in  $S'$  from one point  $A$  to another point  $B$  with the velocity  $u' \ll c$  along the  $x$ -axis (for simplicity). If this process takes time  $\tau$  according to the clocks in  $S'$ , then  $\Delta\xi = u'\tau$  and the inverse Voigt transformations (18) will give

$$\Delta x = \gamma_V^2(u' + V)\tau, \quad \Delta t = \gamma_V^2 \left[ 1 + \frac{u'V}{c^2} \right] \tau. \quad (31)$$

The Voigt transformations (17) imply just the same velocity addition rule as the Lorentz transformations. Therefore, with respect to  $S$ , the clock moving in  $S'$  has the velocity

$$u = \frac{u' + V}{1 + \frac{V}{c^2}u'} \approx (u' + V) \left( 1 - \frac{V}{c^2}u' \right) \approx V + \frac{u'}{\gamma_V^2}. \quad (32)$$

Hence, according to (17) and (31), while moving from  $A$  to  $B$ , the clock will measure the proper time

$$\tau' = \Delta t - \frac{u}{c^2}\Delta x \approx \gamma_V^2\tau \left[ 1 - \frac{V^2}{c^2} - \frac{u'V}{c^2\gamma_V^2} \right] = \left( 1 - \frac{u'V}{c^2} \right) \tau. \quad (33)$$

However, this time interval is measured in time units of the reference frame moving with velocity  $u$ , not  $V$ . Therefore, we need to rescale it in the  $S'$  time units in order to compare to the reading of the clock at rest in  $S'$  at point  $B$ . This rescaling cannot be done on the base of Voigt transformations alone and this is the point where an extra empirical input about the behavior of real clocks is needed. In particular we must remember how time units for Voigt transformations were defined if we stick the interpretation of these transformations given above with either Lorentz or Zahar transformations as the primary source describing the behavior of real clocks. Having all this in mind, we write the rescaling as follows

$$\tilde{\tau} = \tau' \frac{\gamma_u^n}{\gamma_V^n}, \quad (34)$$

where  $n = 1$ , if the real clocks behave as prescribed by special relativity, and  $n = 2$ , if the real clocks are Galilean. But from (32) it is easy to find that

$$\gamma_u \approx \gamma_V \left( 1 + \frac{u'V}{c^2} \right). \quad (35)$$



Therefore, from (33) and (35), we get

$$\tilde{\tau} = \tau \left( 1 - \frac{u'V}{c^2} \right) \left( 1 + n \frac{u'V}{c^2} \right) \approx \tau \left[ 1 + (n - 1) \frac{u'V}{c^2} \right]. \quad (36)$$

As we see,  $\tilde{\tau} = \tau$  if and only if  $n = 1$ . That is only in the Minkowski world the Poincaré-Einstein synchronization and synchronization by slow clock transport agree with each other. In Galilean world these two synchronization methods will give different results.

Robertson [59] and later Mansouri and Sexl [60] developed a general framework for test theories of special relativity which can be considered as a generalization of Voigt, Lorentz and Zahar transformations. These generalized transformations can be written in the form [61]

$$\begin{aligned} x' &= \lambda \frac{c_{\parallel}}{c_{\perp}} \gamma (x - V t), \\ y' &= \lambda y, \\ z' &= \lambda z, \\ t' &= \lambda \frac{c}{c_{\perp}} \gamma \left( t - \frac{V}{c^2} x \right), \end{aligned} \quad (37)$$

where  $c_{\parallel}$  and  $c_{\perp}$  are two-way light speeds in the moving frame  $S'$  in the parallel and perpendicular directions to the the velocity  $\vec{V}$  of  $S'$  with respect to the preferred (æther) frame  $S$ , in which light propagates isotropically with the speed  $c$ . The parameters  $c_{\parallel}/c$ ,  $c_{\perp}/c$  and the conformal factor  $\lambda$  are, in general, functions of  $\gamma$ . For example, if  $\lambda = 1$ ,  $c_{\parallel} = c_{\perp} = c$ , we get from (37) Lorentz transformations; if  $\lambda = 1$ ,  $c_{\parallel} = c/\gamma^2$ ,  $c_{\perp} = c/\gamma$ , we get Zahar transformations, and, finally, if  $\lambda = \gamma^{-1}$ ,  $c_{\parallel} = c_{\perp} = c$ , we get Voigt transformations. It can be shown using (37) that “clock synchronization by clock transport and by the Einstein procedure agree if and only if the time dilatation factor is given exactly by the special relativistic value” [60], that is if  $\lambda c/c_{\perp} = 1$ .

We hope that our above discussions of the Voigt transformations convinced you that a modern reader can find a lot of relativistic content in these transformations. However, all of them are later time readings and we suspect that none were actually possible at Voigt’s time. It would be not true to say that Voigt was at the origin of relativistic revolution. Alas, he missed this opportunity.

## 4 Another derivation of Voigt transformations

One of the reasons why Voigt's 1887 paper remained unnoticed was its clumsy and unwieldy derivation of Voigt's transformations. In this section we will try to give more modern and succinct derivation which will reveal basic physical inputs and assumptions behind it. Inspiration for this derivation and its subsequent generalization comes mostly from one postulate derivation of Lorentz transformations originated in 1910 by von Ignatowsky [62, 63]<sup>4</sup> and further developed by Frank and Rothe [64]. More precisely, our approach rests heavily on the ideas presented in [65]. Eventually we generalize the approach of [65] by relaxing the requirement of spatial isotropy.

Under the assumption that measuring rods do not change their length when gently set into a state of uniform motion, the Galilean transformations (19) are simply a statement that the length of a finite interval is an additive quantity: the length of a union of two intervals is equal to the sum of their lengths. However, it seems not too fantastic to imagine that Voigt in 1887 could question whether the measuring rods really do not change their lengths when moving through æther. At least in 1888 Oliver Heaviside showed that the electric field of a charge in motion relative to the æther is no longer spherically symmetric and becomes distorted in the longitudinal direction. Influenced by the Heaviside's result, FitzGerald and later Lorentz came out with a hypothesis that Heaviside distortion might be applied to the molecular forces too and thus rigid bodies when moving through the æther should experience deformations. In particular, in 1889 FitzGerald published in then little known American journal *Science* a letter in which he wrote:

“I have read with much interest Messrs. Michelson and Morley's wonderfully delicate experiment attempting to decide the important question as to how far the æther is carried along by the earth. Their result seems opposed to other experiments showing that the æther in the air can be carried along only to an inappreciable extent. I would suggest that almost the only hypothesis that can reconcile this opposition is that the length of material bodies changes, according as they are moving through the æther or across it, by an amount depending on the square of the ratio of their velocities to that of light. We know that electric forces are affected by the motion of the electrified bodies relative to the æther,

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<sup>4</sup>An English translation of [62] is available at [https://en.wikisource.org/wiki/Translation:Some\\_General\\_Remarks\\_on\\_the\\_Relativity\\_Principle](https://en.wikisource.org/wiki/Translation:Some_General_Remarks_on_the_Relativity_Principle)

and it seems a not improbable supposition that the molecular forces are affected by the motion, and that the size of a body alters consequently” [66].

Interestingly, this FitzGerald’s publication sank into oblivion and was virtually unknown until 1967 when it was unearthed by Brush [66]. Paradoxically, FitzGerald himself was not aware whether his letter to *Science* was published or not: “FitzGerald did publish the contraction hypothesis before Lorentz, but no one involved in later research on the subject knew about this publication, including FitzGerald himself” [66]. He did promote his deformation idea mostly in lectures and was relieved to know that in 1892 Lorentz came out with essentially the same idea. In 1894 FitzGerald wrote to Lorentz: “I am particularly delighted to hear that you agree with me, for I have been rather laughed at for my view over here” [66].

If we accept the possibility that measuring rods can change their length when in motion, instead of  $x = x' + Vt$  we should write  $x = k_1(V)x' + Vt$ , where  $k_1(V)$  accounts for a possible change of the measuring rod’s length in the moving frame  $S'$ . But from the point of view of an observer in  $S'$ , the reference frame  $S$  moves with velocity  $-V$  (although there are some subtleties in this reciprocity principle [67, 68], we assume that in 1887 Voigt probably would not question it). Therefore we can analogously write  $x' = k_2(-V)x - Vt'$ . If we assume the validity of the relativity principle, then  $k_1(V) = k_2(V)$ , and if we assume the spatial isotropy, then  $k_1(-V) = k_1(V)$ . However, for Voigt, a true believer in æther, relativity principle was not evident, so we do not assume its validity for a while, neither the spatial isotropy. Hence, we have

$$x' = \frac{1}{k_1(V)}(x - Vt), \quad x = \frac{1}{k_2(-V)}(x' + Vt'). \quad (38)$$

From these equations we can express  $t'$  in terms of  $x$  and  $t$ , or  $t$  in terms of  $x'$  and  $t'$ :

$$t' = \frac{1}{k_1(V)} \left[ t - \frac{1 - K(V)}{V} x \right], \quad t = \frac{1}{k_2(-V)} \left[ t' + \frac{1 - K(V)}{V} x' \right], \quad (39)$$

where  $K(V) = k_1(V)k_2(-V)$ . Therefore, we get the transformation

$$\begin{aligned} x' &= \frac{1}{k_1(V)} (x - V t), \\ y' &= y, \\ z' &= z, \\ t' &= \frac{1}{k_1(V)} \left[ t - \frac{1 - K(V)}{V} x \right], \end{aligned} \quad (40)$$

and its inverse

$$\begin{aligned} x &= \frac{1}{k_2(-V)} (x' + V t'), \\ y &= y', \\ z &= z', \\ t &= \frac{1}{k_2(-V)} \left[ t' + \frac{1 - K(V)}{V} x' \right]. \end{aligned} \quad (41)$$

We have assumed that the transverse coordinates do not change. This is not the most general possibility but probably the one which Voigt's contemporaries would infer from Heaviside's result on the moving Coulomb field distortions.

Voigt was interested in wave equation (1) which is obviously invariant under the scale transformations. Considering all coordinates that differ only by a scale change as equivalent, we can choose new coordinates

$$X' = k_1(V)x', \quad Y' = k_1(V)y', \quad Z' = k_1(V)z', \quad T' = k_1(V)t', \quad (42)$$

to regain the Galilean relation  $x = X' + Vt$ . In new coordinates (40) takes the form

$$\begin{aligned} X' &= x - V t, \\ Y' &= k_1(V) y, \\ Z' &= k_1(V) z, \\ T' &= t - \frac{1 - K(V)}{V} x, \end{aligned} \quad (43)$$

while its inverse (41) transforms into

$$\begin{aligned} x &= \frac{1}{K(V)} [X' + V T'],, \\ y &= k_1^{-1}(V) Y', \\ z &= k_1^{-1}(V) Z', \\ t &= \frac{1}{K(V)} \left[ T' + \frac{1 - K(V)}{V} X' \right]. \end{aligned} \tag{44}$$

The velocity addition rule following from (44) has the form

$$\begin{aligned} v_x &= \frac{v'_x + V}{1 + \frac{1 - K(V)}{V} v'_x}, \\ v_y &= \frac{K(V) v'_y}{k_1(V) \left[ 1 + \frac{1 - K(V)}{V} v'_x \right]}, \quad v_z = \frac{K(V) v'_z}{k_1(V) \left[ 1 + \frac{1 - K(V)}{V} v'_x \right]}. \end{aligned} \tag{45}$$

If we now require the universality of the light velocity as it follows from the assumed invariance of the wave equation, and assume  $v'_x = c$ ,  $v'_y = v'_z = 0$ , then we get from (45)  $v_y = v_z = 0$ ,

$$c = v_x = \sqrt{c^2 - v_y^2 + v_z^2} = \frac{c + V}{1 + \frac{1 - K(V)}{V} c}, \tag{46}$$

and therefore,

$$K(V) = 1 - \frac{V^2}{c^2} \equiv \frac{1}{\gamma^2}. \tag{47}$$

On the other hand, if we take  $v'_y = c$ ,  $v'_x = v'_z = 0$ , from (45) we obtain

$$v_x = V, \quad v_y = \frac{K(V) c}{k_1(V)} = \frac{c}{\gamma^2 k_1(V)}, \quad v_z = 0, \tag{48}$$

which together with the condition  $v_x^2 + v_y^2 + v_z^2 = c^2$  then determines  $k_1(V)$ :

$$k_1(V) = \frac{1}{\gamma}. \tag{49}$$

Substituting (47) and (49) into (43) we get indeed the Voigt transformations (17).

