Gauge-invariance and the empirical significance of symmetries

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Abstract

This paper explicates the direct empirical significance (DES) of symmetries. Given a physical system composed of subsystems, such significance is to be awarded to physical differences about the composite system that can be attributed to symmetries acting solely on its subsystems. The debate is: can DES be associated to the local gauge symmetries, acting solely on subsystems, in gauge theory?

In gauge theories, any quantity with physical significance must be a gauge-invariant quantity. Using this defining feature, we can recast the existence of DES as a question of holism: if a larger system is composed of (sufficiently) isolated subsystems, are the individual gauge-invariant states of the subsystems sufficient to determine the gauge-invariant state of the larger system? Or is the relation between the subsystems underdetermined by their physical states, and does the underdetermination carry both empirical significance and a relation to the subsystem symmetries?

To attack the question of DES from this gauge-invariant angle, the straightforward method is gauge-fixing; for the values of gauge-invariant quantities are entirely determined by gauge-fixed representations of the system and subsystem states. However, gauge-fixings are subtle for field theories *in the presence of boundaries*. There are two qualitatively different types of boundary: an internal boundary, dividing the system into subsystems, and an external one, standing outside the entire composite system ('the whole Universe').

We find: (i) for internal boundaries, DES cannot be associated to local gauge symmetries, only to the global components of these symmetries; (ii) here there is no DES-symmetry in vacuum in a simply-connected manifold; but (iii) we *do* recover the recent literature's standard notions of DES for point-particle mechanics, as exemplified by the thought-experiment known as 'Galileo's ship'; finally (iv) we recover previous construals of DES (Greaves & Wallace, 2014; Wallace, 2019b), but only for external boundaries and for sufficiently inhomogeneous configurations of non-Abelian gauge theories.

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Contents

1 Introduction

1.1 The debate

Symmetries of the whole Universe are widely regarded as not being directly observable, that is, as having no direct empirical significance. At the same time, it is widely accepted that some of these symmetries, such as velocity boosts in classical or relativistic mechanics (Galilean or Lorentz boosts), acquire *direct* empirical significance when applied solely to subsystems. Thus Galileo's famous thought-experiment about the ship—that a process involving some set of relevant physical quantities in the cabin below decks proceeds in exactly the same way, whether or not the ship is moving uniformly relative to the shore—is used to show that sub-system boosts have a direct, albeit strictly relational, empirical significance. For while the inertial state of motion of the ship is undetectable by experimenters confined to the cabin, the entire system, composed of ship and shore, registers the difference between two such motions, namely in the relative velocity between ship and sea.

The broad notion of 'direct empirical significance' amounts to the existence of transformations of the universe possessing the following two properties (articulated in this way by (Brading & Brown, 2004), following (Kosso, 2000)):

(i) the transformation applied to the Universe in one state should lead to an empirically different state; and yet

(ii) the transformation should be a symmetry of the subsystem in question (e.g. Galileo's ship), i.e. involve no change in quantities solely about the subsystem.

Whether the concept of 'directly empirically significant subsystem symmetries' (DES henceforth) extends to local gauge theories is less settled.[1] Gauge symmetries are normally taken to encode descriptive redundancy, which suggests gauge theories cannot realize the concept of DES. For surely, a "freedom to redescribe" could not have the content needed for a direct empirical significance, like the one illustrated by Galileo's ship. This argument was developed in detail by Brading and Brown (Brading & Brown, 2004). They take themselves—I think rightly, in this respect—to be articulating the traditional or orthodox answer.

Building on (Healey, 2009) and (Brading & Brown, 2004), Greaves and Wallace (Greaves & Wallace, 2014) resist the orthodoxy by articulating DES for gauge theory differently. They focus on subsystems as given by regions and they identify transformations possessing properties (i) and (ii) by first formulating the putative effects of such transformations on the gauge fields in these regions. Nonetheless, for the question of whether gauge symmetry can be said to have 'empirical significance' in the sense of Galileo's ship—viz. in the sense that certain subsystem symmetries can be used to effect a relational difference between a subsystem (e.g. the ship system) and an environment (e.g. the shore)—Greaves and Wallace's treatment of gauge theory bears many similarities to ours here. But our conclusions will differ greatly: they find DES in many instances we do not.

That is the debate this paper aims to resolve. I will show that there is a general, coherent formalization of DES which yields (a) the aforementioned Galilean symmetries in the ship-scenario (and its relativistic analogue), but which (b) yields no non-trivial realizer of the concept in the case of *local* gauge symmetries.[2] Since the case of gauge theories is the contentious one, it will be my focus in this paper.

1.2 The argument

To earn the title of 'gauge', a theory must accord physical reality only to certain quantities: those that are invariant under a certain class of transformations, labeled gauge. This construal of gauge is business as usual. The crucial novelty introduced by subsystems is that we leave "God's vantage-point" for a more parochial one. That is: we *also* assume restricted access to quantities in each subsystem. Thus

[1] See (Kosso, 2000; Brading & Brown, 2004; Greaves & Wallace, 2014; Teh, 2016; Friederich, 2014, 2017; Gomes, 2019a, 2019b; S. Ramirez & Teh, 2019; Wallace, 2019a, 2019b; S. M. Ramirez, 2019) and references therein.

[2] DES appears only in certain circumstances, and then only related to global (called 'rigid' in (Gomes, 2019b)) gauge transformations. This is in line with Ladyman (Ladyman, 2015), who argues that all explicit examples of DES involve only the global part of the symmetries. Meanwhile, other authors (e.g. (Healey, 2009; Greaves & Wallace, 2014; Wallace, 2019b)) claim that non-trivial realizers exist irrespective of these conditions. For a more complete discussion of circumstances which *would* allow a non-trivial realization of DES in local gauge theories, see (Gomes, 2019b).

the introduction of subsystems brings in what amout to epistemic considerations, additional to ontological ones.

Nonetheless, if we assume that the subsystems in question are intrinsically described by gauge theory, their quantities with physical significance must likewise be gauge-invariant. It is the distinction between subsystem-intrinsic and global gauge-invariant quantities that may imply DES for local gauge theories after all: the question turns on whether or not there is a gap between the (union of) gauge-invariant quantities of the subsystems and those of the entire system.

This was the basis of the argument advanced and explored in (Gomes, 2019b). Building on that argument, the present paper adds a seemingly innocuous remark: gauge-invariance is transparent within a gauge-fixed treatment, and we can describe any *gauge-invariant* quantity as a *gauge-fixed* quantity.

The remark is, in fact, consequential for our topic, for it implies that an approach that stresses gauge-fixed quantities can lay bare the concept of DES. That is, once we assume empirical significance should involve only gauge-invariant quantities, we conclude from the remark that if any empirical significance is to be found, it can be found within a gauge-fixed treatment.

The question of DES is thus shifted to the question of determining which local gauge-symmetries can be 'legally' gauge-fixed, both for the entire system and for its subsystems. And *these* questions are settled, in part, by definitions. That is: given a state space, one can proceed in one of two ways in the characterization of gauge symmetries:

'Kinematical symmetries': we define these symmetries as those that leave relevant structures—the state space, the Lagrangian or the Hamiltonian of the theory—invariant. In this case, the symmetries are entirely determined by the particular features of the state space (e.g. phase space) and of the action functional. So, there is no 'symmetry-principle', simpliciter, in play.

'Fundamental symmetries': A group of transformations is *given* as being gauge. So invariance under transformations of the states constrains the laws to respect the symmetries. In this case, there is a symmetry principle, simpliciter, in play.[3]

I should mention that the distinction above is not hard and fast: a certain set of 'fundamental symmetries' can also appear first dynamically through the invariance of a Lagrangian but then be elevated to fundamental status. Although such a simplistic depiction is not always faithful to how science historically homes in on the correct Lagrangians and associated symmetries, the broad characterization is useful.

I also call attention to a possible departure from the usual use of "kinematical symmetry group". My use aims to emphasize the choice of state space: the choice is usually deemed relatively unproblematic, but in this definition I'm countenancing somewhat gerrymandered examples, with the allowed states differing at boundaries.[4]

In practice, the two definitions are indistinguishable if the universe has no boundaries.

[3]In (S. Ramirez & Teh, 2019, p. 8), this distinction has a different label: (A) and (B), with (A) corresponding to 'Fundamental' and (B) to 'Kinematical', which they describe as "a more refined [...] notion according to which an (A)-type gauge symmetry is further required to encapsulate redundancy for a particular kinematical system, whose states can only be defined after fixing specific boundary conditions".

[4]This is a relatively benign form of gerrymandering the state space: a kinematical view could in principle apply to a contrived state space with different states allowed at different points, points whose union might not form a smooth submanifold, let alone a boundary. I will not countenance such monstrosities here.

1.2.1 Boundaries, subsystems and their symmetries

When considering the entire universe, with the possible exception of some artificial, gerrymandered counter-examples, the known gauge theories make no distinction between 'fundamental symmetries' and 'kinematical symmetries'. But in going from the symmetries of the entire universe to symmetries of its subsystems, there is room for some confusion.

The confusion is engendered by two different ways of introducing subsystems through the use of boundaries. In the first way, one takes the *entire universe* to have a boundary, so that the entire universe is taken as a type of subsystem. This notion of subsystems is more interesting within a kinematical view of symmetries. For in specifiying the state space and dynamics to which the symmetries are subservient, one could impose a fixed representation of the states at the boundary. For instance, one could say: "the configuration space with which I am dealing possesses only one representative of the gauge potential at the boundary". According to the kinematical view of symmetries, this is legitimate; and while it curtails the full set of gauge transformations and gauge representatives at the boundary, it does not *break* gauge-invariance. For there were no gauge transformations acting on the boundary to begin with. Therefore this notion of subsystem—which we will label '*externalist*'—allows gauge theories that can be realized within the kinematical, but not the fundamental, view of symmetries.

In the second way of introducing subsystems through boundaries, a boundary represents only some convenient split of the universe. For fields, one thus specifies the subsystem by drawing boundaries within spacetime, so that the subsystem is composed of fields lying on bounded submanifolds of the original manifold. The subsystem symmetries are then fully determined by the split: the original universal symmetries are restricted to the ones supported on the bounded submanifold. In this case, there is no further question about how symmetries should behave at the boundary, for they behave just as they did in the bulk of the manifold, i.e. before the boundary or split was defined. Since fundamental and kinematical symmetries are indistinguishable in the absence of boundaries, this notion of subsystem also effaces that distinction in the presence of boundaries. We dub this notion of subsystem '*internalist*'.

I should here make a disclaimer. Although I will analyze the externalist notion of subsystem within the kinematical view of symmetries in Section 4, I do not believe this description is as physically relevant as the internalist notion of subsystem. Of course, no one is forbidden from specifying a system where gauge symmetries act differently at the boundary by fiat—as they can in the externalist's notion of subsystem—but the status of such boundaries is not very clear. If it is epistemic, it should still allow for the continuation of the universe beyond the boundary. If it is ontological, it is hard to get a grip on the concept at all. Moreover, we would have to allow a subsystem-quantity that from the fundamental perspective is gauge-*variant*—a quantity such as the boundary value of the gauge potential—to acquire physical significance. That is, these realizations of the externalist notion of subsystems ascribe (kinematical) gauge-invariance to a (fundamentally) gauge-variant quantity.[5]

Indeed, since the externalist's boundaries are taken to enclose the entire universe: there is still considerable conceptual work, to put it charitably, left to relate

[5]In (S. Ramirez & Teh, 2019), the externalist account of DES is labeled 'Type II'. As far as I am aware, they give the only consistent account of DES in these circumstances.

this scenario to the occurrence of *actual* subsystems.[6] I will return briefly to these questions about the externalist's and the internalist's subsystem in Section 5, after we have gleaned the consequences of their differences.

For now, it is useful to summarize: both the Lagrangian and the Hamiltonian formulations of field-theory refer to the fields over the entire universe; we are at first given no subsystems that DES can latch on to. Subsystems must be somehow "conjured into being", and there are two general ways of doing this: one by introducing a boundary external to the entire universe; and the other by one internal to it.

I believe the internalist's subsystem is more physically plausible: start with a universe which is boundary-less and then carve it into subsystems. An added benefit is that the definition of subsystems by splitting the universe rubs out the difference between a fundamental and a kinematical view of subsystem symmetries. This indistinguishability of views on gauge transformations is another reason to hold that the internalist's subsystem should be preferred in evaluating DES: it allows the DES discussion to be detached from other controversies in gauge theory.

Nonetheless, here I will analyze both the internalist and the externalist perspectives. I will first focus on the internal boundary, in sections 2.3 and 3, leaving the treatment of external boundaries to Section 4, where I compare my results with those of (Greaves & Wallace, 2014; Wallace, 2019a, 2019b). They claim to have found non-trivial realizations of DES in local gauge theories under the internalist perspective. There, I will recover their claims—but only for the externalist's subsystem with a kinematical view of symmetries.

The use of gauge-invariant quantities system and subsystem will make the argument outlined above transparent. To describe these quantities unambiguously and to exhibit clearly the role in the argument of the different types of boundaries, we will employ gauge-fixings.

My main thesis will be, as announced at the end of Section 1.1: that in the internalist's 'splitting' notion of subsystem, the gauge-fixing approach fails to find a non-trivial realizer of DES for a subsystem of a simply connected, vacuum Universe. Here space is simply-connected and time is just $\mathbb{R}$; and vacuum means only the gauge fields are present—there is no charged matter anywhere (for the full treatment, with matter and non-simply-connected topology, see (Gomes & Riello, 2019, Sec 4) and (Gomes, 2019b)). At first sight, this may seem a disappointing or negative conclusion. Nonetheless, it is worth stressing, since previous authors have claimed that, for this kind of Universe, a subsystem *can* have DES (cf. footnote 2 and Section 4).

To further buttress my thesis, I will contrast and compare the constructions of the field-theory case with a system of N point-particles. Here, I will show that my physical notion of DES as obtained through gauge-fixings recovers the standard results for Galileo's ship, as summarized at the end of Section 1.1.

[6] This relation is indeed possible, and ubiquitous in gravitational physics. In fact, we today model the solar system in this way: the standard spatially asymptotic flat spacetime we use imposes particular representations of the metric as one approaches the asymptotic boundary; see (Regge & Teitelboim, 1974) for original work discussing these conditions explicitly, and (Belot, 2018; Teh, 2016) for a philosophical discussion. Teh demonstrates that one way of understanding Greaves and Wallace's schema indeed is in terms of 'asymptotic' formulations of such scenarios. Here I am *not* considering asymptotic conditions. I believe one must first consider finite regions, and then obtain the asymptotic regime by progressively enlarging a subsystem; see (Riello, 2019) for an example in the Yang-Mills case. I will come back to this discussion in Section 5.

1.3 Roadmap

In preparation for the remainder of the paper, I will in the final section of this introduction, Section 1.4, introduce the concept of DES in its gauge-invariant version, for all types of symmetries, within the internalist's notion of subsystem. In Section 2, I begin the treatment of gauge theories of fields in earnest.[7] I introduce gauge-fixings and discuss them; this is the most technically advanced section of the paper. I show that no non-trivial realizer of DES exists in the case studied here, i.e. for gauge field theories in a simply-connected, vacuum Universe. Then, in the Abelian case, I confirm the results within the holonomy formalism. As a last consistency check, in Section 3, I apply the same criteria for DES to the case of the theory of particles; and obtain the boosts as in Galileo's ship thought experiment. In Section 4, I reassess the standard derivation of DES in the same context—vacuum, simply-connected Universe, under the internalist's notion of subsystem—and flag its problematic assumptions. I show that, although the standard results do not withstand this criticism for the internalist's notion of subsystem, they are compatible with the externalist's notion of subsystem, but only for the non-Abelian case. In Section 5 I conclude.

1.4 Symmetry-invariance and DES

Let a given group of symmetries, $\mathcal{G}$, have some action on a state space, Φ, i.e. for $g \in \mathcal{G}$ and $\varphi \in \Phi$, there is a map

$$\begin{aligned} \mathcal{G} \times \Phi &\rightarrow \Phi \\ (g, \varphi) &\mapsto \varphi^g. \end{aligned} \tag{1.1}$$

The symmetry group partitions the state space into equivalence classes, $\sim$, where $\varphi \sim \varphi'$ iff for some g, $\varphi' = \varphi^g$. We denote the equivalence classes under this relation by square brackets $[\varphi]$. We call such $[\varphi]$ the *physical state*, and φ is its *representative* (when there is no need to emphasise that φ involves a choice of representative, we call it just 'the state' for short).

Adopting the internalist perspective, we must carve up the system into two subsystems—mutually exclusive, jointly exhaustive—whose state spaces we name Φ_+ and Φ_-, or $\Phi_\pm$, for short. When these subsystems correspond to regions, we will follow the notation and name the regions $R_\pm$.

The universal symmetries bequeath symmetries, through the split, to the subsystems. So we write $\mathcal{G}_\pm$; and similarly, we extend the use of the equivalence class notation and of the square brackets: $\varphi_\pm \sim_\pm \varphi'_\pm$ iff $\varphi'_\pm = \varphi_\pm^{g_\pm}$ for some $g_\pm$, in which case $\varphi'_\pm \in [\varphi_\pm]$.

In section 1.1, we defined DES as transformations of the Universe possessing the following two properties:

(i) the transformation should lead to an empirically different scenario, and

(ii) the transformation should be a symmetry of the subsystem in question.

We also saw that physical quantities in gauge theories are characterized as gauge-invariant quantities, and that this obtains for both the subsystems and for the entire system.

Therefore, to earn the label "Direct Empirical", DES must be construed as referring solely to universal and subsystem gauge-invariant concepts.

[7] The specific form of the action functional will not play a role in what follows, since the statements I am interested in occur already at the level of kinematics: gauge-invariance of the action is all that matters.

Here, properties (i) and (ii) will be taken to apply to a Universe composed of subsystem and environment (as two subsystems). Following the internalist's 'symmetric' treatment of subsystems, (ii) will be taken to apply to all subsystems, i.e. to subsystem and environment.[8] This construal of DES is called *relational*, and it is the only sort we will consider for the internalist's notion of subsystem.

Indeed, we can then translate criteria (i) and (ii) respectively into:

• *Global difference:* $[\varphi] \neq [\varphi']$: the two physical states of the Universe are distinct according to the $\sim$ relation.

• *Regional equivalence:* $[\varphi_\pm] = [\varphi'_\pm]$: regionally the states are physically indistinguishable according to the $\sim_\pm$ relation; that, is for each $(\pm)$ subsystem, the primed and unprimed states are symmetry-related according to their restricted view.

Note that only the 'physically significant', i.e. gauge-invariant, content of the subsystem and Universe states are relevant in the characterization of DES. By introducing some composition of physical states, $\boxplus$, and writing

$$[\varphi_+] \boxplus [\varphi_-] =: [\varphi] \neq [\varphi'] := [\varphi'_+] \boxplus' [\varphi'_-] = [\varphi_+] \boxplus' [\varphi_-] \tag{1.2}$$

we indicate more clearly that the very concept of DES needs to be gauge-invariant, i.e. physical. Note also that the subsystem states are intrinsically identical between the $[\varphi]$ and the $[\varphi']$ Universes, and therefore the difference between them must lie in the relation between the subsystems; this is signalled by (1.2)'s use of $\boxplus$ as well as $\boxplus'$.

Of course, equivalence classes are notoriously resistant to explicit mathematical manipulation. In particular, we cannot articulate a notion of composition using only equivalence classes (see e.g. (Dougherty, 2017; Nguyen, Teh, & Wells, 2018; Gomes, 2019a)). To analyze (1.2) explicitly, we must refer back to local representatives.[9]

I will use the notation $\oplus$ here with the broader meaning of 'composition of *representatives*'; I do not restrict $\oplus$ to mean 'direct sum'. For example, for local, smooth representatives in field theory, there is a straightforward definition of composition, as smooth composition, or gluing, which I will denote $\oplus$ and introduce in section 2. For point-particle systems, as we will see in section 3, $\oplus$ requires an embedding of the subsystems into a common Euclidean space, and then it signifies vector addition. And, like for $\boxplus, \boxplus'$, I will signal different ways of combining—different relations between—representative states by $\oplus, \oplus'$.

Writing $\varphi = \varphi_+ \oplus \varphi_-$, and considering the two given states

$$\varphi = \varphi_+ \oplus \varphi_- \quad \text{and} \quad \varphi' := \varphi'_+ \oplus' \varphi'_-,$$

the condition *regional equivalence* translates into:

$$\varphi' := \varphi'_+ \oplus' \varphi'_- = \varphi_+^{g_+} \oplus' \varphi_-^{g_-} \quad \text{for some pair of elements} \quad g_\pm \in \mathcal{G}_\pm \tag{1.3}$$

[8] Couldn't we allow for a transformation that also changes the physical state of the environment? Invoking a physically significant change in the environment leads to the concern (voiced in, for example, (Friederich, 2014, p. 544)) that the empirical significance intended for the subsystem gauge symmetry is in fact completely due to the change in the environment state, thus leaving no room for the gauge symmetry to do any work. For example, in the Galileo's ship thought experiment, a transformation that leaves the ship and its relation to the shore as they are but changes a grain of sand on the other side of the Universe satisfies (i) and (ii). GW react to this concern by requiring a further condition: there should be a 'principled connection' between the putative change in the environment and the gauge symmetry (Greaves & Wallace, 2014, p. 68, 86 and 87). But they say nothing further about what such a connection might be. On the other hand, if (ii) applies symmetrically to all subsystems, empirical significance is encoded solely in the relations between them. Relational DES would thus exclude from its remit cases such as Faraday's cage. The most comprehensive account for non-relational DES is that of (S. Ramirez & Teh, 2019).

[9] I am sure there is a "throwing the ladder away" metaphor lurking somewhere nearby.

Now, *Global Difference* demands that *no* g exists such that $\varphi' = \varphi^g$. That is, *Global Difference* implies:

$$\text{there is no global } g \text{ such that} \quad g_{|R_+} = g_+, \quad g_{|R_-} = g_-, \tag{1.4}$$

for otherwise $\varphi' = \varphi^g \sim \varphi$. The physical difference between φ and φ' clearly must lie in the different possibilities for composing the two regional states, that is, the difference must lie between $\oplus$ and $\oplus'$.

The point is that there are different domains for the equivalence relations—subsystem or universe—and therefore a Universal empirically significant difference may arise from a transformation that doesn't change the subsystem states, but *does* change their relation.[10] Thus we countenance the possibility that even perfectly isolated systems might bear a variety of relations to each other. And, although we will argue that in the simple case we study here—viz. a simply-connected Universe with just the gauge field—there will be *no* such variety: *it can* exist for non-simply-connected manifolds or in the presence of matter fields (Gomes & Riello, 2019; Gomes, 2019b).

2 The gauge theory of fields

Here I will describe the basic setting with which I will treat the local gauge theory of fields, taking as my model: vacuum Yang-Mills theory on a simply-connected manifold, M. The stated results should be taken as applying to both Abelian and non-Abelian interactions alike; exceptions and differences will be explicitly flagged. Having said this, I will, as a simplification for the technical points about gauge-fixing, only *explicitly* treat Abelian gauge fields (like electromagnetism).[11]

Therefore, let A_μ be the electromagnetic gauge field on a 4-manifold M. The fundamental, or *charge group*, of this theory is $G = \mathrm{U}(1)$, with an associated Lie-algebra $\mathfrak{g} = \mathbb{R}$. With an appropriate choice of units, the gauge transformations are:

$$A_\mu \to A^g_\mu := A_\mu + i\partial_\mu \ln g \tag{2.1}$$

for some U(1)-valued function, i.e. $g \in \mathcal{G}$, where $\mathcal{G} := C^\infty(M, G)$. Here $A \sim A'$ iff $A' = A^g$. And $\Phi \equiv \mathcal{A} := \{A \in \Lambda^1(M, \mathfrak{g})\}$ (the space of Lie-algebra-valued smooth one-forms on M). That is, the φ of the previous section would here be the electromagnetic potential, A. More generally, in the vacuum Yang-Mills case, the representatives φ are identified as the field configurations, A; they are representatives of the equivalence classes, $[A]$.

As dictated by the internalist's notion of subsystem, we introduce subsystems through the splitting of M by internal boundaries. That is, the (Euclidean)[12] spacetime manifold $(M, g_{\mu\nu})$ is taken as simply-connected and boundary-less, and

[10] Here I disagree with (Friederich, 2017), who seems to exclude the possibility of DES almost by assumption. He demands that no difference in relations should be present. E.g. on page 155: "The present article explores the idea that two subsystem ontic variables designate one and the same physical subsystem state only if the states designated by them are empirically indistinguishable both from within the subsystem and from the point of view of arbitrary external observers." and then again (p. 157): "In other words, s and s' designate the same physical state if they are empirically equivalent both from within the subsystem and from the perspectives of arbitrary external observers."

[11] For a more complete treatment, see (Gomes & Riello, 2019; Gomes, 2019b).

[12] To avoid certain complications that only occur for a Lorentzian signature of $g_{\mu\nu}$, but are only incidental, and at least apparently do not pertain to the issues of empirical significance, I will take $g_{\mu\nu}$ to be of Euclidean signature.

is split into two mutually exclusive, jointly exhaustive regions, $R_\pm$. That is, we have the four conditions (respectively): (i) $\partial M = \emptyset$, (ii) M's first fundamental group vanishes, $\pi_1(M) = 0$, (iii) $R_+ \cup R_- = M$, and (iv) $R_+ \cap R_- = \partial R_\pm =: S$.

Since the configuration space of the theory over any spacetime is built out of the configuration spaces of the theory over subregions of that spacetime, the internalist's splitting of the manifold naturally induces an identification of subsystems and regions, and ensuing identifications: $\mathcal{G}_\pm := C^\infty(R_\pm, G)$ and $\Phi_\pm \equiv \mathcal{A}_\pm := \{A^\pm \in \Lambda^1(R_\pm, \mathfrak{g})\}$, where $\Lambda^1(R_\pm, \mathfrak{g})$ are the Lie-algebra-valued (i.e. here $\mathbb{R}$-valued) smooth 1-forms on the spacetime submanifolds $R_\pm$.

In field theory, we can now be more explicit about conditions on the composition operation, $\boxplus$, on physical, i.e. gauge-invariant states. Two subsystem physical states $[A_\pm] \in [\Phi_\pm] := \Phi_\pm / \sim_\pm$ are composable iff they jointly descend from a universal state, $[A]$. Alternatively, they are composable iff there exist representatives, $A_\pm$, such that the value of A_+ and all its derivatives at the boundary S match those of A_-. We call such a notion of composition *gluing*. Given arbitrary representatives, $A_\pm$, the condition of composition can thus be translated into: there exist gauge transformations, $g_\pm \in \mathcal{G}_\pm$, such that the gauge-transformed representatives glue:

$$A_+^{g_+}{}_{|S} = A_-^{g_-}{}_{|S}, \tag{2.2}$$

where the subscript $_{|S}$, restricting to S, is understood as also matching derivatives.[13]

2.1 Gauge-fixing: the general ideas

The physical content of the field, $[A]$, as an equivalence class, is mathematically intractable.[14] But there are many alternative representatives of $[A]$; and so, to obtain a 1-1 representative of the equivalence class, we must eliminate redundancy.