Now we will show that a little generalization of this derivation provides an interesting perspective on the relativity principle. Let us return

to the original coordinates before rescaling (42), but this time we will not assume that the transverse coordinates remain unchanged. So we modify (40) and (41) into

$$\begin{aligned}x' &= \frac{1}{k_1(V)} (x - V t), \\y' &= \lambda(V) y, \\z' &= \lambda(V) z, \\t' &= \frac{1}{k_1(V)} \left[ t - \frac{1 - K(V)}{V} x \right]\end{aligned}\tag{50}$$

and

$$\begin{aligned}x &= \frac{1}{k_2(-V)} (x' + V t'), \\y &= \lambda^{-1}(V) y', \\z &= \lambda^{-1}(V) z', \\t &= \frac{1}{k_2(-V)} \left[ t' + \frac{1 - K(V)}{V} x' \right]\end{aligned}\tag{51}$$

respectively. From (51) we get the following velocity addition rule:

$$\begin{aligned}v_x &= \frac{v'_x + V}{1 + \frac{1-K(V)}{V} v'_x} = F(v'_x, V), \\v_y &= \frac{k_2(-V) v'_y}{\lambda(V) \left[ 1 + \frac{1-K(V)}{V} v'_x \right]}, \quad v_z = \frac{k_2(-V) v'_z}{\lambda(V) \left[ 1 + \frac{1-K(V)}{V} v'_x \right]}.\end{aligned}\tag{52}$$

The reasoning analogous to what followed after (45) makes it clear that the universality of the light velocity will then demand the validity of (47) and

$$\frac{k_2(-V)}{\lambda(V)} = \frac{1}{\gamma}.\tag{53}$$

From (47) and (53) we can get

$$\lambda(V) = \frac{k_2(-V)}{\sqrt{K(V)}} = \sqrt{\frac{k_2(-V)}{k_1(V)}},\tag{54}$$

and

$$\frac{1}{k_1(V)} = \sqrt{\frac{k_2(-V)}{k_1(V)}} \frac{1}{\sqrt{k_1(V)k_2(-V)}} = \frac{\lambda(V)}{\sqrt{K(V)}} = \lambda(V)\gamma.\tag{55}$$

Therefore, (50) takes the form

$$\begin{aligned}x' &= \lambda(V)\gamma(x - Vt), \\y' &= \lambda(V)y, \\z' &= \lambda(V)z, \\t' &= \lambda(V)\gamma\left(t - \frac{V}{c^2}x\right).\end{aligned}\tag{56}$$

Both Einstein [9] and Poincaré [24, 25] got these  $\lambda$ -Lorentz transformations and then both argued that  $\lambda(V) = 1$ . The arguments of Einstein, in section 3 of his paper, were more physical as he correctly identified the physical meaning of  $\lambda(V)$ : it corresponds to a contraction of the length of a moving rod which is perpendicular to the direction of its motion. "He makes then the clever statement that, because the rod is perpendicular to the direction of motion, this possible contraction can well depend on the relative velocity, but not on its sign!" [26]. Therefore,  $\lambda(-V) = \lambda(V)$ . On the other hand, if we adopt the relativity principle so that the æther frame loses its privileged role, then transformations (56) should form a group which implies

$$\lambda(V_1 \oplus V_2) = \lambda(V_1)\lambda(V_2),\tag{57}$$

where  $V_1 \oplus V_2$  denotes the relativistic sum of velocities. In particular, if we take  $V_1 = -V_2 = V$ , then we get  $\lambda(V)\lambda(-V) = \lambda(0) = 1$  which in combination with  $\lambda(-V) = \lambda(V)$  and positivity of  $\lambda(V)$  implies  $\lambda(V) = 1$ .

Poincaré's reasoning, in section 4 of his paper, was more formal and mathematical. He makes a thorough analysis of the Lorentz's group including not only the boosts, but also the spatial rotations and gets  $\lambda(V) = 1$  essentially from the following group theoretical argument (for more complete analysis of Poincaré's derivation, see [69, 70, 71]). Let us denote the  $\lambda$ -Lorentz transformations (56) by  $\hat{L}$  and let  $\hat{R} = \hat{R}^{-1}$  be a rotation by  $\pi$  around the  $y$ -axis which changes signs of the  $x$  and  $z$  coordinates and leaves  $y$  and  $t$  unchanged. Then by the group property  $\hat{R}\hat{L}\hat{R}$  must belong to the same Lorentz group. But, as it can be easily

checked,

$$\begin{aligned}
 \hat{R}\hat{L}\hat{R}(x) &= \lambda(V)\gamma(x + Vt), \\
 \hat{R}\hat{L}\hat{R}(y) &= \lambda(V)y, \\
 \hat{R}\hat{L}\hat{R}(z) &= \lambda(V)z, \\
 \hat{R}\hat{L}\hat{R}(t) &= \lambda(V)\gamma\left(t + \frac{V}{c^2}x\right).
 \end{aligned} \tag{58}$$

Comparing with (56), we conclude that this is a pure boost with velocity  $-V$  instead of  $V$  and, as the boost is uniquely determined by the velocity, we must have  $\lambda(-V) = \lambda(V)$ , which in combination with  $\lambda(V)\lambda(-V) = 1$  and positivity of  $\lambda(V)$  implies  $\lambda(V) = 1$ .

Note that the multiplication law (57) first appeared in Poincaré's May 1905 letter to Lorentz [72]. Einstein in [9] doesn't provide this multiplication law in its full generality — he considers only mutually inverse transformations with  $V_1 = -V_2$ .

In fact, both derivations of  $\lambda(V) = 1$  implicitly assume spatial isotropy. Although natural, this is not the most general possibility. So let us go ahead without this assumption and see what gems were thrown overboard by Einstein and Poincaré when insisting on it.

First of all, Poincaré missed an opportunity to discover that Maxwell equations and wave equation are invariant not only with respect to the Lorentz group but also with respect to much wider class of conformal transformations (we think, for Einstein this feat was unrealistic in 1905). This fact was established several years later, namely in 1910, by Cunningham and Bateman [73] (their road to this discovery is described in [74]). Conformal group is a fifteen parameter Lie group. It contains as a subgroup the ten parameter Poincaré group (inhomogeneous Lorentz transformations). A general Poincaré transformation is a combination of a rotation (three parameters) with a boost (three parameters), followed by a translation in space-time (four parameters). Additional five parameters of the conformal group are:  $\lambda$  of the scale transformation, promoted to an independent parameter representing dilatation, and four parameters of the so called special conformal transformation ( $x_0 = x^0 = ct, x_1 = -x^1 = -x, x_2 = -x^2 = -y, x_3 = -x^3 = -z$ )

$$x'^{\mu} = \frac{x^{\mu} - \alpha^{\mu} x_{\nu} x^{\nu}}{1 - 2\alpha_{\mu} x^{\mu} + \alpha_{\mu} \alpha^{\mu} x_{\nu} x^{\nu}}. \tag{59}$$



Taking  $\alpha^\mu = (0, \vec{a})$ , in the non-relativistic limit  $c \rightarrow \infty$ ,  $|\vec{a}| \rightarrow 0$ ,  $c^2 \vec{a} \rightarrow \vec{g}/2$ , where  $\vec{g}$  is some constant 3-vector, we get

$$t' = t, \quad \vec{r}' = \vec{r} - \frac{\vec{g} t^2}{2}. \quad (60)$$

Therefore, in the non-relativistic limit the transformation (59) describes the transition to a non-inertial frame moving with a constant acceleration  $\vec{g}$ .

Special conformal transformation is tightly related to uniformly accelerated motion in relativistic case too (see the appendix A). It seems therefore tempting to interpret the special conformal transformation (59) as a transition to a uniformly accelerated reference frame. There were long-lasting efforts [75, 76, 77, 78] to make sense of this interpretation however they were not very successful, and the modern consensus is that such interpretation is untenable [79, 80]. Passive interpretation (as a change of the reference frame) is not always advantageous whereas an active interpretation of transformations and the corresponding notion of symmetries is often more fruitful.

The structure of the conformal group (more precisely, the commutation relations of its Lie algebra) indicates that “a Poincaré invariant theory that is also conformally invariant necessarily enjoys scale invariance”, but the contrary is not true: “group theory does not establish conformal invariance from scale and Poincaré invariance” [81]. This fact probably explains why Poincaré missed an opportunity to discover conformal invariance of the Maxwell equations (for a pedagogical exposition of the role of the conformal group in electrodynamics see, for example, [82]). Interestingly, if the dimensionality of space-time is greater than four, the Maxwell equations are no longer conformal invariant even though they remain scale invariant [81].

The second postulate about the universality of light speed in inertial reference frames plays the role of founding principle in Einstein’s version of special relativity. One may wonder what is so special about light that makes its speed a fundamental constant. In the Standard Model there is no compelling theoretical reason to assume that the photon mass is strictly zero [83]. A tiny photon mass can be generated by means of the Stückelberg mechanism that retains gauge invariance, unitarity and renormalizability [84]. Of course, if the Standard Model is a low energy remnant of a grand unification model based on a simple nonabelian group like  $SU(5)$ ,  $SO(10)$ , or  $E_6$ , then the photon will be necessarily truly

massless because one cannot generalize Stückelberg mechanism to a non-Abelian gauge symmetry. However, even in this case the masslessness of the photon originates from the particular pattern of the electroweak symmetry breaking. One may suspect therefore that the fundamental constant  $c$  appearing in the Lorentz transformations is not necessarily light speed in its essence. This is indeed the case as the following simple argument shows [65].

Let us return to the velocity addition rule (52). If we assume the validity of the relativity principle, then  $k_1(V) = k_2(V)$  and hence,  $K(-V) = K(V)$  so that the function  $F(V_1, V_2)$  in (52) is an odd function:

$$F(-V_1, -V_2) = -F(V_1, V_2). \quad (61)$$

Let us combine this oddness of  $F(V_1, V_2)$  with the reciprocity principle  $V_{AB} = -V_{BA}$  for the velocity  $V_{AB}$  of an object  $A$  relative to an object  $B$ . We will have

$$\begin{aligned} V_{AB} = F(V_{AC}, V_{CB}) = -V_{BA} = -F(V_{BC}, V_{CA}) = \\ -F(-V_{CB}, -V_{AC}) = F(V_{CB}, V_{AC}). \end{aligned} \quad (62)$$

Therefore,  $F(V_1, V_2)$  is a symmetric function of its arguments what in light of its definition in (52) implies

$$\frac{1 - K(V_2)}{V_2} V_1 = \frac{1 - K(V_1)}{V_1} V_2. \quad (63)$$

This is only possible if

$$\frac{1 - K(V)}{V^2} = k = \text{const.} \quad (64)$$

If  $k < 0$ , then we will face a number of strange features like positive velocities that sum up in a negative velocity. Most importantly, in this case it will be difficult if not impossible to define a causal structure on the corresponding space-time [65, 67, 85].