The procedure for fixing the representative of the state is called 'fixing the gauge': it extricates physically significant properties from spurious ones—those associated merely with a difference of representation. In other words, by fixing the gauge, no physical property is lost. Thus important physical effects, such as the Aharonov-Bohm effect, quantum anomalies, interference, are all perfectly expressible in a gauge-fixed formalism, e.g. in Lorenz gauge.

In the absence of matter fields, fixing the gauge is a non-local procedure, much as in other gauge-invariant treatments, such as in the holonomy interpretation, where these features are already present at the ground level. This is just a reflection of the non-local aspects of gauge-invariant functions (cf. (Earman, 1987, p. 460), (Healey, 2007, Ch. 4.5), and (Strocchi, 2015; Gomes, 2019a)).

It is important to note that a gauge-fixing is *not* analogous to choosing an arbitrary coordinate representation of a single configuration. The latter is mostly useful when we do *not* need to compare the totality of physical attributes of different configurations. We use a single arbitrary representative A, (and similalry, in the case of general relativity and other spacetime theories, an arbitrary choice of spacetime coordinates), if all we are interested in is to describe certain properties of a single physical situation, e.g. if there are no counter-factual considerations in play. We can then calculate some quantity and show that it was independent of the coordinate choice we made at the beginning.

[13] For the standard notion of continuity, i.e. when all we require is the value of f at the boundary, and not also of its derivatives, we employ no bar, i.e: $f_S = f'_S$ iff $f(x) = f'(x)\ \forall x \in S$.

[14] It also has other disagreeable properties vis à vis the composition of subsystems. See (Dougherty, 2017; Gomes, 2019a). We will come back to this point in Section 2.2.1.

But how are we to judge whether two given representatives, A, A' are physically the same, i.e. give the same value for *all* gauge-invariant quantities? These are the kinds of modally loaded questions that we face when investigating DES. When we need to compare the totality of physical facts, we need a gauge-fixing: for it provides, in a manner of speaking, the *same* 'coordinates' with which to compare different states. In other words, two configurations are physically the same if and only if they are identical once gauge-fixed.

More formally, the gauge-fixing procedure relies on the given state—i.e. the given physical state $[\varphi]$ *according* to some representative φ—satisfying some auxiliary condition. That is, we impose further functional equations that the state *in a specific representation* must satisfy, so that the freedom to change the representation is constrained.

In the language of fiber bundles, a gauge-fixing is a choice of *section* of the bundle, i.e. of the extended configuration space. For example: given the space of all possible gauge potentials, call it $\mathcal{A} := \{A \in \Lambda^1(M, \mathfrak{g})\}$, a section gives a local submanifold of $\mathcal{A}$ which intersects each gauge-orbit, i.e. fibre, only once. The manner by which we define such a surface is a *gauge-fixing*.

That is, we fix the gauge freedom by imposing conditions on the gauge potential, i.e. by imposing a local functional equation $\sigma(A) = 0$,[15] for some σ which, ideally, must satisfy two conditions:

- *Universality*: $\sigma(A^g) = 0$ must be solvable by a functional $g_\sigma[A]$, such that $A^{g_\sigma[A]}$ lies in the section, i.e. $\sigma(A^{g_\sigma[A]}) = 0$, for any $A \in \mathcal{A}$. This condition ensures σ doesn't forbid certain states, i.e. that σ is not over-constraining, and that each orbit possesses at least one intersection with the gauge-fixing section. Here, $g_\sigma[A]$ represents the gauge transformation required to transform A to another configuration, one belonging to the gauge-fixing section and therefore satisfying $\sigma(A^{g_\sigma[A]}) = 0$.
- *Uniqueness*: If $g_\sigma[A]$ as above satisfies $\sigma(A^{g_\sigma[A]}) = 0$, then $A^{g_\sigma[A]}$ is unique. That is: $A^{g_\sigma[A]} = A'^{g_\sigma[A']}$ if and only if $A \sim A'$.

Since $\sigma(A^{g_\sigma[A]}) = 0$, it is useful to define $h_\sigma[A] := A^{g_\sigma[A]}$, the projection of A onto the gauge-fixing surface. This can be seen as a functional: $h_\sigma : \mathcal{A} \to \mathcal{A}$, a projection onto a submanifold of $\mathcal{A}$; it is a functional on $\mathcal{A}$ taking as values elements of $\mathcal{A}$. Moreover, $h_\sigma[A]$ is a gauge-invariant functional, in the sense that $h_\sigma[A^{g'}] = h_\sigma[A]$, i.e. it is invariant under the group action on its domain (we will return to the group action on its image later). A gauge-fixing yields a one-to-one relation: $[A] \leftrightarrow h[A] := A^{g_\sigma[A]}$, which is what is meant when we say that the entire gauge-invariant content of a configuration is contained in its gauge-fixed form.

We will see explicit examples of $h_\sigma[A]$ soon. It is also important to distinguish $h_\sigma[A]$, which is itself a gauge potential, from $g_\sigma[A]$, which is a group transformation taking A to $h_\sigma[A]$. To unclutter notation, we will remove the subscript σ from all functionals unless explicit reference to σ is needed as a reminder.

Assume both of these conditions hold good. Then we can describe any element of $\mathcal{A}$ as a doublet: (h, g), where h is any A satisfying the condition $\sigma(h) = 0$ and $g \in \mathcal{G}$ describes a gauge transformation as applied to the given element of the section.[16] A gauge-fixing "fixes the same coordinates" for different configurations in the following sense: it gives a transformation of coordinates such that $[A] = [A']$

[15] I will denote local functional equations, i.e. determined separately at each point in terms of values of the field and its derivatives, by round brackets, and more general functional dependence by square brackets.

[16] Wallace calls the doublet (O, g) instead of (h, g), and instead of gauge-invariant information, he uses the term "subsystem-intrinsic" (Wallace, 2019a).

(or, equivalently, $A \sim A'$) if and only if the field values in the shifted 'coordinates' coincide point by point.

Here I will only look at σ's that satisfy *'Universality'*.[17] The challenge will be to ensure σ also satisfies *Uniqueness*.

But note that rising to this challenge does not require the solution $g_\sigma[A]$ to be unique. For, even if $\sigma(A) = 0$ underdetermines $g_\sigma[A]$, the gauge-fixed representative of $[A]$ need not be underdetermined.

Underdetermination is unavoidable only in the presence of *stabilizers*. A a stabilizer is a gauge transformation that leaves A invariant, or more precisely, the set (subgroup) of such transformations. Thus the stabilizer for the representative φ is defined as the subgroup

$$\mathrm{Stab}(\varphi) := \{\tilde{g} \in \mathcal{G} \mid \varphi^{\tilde{g}} = \varphi\}, \tag{2.3}$$

Since, for general non-Abelian gauge-fields, the stabilizer depends on the representative, it is also useful to denote this dependence explicitly as e.g. $\tilde{g}(A)$. Since stabilizers transform as $\tilde{g}(A^g) = g^{-1}g(A)g$, stabilizers of two elements of a single fiber are conjugate subgroups. The presence of stabilizers implies that features of $[A]$ do not possess enough variety—"wrinkliness"—to completely fix the *gauge transformations* that carry an $A \in [A]$ to an $h[A]$.

But for non-Abelian groups, $\mathcal{A}$ is generically stabilizer-free, i.e. stabilizer groups are trivial, i.e. just the identity. Nonetheless, particular physical states, such as the physical state of "no A field", i.e. $A \sim 0$, allow stabilizers, and thus don't allow gauge transformations to be uniquely fixed. In the Abelian case, all configurations share the same stabilizer: viz. the group of constant gauge transformations (cf. discussion in Section 2.2).

Indeed, for the purposes of this paper, this is the most important distinction between Abelian and non-Abelian theories. Namely: stabilizers are the same for all Abelian field configurations—they are the constant transformations—and, on the other hand, are trivial for generic non-Abelian field configurations.

In any case, if the only "slack" left in the determination of $g_\sigma[A] \in \mathcal{G}$ is due to stabilizers, it is idle: there is no effect on the resulting gauge-fixed $h[A]$. That is because the slack in the group elements g has a trivial action on the configuration. In other words, if $\sigma(A) = 0$, satisfies *Uniqueness*, then if a solution to $\sigma(A^g) = 0$ is not unique, the degeneracy lies in $\mathrm{Stab}(A)$ and thus cannot be detected by any representative. That is, for two solutions $g[A], g'[A]$ to $\sigma(A^g) = 0$, we demand $A^{g[A]} = A^{g'[A]}$, $\forall A$.

2.2 Details for Coulomb gauge

Now we will illustrate the previous definitions explicitly, by employing an explicit gauge-fixing functional σ. In the first subsection, 2.2.1, I will describe how this works out when the manifold is closed but without boundary. Formally, this is simpler than the bounded case, which we study in Section 2.2.2. Nonetheless, the simpler case already suffices to illustrate many of the intricacies of a gauge-fixing.

[17] And thereby ignore the Gribov problem (Gribov, 1978), that occurs in the non-Abelian case. Our conclusions also apply there, but do so only in a perturbative regime.

2.2.1 For the unbounded case

Let us start by introducing a standard gauge-fixing for the entire manifold: Coulomb gauge.[18] Following the nomenclature of Section 2.1 for the gauge-fixing section σ, we define:

$$\sigma(A) := \mathrm{div}(A) = 0. \tag{2.4}$$

It is easy to see that this gauge-fixing satisfies *Universality* and *Uniqueness*. First, *Universality*: given a general A, not necessarily belonging to the gauge-fixing section, i.e. A such that $h[A] \neq A$, we must ensure that there exists a gauge transformation that takes A to that section. Second, *Uniqueness*: we must ensure that whatever slack remains in the determination of this transformation, it cannot be "detected" by any A.

As to the first demand, equation (2.1) yields:

$$\mathrm{div}(A^g) = \mathrm{div}(A) + i\nabla^2(\ln g) = 0 \tag{2.5}$$

$$\therefore \;\; g[A] = \exp(i\nabla^{-2}(\mathrm{div}(A))) \tag{2.6}$$

where ∇^{-2} are the Green's functions associated to ∇^2.[19] For all A, we can find a solution $g[A]$ and thus a projection, $h[A]$. Therefore the gauge-fixing is *Universal*.

And although it doesn't determine $g[A]$ uniquely, the underdetermination is solely due to stabilizers. Here in the Abelian case, $g[A]$ and $g'[A]$ are solutions to (2.6), if and only if $g'[A] = g[A] + c$, where c is a constant.[20] Nonetheless, since here stabilizers have a trivial action on the gauge potential (from (2.1), since $\partial \ln c = 0$), the gauge-fixing still satisfies *Uniqueness*; i.e. $A^{g_\sigma[A]} = A^{g'_\sigma[A]}$.

Consistently, the 1-form $h_\mu[A] := A_\mu^{g_\sigma[A]}$, defined through (2.6) does satisfy (2.5). That is, it is easy to verify:

$$h[A] := A + i\,\mathrm{grad}(i\nabla^{-2}(\mathrm{div}(A))), \tag{2.7}$$

satisfies

$$\mathrm{div}(h[A]) = 0. \tag{2.8}$$

The projection h is invariant under gauge transformations: $h[A^{g'}] = h[A]$:

$$\begin{aligned} h[A^{g'}] &= A + i\,\mathrm{grad}(\ln g') + i\,\mathrm{grad}((i\nabla^{-2}(\mathrm{div}(A + i\mathrm{grad}(\ln g'))) && (2.9)\\ &= A + i\,\mathrm{grad}(i\nabla^{-2}(\mathrm{div}(A))) && (2.10)\\ &= h[A] && (2.11)\end{aligned}$$

where in passing from the first to the second line, we used $\mathrm{div}(\mathrm{grad}(\ln g')) = \nabla^2 \ln g'$. In this way, $h[A]$ captures the full gauge-invariant content of A.

[18] Since we are in the Euclidean setting, we actually mean a Landau gauge, which is Lorenz gauge in the Euclidean signature. Our specific findings do not depend on the particular gauge-fixing, as long as it adheres to the definitions in Section 2.1; these definitions imply that some non-locality, or integration over the spacetime region, will be involved in finding the particular gauge-fixed representation. And it will also necessarily satisfy '*Uniqueness*', as defined at the end of Section 2.1. All the constructions presented here have an exact analogue in the non-Abelian case. Essentially, the analogue replaces ∂_μ by the gauge-covariant $\mathrm{D}_\mu = \partial_\mu + [A, \cdot]$, and constant gauge transformations are replaced by the more general concept of stabilizer (2.3). See (Gomes, Hopfmüller, & Riello, 2019; Gomes & Riello, 2019; Gomes, 2019b).

[19] Roughly, as an operator on the space of functions, for $f \in C^\infty(M)$ we have $\int_M \nabla_x^{-2} y((\nabla^2 f)(x) = f(y)$. For a closed manifold, the operator exists and is unique on the space of non-constant functions, see e.g. (Gilbarg & Trudinger, 2001). When the metric is sufficiently homogeneous, Green's functions can be obtained in explicit form.

[20] To both be solutions to (2.5), we must have $\nabla^2(g[A] - g'[A]) = 0$. But it is easy to show that $\nabla^2 f = 0$ iff f is constant: $0 = \int f\nabla^2 f = \int |\mathrm{grad} f|^2 \Leftrightarrow \mathrm{grad} f = 0 \Leftrightarrow f = $const.

It is important to note that fixing the representative requires finding something like $g[A]$, and this is always a non-local process. That is, since A is related to g through a derivative, going the other way—tying g to A—always requires integration.[21]

2.2.2 In the bounded case

Now we want to generalize (2.4) to the bounded case, i.e. for regions $R_\pm$ bounded by S, as in the introduction to Section 2.