If  $k \geq 0$ , we can take  $k = 1/c^2$ , where  $c$  is some constant with a dimensionality of velocity. Nothing however indicates in such a derivation that  $c$  is the light velocity in vacuum. In this case  $K(V)$  is given by (47) and the first equation in (52) coincides to the relativistic addition law for parallel velocities. To promote  $c$  to the invariant velocity for perpendicular velocities too, we will need (53). Then (52) will completely coincide

with the special-relativistic velocity-addition law.  $k = 0$  corresponds to the  $c \rightarrow \infty$  limit and gives Galilean velocity-addition law. Whether the light velocity in vacuum is the invariant velocity  $c$  is the question about the properties of light and it has nothing to do with the logical structure of special relativity. However, experimental limits on the nonzero photon mass are very tough: the photon Compton wavelength is already known to be at least comparable in dimension to the Earth-Sun distance [86]. So for all laboratory-scale purposes the photon mass safely may be taken as zero and therefore  $c$  may be considered as light velocity for all practical purposes.

## 5 Lalan-Alway-Bogoslovsky transformations and anisotropic special relativity

It was hinted in the previous section that  $\lambda$ -Lorentz transformations (56) with  $\lambda(V)$  satisfying the functional equation (57) are the most general transformations that follow from Einstein's two postulates under some natural assumptions (homogeneity of space and time, the reciprocity principle). To solve the functional equation (57) it is beneficial to realize, as it is well known, that the natural parameter for special Lorentz transformations is not the velocity  $V$  but the rapidity  $\psi$  [87] defined through

$$\tanh \psi = \beta = \frac{V}{c}. \quad (65)$$

Rapidities are additive for the special Lorentz transformations, and the functional equation (57) takes the form of the Cauchy exponential functional equation

$$\lambda(\psi_1 + \psi_2) = \lambda(\psi_1) \lambda(\psi_2). \quad (66)$$

Although there exist infinitely many wildly discontinuous solutions of (66), the continuous solutions, which are the only ones acceptable in the context of  $\lambda$ -Lorentz transformations, all have the form [88]

$$\lambda(\psi) = e^{-b\psi} = \left( \frac{1 - \beta}{1 + \beta} \right)^{b/2} \quad (67)$$

with some real constant  $b$ . Indeed, we will have from (66),  $\lambda(\psi + d\psi) = \lambda(\psi)\lambda(d\psi)$ , and because  $\lambda(0) = 1$ ,  $\lambda(\psi) + \lambda'(\psi)d\psi \approx (1 + \lambda'(0)d\psi)\lambda(\psi)$ , which leads to the differential equation  $\lambda'(\psi) = \lambda'(0)\lambda(\psi)$  with the solution (67) with  $\lambda'(0) = -b$ .

Therefore,  $\lambda$ -Lorentz transformations (56) take the form

$$\begin{aligned}x' &= \left(\frac{1-\beta}{1+\beta}\right)^{b/2} \gamma(x - Vt), \\y' &= \left(\frac{1-\beta}{1+\beta}\right)^{b/2} y, \\z' &= \left(\frac{1-\beta}{1+\beta}\right)^{b/2} z, \\t' &= \left(\frac{1-\beta}{1+\beta}\right)^{b/2} \gamma\left(t - \frac{V}{c^2}x\right).\end{aligned}\quad (68)$$

Let us introduce light-cone coordinates [89]

$$u = ct + x, \quad v = ct - x. \quad (69)$$

Then we get from (68)

$$u' = e^{-(1+b)\psi} u, \quad v' = e^{(1-b)\psi} v, \quad y' = e^{-b\psi} y, \quad z' = e^{-b\psi} z. \quad (70)$$

Using

$$\frac{u'}{v'} = e^{-2\psi} \frac{u}{v}, \quad u'v' = e^{-2b\psi} uv, \quad (71)$$

we can easily find invariant quantities

$$\begin{aligned}\left(\frac{v'}{u'}\right)^b u'v' &= \left(\frac{v}{u}\right)^b uv, \quad \left(\frac{v'}{u'}\right)^b y'^2 = \left(\frac{v}{u}\right)^b y^2, \quad \left(\frac{v'}{u'}\right)^b z'^2 = \left(\frac{v}{u}\right)^b z^2, \\ \frac{u'v'}{y'^2} &= \frac{uv}{y^2}, \quad \frac{u'v'}{z'^2} = \frac{uv}{z^2}.\end{aligned}\quad (72)$$

Only first three of them are independent. Therefore, the following quantity, which was considered in the literature as a generalization of the relativistic interval, is invariant under  $\lambda$ -Lorentz transformations (56):

$$\begin{aligned}s^2 &= \left(\frac{v}{u}\right)^b (uv - y^2 - z^2) \left(\frac{uv}{uv - y^2 - z^2}\right)^b = \\ &v^{2b}(uv - y^2 - z^2)^{1-b} = (ct - x)^{2b} (c^2t^2 - x^2 - y^2 - z^2)^{1-b}.\end{aligned}\quad (73)$$

Since the  $\lambda$ -Lorentz transformations (56) are linear, the differentials of coordinates are transformed as the coordinates themselves. Consequently, the metric of the space-time, invariant with respect to the  $\lambda$ -Lorentz transformations (56) has the form

$$ds^2 = (cdt - dx)^{2b} (c^2dt^2 - dx^2 - dy^2 - dz^2)^{1-b}. \quad (74)$$

Introducing the fixed null-vector  $n^\mu = (1, 1, 0, 0) = (1, \vec{n})$ ,  $\vec{n}^2 = 1$ , which defines a preferred null-direction in the space-time, (74) can be rewritten as follows

$$ds^2 = (n_\nu dx^\nu)^{2b} (dx_\mu dx^\mu)^{1-b}, \tag{75}$$

As we see, in general, if the spatial isotropy is not assumed, Einstein's two postulates do not lead to Lorentz transformations and Minkowski metric. Instead they lead to more general  $\lambda$ -Lorentz transformations and Finslerian metric (75). In the Finsler geometry the squared line element is homogeneous of second degree in the coordinate increments:  $ds^2 = f(x_\mu, dx_\nu)$ ,  $f(x_\mu, \alpha dx_\nu) = \alpha^2 f(x_\mu, dx_\nu)$ . Although Finsler geometry is more general geometry than the Riemann geometry, which is obtained than  $f(x_\mu, dx_\nu)$  is a quadratic form in coordinate increments, almost all the results of Riemannian geometry can be developed in the Finsler case too [90]. For applications of Finsler geometry in physics see, for example, [91, 92].

Note that the way how we obtained (75) is highly non-unique. However (75) is the one of simplest Finsler metric which can be constructed from the invariants of the  $\lambda$ -Lorentz transformations. That it may have a physical meaning is reinforced by the fact that (75) emerges also in the process of deformation of the very special relativity symmetry group  $ISIM(2)$  [93].

$\lambda$ -Lorentz transformations (56) give the coordinate transformations between inertial frames in the case when one of them moves with respect to other with relative velocity  $V$  along the  $x$ -axis (along the preferred direction  $\vec{n}$ ). What about other inertial frames?

The corresponding generalized Lorentz transformations were obtained in [94] by using the Lobachevsky geometry of the special-relativistic velocity space. In the appendix B we give a different derivation of these transformations. To summarize the results of this appendix, the Finsler metric (75) is invariant under the generalized Lorentz transformations [94]

$$x'^\mu = D(\lambda) R^\mu_\nu(\vec{m}; \alpha) L^\nu_\sigma(\vec{V}) x^\sigma, \tag{76}$$

where  $D(\lambda)$ ,  $R^\mu_\nu(\vec{m}; \alpha)$  and  $L^\nu_\sigma(\vec{V})$  represent, respectively, the dilatation transformation, the spacial rotation and the Lorentz boost each defined in the appendix.

Using (116) and (119), we can get an explicit form of these generalized

Lorentz transformations:

$$\begin{aligned} x'_0 &= \left[ \gamma(1 - \vec{\beta} \cdot \vec{n}) \right]^b \gamma(x_0 - \vec{\beta} \cdot \vec{r}), \\ \vec{r}' &= \left[ \gamma(1 - \vec{\beta} \cdot \vec{n}) \right]^b \left\{ \vec{r} - \frac{\vec{\beta}(x_0 - \vec{n} \cdot \vec{r})}{1 - \vec{\beta} \cdot \vec{n}} - \right. \\ &\left. \vec{n} \left[ \gamma \vec{\beta} \cdot \vec{r} + \frac{\gamma - 1}{\gamma} \frac{\vec{n} \cdot \vec{r}}{1 - \vec{\beta} \cdot \vec{n}} + \frac{(\gamma - 1)\vec{\beta} \cdot \vec{n} - \gamma\beta^2}{1 - \vec{\beta} \cdot \vec{n}} x_0 \right] \right\}. \end{aligned} \quad (77)$$

We have checked that these expressions are equivalent to Eq.29 of [94]. If  $\vec{\beta}$  and  $\vec{n}$  are parallel and the  $x$ -axis is along  $\vec{\beta}$ , (77) are reduced to the  $\lambda$ -Lorentz transformations (68). Another interesting limiting case is when  $\vec{\beta}$  and  $\vec{n}$  are perpendicular to each other. Assuming that the  $x$  and  $y$  axes are respectively along  $\vec{\beta}$  and  $\vec{n}$ , we get from (77)

$$\begin{aligned} x' &= \gamma^b [x - Vt + \beta y], \\ y' &= \gamma^{1+b} [(1 - \beta^2)y - \beta(x - Vt)], \\ z' &= \gamma^b z, \\ t' &= \gamma^{1+b} \left[ t - \frac{V}{c^2} x \right]. \end{aligned} \quad (78)$$

It may appear that transformations (78) violate the reciprocity principle because according to (78) the origin  $x = 0, y = 0, z = 0$  of the frame  $S$  moves in the frame  $S'$  with velocity  $V'_x = -V/\gamma, V'_y = \beta V, V'_z = 0$ , not with  $V'_x = -V, V'_y = 0, V'_z = 0$  as one might expect. However, remember that the generalized Lorentz transformations (76) contain the rotation  $R(\vec{m}; \alpha)$ . When passively interpreted, this rotation implies in the particular case of (78) that the  $x$  and  $y$ -axes of  $S'$  are rotated around the  $z$ -axis with an angle  $\alpha$  such that  $\sin \alpha = \beta, \cos \alpha = 1/\gamma$  according to (111) and (113). No wonder that the vector  $-\vec{V}$  has the coordinates  $V'_x = -V \cos \alpha = -V/\gamma$  and  $V'_y = V \sin \alpha = \beta V$  in  $S'$ .