Although we want to define a gauge-fixing through the *gauge-potential*, $\sigma(A)$, we still aim to fix a particular value of the *gauge transformations*: the transformation that takes each representative to the gauge-fixing surface. Again, as in the discussion in Section 2.1, we want to fix the gauge using functions that exploit the 'wrinkliness' of the states.

Since in the bulk of the manifold we have a scalar second-order differential equation for g viz. equation (2.5), we want to impose scalar conditions for g on the boundary: either Dirichlet or Neumann boundary conditions, which are, respectively, of zeroth and first order in derivatives. However, as I commented above, these conditions should descend to g from $\sigma(A)$; we do not simply impose Neumann or Dirichlet boundary conditions on g. Apart from being conceptually uncouth, such imposed conditions would have the rather unwanted effect that the boundary conditions would not be gauge-covariant; they would depend on the initial, arbitrary choice of A that we assumed to represent $[A]$.[22]

Dirichlet or Neumann (scalar) boundary conditions imply that we must fix one, and only one, of the components of A at the boundary. Since A can only constrain the gradients of g through their relation in equation (2.1), this rules out Dirichlet conditions. And since only the spacetime direction normal to the boundary is singled out by the introduction of a boundary, we can only naturally introduce Neumman boundary conditions by fixing the normal component A_n of the gauge potential (see Section 4 for more on this).

Lastly, if we do not want to introduce further arbitrary parameters in our gauge-fixing, the simplest choice for $\sigma(A_\pm)$ as given by:

$$\sigma(A_\pm) \equiv \begin{cases} \mathrm{div}(A^\pm) & = 0 \\ A_n^\pm & = 0. \end{cases} \tag{2.12}$$

Given arbitrary regional configurations $A_\pm$, we solve the following second-order set of differential equations with field-dependent, covariant boundary conditions:

$$\nabla^2(\ln g_\pm) \;=\; i\mathrm{div}(A_\pm) \tag{2.13}$$
$$\partial_n(\ln g_\pm) \;=\; \pm i A_n \tag{2.14}$$

where the $\pm$ signs on the rhs of (2.14) come from the opposite directions of the normal at S. The solution is as in (2.6), namely,

$$g_\pm[A_\pm] = \exp(\pm i \nabla^{-2}_{\mathrm{Neu}(\pm A_n)}(\mathrm{div}(A_\pm))) \tag{2.15}$$

with the difference that now the Green's functions (inverse Laplacian), $\nabla^{-2}_{\mathrm{Neu}(A_n)}$ is defined for the field-dependent, Neumann boundary conditions (2.14); therefore $\partial_n \ln g_\pm[A_\pm] = \pm i A_n$ holds automatically.

[21] A simple example: radial, or axial gauge, $A_r = 0$. This is not a complete gauge-fixing, but we still find $g(x,r) = \int_0^r dr' A(r', x)$, where r are radial coordinates, and x are the remaining coordinates.

[22] Unless there is no choice for the boundary A, as in the externalist scenario.

As in the previous, unbounded case, the only ambiguity in solutions is due to stabilizers. Again, in the Abelian case, this means $g_\pm[A_\pm]$ and $g'_\pm[A_\pm]$ are solutions to (2.13) and (2.14) if and only if $g'_\pm[A_\pm] = g_\pm[A_\pm] + c$; as is easy to check.[23]

And again, for the same reasons, this ambiguity will have no effect on the representative. In other words, the associated projected potentials, $h[A_\pm] := A^{g[A]}$ and $h'[A_\pm] := A^{g'[A]}$, are identical, $h'[A] = h[A]$. Moreover, they are, as expected, gauge invariant: $h[A^g_\pm] = h[A_\pm]$, $\forall g \in \mathcal{G}$. Thus σ satisfies both *Universality* and *Uniqueness* and provides a bona-fide *gauge-fixing.*

The projected field $h_\pm$, defined as in (2.7), also satisfies (2.12). That is, $h[A_\pm]_n = 0$, even if $A^\pm_n \neq 0$. In other words, even though we have not restricted the set of $A_\pm$'s, any $A_\pm$, independent of its behavior at the boundary, through a gauge transformation—also generically non-trivial at the boundary—can be brought to satisfy equations (2.12). The reason is simple: the system of equations (2.13) and (2.14) always has solutions (existence).[24]

In sum: we now have our set of physical representatives; we have eliminated redundancy. Connecting back to the notation of Section 1.4: $h[A]$ and $h[A_\pm]$ carry the same information, respectively, as $[\varphi] \equiv [A]$ and $[\varphi_\pm] \equiv [A_\pm]$. So, not only is the configuration space $\mathcal{A}$ defined by the spaces $\mathcal{A}_\pm$, but each such space has its own principal fiber bundle structure.[25] But this says nothing about how $\mathcal{A}/\mathcal{G}$ is related to $\mathcal{A}_\pm/\mathcal{G}_\pm$, the question surrounding DES.

2.3 Finding DES in gauge theories

The question of DES amounts to whether there are *physically distinct* ways that the composition of *physically identical* subsystems states can go. We need to assess the possibilities for satisfying (1.2), finding $[\varphi] = [\varphi_+] \boxplus [\varphi_-] \neq [\varphi'] = [\varphi'_+] \boxplus' [\varphi'_-]$ and non-trivially satisfying DES according to Section 1.4.

Gauge-fixings have simplified all questions of identity: $h[A] = h[A']$ iff $A \sim A'$ (or $[A] = [A']$). The strategy now to find DES is quite simple: we are given two regional gauge-fixed states, and want to find out in how many globally physically distinct ways we can compose them (see Figure 1).

But whether two regional physical states are composable must be determined in terms of their representatives. And given the gauge-fixed representatives, the question is *not* whether they smoothly join *simpliciter*. Whether two regional gauge-fixed states $h_\pm$ are composable turns on whether there are gauge transformations on each region such that the transformed states—no longer of the form $h_\pm$—smoothly join, or glue. But if we have already shown that $h[A]$ is gauge-invariant, how do we proceed?

Indeed, we have at several turns stressed the importance of using a projection, $h : \mathcal{A} \to \mathcal{A}$ as opposed to a reduction, $\mathcal{A} \to [\mathcal{A}]$. In (Gomes, 2019a, 2019b), the construal of a gauge-fixing as a projection, and not as a quotienting, was argued to

[23] As in footnote 20, to be solutions to (2.5), we must have $\nabla^2(g - g') = 0$ and $\partial_n(\ln g - g') = 0$. Again, calling $f = (g - g')$, we have $0 = \int f\nabla^2 f = \int |\mathrm{grad} f|^2 + 2\oint f\partial_n f \Leftrightarrow \mathrm{grad} f = 0$, and since $\partial_n f = 0$, $f =$const.

[24] There is also the added benefit, in the dynamical 3+1 setting of Yang-Mills gauge theories, that such gauge-fixings correspond to *Helmholtz decompositions* separating the Coulombic from the radiative degrees of freedom of the region. Radiative degrees of freedom are those that are intrinsic to a region; they do not depend on further incoming information at the boundary. See (Gomes & Riello, 2019, Sec. 3) for more on this point.

[25] What we have just shown for each space is essentially equivalent to the existence of a local *slice*, which is the mathematical jargon for a gauge-fixing section (local on field-space) on infinite-dimensional configuration spaces. The existence of a local slice is *the* characterizing feature for (the closest analogues of a) principal fiber bundle structure in this context. See, e.g. (Mitter & Viallet, 1981; Kondracki & Rogulski, 1983; Wilkins, 1989).

be fundamental for the gluing of regions: for both are gauge-invariant with respect to gauge-transformations on the common domain, $\mathcal{A}$, but only one allows further transformations to be enacted on its range. Therefore it is useful to distinguish two sorts of action of $\mathcal{G}$:

Subsystem-intrinsic gauge transformations Given $h : \mathcal{A} \to \mathcal{A}$, a subsystem-intrinsic gauge-transformation acts solely on the domain of h. The projection h is invariant under subsystem-intrinsic gauge transformations: $h[A^{g'}] = h[A]$.

The label 'intrinsic' stands in opposition to 'extrinsic'. Such gauge transformations are all that is needed for unique description of the entire Universe. But if we have more than one subsystem and we want to satisfy (2.2), we may need to change the representative of $[A]$—from the outside, as it were.

Subsystem-extrinsic gauge transformations Given that subsystem-intrinsic gauge transformations are defined as those acting on the field configurations in the domain of the projection $h : \mathcal{A} \to \mathcal{A}$, we can define *external* gauge transformations g'_{ext}, as those transformations acting on the *range* of h as

$$h[A] \mapsto h[A] + i\,\text{grad}(g'_{\text{ext}}). \tag{2.16}$$

Of course such a transformed field would no longer satisfy (2.8).[26]

2.3.1 The set-up

We are now finally ready to describe uniqueness conditions on the composition of physical states. When I introduced subsystem-extrinsic transformations, in (2.16), as the group action on the range of the projection map $h : \mathcal{A} \to \mathcal{A}$, I mentioned the crucial role they played in the gluing of regions. They are needed because the only criteria for gluing quotients employs representatives.[27] That is, there is no composition of physical states, $\boxplus$ of (1.2), that is not formulated in terms of $\oplus$. Once we have eliminated redundancy and fixed a 1-1 correspondence with $[A_\pm]$ by the use of a gauge-fixing section, the image of h, i.e. $h[\mathcal{A}] \subset \mathcal{A}$, is invariant with respect to gauge transformations acting on its domain, but we can still change representatives by acting on its range, $\mathcal{A}$.

That is, we are given the physical content of the configuration as input, and that is enough for our purposes; but $h_\pm$ might not smoothly join, and yet they still correspond to a physically possible global state. The existence of subsystem-extrinsic gauge transformations smoothening out the transition between h_+ and h_- is a necessary and sufficient condition for their compatibility.

In sum, we have agreed to employ $h_\pm$ to eliminate redundancy: for the regional physics, we have extricated physical information from matters of representation. Moreover, these $h_\pm$ are all we need as input to determine the compatibility of different regional physical states. That is: $h_\pm$ determine whether they can be joined

[26] These two possibilities of action of $\mathcal{G}$ have clear relations to the employment of homotopy type theory in gauge theory, as advocated by (Ladyman, 2015). Ladyman seeks a representation that "both (a) distinguishes states conceived of differently even if they are subsequently identified, and (b) distinguishes the identity map from non-trivial transformations that nonetheless might be regarded as delivering an identical state". Here we have two sorts of transformations: the subsystem-intrinsic one, $A \mapsto A^g$, and this does not change $h[A]$—satisfying Ladyman's (b)—, and the external one, that does the work of Ladyman's (a).

[27] See (Gomes, 2019a, Sec 2) for more on the fundamental difference between the sorts of projection implied by h and the quotienting procedure.

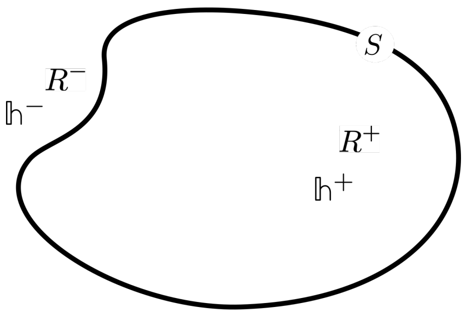

Figure 1: The two subregions of M, i.e. $R_\pm$, with the respective horizontal perturbations $h^\pm$ on each side, along with the separating surface S.

by subsystem-extrinsic gauge transformations. The condition is that subsystem-extrinsic gauge transformations exist such that (in spacetime index-free notation[28]):

$$(h_+ - h_-)_{|S} = i\text{grad}(\ln g_+^{\text{ext}} - \ln g_-^{\text{ext}})_{|S}; \tag{2.17}$$

which is the appropriate rewriting of (2.2).

So far so good. Of course, there could be many such possible "adjustments" of h; we need to partition them into physical equivalence classes. For the remaining question—whether the composition is physically unique—we employ a gauge-fixing of the global state, i.e. we demand that the global state is also given by some h. That is, after gluing we must again project into a gauge-fixing section. To repeat: the question of DES amounts to whether there are *physically distinct* ways the composition of *physically identical* subsystems states can go.

Thus we are given $h_\pm$ that satisfy (2.12) (in place of $A_\pm$) and (2.17) (necessary if $h_\pm$ are to be composable), and want to modify them "subsystem-extrinsically", so as to find an h:[29]

$$h := (h_+ + i\text{grad}(\ln g_+^{\text{ext}}))\Theta_+ + (h_- + i\text{grad}(\ln g_-^{\text{ext}}))\Theta_- \tag{2.18}$$

satisfying (2.4) (so that we uniquely determine the universal physical state). The solution is obtained by finding the appropriate subsystem-extrinsic gauge transformations $g_\pm^{\text{ext}}$ that also satisfy (2.17). The $\Theta_\pm$ in (2.18) are the (Heaviside) characteristic functions of regions $R_\pm$.[30]

It is important to emphasize that the use of the gauge-fixed fields has eliminated redundancy; no further demands on the states are allowed. For instance, imposing $(h_+ - h_-)_{|S} = 0$ would restrict our analysis to a subset of compatible physical configurations (cf. footnote 29).[31]

That is, since we have also partitioned the global state space, and identified 1-1 representatives of the physical equivalence classes: whatever information beyond

[28]Using indices, the equation is: $(h_+^\mu - h_-^\mu)_{|S} = i\partial^\mu(\ln g_+^{\text{ext}} - \ln g_-^{\text{ext}})_{|S}$.

[29]Notice that here the requirement of finding adjustments is a consequence of the non-locality of the gauge-fixing, implicit in the inverse Laplacian. That is, the restrictions of universal h's over M—satisfying (2.4)—to the regions $R_\pm$ are not necessarily themselves of the form of $h_\pm$, i.e. do not necesarily satisfy Neumann boundary conditions. To repeat: there may be global h's that do not themselves restrict to $h_\pm$'s. Therefore, even if they are to form a global h, $h_\pm$ need not smoothly join.

[30]The assumption of states as supported on the regions $R_\pm$ and adjacency of the regions fixes the embedding of the subsystems through the distributions $\Theta_\pm$. Then conditions for gluing become simply smoothness conditions.

[31]To get a bit ahead of ourselves: this is precisely what will happen in (4.3), when we apply a fundamental view of symmetries to the externalist's notion of boundary.

the specification of $h_\pm$ is required to determine a solution will reveal a gap between $[\mathcal{A}]$ and the union of $[\mathcal{A}_\pm]$. If we find underdetermination here, it will no longer be descriptive; it will have physical significance, fulfilling the notion of DES. In other words, the DES transformations—satisfying criteria (i) and (ii) for DES, of Section 1.1—will take the form of 'externally applied' symmetries on the subsystems—subsystem-extrinsic symmetries—, that lead to real physical difference.

2.3.2 A sketch of the solution

After this stage setting, we sketch the solution to our original problem: in the type of systems we have focussed on—vacuum, simply connected Universe—do regional physical states uniquely determine the entire physical state? Gauge-fixing has disentangled the issue of redundant representation from empirical significance, and thus cleared the way to a full resolution.

Essentially, to find $g^{\text{ext}}_\pm$, we obtain, from (2.8), i.e. from $\text{div}(h) = 0$ (and $\text{div}(h_\pm) = 0$) that $\nabla^2(g^{\text{ext}}_\pm) = 0$; and the action of the divergence operator on the Heaviside functions on (2.18) (and the Neumann conditions $h^\pm_n = 0$) enforces a continuity equation for $g^{\text{ext}}_\pm$ in terms of $h_\pm$. When this is conjoined with the compatibility equation, (2.17), we have enough information to fix the appropriate boundary conditions for the solutions $g^{\text{ext}}_\pm$ (see (Gomes & Riello, 2019, Sec.4, p.30-33)).

When all the chips have fallen, one can prove existence and almost uniqueness for the $g^{\text{ext}}_\pm$ of (2.18). Unsurprisingly, the only degeneracy left is again made up of regional stabilizers.[32] In other words, we find *unique* $g^{\text{ext}}_\pm$ *strictly as functionals of* the boundary values of $h_\pm$ (no derivatives at the boundary are necessary, cf. footnote 13) and of regional stabilizers $c_\pm$:

$$g^{\text{ext}}_+ = g^{\text{ext}}_+[(h_\pm)_S, c_+] \qquad \text{and} \qquad g^{\text{ext}}_- = g^{\text{ext}}_-[(h_\pm)_S, c_-]. \tag{2.19}$$

And given e.g. c_+ and c'_+,

$$g^{\text{ext}}_+[(h_\pm)_S, c'_+] = g^{\text{ext}}_+[(h_\pm)_S, c_+] + (c'_+ - c_+);$$

that is: the difference between two solutions is entirely due to stabilizers.[33]

But, as before, since we are in vacuum, stabilizers—for electromagnetism, constant gauge transformations—do not affect the gauge potential. That is, some internal directions are not fixed by gluing, but they also do not change the vacuum states; this underdetermination of $g^{\text{ext}}_+[h_\pm]$ and $g^{\text{ext}}_-[h_\pm]$ *cannot* be converted into *physical* variety (Gomes, 2019b).[34] Therefore, given $h_\pm$, there is a unique h which

[32] In the non-Abelian case, the proof is perturbative, for reasons mainly to do with the Gribov problem (Gribov, 1978) (the Gribov problem says there is no gauge-fixing section that covers the entire configuration space).

[33] For illustration purposes, I display the solution here:

$$\ln g^{\text{ext}}_\pm = \zeta^{\pm\Pi}_{(\pm)} \quad \text{with} \quad \Pi = \left(\mathcal{R}_+^{-1} + \mathcal{R}_-^{-1}\right)^{-1} \left((\nabla^2_S)^{-1}\text{div}_S(h_+ - h_-)_S\right),$$

where the subscript S denotes operators and quantities intrinsic (i.e. pulled-back) to the interface surface S; $\zeta^u_{(\pm)}$ is a harmonic function on (resp.) $R_\pm$ with Neumann boundary condition $\partial_n \zeta^u_{(\pm)} = u$, and $\mathcal{R}$ is the Dirichlet-to-Neumann operator. For the meaning of these operators, and also the analogous solution for the general non-Abelian Yang-Mills gauge theories, see (Gomes & Riello, 2019, Sec. 4), and (Gomes, 2019b, Appendix D).

[34] Both for Abelian and non-Abelian, in the presence of matter, the stabilizer redundancy *can* lead to real physical difference, since it acts non-trivially on the matter fields. In case M is not simply-connected, there is more freedom in how one embeds, or puts together, the regions. This topological redundancy produces physical variety even in the absence of matter (cf. (Gomes & Riello, 2019; Gomes, 2019b)). This variety is more akin to the standard Galileo ship case, as we will see in Section 3.2.

can be obtained from their union. In this particular case, we are left *without* DES.

The case of point-particles—which, in effect, formalizes the Galileo's ship thought-experiment with which we began—is different. But before we turn to it, it is worthwhile to illustrate how this analysis plays out in the holonomy formulation of gauge theories.

2.4 Comparison with the holonomy formalism

The holonomy interpretation of electromagnetism takes as its basic elements assignments of unit complex numbers to loops in spacetime. A loop is the image of a smooth embedding of the oriented circle, $\gamma : S^1 \to \Sigma$; the image is therefore a closed, oriented, non-intersecting curve, One can form a basis of gauge-invariant quantities for the holonomies (e.g. (Barrett, 1991) and (Healey, 2007, Ch.4.4) and references therein),[35]

$$hol(\gamma) := \exp\,(i \int_\gamma A). \tag{2.20}$$

2.4.1 The basic formalism

Let us look at this in more detail. By exponentiation (path-ordered in the non-Abelian case), we can assign a complex number (matrix element in the non-Abelian case) $hol(C)$ to the oriented embedding of the unit interval: $C : [0,1] \mapsto M$. This makes it easier to see how composition works: if the endpoint of C_1 coincides with the starting point of C_2, we define the composition $C_1 \circ C_2$ as, again, a map from $[0,1]$ into M, which takes $[0,1/2]$ to traverse C_1 and $[1/2,1]$ to traverse C_2. The inverse C^{-1} traces out the same curve with the opposite orientation, and therefore $C \circ C^{-1} = C(0)$.[36] Following this composition law, it is easy to see from (2.20) that

$$hol(C_1 \circ C_2) = hol(C_1)hol(C_2), \tag{2.21}$$

with the right hand side understood as complex multiplication in the Abelian case, and as composition of linear transformations, or multiplication of matrices, in the non-Abelian case.

For both Abelian and non-Abelian groups, using the above notion of composition, holonomies are conceived of as smooth homomorphisms from the space of loops into a suitable Lie group. One obtains a representation of these abstractly defined holonomies as holonomies of a connection on a principal fiber bundle with that Lie group as structure group; the collection of such holonomies carry the same amount of information as the gauge-field A. However, only for Abelian theory can we cash this relation out in terms of gauge-invariant functionals. That is, while (2.20) is gauge-invariant, the non-Abelian counterpart (with a path-ordered exponential), is not.[37]

[35] Of course, any discussion of matter charges and normalization of action functionals would require e and $\hbar$ to appear. But, I am not treating matter, so these questions of choice of unit don't become paramount. As before, if needed, I set my units as $e = \hbar = 1$; as is the standard choice in quantum chromodynamics (or as in the so-called Hartree convention for atomic units).

[36] It is rather intuitive that we don't want to consider curves that trace the same path back and forth, i.e. *thin* curves. Therefore we define a closed curve as *thin* if it is possible to shrink it down to a point while remaining within its image. Quotienting the space of curves by those that are thin, we obtain the space of *hoops*, and this is the actual space considered in the treatment of holonomies. I will not call attention to this finer point, since it follows from a rather intuitive understanding of the composition of curves.

[37] For non-Abelian theories the gauge-invariant counterparts of (2.20) are Wilson loops, see e.g. (Barrett,

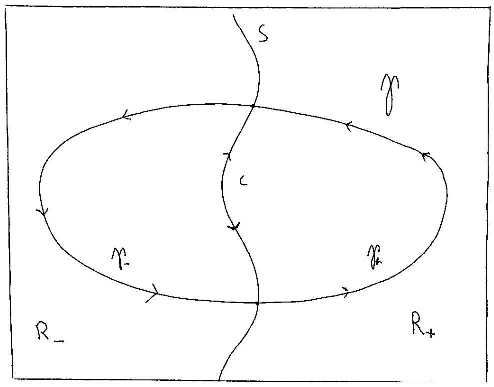

Figure 2: Two subregions, i.e. $R_{\pm}$, with the separating surface S. A larger loop γ crosses both regions. But, since γ_1 and γ_2 traverse S along C in opposite directions, $\gamma = \gamma_1 \circ \gamma_2$.

2.4.2 DES and separability

As both Healey (Healey, 2007, Ch. 4.4) and Belot ((Belot, 2003, Sec.12) and (Belot, 1998, Sec.3)) have pointed out, even classical electromagnetism, in the holonomy interpretation, evinces a form of non-locality, which one might otherwise have thought was a hallmark of non-classical physics. But is it still the case that the state of a region supervenes on assignments of intrinsic properties to patches of the region (where the patches may be taken to be arbitrarily small)? This is essentially the question of *separability* of the theory (see (Healey, 2007, Ch.2.4), (Belot, 2003, Sec.12), (Belot, 1998, Sec.3), and (Myrvold, 2010)).

Clearly, the question of DES asked in this paper is intimately related to the one of separability. For DES, in all its incarnations, e.g. (Brading & Brown, 2004; Greaves & Wallace, 2014; Teh, 2016; Friederich, 2014), is conditional on the existence of global gauge-invariant quantities that are *not* specified by the regional gauge-invariant content. But we are not here interested in cases of "topological holism", as related to the Aharonov-Bohm effect. We are asking whether a vacuum, simply-connected universe still displays non-separability. For this topic, elaborating on Healey's notion of Weak Separability (Healey, 2007, p. 46), and, equivalently, Belot's Synchronic Locality (Belot, 1998, p 540), we can directly follow Myrvold's definition (Myrvold, 2010, p.427):

- *Patchy Separability for Simply-Connected Regions.* For any simply-connected spacetime region R, there are arbitrarily fine open coverings $\mathcal{N} = \{R_i\}$ of R such the state of R supervenes on an assignment of qualitative intrinsic physical properties to elements of $\mathcal{N}$.

If *Patchy Separability for Simply-Connected Regions* holds, there will be no room for DES. And indeed, in vacuum, it is easy to show that it *does* hold. In Figure

1991), $W(\gamma) := \mathrm{Tr}\,\mathcal{P}\exp\left(i\int_\gamma A\right)$, where one must take the trace of the (path-ordered) exponential of the gauge-potential. It is true that all the gauge-invariant content of the theory can be reconstructed from Wilson loops. But, importantly for our purposes, it is no longer true that there is a homomorphism from the composition of loops to the composition of Wilson loops. That is, it is no longer true that (2.21) holds, $W(\gamma_1 \circ \gamma_2) \neq W(\gamma_1)W(\gamma_2)$. This is due solely to the presence of the trace. The general composition constraints—named after Mandelstam—come from generalizations of the Jacobi identity for Lie algebras, and depends on N for SU(N)-theories; e.g. for $N = 2$, it applies to three paths and is: $W(\gamma_1)W(\gamma_2)W(\gamma_3) - \frac{1}{2}(W(\gamma_1\gamma_2)W(\gamma_3) + W(\gamma_2\gamma_3)W(\gamma_1) + W(\gamma_1\gamma_3)W(\gamma_2)) + \frac{1}{4}(W(\gamma_1\gamma_2\gamma_3) + W(\gamma_1\gamma_3\gamma_2) = 0$.

2, we see a loop γ not contained in either R_+ or R_-. However, we can decompose it as $\gamma = \gamma_+ \circ \gamma_-$, where each regional loop $\gamma_\pm$ does not enter the complementary region ($R_\mp$, respectively). Following (2.21), it is then true that, since holonomies form a basis of gauge-invariant quantities, the global gauge-invariant content of the theory supervenes on the regional gauge-invariant content of the theory.[38]

Unfortunately, this holonomy-based analysis cannot be reproduced for non-Abelian theories (see footnote 37); and it does not apply to an externalist's notion of boundaries; and it cannot be translated to the point particle language. Since we will have to analyse point-particles and the externalist's notion of boundaries, and since we want our formalism to apply also to the non-Abelian case, a treatment with holonomies—even if good for illustration—will not do. We therefore now revert back to our gauge-fixed approach, for the analysis of point-particles.

3 Point-particle systems

To compare the local gauge theory discussed above to the case that originally motivated the notion of DES—Galileo's ship—we introduce gauge-fixings to the study of point particles in Euclidean space.[39] We would like here to gauge-fix the group of translations and rotations, for two subsystems, replacing ship and shore respectively. After some prescription for composing the system, we would still like to evaluate whether different compositions are physically distinguishable or not, and therefore we must again apply a gauge-fixing to the global state.

3.1 Gauge-fixing

For particle systems, it is straightforward to fix translations by the center of mass and rotations by diagonalizing the moment of inertia tensor around the center of mass. It is again true that these gauge-fixing choices may not satisfy '*uniqueness*'. In the case of translations, this can happen for infinite, homogeneous mass distributions; there just is no unique center of mass to speak about. For rotations, the lack of uniqueness will obtain when the configuration has some rotational symmetry along an axis. We will only consider a finite number of pointlike mass particles, leaving only the degeneracy of rotations as relevant.