To our best knowledge, it was Lalan [95, 96, 97] who first derived  $(1 + 1)$ -dimensional  $\lambda$ -Lorentz transformations and demonstrated that the metric invariant under such transformations was of pseudo-Finslerian type rather than of the Minkowski pseudo-Riemannian.  $(1 + 3)$ -dimensional generalized Lorentz transformations (77) were first discovered by Alway [98]. He gives no details on how the transformations were obtained but it seems that the spirit of the derivation was somewhat

similar to the one presented above as he cites Pars [99] and mentions that one of the Pars' assumptions, namely isotropic behavior of clocks, is not guaranteed in general. Pars like von Ignatowsky [62] provides the derivation of the Lorentz transformations without Einstein's second postulate. He doesn't cite neither von Ignatowsky nor anybody else in his paper. The fate of Pars's and von Ignatowsky's innovative derivations of Lorentz transformations were similar: "like numerous others that followed, these have gone largely unnoticed" [100]. Alway's generalized Lorentz transformations and the corresponding Finslerian space-time metric discovered by him also fell into oblivion immediately after it was recognized that experimental observations severely constrain spatial anisotropy essentially making the parameter  $b$  zero [101]. Fortunately, generalized Lorentz transformations (77) were soon rediscovered by Bogoslovsky [94, 102] who then thoroughly investigated their physical consequences [103, 104, 105].  $(1+1)$ -dimensional  $\lambda$ -Lorentz transformations for the case  $b = 1/2$  was independently rediscovered by Brown [106] and then generalized to any value of  $b$  by Budden [107]. Apparently, initially they were unaware of Bogoslovsky's pioneering contributions. Later they cite Bogoslovsky in [108, 109] and [110], but neither Lalan nor Alway are mentioned in their work. Slightly different perspective on Finslerian relativity is given by Asanov [111].

As was already mentioned, observations severely constrain the Finslerian parameter  $b$ . Some æther-drift experiments imply  $|b| < 10^{-10}$ , while the Hughes-Drever type limits on the anisotropy of inertia can potentially lower the limit on the Finslerian parameter  $b$  up to  $|b| < 10^{-26}$ , though in a model-dependent way [93]. It may appear therefore that Strnad's conclusion [101] that the special relativity is valid for all practical purposes, and the Alway's extension of Lorentz transformations [98] is superfluous, is well justified. However, notwithstanding our firm belief that special relativity is indeed valid for all practical purposes, its Finslerian extension described above, in our opinion, has a very significant conceptual value. Let us explain why.

Besides the generalized pure Lorentz boosts (77), the Finsler metric (75) is left invariant by rotations about preferred spatial direction  $\vec{n}$  and by space-time translations  $x'^{\mu} = x^{\mu} + a^{\mu}$ . Together these transformations constitute an 8-parameter group of isometries of the Finsler space-time with metric (75). This group is given a fancy name  $\text{DISIM}_b(2)$  (deformed ISIM(2),  $b$  being the deformation parameter) in [93]. Both  $\text{DISIM}_b(2)$  and the 10-parameter Poincaré group are subgroups of the 11-parameter

Weyl group (Poincaré transformations augmented with arbitrary dilations). It may appear therefore that  $\text{DISIM}_b(2)$  plays the role similar to the Poincaré group but it doesn't. The role analogous to the Poincaré group is played by  $\text{ISIM}(2)$  which is the  $b \rightarrow 0$  limit (or better to say contraction in the sense of İnönü and Wigner [112]) of  $\text{DISIM}_b(2)$ . Although the Finsler metric (75) is reduced to the Minkowski metric in the  $b \rightarrow 0$  limit, the generalized Lorentz transformations (77) are not reduced to the ordinary Lorentz transformations. Instead in this limit the transformations (77) augmented with rotations about the preferred direction  $\vec{n}$  constitute a 4-parameter subgroup of the Lorentz group called  $\text{SIM}(2)$  (similitude group of the plane consisting of dilations, translations and rotations of the plane). If we further add space-time translations, we get a 8-parameter subgroup of the Poincaré group  $\text{ISIM}(2)$  which is a semi-direct product of  $\text{SIM}(2)$  with the group of space-time translations. Very special relativity developed by Cohen and Glashow [113] assumes that the exact symmetry group of nature may be not the Poincaré group but its subgroup  $\text{ISIM}(2)$ .

It is known [114, 115] that the Lorentz group has no 5-parameter subgroup and only one 4-parameter subgroup  $\text{SIM}(2)$  up to isomorphism. As a result, very special relativity breaks Lorentz symmetry in a very mild and minimal way. For amplitudes satisfying appropriate analyticity properties,  $CPT$  discrete symmetry follows from  $\text{SIM}(2)$ , but  $\text{SIM}(2)$  violates  $P$  and  $T$  discrete symmetries [113]. However, the incorporation of either  $P$ ,  $T$  or  $CP$  discrete symmetries enlarges  $\text{SIM}(2)$  subgroup to the full Lorentz group and thus, for particle physics theories conserving any one of these three discrete symmetries Lorentz-violating effects in very special relativity are absent [113]. The existence of the preferred light-like direction  $n^\mu$  can be interpreted as the existence of light-like æther. However, this peculiar form of æther is very difficult to detect because it doesn't single out any preferred inertial reference frame and, since  $CP$  violating effects are small, Lorentz-violating effects in very special relativity are expected to be also small.

Although the  $b \rightarrow 0$  limits of (75) and (77) lead at the first sight naturally to the very special relativity, it is not so. When  $b = 0$ , or when the space is isotropic, we have no reason to introduce the preferred light-like direction  $n^\mu$ . Of course, we can do this artificially and consequently arrive to the generalized Lorentz transformations (77) instead of usual Lorentz transformations. However, in this case  $\vec{n}$  has no physical meaning and just serves to calibrate the orientations of space axes of



the inertial frames of reference in such way that if in one such frame of reference the beam of light has the direction  $\vec{n}$ , it will have the same direction in all inertial reference frames [94]. The resulting theory will be not the very special relativity but the ordinary special relativity under the indicated special convention about the spatial axes orientations. This is true if we can choose  $n^\mu$  arbitrarily and all such choices are equivalent. In fact, it is sufficient to require that the choices  $\vec{n}$  and  $-\vec{n}$  are equivalent. Indeed, when  $\vec{n} \rightarrow -\vec{n}$ , then  $\vec{m} \rightarrow -\vec{m}$  in (115) and, since  $R^\mu_\lambda(-\vec{m}, \alpha)R^\lambda_\nu(\vec{m}, \alpha) = \delta^\mu_\nu$ , if we require the symmetry under rotations about  $-\vec{m}$  and if  $b = 0$ , we can easily transform the generalized Lorentz transformations (77) into the ordinary Lorentz transformations by an additional rotation  $x''^\mu = R^\mu_\nu(-\vec{m}, \alpha)x'^\nu$ .

Therefore, if the space-time is Minkowskian, the very special relativity (ISIM(2) symmetry) is not natural in light of relativity principle and will indicate its breaking, although very subtle, if it is indeed realized in nature. In our opinion far more natural possibility which completely respects the relativity principle, is that the space-time is Finslerian with the Lalan-Alway-Bogoslovsky metric (75) but the parameter  $b$  of this metric is very small. This conclusion is reinforced by the following analogy with the cosmological constant.

A surprising fact about Minkowski's "Raum und Zeit" lecture is that it never mentions Klein's Erlangen program of defining a geometry by its symmetry group [27]. A link between Minkowski's presentation of special relativity and Erlangen program was immediately recognized by Felix Klein himself [116] who remarked: "What the modern physicists call the theory of relativity is the theory of invariants of the 4-dimensional space-time region  $x, y, z, t$  (the Minkowski 'world') under a certain group of collineations, namely, the 'Lorentz group' ". Untimely death of Minkowski presumably hindered the appreciation of this important fact by physicists. Only in 1954 Fantappi  rediscovered the connection and put forward an idea which he himself calls "an Erlangen program for physics": a classification of possible physical theories through their group of symmetries [117, 118].

Interestingly, we have one more would-be "missed opportunities" story here [3]. Fantappi  discovered that the "final relativity" group is not the Poincar  group but the De Sitter group, the Poincar  group being just a limit of the latter when the radius of curvature of the De Sitter space-time goes to infinity, much like the Galilei group being a limit of the Lorentz group when the speed of light goes to infinity. In fact, the

full story of possible kinematic groups and their interconnections under different limits was given later by H. Bacry and J. Lévy-Leblond [119]. This picture seems quite natural in light of the Erlangen program and of ideas of group contractions and deformations put forward by Inönü and Wigner [112] and Irving Segal [120] about the same time Fantappiè found his “final relativity”. Dyson argues [3] that Minkowski, or any other pure mathematician, were in a position to conjecture already in 1908 that the true invariance group of the universe should be the de Sitter group rather than the Poincaré group and thus suggest a cosmological space expansion much earlier than Hubble discovered it, if only they would have brought Minkowski’s argument to its logical conclusion. However, “to be honest, it would be rather anachronistic to expect an approach like the one of Fantappiè at the beginning of the 20th century, as Dyson advocates” [118]. Even Fantappiè himself “missed the chance to anticipate the theory of Lie group contractions and deformations” [118].

Segal’s principle [120, 121, 122] states that a true physical theory should be stable against small deformations of its underlying algebraic (group) structure (a nice informal review of Lie algebra contractions and deformations can be found in [123]). Lie algebra of inhomogeneous Galilei group is not stable and its deformation leads to Lie algebra of the Poincaré group. Consequently, the relativity theory based on the Poincaré group has a greater range of validity than Galilean relativity. However, the Poincaré Lie algebra is by itself unstable and its deformation leads to either de Sitter or anti-de Sitter Lie algebras [119]. Therefore, it is not surprising that the cosmological constant turned out to be not zero and correspondingly, the asymptotic vacuum space-time is not Minkowski but de Sitter space-time. What is really surprising is why the cosmological constant is so small that makes special relativity valid for all practical purposes.

Lie algebra of the very special relativity symmetry group  $ISIM(2)$  is not stable against small deformations of its structure and a physically relevant deformation  $DISIM_b(2)$  of it does exist [93]. Therefore, in light of Segal’s principle, we expect that the very special relativity cannot be a true symmetry of nature and should be replaced by  $DISIM_b(2)$  and the corresponding Finslerian space-time. Drawing an analogy with the cosmological constant, it can be argued that  $b$  is really not zero but very small. In this case, to detect the effects of Finslerian nature of space-time

in laboratory experiments will be almost impossible<sup>5</sup>. Nevertheless, the question of the true value of the parameter  $b$  has the same fundamental significance as the question why the cosmological constant is so small [93]. Perhaps, both questions are just different parts of the same mystery.

## 6 Concluding remarks

In parallel to the advance in modern physics, in the middle of the twentieth century it became increasingly evident that Poincaré's contribution to relativity was unjustly downplayed. As a result, some attempts to restore the justice followed. Unfortunately it was forgotten in the majority of these attempts that "injustice cannot be corrected by committing another injustice" [125] and recurrent attempts to deny Einstein the discovery of special relativity took place [126]. A great amount of these attempts don't deserve even to be mentioned. However, there are some of them that are worth our attention and which are desirable to be clarified.

Renowned English mathematician Edmund Whittaker wrote a book *A History of the Theories of Aether and Electricity*, first edition of which was published in 1910. This is an excellent and very detailed book describing the development of field theories from Descartes up to the year 1900. In the final chapter of the book there is some survey of the work done after 1900. It includes a note about Minkowski's *Raum und Zeit* paper. Then little known Einstein is mentioned twice in footnotes [127]. First footnote says that Einstein completed Lorentz's work on Lorentz transformations. Then the footnote cites Voigt and says that "It should be added that the transformation in question had been applied to the equation of vibratory motions many years before by Voigt" [127] that is not quite correct as we have explained earlier. The next footnote correctly attributes to Einstein the first clear expression of the opinion that all inertial frames are equivalent and no one is granted a primacy by having an absolute relation to the æther. Poincaré is mentioned three times, also in footnotes, but never in the context of relativity.

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<sup>5</sup>In [124] a different Finslerian modification of space-time and its effect on the Maxwell's equations were considered along with the corresponding experimental consequences due to modifications of the Coulomb potential. Although we expect that the true modification of the Minkowski space-time is the Lalan-Alway-Bogoslovsky metric with the parameter  $b$  so small that probably it will be impossible to detect its presence in laboratory experiments, the ultimate judge is real experiments of course and any experimental efforts to place bounds on proposed Finsler parameters are worthwhile.

Forty-three years later, in 1953, Whittaker published a revised edition of his book, which now included a second volume covering the years from 1900 to 1926 [128]. The chapter devoted to relativity was called “The relativity theory of Poincaré and Lorentz” and its content left little doubt that Whittaker attributes the whole credit for the discovery of special relativity almost exclusively to Poincaré and Lorentz. About Einstein’s groundbreaking 1905 paper it was said that “Einstein published a paper which set forth the relativity theory of Poincaré and Lorentz with some amplifications, and which attracted much attention”. Why such injustice to Einstein? Maybe the following long excerpt from Max Born’s 26 September, 1953 letter to Einstein can shed some light on this difficult question.

“Very often I feel the need to write to you, but I usually suppress it to spare you the trouble of replying. Today, though, I have a definite reason — that Whittaker, the old mathematician, who lives here as Professor Emeritus and is a good friend of mine, has written a new edition of his old book *History of the Theory of the Ether*, of which the second volume has already been published. Among other things it contains a history of the theory of relativity which is peculiar in that Lorentz and Poincaré are credited with its discovery while your papers are treated as less important. Although the book originated in Edinburgh, I am not really afraid you will think that I could be behind it. As a matter of fact I have done everything I could during the last three years to dissuade Whittaker from carrying out his plan, which he had already cherished for a long time and loved to talk about. I re-read the originals of some of the old papers, particularly some rather off-beat ones by Poincaré, and have given Whittaker translations of German papers (for example, I translated many pages of Pauli’s *Encyclopedia* article into English with the help of my lecturer, Dr. Schlapp, in order to make it easier for Whittaker to form an opinion). But all in vain. He insisted that everything of importance had already been said by Poincaré, and that Lorentz quite plainly had the physical interpretation. As it happens, I know quite well how skeptical Lorentz was and how long it took him to become a relativist. I have told Whittaker all this, but without success” [129].

Born’s role in this affair is more ambivalent than it can appear from this letter [130]. At first sight Born was in a good position to be a competent judge in the relativity priority dispute. He recalls [131] that at the time when Einstein’s 1905 paper appeared he was “in Göttingen and well acquainted with the difficulties and puzzles encountered in the

study of electromagnetic and optical phenomena in moving bodies, which we thoroughly discussed in a seminar held by Hilbert and Minkowski. We studied the recent papers by Lorentz and Poincaré, we discussed the contraction hypothesis brought forward by Lorentz and Fitzgerald, and we knew the transformations now known under Lorentz's name. Minkowski was already working on his 4-dimensional representation of space and time, published in 1907, which became later the standard method in fundamental physics" and all this was before Born became aware of Einstein's 1905 paper. Therefore, Born was well versed in pre-Einstein relativity of Lorentz and Poincaré. In spite of this, Einstein's paper greatly impressed Born. In his book *Physics in my generation* [131] he wrote:

"A long time before I read Einstein's famous 1905 paper, I knew the formal mathematical side of the special theory of relativity through my teacher Hermann Minkowski. Even so, Einstein's paper was a revelation to me which had a stronger influence on my thinking than any other scientific experience. . . Einstein's simple consideration, by which he disclosed the epistemological root of the problem. . . made an enormous impression, and I think it right that the principle of relativity is connected with his name, though Lorentz and Poincaré should not be forgotten" [130, 131].

Born's letter to Einstein leaves an impression that "Born's attention was drawn to Poincaré's papers by Whittaker, rather than the other way around, in contradiction to Born's recollection of having read Poincaré's papers in his student days before reading Einstein's paper" [130]. We think the contradiction is only apparent here. There is little doubt that Poincaré's papers were discussed at Minkowski's seminars and they were well known to Born. The only explanation why he needed to re-read them in fifties to discover that several of Poincaré's pre-1905 statements sound very much like special relativity is that in 1905 or before Poincaré's papers have not impressed him as much as Einstein's 1905 paper which was "a revelation" to him. Whittaker also was not aware of the importance of Poincaré's papers in 1910. Only in fifties he was in a position to acknowledge their importance.

It is not surprising that modern physicists like, for example, Logunov [18], find Poincaré's papers important and rise doubts about Poincaré's priority in discovery of special relativity. Even Born, somewhat contrary to what he writes in the letter to Einstein, hesitated in giving the priority: "Does this mean that Poincaré knew all this before Einstein? It is

possible, but the strange thing is that this lecture [Poincaré's 1909 lecture *La mécanique nouvelle*] definitely gives you the impression that he is recording Lorentz's work... On the other hand, Lorentz himself never claimed to be the author of the principle of relativity" [130, 131]. Further he writes about Einstein's 1905 paper: "The striking point is that it contains not a single reference to previous literature. It gives you the impression of quite a new venture. But that is, of course, as I have tried to explain, not true" [131]. And this is, of course, "precisely the conclusion that Whittaker reached... after long discussions of the subject with Born" [130].

However, Whittaker's conclusion, implicitly shared by many modern physicists who had taken a trouble to indeed read Poincaré's papers, is wrong. This conclusion is based on the retrospective reading of Poincaré's papers. However, in retrospective reading of really deep papers, and there is no doubt that Poincaré's relativity papers are really very deep, the reader can see much more than it was possible to see by contemporaries, including the author, who were bound by concrete historical context and prejudices of those days. If we inspect the physical literature of the founding period of relativity (1905-1918), we will clearly see that the scientific community never hesitated to give Einstein a due credit, and there was no such a thing as Poincaré's version of relativity "accessible and perceived as such by physicists of the times" [35]. It is a historical fact that the relativistic revolution, as seen in the contemporary physical literature of those days, rightly or wrongly is dominated by only one name, the Einstein's papers playing the major role, while Poincaré's ones being left virtually unnoticed [35]. The importance of these papers was recognized only later. In this respect, Poincaré-Einstein mystery "is an artefact of projecting backward a particular reading of scientific papers that does not correspond to what the actors of the time saw in them" [35]. We cannot rewrite the history to restore justice towards Poincaré. This is not needed indeed. A scientific rehabilitation of Poincaré's contribution to relativity is an accomplished fact now. This contribution, being far ahead of his time, is still alive and modern scholars can still find inspiration in it. Isn't this a wonderful miracle?

Interestingly, it seems like Born in his letter to Einstein is trying to provoke Einstein over the priority issue [130]. He writes: "I am annoyed about this, for he is considered a great authority in the English speaking countries and many people are going to believe him" [129]. Einstein's response in his 12 October, 1953 letter was short and priceless in its

wisdom:

“Don’t lose any sleep over your friend’s book. Everybody does what he considers right or, in deterministic terms, what he has to do. If he manages to convince others, that is their own affair. I myself have certainly found satisfaction in my efforts, but I would not consider it sensible to defend the results of my work as being my own ‘property’, as some old miser might defend the few coppers he had laboriously scraped together. I do not hold anything against him, nor of course against you. After all, I do not need to read the thing” [129].

As witnessed by Robert Oppenheimer, Einstein in his last years “was a twentieth-century Ecclesiastes, saying with unrelenting and indomitable cheerfulness, ‘Vanity of vanities, all is vanity’ ” [132].

Of course we can wonder why Einstein, and not then much famous and educated Poincaré, was at the origin of relativistic revolution. Undoubtedly some peculiar personal character traits of Einstein played the role. “He was almost wholly without sophistication and wholly without worldliness. I think that in England people would have said that he did not have much ‘background’ and in America that he lacked ‘education’”. This may throw some light on how these words are used. I think that this simplicity, this lack of clutter and this lack of cant, had a lot to do with his preservation throughout of a certain pure, rather Spinoza-like, philosophical monism, which of course is hard to maintain if you have been ‘educated’ and have a ‘background’. There was always with him a wonderful purity at once childlike and profoundly stubborn” [132].

From the scientific point of view, the fundamental difference between the conceptual foundations of the Lorentz-Poincaré æther based theory from one side, and Einstein’s special relativity from another, played the crucial role. Most succinctly this difference was expressed by Lorentz himself: ‘the chief difference being that Einstein simply postulates what we have deduced, with some difficulty, and not altogether satisfactorily, from the fundamental equations of the electromagnetic field’ [133]. It is clear that in contrary to Einstein, Lorentz and Poincaré considered special relativity as a derivative, not fundamental theory and tried to base it on a more general premises. Although very modern in its spirit, such an approach had no chance in 1905. Even now we have little clue how special relativity can be considered as an emergent and not a fundamental phenomenon.

Whittaker was not the only prominent mathematician who ill treated

Einstein's contribution to relativity. Outstanding Russian mathematician Vladimir Arnold also gives a quite distorted picture:

“Minkowski, being a teacher of Einstein and a friend of Poincaré, had early suggested to Einstein that he should study Poincaré's theory, and Einstein did (though never referring to this until a 1945 article)” [134].

It is true that Minkowski was Einstein's former mathematics professor at the Federal Polytechnic School of Zürich. However, at that time Einstein was less interested in abstract mathematics and often skipped Minkowski's classes during his studies at the Polytechnic [30]. Needless to say, it is simply not true that Minkowski ever supervised Einstein in his scientific research. According to Arnold Sommerfeld's recollections “Strangely enough no personal contacts resulted between his teacher of mathematics, Hermann Minkowski, and Einstein” [30].

At first sight it seems difficult to explain why Poincaré and Einstein almost did not refer to each other when writing about relativity. First we consider Poincaré's silence which is, perhaps, easier to explain. As was mentioned above, Poincaré's objective was much more ambitious than Einstein's as he wanted to derive special relativity as an emergent phenomenon. It is quite possible therefore that Poincaré simply considered Einstein's contribution as being too trivial in light of this bigger goal. “To Poincaré, Einstein's theory must have been seen as a poor attempt to explain a small part of the phenomena embraced by the Lorentz theory” [135]. Perhaps, later in his life Poincaré became aware of the importance of Einstein's work. In November of 1911, one year before his unexpected death, when he was asked to write a recommendation letter for Einstein who was looking to become a professor of theoretical physics at Swiss Federal Institute of Technology in Zurich, Poincaré wrote:

“Monsieur Einstein is one of the most original minds I have known; in spite of his youth he already occupies a very honorable position among the leading scholars of his time. We must especially admire in him the ease with which he adapts himself to new concepts and his ability to infer all the consequences from them. He does not remain attached to the classical principles and, faced with a physics problem, promptly envisages all possibilities. This is translated immediately in his mind into an anticipation of new phenomena, susceptible some day to experimental verification. I would not say that all his expectations will resist experimental check when such checks will become possible. Since he is probing in all directions, one should anticipate, on the contrary, that most of the roads he is following will lead to dead ends; but, at the same time, one



must hope that one of the directions he has indicated will be a good one; and that suffice" [29].