To be more explicit, the total system is given by N particles of mass m_α, $\alpha \in I = \{1, \cdots N\}$, with position vectors $\mathbf{q}_\alpha$ in some arbitrary inertial frame of $\mathbb{R}^3$, constituting the configuration space $\mathcal{Q} = \mathbb{R}^{3N}$. The subsystems are defined by

[38] It is also easy to see how *Patchy Separability for Simply-Connected Regions* fails when charges are present within the regions but absent from the boundary S (see in particular (Gomes & Riello, 2019, Sec. 4.3.2), and footnote 70 in (Gomes, 2019b)). For, in the presence of charges, we can form *gauge-invariant* functions from a non-closed curve C' that crosses S and e.g. one positive and one negative charge, $\psi_\pm(x_\pm)$, capping off C' at $x_\pm \in R_\pm$. That is, the following quantity is a gauge-invariant function:

$$Q(C', \psi_\pm) = \psi_-(x_-) hol(C') \psi_+(x_+)$$

for $C'(0) = x_-, C'(1) = x_+$. It is easy to check from the transformation property $\psi \mapsto g\psi$, that Q is gauge-invariant. Moreover, we cannot break this invariant up into the two regions, since we have assumed no charges lie at the boundary. Indeed, this is related to the thought-experiment known as "the 't Hooft beam splitter" (cf. ('t Hooft, 1980, p.110) and (Brading & Brown, 2004, p. 651)). See (Gomes & Riello, 2019, Sec. 4.3.2), and (Gomes, 2019b) for a complete analysis for how DES *does* emerge in these cases.

[39] This discussion echoes (Rovelli, 2014), which considers precisely the question of matching physical information about point-particle subsystems. The thought-experiment is made more explicit in the context we are exploring here in (Gomes, 2019a, Sec 2). For an enlightening discussion of the topic, see also (Teh, 2016).

selecting two subsets of these particles, $I_\pm \subset I$, so that $I_+ \cap I_- = \emptyset$ and $I_+ \cup I_- = I$; that is, they are mutually exclusive and jointly exhaustive. The subsets are the analogues of $R_\pm$, whereas the relevant configuration space, $\mathcal{Q}$, is analogous to $\mathcal{A}$, and $\mathcal{Q}_\pm$ to $\mathcal{A}_\pm$. The translations act as $T : \mathbf{q}_\alpha \mapsto \mathbf{q}_\alpha + \mathbf{t}$, for a given vector $\mathbf{t}$. The rotations act as $R : \mathbf{q}_\alpha \mapsto \mathbf{R}\mathbf{q}_\alpha$, where $\mathbf{R} \in SO(3)$, acting in coordinates as $R : q^i_\alpha \mapsto R^i_j q^j_\alpha$.[40] The semi-direct product of these two groups plays the role of $\mathcal{G} = SO(3) \ltimes \mathbb{R}^3$, with group action $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ denoted by '$\cdot$', i.e. $(g, g') \mapsto g \cdot g'$. We will denote $g_\pm = (R_\pm, t_\pm) \in \mathcal{G}_\pm$ and $\circ$ will be the action of rotations and translations on the configurations, e.g. $\circ : (g, \mathbf{q}) \mapsto g \circ \mathbf{q}$.

For each system, $J = I, I_+$ or I_-, we first fix center of mass coordinates through the gauge-fixing $\sigma_t(q) = 0$, as:

$$\sum_{\alpha \in J} m_\alpha \mathbf{q}_\alpha + \mathbf{t} = 0 \tag{3.1}$$

and so define $\mathbf{t}_\sigma(q) = \sum_{\alpha\in J} m_\alpha \mathbf{q}_\alpha$. Fixing the rotations is slightly more complicated. We first define the translationally fixed positions through the translationally fixed coordinates, as $\overline{\mathbf{q}}_\alpha := \mathbf{q}_\alpha + \mathbf{t}_\sigma(q)$. Now we can define the moment of inertia tensor as $\mathbf{L}$ with components:

$$L^{ij} := \sum_{\alpha \in J} m_\alpha (\|\overline{\mathbf{q}}_\alpha\|^2 \delta^{ij} - \overline{q}^i_\alpha \overline{q}^j_\alpha)$$

L^{ij} is a real symmetric matrix. A real symmetric matrix has an almost unique eigendecomposition into the product of a rotation matrix and a diagonal matrix. We therefore fix rotations $\sigma_R(q) = 0$:

$$\mathbf{R}^T \mathbf{L} \mathbf{R} - \boldsymbol{\Lambda} = 0 \tag{3.2}$$

where $\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2, \Lambda_3)$ is a diagonal matrix, whose non-zero elements are called the principal moments of inertia. When all principal moments of inertia are distinct, the principal axes through the center of mass are uniquely specified. If two principal moments are the same, there is no unique choice for the two corresponding principal axes. If all three principal moments are the same, the moment of inertia is the same about any axis. These constitute the possible degeneracies in the determination of $\boldsymbol{\Lambda}$. And so we find the configuration-dependent rotation matrix $\mathbf{R}_\sigma(q)$. As with the translation element $t_\sigma(q)$, it depends on the positions of all the particles, $\{\mathbf{q}_\alpha\}$, a dependece we denote simply by (q).

We have thus completely fixed the coordinate system for the particles, and therefore a complete gauge-fixing of the configurations is given by the n-tuples:

$$\mathbf{h}(q)_\alpha = \mathbf{R}_\sigma(q)(\mathbf{q}_\alpha + \mathbf{t}_\sigma(q)) = g_\sigma(q) \circ \mathbf{q}_\alpha \tag{3.3}$$

in perfect analogy with our definition of $h[A]$ in (2.7); e.g.

$$\mathbf{h}(g \circ q)_\alpha = \mathbf{h}(q)_\alpha \tag{3.4}$$

[40] We should remind ourselves that we here adopt the 'fundamental' view of symmetries (Section 1.2). The kinematical considerations here suffice to examine the role of DES in particle systems, and I will therefore not need to introduce any dynamical considerations. The full dynamical group of Galilean translations or boosts could easily be accommodated in what follows. For the same reasons, we do not concern ourselves with rigid bodies.

where $g_\sigma(q) = (\mathbf{R}_\sigma(q), \mathbf{t}_\sigma(q))$ is the necessary translation and rotation to bring the configurations to the frame chosen by σ. This configuration-dependent group element obeys:

$$g_\sigma(g' \circ q) = g_\sigma(q) \cdot g'^{-1} \tag{3.5}$$

which is what guarantees (3.4). Again we can see $\mathbf{h} : \mathcal{Q} \to \mathcal{Q}$ as a projection from configuration space to configuration space, and such that the image of $\mathbf{h}$ is the gauge-fixing surface and the map is invariant under gauge transformations on the domain. But, as before, in equation (2.16), we can apply subsystem-extrinsic gauge transformations to the image of $\mathbf{h}$.

3.2 Finding DES

Again, the idea is:—Assume each subsystem employs these coordinates so that there is no more descriptive redundancy. Then we ask: in how many physically distinct ways can we compose given physical states of the subsystems?

In Section 2.3 we saw that a global gauge-fixed field, h, did not necessarily restrict to the corresponding regional gauge-fixed fields, $h_\pm$, because of non-locality—this is why subsystem-extrinsic gauge transformations were required. Again it is true that, given a global configuration in the preferred coordinates, $\mathbf{h}_\alpha(q)$, restriction to subsystems—to the analogue of $R_\pm$—will not be in their center of mass and diagonalized moment of inertia. Therefore, again: in order to relate an h to the $h_\pm$, we must allow subsystem-extrinsic transformations so that (omitting the particle indices):

$$\mathbf{h} = (g_+^{\text{ext}} \circ \mathbf{h}_+) \oplus (g_-^{\text{ext}} \circ \mathbf{h}_-) \tag{3.6}$$

Previously, we nailed down the composition $\oplus$ in terms of the embeddings of the manifolds: it amounted to smooth composition along a shared boundary. What is it here? No criterion seems readily available.

Indeed, this points to the real difference between the present point-particle case and the previous one of local gauge theories. For fields, the splitting of the universe into adjacent regions nails down the embedding of the regions supporting the subsystems into the larger spacetime manifold. Consider: were the two regions $R_\pm$ not adjacent, we would have had a further freedom of composition given by the possible embeddings of one submanifold with respect to the other (of course, the two regional states would then not have determined the global state). In the field theory example, stipulating that the two regional subsystems descended from a splitting of the universe and were to jointly determine the state of the global system, we topologically fixed the embeddings of the regions.[41]

Here in the particle case, we are missing an analogue of the gluing condition, (2.17). Even if we hold that the two subsystems should jointly describe the state of the universe, we have the extra step of stipulating how to embed the subsystems. It is *this* freedom that gives rise to Galileo's ship realizations of DES.

Therefore, instead of finding explicit $g_\pm^{\text{ext}}$ in (3.6), we divide the process into two parts: we first arbitrarily embed the subsystems into the same Euclidean space, and then we find a transformation that brings the newly defined composite system to its gauge-fixed frame. At the end, we want to determine what information is required to determine $\mathbf{h}$ beyond that provided by $\mathbf{h}_\pm$.

[41] Indeed, for non-simply connected manifolds, adjacency does *not* fix the topological embedding uniquely, giving rise to a DES for gauge systems associated to the Aharonov-Bohm effect; see (Gomes & Riello, 2019, Sec.4.5).

Here $\oplus$ defines an embedding of the two frames into the larger universe. We embed them by defining a new frame, which is related to the ones used in $\mathcal{Q}_\pm$ by arbitrary transformations $g_\pm^{\text{emb}} \in \mathcal{G}_\pm$. We thus obtain a global configuration,

$$\mathbf{q}_\alpha = \begin{cases} g_+^{\text{emb}} \circ \mathbf{h}_\alpha^+ & \text{for} \quad \alpha \in I_+ \\ g_-^{\text{emb}} \circ \mathbf{h}_\alpha^- & \text{for} \quad \alpha \in I_- \end{cases} \; ; \tag{3.7}$$

with the understanding that α runs through the appropriate domains for $I_\pm$, we can replace those indices by $\pm$. The positions of the particles are now all seen to inhabit the same Euclidean 3-space, and $\oplus$ becomes simple vector addition.

Of course, this $\mathbf{q}_\alpha$ is not yet in the form of $\mathbf{h}_\alpha$; that is, it is not in a global center of mass and eigenframe of the moment of inertia coordinate system. As above, a gauge-fixing yields $g(q_\alpha)$, and therefore, by linearity (omitting particle indices):

$$\mathbf{h} := g(q) \circ \mathbf{q}_\alpha = (g(q) \circ (g_+^{\text{emb}} \circ \mathbf{h}_+)) + (g(q) \circ (g_-^{\text{emb}} \circ \mathbf{h}_-)).$$

But we can put this in a slightly more economical form. Since here the symmetries act globally, and we know the covariance property (3.5) holds (this is what guarantees (3.4)), there is no loss of generality if we replace (3.7) by:

$$\mathbf{q}'_\alpha = \begin{cases} g_+^{\text{emb}'} \circ \mathbf{h}_\alpha^+ & \text{for} \quad \alpha \in I_+ \\ \mathbf{h}_\alpha^- & \text{for} \quad \alpha \in I_- \end{cases} \tag{3.8}$$

where $g_+^{\text{emb}'} := (g_-^{\text{emb}})^{-1} \cdot g_+^{\text{emb}}$ (we can compose them since they all act on the same Euclidean space). Equation (3.8) is merely a justification for fixing the frame of one of the subsystems of the original embeddings as coincident with the frame of the universe, before global gauge-fixing. Thus, finally, we can write our solution (again omitting the index α):

$$\mathbf{h} = (g(q') \circ (g_+^{\text{emb}'} \circ \mathbf{h}_+)) + (g(q') \circ \mathbf{h}_-) \tag{3.9}$$

where '+' is now simply vector addition in the center of mass frame.

We can unpack $g(q') = g(\mathbf{h}_\pm, g_+^{\text{emb}'})$. Therefore the solution is uniquely defined, in terms of $g_+^{\text{emb}'}$ and $\mathbf{h}_\pm$. Although $g_+^{\text{emb}'}$ is globally gauge invariant—we can no longer rid ourselves of it by a global change of coordinates—it is *not* solely determined by $\mathbf{h}_\pm$. This is in contrast to what we found for the field theory, in equation (2.19), where, for a simply-connected, vacuum universe, up to stabilizers, the transformations were uniquely determined by $h_\pm$. Here there is no way to associate $g_+^{\text{emb}'}$—the information required beyond $\mathbf{h}_\pm$—with stabilizers of the configurations.

The physical variety, i.e. the variety of ways to compose physical states of subsystems, is therefore given by $g_+^{\text{emb}'}$: namely, *how we embed one of the subsytems with respect to the other*. Everything else is uniquely determined by $\mathbf{h}_\pm$; for two ships, these would be the description of all the particles of each ship with respect to its own gauge-fixed coordinates (center of mass and diagonal moment of inertia).[42]

4 Comparison with the common derivation of DES

This paper started with the question: it is widely ackowledged that rigid symmetries in particle mechanics can have a (relational) direct empirical significance (DES)

[42] I should also mention that dynamical considerations are secondary: we can apply this scheme to whatever symmetries of the N-particle system emerge from the dynamics, such as Gallilean symmetries (see footnote 40).

when applied to subsystems; do local gauge theories also realize the concept of DES?