Einstein's silence is more difficult to explain. It is reasonable to assume that young Einstein, like most other physicists of those times, just lacked the proper mathematical education to dully appreciate Poincaré's work. "More surprisingly, Einstein continued to ignore Poincaré's contribution in all his later writings on special relativity and in his autobiographical notes" [136]. For late Einstein it is impossible to assume that he was not capable to recognize the importance of Poincaré's work. How can we then explain Einstein's silence? We think he simply had not read Poincaré's key papers on relativity until perhaps later times. Contrary to what can be expected, most scholars do not have a habit to read the papers of their fellow scientists attentively. Empirical studies of misprint distributions in citations indicate that only about 20% of scientists read the original that they cite [137]. In case of Einstein we have his own confession to Ehrenfest: "My dear friend, do you think that I make a habit of reading papers written by others?" [138]. In the early 1950s Abraham Pais asked Einstein how Poincaré's Palermo paper had affected his thinking. Einstein answered that he had never read the paper. Then Pais lent to Einstein his own copy of the paper which afterwords vanished [29]. "Perhaps he did read it. In 1953 Einstein received an invitation to attend the forthcoming Bern celebration of the fiftieth anniversary of special relativity. Einstein wrote back that his health did not permit him to plan such a trip. In this letter Einstein mentions for the first time (as far as I know) Poincaré's role in regard to the special theory: 'I hope that one will also take care on that occasion to honor suitably the merits of H. A. Lorentz and H. Poincaré' " [29]. Besides, two months before his death, Einstein wrote to his biographer Carl Seelig that Poincaré further deepened Lorentz's insight about the essential role of Lorentz's transformations for the analysis of Maxwell's equations [136].

There is still another aspect which makes Einstein-Poincaré priority dispute pointless. Modern understanding of relativity is significantly different from the one that was cultivated at the beginning of the twentieth century. Two examples are the notions of æther and relativity of simultaneity which are often used in the priority dispute.

Einstein proponents stress that Poincaré never abandoned the æther. Indeed, in 1912, a few months before his death, he gave a talk at a conference at the University of London in which he said: "Everything happens

as if time were a fourth dimension of space, and as if 4-dimensional space resulting from the combination of ordinary space and of time could rotate not only around an axis of ordinary space in such a way that time were not altered, but around any axis whatever. For the comparison to be mathematically accurate, it would be necessary to assign purely imaginary values to this fourth coordinate of space" [139, 140]. Taken out of context, it is tempting to consider this statement as an evidence that Poincaré was accepting the Einstein-Minkowski concept of space-time. Especially when he continues: "the essential thing is to notice that in the new conception space and time are no longer two entirely distinct entities which can be considered separately, but two parts of the same whole, two parts which are so closely knit that they cannot be easily separated" [139, 140]. However, it is clear from the full text of his talk that he is not at all ready to accept this new conception, as he further continues with the conclusion: "What shall be our position in view of these new conceptions? Shall we be obliged to modify our conclusions? Certainly not; we had adopted a convention because it seemed convenient and we had said that nothing could constrain us to abandon it. Today some physicists want to adopt a new convention. It is not that they are constrained to do so; they consider this new convention more convenient; that is all. And those who are not of this opinion can legitimately retain the old one in order not to disturb their old habits. I believe, just between us, that this is what they shall do for a long time to come" [139, 140].

Therefore, clearly, "Poincaré never believed in the physical relevance of the conceptual revolution brought by Einstein in the concept of time (and extended by Minkowski to a revolutionary view of the physical meaning of space-time)" [140]. In particular, for Poincaré the æther, even unobservable by physical experiments, remained an important conceptual element.

Usually this stubbornness of Poincaré with respect to the æther is considered as his weak point, as an evidence that he didn't really understand relativity. It is historically true that the abolishment of the æther by Einstein played a crucial role and revolutionized physics. However, frankly speaking, in retrospect, when this revolution came to its logical end in modern physics, we can equally well consider Poincaré's attitude as prophetic.

As modern physics has progressed in the twentieth century, it became increasingly evident that the vacuum, the basic state of quantum

field theory, is anything but empty space. In fact, at present an æther, “renamed and thinly disguised, dominates the accepted laws of physics” [141]. It is clear that only “intellectual inertia” [142] prevents us from using historically venerable word “æther” instead of “vacuum state” when referring to the states with such complex physical properties as vacuum states of modern quantum theories.

However, the æther of modern physics is Lorentz invariant and hardly the one which Poincaré had in mind. Interestingly both in classical mechanical systems [143] and in condensed matter physics [144] an effective Lorentz symmetry may arise from non-relativistic background physics. This is reminiscent of Lorentz-Poincaré program of deriving the Lorentz symmetry rather than postulating it. This program was never completely abandoned. For most notable attempts in this direction see the books of Brown [110] and Jánossy [145]. Such attempts had, and still have, little chance of complete success. It is not excluded however that they will get a better chance when our knowledge of the Planck scale physics progresses. Therefore, in this respect Poincaré’s idea of the æther in which an effective Lorentz symmetry emerges in some degrees of freedom at low energy limit is still alive and modern.

Now about relativity of simultaneity. Relativity of simultaneity was considered by Einstein himself and his contemporaries as the most crucial and revolutionary aspect of special relativity. For example Max Planck in his lectures given at Columbia University in 1909 (consequently published by him in 1910) so describes Einstein’s ideas on space and time: “It need scarcely be emphasized that this new conception of the idea of time makes the most serious demands upon the capacity of abstraction and the power of imagination of the physicist. It surpasses in boldness everything previously suggested in speculative natural phenomena and even in the philosophical theories of knowledge: non-euclidean geometry is child’s play in comparison. And, moreover, the principle of relativity, unlike non-euclidean geometry, which only comes seriously into consideration in pure mathematics, undoubtedly possesses a real physical significance. The revolution introduced by this principle into the physical conceptions of the world is only to be compared in extent and depth with that brought about by the introduction of the Copernican System of the universe” [15].

As was already mentioned above, relativity of simultaneity was originally Poincaré’s invention. Surprisingly, he did not attach a particularly fundamental importance to it [15], as witnessed by Poincaré’s conclusion

in the chapter *The measure of time* of his book *The Value of Science* first published in 1905. We provide here this conclusion in its entirety because of its importance in revealing Poincaré's position (the fragment is also reproduced in [15]):

“To conclude: we do not have a direct intuition of simultaneity, nor of the equality of two durations. If we think we have this intuition, this is an illusion. We replace it by the aid of certain rules which we apply almost always without taking count of them.

But what is the nature of these rules? No general rule, no rigorous rule; a multitude of little rules applicable to each particular case.

These rules are not imposed upon us and we might amuse ourselves in inventing others; but they could not be cast aside without greatly complicating the enunciation of the laws of physics, mechanics and astronomy.

We therefore choose these rules, not because they are true, but because they are the most convenient, and we may recapitulate them as follows: *The simultaneity of two events, or the order of their succession, the equality of two durations, are to be so defined that the enunciation of the natural laws may be as simple as possible. In other words, all these rules, all these definitions are only the fruit of an unconscious opportunism*” [146].

Poincaré proponents in the priority dispute argue that Einstein synchronization, which Einstein himself considered as the crucial element of special relativity, has in fact originated from Poincaré's work. Above we had already commented on this difficult issue. Now we would like to consider another aspect of it. The fact is that in light of the modern view on relativity, Einstein synchronization can no longer be considered as a crucial element. What is really essential is Minkowski geometry (or slightly Finslerian) of space-time in the absence of gravity (and cosmological constant). Einstein synchronization is just a convention, or as Einstein himself put it, a “stipulation” which allows to introduce a convenient coordinate chart in Minkowski space-time. However, an introduction of any other coordinate charts, although probably less convenient, is equally possible. This modern view is very close to the above given Poincaré's attitude.

For example, we can use 'everyday' clock synchronization when clocks are adjusted by using time signal broadcasted by some radio-station telling that, for example, “At the six stroke, the time will be

12 o'clock exactly" [147]. All coordinate-dependent quantities, including the Lorentz transformations too, can experience drastic changes under the change of synchronization convention. For example, such once thought intrinsic feature of relativity as relativity of simultaneity is no longer true when 'everyday' clock synchronization is used [147]. Under such synchronization the "Lorentz" transformations relating the two 'everyday' observers  $S$  and  $S'$  with relative velocity  $V$  (as measured in the frame  $S$ ) along the  $x$ -axis have the following form [147]

$$\begin{aligned} x' &= \frac{x-Vt}{\sqrt{1+2\beta}}, \\ y' &= y, \\ z' &= z, \\ t' &= \sqrt{1+2\beta}t. \end{aligned} \tag{79}$$

where  $\beta = V/c$ . It is clear from these transformations that simultaneity under 'everyday' synchronization is absolute: if  $\Delta t = 0$  in some system  $S$  then in all other inertial systems  $S'$  we also shall have  $\Delta t' = 0$ .

Conventionality of simultaneity was systematically investigated by Reichenbach (see, for example, [148]). Suppose that two distant clocks  $A$  and  $B$  are motionless in a common inertial frame  $S$ . If a light signal is sent from  $A$  at time  $t_A$ , is instantaneously reflected by  $B$  at time  $t_B$ , and arrives back at  $A$  at time  $t'_A$ , then  $A$  and  $B$  are Poincaré-Einstein synchronized if  $t_B - t_A = t'_A - t_B$ , or

$$t_B = \frac{1}{2}(t_A + t'_A) = t_A + \frac{1}{2}(t'_A - t_A). \tag{80}$$

Reichenbach modifies this definition of synchronization as follows

$$t_B = t_A + \epsilon(t'_A - t_A), \tag{81}$$

where now  $\epsilon$  is some real parameter and  $0 \leq \epsilon \leq 1$  (because for causality reasons we need  $t_B \geq t_A$  and  $t_B \leq t'_A$ ), the equality corresponding to the infinite one-way velocity of the light signal. In fact, these limiting cases were excluded by Reichenbach).

The modification of Lorentz transformations needed to meet this general  $\epsilon$ -synchronization, was considered by Winnie [149]. If  $\epsilon$ -synchronization is adopted in the reference frame  $S$  while  $\epsilon'$ -synchronization is adopted in the reference frame  $S'$ , Winnie transformations have the

form [149]

$$\begin{aligned} x' &= \frac{x-Vt}{\alpha}, \\ y' &= y, \\ z' &= z, \\ t' &= \frac{1}{\alpha} \left\{ [1 + 2\beta(1 - \epsilon - \epsilon')]t - [2(\epsilon - \epsilon') + 4\beta\epsilon(1 - \epsilon)] \frac{x}{c} \right\}, \end{aligned} \quad (82)$$

where

$$\alpha = \sqrt{[1 - (2\epsilon - 1)\beta]^2 - \beta^2}. \quad (83)$$

If  $\epsilon = \epsilon' = 1/2$ , we recover the ordinary Lorentz transformations, and if  $\epsilon = \epsilon' = 0$  we get the transformations (79) which correspond to 'everyday' synchronization.

It is interesting to note that the 'everyday' synchronization is not the only choice that corresponds to absolute simultaneity. Historically Frank Robert Tangherlini was the first to realize that absolute simultaneity doesn't contradict special relativity in his 1958 PhD dissertation supervised by Sidney Drell and, at the initial stage of the work, Donald Yennie [150, 151, 152]. Tangherlini's transformations have the form

$$\begin{aligned} x' &= \gamma(x - Vt), \\ y' &= y, \\ z' &= z, \\ t' &= \gamma^{-1}t, \end{aligned} \quad (84)$$

and they correspond to the special case of the Winnie transformations (79) when  $\epsilon = 1/2$  and  $\epsilon' = (1 + \beta)/2$ . External synchronization provides a possible realization of the Tangherlini's simultaneity [60, 153]. First clocks in the "æther" frame  $S$  are Poincaré-Einstein synchronized with  $\epsilon = 1/2$ . Then these clocks are used to synchronize nearby clocks in the moving frame  $S'$  by adjusting clocks from  $S'$  to  $t' = 0$  whenever they fly past a clock in  $S$  which shows  $t = 0$ .

Note that the Tangherlini transformations (84) were obtained twenty years before his thesis by English mathematician Albert Eagle, although from erroneous anti-relativistic premises [153, 154] — one more example that correct mathematics doesn't guarantee a correct physical interpretation.

Some other simultaneity conventions, not necessarily based on light signals and their two-way universal velocity  $c$ , are also feasible. For examples, inhabitants of the homogeneous isotropic ocean can find convenient to use for synchronization light signals with two-way velocity  $c' < c$

[155], and dolphins can even prefer sound waves for this goal [156]. Of course, coordinate-independent quantities are not affected by synchronization conventions. Various synchronization conventions are useful to separate the effects due to synchronization and the real contraction of moving bodies and slowing down of moving clocks. Such investigations allow us to come to the conclusion that “the results of relativistic experiments have their origin in the length contraction and time dilatation effects which are so real as a change of the length of a rod caused by the change of temperature” [157, 158].

Rosen’s  $c'$ -relativity gives an interesting perspective on superluminal objects [155]. Effective  $c'$ -relativity (as far as the electromagnetic phenomena are concerned) precludes charged particles in the Rosen’s ocean to acquire speeds greater than or equal to  $c'$ . However, one might have an energetic cosmic muon enter the ocean from outside with a speed greater than  $c'$ . The behavior of such a particle could not be described in the framework of  $c'$ -relativity. For example, the inhabitants of the ocean will observe “vacuum” Cherenkov radiation emanated from such a particle — a phenomenon they certainly think impossible on the base of  $c'$ -relativity. It was speculated in [159] that likewise  $c$ -relativity might also not embrace some hidden sectors weakly coupled to our visible sector of the world. Then superluminal objects called elvisebrions in [159] can enter the visible sector and their behavior could not be described in the framework of special relativity.

Tangherlini was guided by the logic of general relativity when undertaking his investigation of absolute simultaneity [152]. According to this logic, equations (84) represent just a coordinate transformation which makes the metric tensor non-diagonal. Indeed, it is easy to get from (84) that

$$dx = \gamma^{-1} dx' + \gamma V dt', \quad dt = \gamma dt', \quad (85)$$

and the relativistic interval (line element) takes the form

$$ds^2 = c^2 dt'^2 - 2V dx' dt' - \gamma^{-2} dx'^2 - dy'^2 - dz'^2. \quad (86)$$

From the point of view of general relativity, these coordinates are as good as any others because of general covariance. However, as was already shown by Kretschmann in 1917, any space-time theory can be expressed in general covariant manner, not only general relativity. Therefore, the coordinates (84) are as good as any others in special relativity too. It is just an other expression of “intellectual inertia” that standard textbook

presentations of special relativity are based solely on inertial observers and diagonal metric. Another misbelief, which still prevails, is that accelerated observers cannot be treated by special relativity, and that one needs general relativity for their incorporation. In fact, there is no principal difficulty to consider Minkowski space-time from the point of view of non-inertial observers at the expense of somewhat more sophisticated mathematics [162] (for more traditional exposition of special relativity see, for example, an excellent textbook [161]).

We hope that the above given examples show clearly how different the present day interpretation of key features of special relativity is from what had in mind founding fathers of this magnificent theory. And the story of special relativity is not finished yet. For example, the debate about conventionality of simultaneity still continues and seems to be far from being settled (see, for instance, [163] and references therein). The initial observer-centered presentation of special relativity was changed to the modern geometry-centered presentation. However, it seems that the full potential of the notion of an observer as ontologically prior to either space or space-time is not yet exhausted [164, 165] and the initial presentation of special relativity, although renewed and enriched by our experience with space-time picture, can still strike back.

In light of this immense and still continuing progress of modern physics, attempts to retrospectively induce an artificial Poincaré-Einstein priority dispute and rewrite the history seem minute. We will be happy if this arid and futile dispute will come to its end. There is nothing scientific in it and its presence only emphasizes hideous traits of human nature. However, there is no single thing we can do, except writing this article, to help to end this unfortunate dispute. It remains, like aging Einstein, to repeat after Ecclesiastes 'Vanity of vanities, all is vanity'.

## appendixes

### A Uniformly accelerated motion

Uniformly accelerated motion is defined as having a zero jerk (time derivative of the acceleration 3-vector) in instantaneous rest frame [75]. Differentiating velocity 4-vector

$$u^\mu = \gamma(c, \vec{V}), \quad (87)$$



we get successively the acceleration 4-vector

$$\begin{aligned} \dot{u}^\mu &= \frac{du^\mu}{d\tau} = \gamma \frac{du^\mu}{dt} = \gamma^4 \left( \vec{\beta} \cdot \vec{a}, \frac{\vec{a}}{\gamma^2} + (\vec{\beta} \cdot \vec{a})\vec{\beta} \right) = \\ &\gamma^4 \left( \vec{\beta} \cdot \vec{a}, \vec{a} + \vec{\beta} \times (\vec{\beta} \times \vec{a}) \right), \end{aligned} \quad (88)$$

and the naive jerk 4-vector

$$\begin{aligned} \ddot{u}^\mu &= \frac{d\dot{u}^\mu}{d\tau} = \gamma^5 \left( \frac{\vec{a} \cdot \vec{a}}{c} + \vec{\beta} \cdot \vec{b} + \frac{4\gamma^2}{c} (\vec{\beta} \cdot \vec{a})^2, \frac{\vec{b}}{\gamma^2} + 3 \frac{(\vec{\beta} \cdot \vec{a})}{c} \vec{a} + \right. \\ &\left. \left[ \vec{\beta} \cdot \vec{b} + \frac{\vec{a} \cdot \vec{a}}{c} + 4 \frac{\gamma^2}{c} (\vec{\beta} \cdot \vec{a})^2 \right] \vec{\beta} \right), \end{aligned} \quad (89)$$

where  $\vec{a} = \frac{d\vec{V}}{dt}$  and  $\vec{b} = \frac{d\vec{a}}{dt}$ . In the instantaneous rest frame

$$\ddot{u}_{(0)}^\mu = \left( \frac{\vec{a}_{(0)} \cdot \vec{a}_{(0)}}{c}, \vec{b}_{(0)} \right), \quad (90)$$

which indicates that  $\ddot{u}^\mu$  is not necessarily spacelike and could be nonzero even if 3-jerk satisfies  $\vec{b}_{(0)} = 0$ . Due to these properties  $\ddot{u}^\mu$  can not be really considered as a relativistic jerk 4-vector [166]. To find a better candidate for the role of the relativistic jerk 4-vector let us note that

$$u_\mu \ddot{u}^\mu = \vec{a}_{(0)} \cdot \vec{a}_{(0)} = -\frac{\dot{u}_\nu \dot{u}^\nu}{c^2} u_\mu u^\mu, \quad (91)$$

which implies

$$u_\mu \left( \ddot{u}^\mu + \frac{\dot{u}_\nu \dot{u}^\nu}{c^2} u^\mu \right) = 0. \quad (92)$$

Therefore, the 4-vector

$$\Gamma^\mu = \ddot{u}^\mu + \frac{\dot{u}_\nu \dot{u}^\nu}{c^2} u^\mu \quad (93)$$

is spacelike because it is orthogonal to the timelike 4-vector  $u^\mu$ . Besides, in the instantaneous rest frame  $\Gamma_{(0)}^\mu = (0, \vec{b}_{(0)})$ . Hence, it makes more sense to define the relativistic jerk to be  $\Gamma^\mu$  [166]. Then the covariant condition of uniformly accelerated motion is the vanishing of jerk 4-vector  $\Gamma^\mu$  [75] (in [75]  $\Gamma^\mu$  is called the Abraham 4-vector due to its relation to the Lorentz-Abraham radiation reaction force). Using

$$\dot{u}_\nu \dot{u}^\nu = -\gamma^4 \left[ \vec{a} \cdot \vec{a} + \gamma^2 (\vec{\beta} \cdot \vec{a})^2 \right], \quad (94)$$

which follows from (88), and

$$\vec{b} + 3 \frac{\gamma^2}{c} (\vec{\beta} \cdot \vec{a}) \vec{a} = \frac{1}{\gamma^3} \frac{d}{dt} (\gamma^3 \vec{a}), \quad (95)$$

we get after some algebra

$$\begin{aligned} \Gamma^\mu &= \gamma^2 \left( \vec{\beta} \cdot \frac{d}{dt} (\gamma^3 \vec{a}), \frac{1}{\gamma^2} \frac{d}{dt} (\gamma^3 \vec{a}) + \left( \vec{\beta} \cdot \frac{d}{dt} (\gamma^3 \vec{a}) \right) \vec{\beta} \right) = \\ &\gamma^2 \left( \vec{\beta} \cdot \frac{d}{dt} (\gamma^3 \vec{a}), \frac{d}{dt} (\gamma^3 \vec{a}) + \vec{\beta} \times \left( \vec{\beta} \times \frac{d}{dt} (\gamma^3 \vec{a}) \right) \right). \end{aligned} \quad (96)$$

Therefore, a uniformly accelerated motion is characterized by the condition [75]

$$\frac{1}{\gamma^3} \frac{d}{dt} (\gamma^3 \vec{a}) = \vec{b} + 3 \frac{\gamma^2}{c} (\vec{\beta} \cdot \vec{a}) \vec{a} = 0, \quad (97)$$

which is equivalent to:

$$\gamma^3 \vec{a} = \vec{g}, \quad (98)$$

with some constant vector  $\vec{g}$ .