Gauge symmetries are normally taken to encode descriptive redundancy: a view I endorse. This descriptive redundancy means that the natural answer to our question is 'No'. This 'No' answer was developed in detail by Brading and Brown (Brading & Brown, 2004). The 'Yes' answer has been argued for by Greaves and Wallace (Greaves & Wallace, 2014). Building on (Healey, 2009), they articulate DES for gauge theory differently, fostering their 'Yes' answer to the above. Although our answers differ, their treatment bears many similarities to ours here: they focus on subsystems as given by regions; they think of subsystems as given by a splitting of the universe; they endorse the 'fundamental' view of symmetries; they identify transformations possessing properties (i) and (ii) in Sections 1.1 and 1.4 by first formulating the putative effects of such transformations on the gauge fields in these regions; and they construe DES essentially as a relational property.[43]

But *unlike* our results, for a given subsystem state φ_+, they claim that there is relational DES transformations in 1-1 correspondence with the following quotient:

$$\mathcal{G}^{\text{GW}}_{\text{DES}}(\varphi) \simeq \mathcal{G}_S(\varphi_+)/\mathcal{G}^S_{\text{Id}}, \tag{4.1}$$

where $\mathcal{G}_S(\varphi_+)$ are the gauge transformations of the region which preserve the state φ_+ *at the boundary* of the region, and $\mathcal{G}_{\text{Id}}$ are the gauge transformations of the region which are the identity at the boundary.

Below, I will describe these claims in detail, comparing the results of this paper with those of (Greaves & Wallace, 2014; Wallace, 2019b).[44]

4.1 The standard derivation of DES

The result (4.1) requires an assumption: when looking for the realizers of the conditions *Global Difference* and *Regional Equivalence* (cf. Section 1.4), one may keep one of the regional subsystems—labeled 'the environment'—not only physically fixed (both are physically 'fixed' according to *Regional Equivalence*), but also representionally fixed. In other words, there is an assumption that we have a fixed representative φ_- of the physical environment state $[\varphi_-]$ which we can employ as a reference to externally assess the capacity of regional gauge transformations g_+ to produce empirically distinguishable differences. It is usually taken for granted that we can restrict attention to g_- being the identity transformation.

That is, *if* some physical states already satisfy *Global Difference* and *Regional Equivalence*, instead of (1.3) it is often assumed we can have representatives of the states fulfilling:

$$\varphi = \varphi_+ \oplus \varphi_- \quad \text{and} \quad \varphi' := \varphi_+^{g_+} \oplus \varphi_-. \tag{1.3'}$$

If (1.3') is assumed, we can similarly rewrite (1.4) as follows:

$$\text{there is no } g \in \mathcal{G} \text{ such that} \quad g_{|R_+} = g_+, \quad g_{|R_-} = \text{Id}, \tag{1.4'}$$

The assumption is consequential for the issue at hand: we explicitly rejected it in Section 2.2.2 (and we will see below in detail why that rejection was warranted).

[43] Although GW allow for the larger, non-strictly relational quotient group, of all subsystem symmetries quotiented by the interior ones, they do not investigate this larger (infinite dimensional) group, whose physical meaning—if any—is unclear (Greaves & Wallace, 2014, p.86,87).

[44] See (Gomes & Riello, 2019, Sec 4.3.2), and (Gomes, 2019b) for a more thorough treatment with matter and non-trivial topology.

Without this assumption—that g_- can be legitimately set to the identity—no relational empirical significance following the present route can be found, thus recovering our own results.

For now, let us take the assumption at face value: if two states satisfy *Global Difference* and *Regional Equivalence*, we assume representatives exist so that (1.3') and (1.4') are satisfied. From this point, if we ignore questions about the gauge-invariance of the procedure, we can easily derive relational DES of the form (4.1).

To see how this goes, let us define in more detail the groups appearing in (4.1): first, the *interior symmetries*, the elements of $\mathcal{G}_S^{\rm Id}$ are those g_+ such that $g_{+|S} = \text{Id}$, *and such that all derivatives* of g_+ vanish at S. Elements of $\mathcal{G}_S^{\rm Id}$ leave the boundary values of any state φ_+ fixed.[45] By exploring such symmetries we cannot find states that satisfy *Global Difference*; for interior symmetries, (1.4') fails by fiat.

On the other hand, since we are obliged to keep g_- fixed to the identity according to (1.3'), if we find any g_+ satisfying three conditions:
(i) $g_+ \notin \mathcal{G}_S^{\rm Id}$,
(ii) $\varphi' := \varphi_+^{g_+} \oplus \varphi_-$ is a valid smooth state of the universe, and
(iii) $\varphi_+^{g_+} \neq \varphi_+$ (i.e. $g_+ \notin \text{Stab}(\varphi_+)$);[46]
then there will be no *smooth* global gauge transformation joining g_+ and $g_- =$Id and the smooth $\varphi' \neq \varphi$ satisfies both (1.3') and (1.4'). Only if the three conditions are satisfied can we say such g_+ has a *relational empirical significance*.

For a given state φ_+, the group of boundary-preserving symmetries of φ_+, $\mathcal{G}_S(\varphi_+)$, was introduced in (4.1): under the action of elements of this group, the representative at the boundary $\varphi_{+|S}$ and all its derivatives are preserved. Such g_+ are the only ones that allow (ii) to be satisfied, because the near-boundary state of φ_+ is the unique one to smoothly join to φ_-, and therefore the transformation must leave it intact. Then, if we postulate (iii) is satisfied and set aside questions of regional and global gauge-invariance of the entire procedure, the group of relationally empirically significant symmetries is given by the quotient in (4.1):

$$\mathcal{G}_S(\varphi_+)/\mathcal{G}_S^{\rm Id}.$$

This is essentially the conclusion of (Greaves & Wallace, 2014; Wallace, 2019b) (for a proof along the lines just shown, see (Gomes, 2019b, Sec 4.1.)).[47]

But problems lie ahead on this road to DES. First, exceptions are easy to find: according to condition (iii) we only obtain (4.1) if the *stabilizer group*,

$$\text{Stab}(\varphi_+) := \{g_+ \in \mathcal{G}_+ \mid \varphi_+^{g_+} = \varphi_+\},$$

defined in (2.3), is the identity, $\text{Stab}(\varphi_+) = \{\text{Id}\}$. If it is not the identity, then item (iii) of the conditions on g_+ above demands we also quotient by this group, and it is not clear this double-quotient always makes sense.[48]

[45]Since the condition of composition of states requires continuity of all the regional derivatives of the states. If we were taking the gauge group as analytic, and not just smooth, such a g would have to be the identity everywhere within R_+.

[46]Even though (i) implies $g_+ \neq \text{Id}$, there could stabilizers; see (2.3).

[47]A few technicalities are omitted in (Greaves & Wallace, 2014; Wallace, 2019b): for instance, one must also consider the above conditions on the derivatives of the group element at the boundary. Neglecting these conditions can create discrepancies in the claimed results (in the analytic case, the only surviving quotient corresponds to non-trivial stabilizers on the entire region). Moreover, even asymptotically, it is claimed that the Poincaré transformations are the left-over ones, with associated charges. But it is well-known that asymptotic boundary conditions are more 'layered', at least naively allowing supertranslations to remain; (see e.g. (Strominger, 2018) and references therein).

[48]To take a quotient of group actions, the two groups are usually assumed to act on the same space. Here, a quotient doesn't necessarily have a well-defined, canonical action on the same space as the original groups do.

One way to make sense of the double quotient is if the group to be quotiented has a product structure in terms of the divisor groups. But this does not always help with finding non-trivial DES.

For instance, in the case of vacuum electromagnetism, $\varphi = A$, we always have the constant gauge transformations as the stabilizers. Likewise, for any boundary state without charges, the elements of $\mathcal{G}_S(A_+)$—i.e. those g_+ that preserve said boundary state—are the boundary-constant gauge transformations. Therefore elements of $\mathcal{G}_S^{\rm Id}$ and $\mathrm{Stab}(A_+)$ generate all of $\mathcal{G}_S(A_+)$.[49] This means we cannot simultaneously satisfy (i, ii, iii), and therefore cannot fulfill (1.3') and (1.4'). Indeed, in the vacuum case, for both Abelian and non-Abelian theories, if boundary stabilizers always correspond to global stabilizers, the action of $\mathcal{G}_S(A_+)$ can be decomposed as the action of $\mathcal{G}_S^{\rm Id}$ and $\mathrm{Stab}(A_+)$. And none of these two subgroups has a DES-producing effect on A_+, according to the criteria set forth in (Greaves & Wallace, 2014).

Another trivial counter-example: suppose the differentiability class of our gauge transformations and states is analytic. In the analytic case, a non-trivial boundary stabilizer will also stabilize the state throughout the region, implying $\varphi_+^{g+} = \varphi_+$, thus all elements of $\mathcal{G}_S$ also belong to Stab, foiling (iii). These exceptions should give us pause: (4.1) certainly does *not* always hold.

More importantly, it is clear that this argument cannot rely solely on φ_+: whatever g_+ is, if $\varphi_- \sim \varphi_-'$, I can always analytically continue g_+ towards R_- (or extend it in any other way) obtaining some g_- and some g such that $\varphi' := \varphi_+^{g+} \oplus \varphi_-^{g-}$ and therefore $\varphi' \sim \varphi$, thus foiling condition *Global Difference*. Therefore, even if $\mathrm{Stab}(\varphi_+) = \{\mathrm{Id}\}$ and we demand smooth (i.e. C^∞) differentiability, we stumble on the issue that we momentarily set aside above: our assumption that the representation of the environment state, φ_-, can be fixed without loss of generality, yielding (1.3').

It smells like an implicit "gauge-fixing" of sorts was assumed in the construction. But setting $g_{-|S}$ =Id is not a gauge-covariant condition: it depends on the initial *choice of representative* φ_-. It is not clear we would we have obtained the same results had we chosen another representative of the regional reference states (e.g. $\varphi_\pm' \neq \varphi_\pm$ but such that $\varphi_\pm' \sim \varphi_\pm$).The only way we can guarantee that the initial choice of φ_- at the boundary did not play a role, is to eliminate other choices! That is what we will do in the next Section, 4.2, as it entails an externalist's notion of subsystems.

In sum: without the explicit characterization of a bona-fide gauge-fixing, one is liable to be led astray in the internalist case. Indeed, even such expert authors as (Greaves & Wallace, 2014) *were* led astray. The lesson is that, to be sure that the end result is fully regionally and globally gauge-invariant, any approach to these questions must take due care to employ only gauge-invariant raw materials; as I have done in Section 2.3 by the use of gauge-fixed quantities. Let us now try to fit the assumptions above with the gauge-fixing paradigm.

4.2 Using gauge-fixings in the kinematical, externalist's subsystem

Let us now see in more detail how our analysis through gauge-fixing, when applied to the 'kinematical' view of the externalist's notion of subsystem, as in Section 1.2.1, recovers the results of (Greaves & Wallace, 2014; Wallace, 2019b).

[49] There is a direct sum decomposition of the Lie algebra of $\mathcal{G}_S(A_+)$ in terms of those of $\mathcal{G}_S^{\rm Id}$ and $\mathrm{Stab}(A_+)$. Not all decompositions leave the state A_+ fixed. But to recover (4.1) one must argue why one decomposition should be preferred over another.

4.2.1 The externalist's and fundamental symmetries: what goes wrong

First, let us see what goes wrong when we mix the externalist's notion of subsystems with the fundamental view on gauge symmetries. Suppose, instead of the covariant boundary conditions (2.12), we tried to implement $A_{|S}=\lambda$, for some fixed boundary 1-form, λ. That is:

$$\sigma(A)\equiv\begin{cases}\mathrm{div}(A^{\pm}) &=0\\ A_{|S} &=\lambda\end{cases}\tag{4.2}$$

This is what fixing the reference state in equation (1.3')—one of the dubious suppositions at stake in Section 4—demands.

Then, instead of (2.14), we would obtain: $\partial_\mu(\ln g)_{|S}=i(A_\mu-\lambda_\mu)_{|S}$, which in particular, contains Neumann boundary conditions, like (2.14), but now dependent on λ; namely $\partial_n(\ln g)_{|S}=i(A_n-\lambda_n)_{|S}$. From here, it is easy to see we have overconstrained our system, for the solution to the Neumann boundary problem $g[A,\lambda]$, is now fully determined at the boundary, irrespective of the tangential components of $A_{|S}$ and λ. Thus we can only satisfy $A_{|S}=\lambda$ for those A_μ such that

$$A_{\|}=\lambda_{\|}+\partial_{\|}(\ln g[A,\lambda])\tag{4.3}$$

where $\|$ denote the directions parallel to S. Nothing in the solution $g[A,\lambda]$ relates these tangential derivatives to the boundary. Therefore, conditions such as $A_{|S}=\lambda$ are overconstraining if we are to employ a 'fundamental' view of symmetries, or, equivalently, to employ a view of the subsystem as descending from some splitting of the universe. That is, such attempts at gauge-fixings will certainly not fulfill condition *'Universality'*; a sine qua non for gauge-fixings according to Section 2.1.

The lesson is one we had already learned in Section 2.3.1 (see footnotes 29 and 31 and equation (2.18)). To investigate the possible gluing of two given regional physical states, we *cannot* previously fix the boundary representives. The representative of the state on the boundary will depend on the rest of the state on the bulk: this is just a sign of the non-locality of gauge-invariant observables, and, in particular, of the non-locality of the projection $h[A]$ (see e.g. (Strocchi, 2015)).[50]

4.2.2 The externalist's subsystem and kinematical symmetries

The conclusion is that the boundary condition $A_{|S}=\lambda$ is only compatible with the '*kinematical*' view on symmetries, and with a boundary taken as external to the whole universe, as discussed in Section 1.2.1.

To require $A_{\pm|S}=\lambda$ as a boundary condition, we must use a different gauge-fixing than that determined by (4.2). First, we must appropriately truncate configuration space, so that only $A_{\pm|S}\equiv\lambda$ are allowed, i.e. $\mathcal{A}=\{A_\mu\in\Lambda^1(M,\mathfrak{g})\ ,\ A^I_{\mu|S}\equiv\lambda^I_\mu\}$. Such a violent truncation would demand that $\partial_\mu(\ln g)_{|S}\equiv 0$. More generally, also in the non-Abelian case, it implies that the gauge transformations must be boundary-stabilizers, called $\mathcal{G}_S(A)$ in Section 4.

After suitably truncating configuration space, we can then choose a non-covariant—there is no need for covariance, since A is fixed at the boundary—Dirichlet boundary

[50]The reader should note that this statement is not anathema to the criterion of gluing through smooth composition, as per equation (2.2). Of course, if the two regional states are composable, it will still be true that there are representatives that smoothly compose accross the boundary; they just need not be the gauge-fixed ones.

condition for the gauge transformations, $g_{|S} = \tilde{g}_{|S} =: \kappa$, for some arbitrary $\tilde{g} \in \mathcal{G}_S$. So we have, in analogy to (2.13) and (2.14), the system:

$$\nabla^2(\ln g) = i\mathrm{div}(A) \tag{4.4}$$
$$g_{|S} = \kappa \tag{4.5}$$

Does this now, finally, satisfy both *'Universality'* and *'Uniqueness'*, thereby yielding a bona-fide gauge-fixing as seen in Section 2.1? Not yet. For as just seen, these conditions demand g stabilizes the boundary state; but we have many choices here: any boundary stabilizer will do. And each choice $g_{|S} = \kappa$ can yield a different gauge-fixed A. That is, for $\kappa \neq \kappa'$, we can have substantially different solutions, $g_\kappa[A] \neq g_{\kappa'}[A]$, and perhaps even such that their difference is not due to stabilizers, and therefore $h_\kappa(A) \neq h_{\kappa'}(A)$.

To see this in generic terms, let us look at the non-Abelian case and suppose A has only a boundary, but no global, stabilizer.[51] The system (4.4) and (4.5) has a different unique solution $g_\kappa[A]$ for each κ. The challenge is solely to determine whether these solutions are related by a global stabilizer (in which case the difference between $g_\kappa[A]$ and $g_{\kappa'}[A]$ would not affect h). However, since A is assumed not to have a stabilizer, this cannot be the case. Therefore, generically, the gauge-fixed, or projected states h_κ (cf. (2.7)), will differ, depending on the boundary value of the gauge-group: $h_\kappa(A) \neq h_{\kappa'}(A)$. Generically, each A corresponds to a collection of $h_\kappa(A)$'s, parametrized by a choice of boundary stabilizer, κ.

Since the group of boundary stabilizers is isomorphic to the quotient (4.1), it is then true that we have leftover physically inequivalent configurations in that same amount. They can be taken to possess 'non-relational DES' if you will, because these inequivalent possibilities are related by 'gauge transformations' of the boundary conditions: κ and κ' *are* gauge-related after all.[52]

If A has not even a boundary stabilizer, there is no DES, for $\kappa =$Id (the same conclusion holds from (4.1)). But what happens when A has not only boundary stabilizers but global stabilizers as well? This was the situation we posed as a counter-example to the standard derivation of DES in Section 4.1. In that case, we *can* relate the gauge-fixings for different choices of κ by *global* stabilizers. E.g. for electromagnetism: $g_\kappa[A] = g_{\kappa'}[A] + (\kappa - \kappa')$. In this case, the different constants κ and κ' at the boundary are immaterial: we still have a gauge-fixing, as defined at the end of Section 2.1; each A corresponds to a unique $h[A]$.

We can now summarize our findings: in the externalist scenario, in vaccuum and in the simply-connected case, we find that boundary stabilizers that do not correspond to global stabilizers yield DES. If the global and boundary stabilizers match-up, there is no discrepancy with respect to the internalist scenario: neither obtains DES. Moreover, if they do match-up, the internalist and the externalist also agree about non-trivial DES in the presence of charged matter within the region(s) (cf. footnotes 34 and 38). These findings are expressed in Table 1.

[51] We cannot proceed in precise analogy to footnotes 20 and 23 here. Writing g as the (path)exponential of an infinitesimal ξ for simplification, we have $\mathrm{D}^2(\xi_\kappa[A] - \xi'_\kappa[A]) = 0$, in the non-Abelian analogue. But now the integration by parts trick of footnote 23 no longer works, because we are using Dirichlet, not Neumann conditions.

[52] The most comprehensive treatment of 'non-relationa' DES is given in (S. Ramirez & Teh, 2019). There, they are able to *dynamically* treat the κ's that emerge here.

5 Conclusions

In gauge theories, empirical significance can be obscured by redundancy of representation. Ultimately, that is why DES continues to be a debated question. Nonetheless, the standard treatment of DES is almost silent on all issues of fixing representation,[53] with the exception of (Gomes, 2019b) and a recent paper by Wallace, where the assumption is partially flagged, but not fully examined.[54]

The main question I have investigated here is precisely how to establish a choice of 'representational convention' (in Wallace's usage, cf. footnote 54) in the context of our search for DES. This approach to DES reveals the inadequacy of the standard construction of Section 4 and provides a straightforward alternative. And, while I agree with some aspects of Greaves and Wallace's analysis of symmetries, recasting the topic to focus on gauge-*invariant* information about regions yields entirely different results.

The gauge-invariant construal of DES was expressed by a gauge-fixed formalism: the upshot is that gauge-fixing disentangles the issue of redundant representation from empirical significance, for *all* types of systems and their subsystems.

In this context, it was fruitful to divide the notion of subsystems into two categories: an internalist notion—where subsystems arise from internal boundaries of the universe—and an externalist notion: where the subsystem is seen as the entire universe, truncated by a boundary.

Let us start with the findings of Section 2, for the internalist definition—which views subsystems as inherited from a decomposition of spacetime into mutually exclusive, jointly exhaustive regions. The basic building blocks of the construction, $h_\pm$, were all gauge-invariant, and therefore, to glue such regional states, we required subsystem-*extrinsic* gauge-transformations (cf. (2.16) and (2.18)).[55] These transformations are functionals of the input $h_\pm$, and adjust the regional representatives so that they form a smooth global state.

But global gauge-invariant quantities may not be uniquely determined from the (composition of) subsystem gauge-invariant quantities. When they are not so determined, the subsystem-extrinsic transformations, as functions of $h_\pm$, do *not* uniquely determine the joint global physical state (cf. (Gomes, 2019b, Sec. 5.2.3) and (Gomes & Riello, 2019, Sec. 4.1)). This analysis thus forges the link between a

[53] As noted in Section 4 yielding equation (4.1), if φ_- is not kept fixed, one can always extend g_+. Such extensions have caused some confusion in the literature (see (Friederich, 2014)).

[54] The most indicative quote I could find is (Wallace, 2019a, p. 18), translated to my nomenclature:

> Given configurations $[\varphi_+]$ and $[\varphi_-]$, of the systems separately, we have not been given enough information to describe their joint configuration: that requires, in addition, a representational convention as to how points in the two configuration spaces are to be compared. Such a convention is inevitably required whenever we combine subsystems into a joint system. (In practice, the convention is often given by a choice of coordinate systems, and/or of reference frames, in the two subsystems.) Prior to stipulating any such convention, there is no sense in which $[\varphi_+] \boxplus [\varphi_-]$ specifies a different joint configuration from $[\varphi_+^g] \boxplus [\varphi_-]$, since φ_+^g and φ_+ are representationally equivalent. Given a choice of representational convention, though, it is clear that applying the symmetry transformation to one system gives rise to a different total configuration (and that this is true independent of what the actual representational convention is). So: symmetry-related configurations can be understood as representing different possible configurations if we hold fixed the choice of representational convention.

[55] It is precisely to relate the different regional gauge-invariant states that one should *not* quotient. As is argued in (Gomes, 2019a), the alternative is to gauge-fix, but to allow subsystem-extrinsic gauge-transformations for gluing.

global physical variety (*Global Difference*, cf. Sec. 1.4) and subsystem-symmetries, i.e. *DES*. Moreover, since only gauge-invariant quantities were deployed for the analysis, DES as construed here is therefore relational—i.e. obtained from the physical relations between two complementary subsystems of the universe.

Nonetheless, this possibility is not always fulfilled: in the case of bounded gauge systems in a simply-connected, vacuum universe, DES is *not* realized. In the presence of matter, the internalist's boundary only allows (relational) DES to be associated to global symmetries; that is, only *if* global symmetries can can be singled out (see Table 1).[56] These findings were confirmed by a treatment of the issue within the holonomy interpretation of electromagnetism, in Section 2.4.

As a consistency-check, I then applied the gauge-fixed approach to particle mechanics (Section 3). Thus, using precisely the same type of constructions as for gauge theories, I recovered the standard DES associated to Galileo's ship. In that context, DES arises from the different ways to embed intrinsically identical subsystems into the universe.[57]

For the externalist's notion of subsystems, the findings *differed*. Indeed, then there is room for a non-trivial realizer of "non-relational DES". This route requires configuration space to be (non-covariantly) truncated. That is, the truncation does not limit solely the set of physical possibilities, $[\varphi]$, but also the set of representatives, φ. By abandoning the requirement that gauge symmetries act equably on all configurations, the truncation yields a conceptually dubious notion of subsystem, that I denounced already in Section 1.2.1.

In this case, gauge-fixings satisfy *Uniqueness* and *Universality* as described at the end of Section 2.1 *only if* boundary stabilizers correspond to global stabilizers. Otherwise, the choice of boundary stabilizer has a physical effect. But since these choices do not belong to the configuration space $\mathcal{A}$, they should be seen as additional degrees of freedom, which, ultimately, represent the externalist's version of DES.[58] Here, different physical states are related by different boundary stabilizers, matching the findings of (Greaves & Wallace, 2014) (cf. equation (4.1)).

In Table 1, I summarize the presence of DES in different types of systems for simply-connected manifolds. In the *non-simply connected* case, DES always appears, but then it is not related to the structure group of the theory, G.

As a last remark, I admit that the externalist's notion of boundaries is ubiquitous in asymptotic treatments of symmetries. In fact, we model the solar system in this way: the standard spatially asymptotically flat spacetime imposes a particular form of the metric as one approaches the asymptotic boundary; it is not a diffeomorphism-invariant, geometric boundary condition.

Note, moreover, that the non-trivial 'asymptotically homogeneous' cases correspond precisely to the problematic ones above; they have a gap between their

[56]For non-Abelian theories, one can only single out global symmetries for configurations with stabilizers, i.e. 'isotropic'. In the nomenclature of (Gomes, 2019b), these symmetries are dubbed 'rigid'. For a more general treatment including matter and non-trivial topology, where DES does emerge, see (Gomes & Riello, 2019) and (Gomes, 2019b).

[57]No such possibilities exist in the field theoretic case, for the simply-connected manifold partitioned into regions; the embedding is determined by the partition.

[58]The only way I know to make this construction kosher, is that explored in (S. Ramirez & Teh, 2019), which endows these boundary 'gauge' degrees of freedom with their own dynamics, following the work of (Donnelly & Freidel, 2016). One should not confuse these degrees of freedom with the generalized (non-Abelian) electric flux and its conjugate. These latter quantities emerge in the *dynamical*—i.e. Lorentzian signature—setting studied in (Gomes & Riello, 2019); and they emerge even in the internalist's subsystem. They should not be interpreted as new degrees of freedom and do not contribute to the issue of DES (cf. (Gomes & Riello, 2019, Section 5)).

	Internalist	Externalist
Vaccum, isotropic	✗	✗
Vaccum, generic	✗	✓
Bulk charges, isotropic	✓	✓
Bulk charges, generic	✗	✓

Table 1: **The presence of DES.** Here 'isotropic' means, for the internalist's notion, that the regional configurations have stabilizers; and for the externalist's notion it means the boundary stabilizers are matched to the bulk stabilizers. 'Generic' stands for the negation of 'isotropic'—which happen to be generic conditions in the non-Abelian case. Abelian theories fall under isotropic. A case such as 't Hooft's beam splitter would correspond to the internalist's third line. This paper focused on the vacuum case.

boundary and their bulk stabilizers.[59] Therefore I grant that, even if ideally boundary conditions should be gauge-covariant, the externalist approach—which does not abide by that ideal—may work well for some purposes.

But, to obtain a solid conceptual footing, the externalist's notion os subsystem requires further conceptual analysis (see (Belot, 2018) which attempts such an analysis). Thus, echoing the doubts announced in Section 1.2.1: I believe we should not leave the underlying assumptions about asymptotic symmetries unexamined simply because they are useful; lest we acquiesce to what amounts to a 'shut up and calculate' mentality in the treatment of gauge-field asymptotics.

A more conceptually grounded approach is also a more conservative one: it goes from small systems to big ones. We should first properly understand gauge systems in finite regions and then move to the asymptotic regime by progressive enlargement, keeping careful track of how objects and relations maintain or lose their properties in the (singular) limit.[60]

Thus, in sum: this study has shown that previous literature on DES (Healey, 2009; Greaves & Wallace, 2014; Wallace, 2019a, 2019b) errs in large part by conflating two conflicting scenarios for boundaries: as boundaries of the entire universe—the externalist's notion—, or as an artefactual surface due to a choice of splitting.[61]

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[59] Only Minkowski space shares all the stabilizers that asymptotically flat spacetime possesses, and the asymptotic treatment is only interesting—has non-zero charges—for spacetimes that are *not* Minkowski.

[60] This is how Riello relates the subsystem divisions used here to the asymptotic regime, in the case of Yang-Mills theory (Riello, 2019). By doing so, he finds a singular limit for the asymptotic charges, recovering precisely the results of (Ashtekar A., 1981), who also treats boundary conditions in a diffeomorphism-invariant manner in general relativity, through Penrose compactification.

[61] (Teh, 2016) is a notable exception: it focuses on the externalist scenario, but ackowledges its limitations.

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