Suppose we want to interpret special conformal transformation (59) as a transition to a new reference frame. How does the origin  $\vec{r}' = 0$  of this frame move? When  $\vec{r}' = 0$ , we will have from (59)

$$\vec{r} = \vec{\alpha} x \cdot x, \quad x \cdot x = x_\mu x^\mu = c^2 t^2 - \vec{r} \cdot \vec{r}. \quad (99)$$

Differentiating this relation we find

$$\vec{V} = 2\vec{\alpha} (c^2 t - \vec{r} \cdot \vec{V}), \quad \vec{a} = 2\vec{\alpha} (c^2 - V^2 - \vec{r} \cdot \vec{a}) = \frac{2\vec{\alpha}}{\gamma^2} (c^2 - \gamma^2 \vec{r} \cdot \vec{a}). \quad (100)$$

On the other hand, taking the scalar products of (100) and (99), we get the relations

$$\vec{r} \cdot \vec{V} = 2\vec{\alpha} \cdot \vec{\alpha} x \cdot x (c^2 t - \vec{r} \cdot \vec{V}), \quad \vec{r} \cdot \vec{a} = 2\vec{\alpha} \cdot \vec{\alpha} x \cdot x \left( \frac{c^2}{\gamma^2} - \vec{r} \cdot \vec{a} \right). \quad (101)$$

From these equations  $\vec{r} \cdot \vec{V}$  and  $\vec{r} \cdot \vec{a}$  can be determined, and substituting the results back in (100) we get

$$\vec{V} = \frac{2\vec{\alpha} c^2 t}{1 + 2\vec{\alpha} \cdot \vec{\alpha} x \cdot x}, \quad \vec{a} = \frac{2\vec{\alpha} c^2}{\gamma^2} \frac{1}{1 + 2\vec{\alpha} \cdot \vec{\alpha} x \cdot x}. \quad (102)$$

Therefore,

$$1 - \beta^2 = 1 - \frac{4\vec{\alpha} \cdot \vec{\alpha} c^2 t^2}{(1 + 2\vec{\alpha} \cdot \vec{\alpha} x \cdot x)^2} = \frac{1}{(1 + 2\vec{\alpha} \cdot \vec{\alpha} x \cdot x)^2}, \quad (103)$$

where we have taken into account that in light of (99)

$$c^2 t^2 = x \cdot x + \vec{r} \cdot \vec{r} = x \cdot x + \vec{\alpha} \cdot \vec{\alpha} (x \cdot x)^2. \quad (104)$$

As we see,

$$\gamma = 1 + 2\vec{\alpha} \cdot \vec{\alpha} x \cdot x, \quad (105)$$

and then the second equation in (102) gives  $\gamma^3 \vec{a} = 2c^2 \vec{\alpha}$  what implies that the point  $\vec{r}' = 0$  experiences uniformly accelerated motion.

## B Lalan-Alway-Bogoslovsky transformations

Let us consider an inertial reference frame  $S'$  which moves with the velocity  $\vec{V}$ . If for a moment we consider  $n^\mu$  as a normal 4-vector, not the fixed one, we may write

$$(n_\nu dx^\nu)^{2b} (dx_\mu dx^\mu)^{1-b} = (n'_\nu dx'^\nu)^{2b} (dx'_\mu dx'^\mu)^{1-b}, \quad (106)$$

where

$$n'_0 = \gamma \left( n_0 - \beta \frac{\vec{n} \cdot \vec{V}}{V} \right) = \gamma (1 - \vec{n} \cdot \vec{\beta}), \quad \vec{\beta} = \frac{\vec{V}}{c}, \quad (107)$$

and

$$\begin{aligned} \vec{n}' &= \frac{\vec{V}}{V} \gamma \left( \frac{\vec{n} \cdot \vec{V}}{V} - \beta n_0 \right) + \left( \vec{n} - \frac{\vec{n} \cdot \vec{V}}{V} \frac{\vec{V}}{V} \right) = \\ &= \vec{n} + \frac{\vec{V}}{V} \left[ (\gamma - 1) \frac{\vec{n} \cdot \vec{V}}{V} - \gamma \beta \right]. \end{aligned} \quad (108)$$

Using

$$\gamma - 1 = \frac{\gamma^2 - 1}{\gamma + 1} = \frac{\gamma^2 \beta^2}{\gamma + 1}, \quad (109)$$

we can rewrite (108) as follows

$$\vec{n}' = \vec{n} - \gamma \vec{\beta} \left[ 1 - \frac{\gamma}{\gamma + 1} \vec{n} \cdot \vec{\beta} \right] = \vec{n} - \frac{\gamma \vec{\beta}}{\gamma + 1} \left[ 1 + \gamma (1 - \vec{n} \cdot \vec{\beta}) \right]. \quad (110)$$

If  $\vec{V}$  is not parallel to  $\vec{n}$ ,  $\vec{n}'$  will be not parallel to  $\vec{n}$  neither. If  $\alpha$  is an angle between  $\vec{n}'$  and  $\vec{n}$ , we will have

$$\sin \alpha = \frac{|\vec{n} \times \vec{n}'|}{n'_0} = \frac{|\vec{n} \times \vec{\beta}|[1 + \gamma(1 - \vec{n} \cdot \vec{\beta})]}{(\gamma + 1)(1 - \vec{n} \cdot \vec{\beta})}, \quad (111)$$

where we have used that  $|\vec{n}'| = n'_0$  because the Lorentz transformation leaves  $n'_\mu$  lightlike. Alternatively, we can write

$$\begin{aligned} \cos \alpha &= \frac{\vec{n} \cdot \vec{n}'}{n'_0} = \frac{1 - \gamma(\vec{n} \cdot \vec{\beta} - 1 + 1) + \frac{\gamma^2}{\gamma+1} (\vec{n} \cdot \vec{\beta})^2}{\gamma(1 - \vec{n} \cdot \vec{\beta})} = \\ &1 + \frac{1 - \gamma + \frac{1}{\beta^2} (\gamma - 1)(\vec{n} \cdot \vec{\beta})^2}{\gamma(1 - \vec{n} \cdot \vec{\beta})} = 1 - \frac{\gamma - 1}{\gamma\beta^2} \frac{\beta^2 - (\vec{n} \cdot \vec{\beta})^2}{1 - \vec{n} \cdot \vec{\beta}}. \end{aligned} \quad (112)$$

But  $\beta^2 - (\vec{n} \cdot \vec{\beta})^2 = [\vec{n} \times \vec{\beta}]^2$ , and finally

$$\cos \alpha = 1 - \frac{\gamma - 1}{\gamma\beta^2} \frac{[\vec{n} \times \vec{\beta}]^2}{1 - \vec{n} \cdot \vec{\beta}} = 1 - \frac{\gamma}{\gamma + 1} \frac{[\vec{n} \times \vec{\beta}]^2}{1 - \vec{n} \cdot \vec{\beta}}. \quad (113)$$

We can make  $\vec{n}'$  again parallel to  $\vec{n}$  by an additional rotation around the axis  $\vec{n}' \times \vec{n} \parallel \vec{n} \times \vec{\beta}$  by certain angle  $\alpha$ . For the radius vector  $\vec{r}'$  the result of this additional rotation is given by the Euler-Rodrigues formula [167] (for proofs of this formula see, for example, [168, 169])

$$\vec{r}'' = \vec{r}' + (\vec{m} \times \vec{r}') \sin \alpha + [\vec{m} \times (\vec{m} \times \vec{r}')] (1 - \cos \alpha), \quad (114)$$

where

$$\vec{m} = \frac{\vec{n} \times \vec{\beta}}{|\vec{n} \times \vec{\beta}|} \quad (115)$$

is the unit vector along the axis of the rotation and

$$\vec{r}' = \vec{r} + \frac{\vec{\beta}}{\beta} \left[ (\gamma - 1) \frac{\vec{r} \cdot \vec{\beta}}{\beta} - \gamma \beta x_0 \right] \quad (116)$$

is the result of the preceding Lorentz transformation on the spatial part of the  $x^\mu$  4-vector. Using (111) and (113) along with

$$\vec{m} \times \vec{r}' = \frac{(\vec{n} \times \vec{\beta}) \times \vec{r}'}{|\vec{n} \times \vec{\beta}|} = \frac{\vec{\beta}(\vec{n} \cdot \vec{r}') - \vec{n}(\vec{\beta} \cdot \vec{r}')}{|\vec{n} \times \vec{\beta}|} \quad (117)$$

and

$$\begin{aligned} \vec{m} \times (\vec{m} \times \vec{r}') &= \frac{(\vec{n} \times \vec{\beta}) \times [\vec{\beta}(\vec{n} \cdot \vec{r}') - \vec{n}(\vec{\beta} \cdot \vec{r}')] =}{[\vec{n} \times \vec{\beta}]^2} \\ &= \frac{\vec{\beta} [(\vec{n} \cdot \vec{r}')(\vec{n} \cdot \vec{\beta}) - \vec{\beta} \cdot \vec{r}'] + \vec{n} [(\vec{\beta} \cdot \vec{r}')(\vec{\beta} \cdot \vec{n}) - \beta^2(\vec{n} \cdot \vec{r}')] }{[\vec{n} \times \vec{\beta}]^2}, \end{aligned} \quad (118)$$

we get

$$\begin{aligned} \vec{r}'' &= \vec{r}' + \frac{\vec{\beta}}{1 - \vec{n} \cdot \vec{\beta}} \left[ \vec{n} \cdot \vec{r}' - \frac{\gamma}{\gamma + 1} \vec{\beta} \cdot \vec{r}' \right] + \\ &= \frac{\vec{n}}{1 - \vec{n} \cdot \vec{\beta}} \left[ (\vec{\beta} \cdot \vec{r}') \left( \frac{2\gamma}{\gamma + 1} \vec{\beta} \cdot \vec{n} - 1 \right) - \frac{\gamma - 1}{\gamma} \vec{n} \cdot \vec{r}' \right]. \end{aligned} \quad (119)$$

This result coincides with the one that follows from Eq.34 of [94]. As an other check, if we take  $\vec{r}' = \vec{n}'$  and use the relations

$$\vec{\beta} \cdot \vec{n}' = \gamma[\vec{\beta} \cdot \vec{n} - \beta^2], \quad \vec{n} \cdot \vec{n}' = 1 - \gamma \vec{\beta} \cdot \vec{n} + \frac{\gamma^2}{\gamma + 1} (\vec{\beta} \cdot \vec{n})^2 \quad (120)$$

that follow from (110), we get after some calculations the correct result

$$\vec{n}'' = \gamma(1 - \vec{\beta} \cdot \vec{n}) \vec{n}. \quad (121)$$

Consequently, after the combined transformations

$$x''^\mu = R_\nu^\mu(\vec{m}; \alpha) L_\sigma^\nu(\vec{V}) x^\sigma, \quad n''^\mu = R_\nu^\mu(\vec{m}; \alpha) L_\sigma^\nu(\vec{V}) n^\sigma, \quad (122)$$

where the matrix  $R_\nu^\mu(\vec{m}; \alpha)$  represents a rotation by the angle  $\alpha$  around the vector  $\vec{m}$  and the matrix  $L_\sigma^\nu(\vec{V})$  corresponds to a Lorentz boost with the velocity  $\vec{V}$ , we will obtain

$$\begin{aligned} (n_\nu dx^\nu)^{2b} (dx_\mu dx^\mu)^{1-b} &= (n''_\nu dx''^\nu)^{2b} (dx''_\mu dx''^\mu)^{1-b} = \\ &= \left[ \gamma(1 - \vec{\beta} \cdot \vec{n}) \right]^{2b} (n_\nu dx''^\nu)^{2b} (dx''_\mu dx''^\mu)^{1-b}. \end{aligned} \quad (123)$$

The unwanted conformal factor  $\left[ \gamma(1 - \vec{\beta} \cdot \vec{n}) \right]^{2b}$  can be compensated by final dilatation

$$x''''^\mu = D(\lambda) x''^\mu \equiv \lambda x''^\mu, \quad (124)$$

with the suitably chosen scale-factor

$$\lambda = \left[ \gamma(1 - \vec{\beta} \cdot \vec{n}) \right]^b, \quad (125)$$

and we finally get the desired invariance

$$(n_\nu dx^\nu)^{2b} (dx_\mu dx^\mu)^{1-b} = (n_\nu dx''^\nu)^{2b} (dx''_\mu dx''^\mu)^{1-b}. \quad (126)$$

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