# The Metaphysics of Invariance

David Schroeren
Princeton University*
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**Abstract**

Fundamental physics contains an important link between properties of elementary particles and continuous symmetries of particle systems. For example, properties such as mass and spin are said to be 'associated' with specific continuous symmetries. These 'associations' have played a key role in the discovery of various new particle kinds, but more importantly: they are thought to provide a deep insight into the nature of physical reality.

The link between properties and symmetries has been said to call for a radical revision of perceived metaphysical orthodoxy. However, if we are to use claims about an 'association' between properties and symmetries in the articulation of metaphysical views, we first need to develop a sufficiently precise understanding of the content of these claims. The goal of this paper is to do just that.

## 1 Introduction

Fundamental physics posits an important link between properties of elementary particles and continuous symmetries of particle systems. As Steven Weinberg puts it, properties such as mass and spin are each "associated with" specific continuous symmetries.[1] This link has played a key role in the discovery of various new particle kinds, but more importantly: it is thought to provide a deep insight into the nature of physical reality. For example, Weinberg claims that the relevant physical properties "are what they are *because* of the [associated] symmetries of the laws of nature."[2]

According to an influential line of scholarship, this link between properties and symmetries calls for a radical revision of metaphysics—away from a perceived orthodoxy

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*The Department of Philosophy, 1879 Hall, Princeton University, Princeton, NJ 08544, United States. E-mail: `schroeren@princeton.edu`.

[1]Weinberg (1993, p. 146). Other physicists have made similar claims; for example, Heisenberg (1976, p. 924), Penrose (2004, p. 568), and Ne'eman and Sternberg (1991, p. 327). The link between properties and symmetries was first hypothesized by Eugene Wigner (1939).

[2](Weinberg, 1993, p. 138n, emphasis added). Elsewhere, Weinberg claims that the relevant link between properties and symmetries shows that "at the deepest level, all we find are symmetries and responses to symmetries" (Weinberg, 1987, p. 80) and that "matter thus loses its central role in physics" (Weinberg, 1993, p. 139).

according to which the world fundamentally consists of objects instantiating properties, and toward a metaphysics which conceives of fundamental reality as purely 'structural' in a sense to be elucidated (at least in part) by appeal to just this link between properties and symmetries. Proponents of this line—also known as *ontic structural realism*—claim that the link between physical properties and symmetries is "the basis of the claim that such properties should be conceived of structurally,"[3] in the sense that "the properties that particles have [...] appear to be explicable via considerations of [symmetry] structure" so that "what should be regarded as properly fundamental is the symmetry structure [...] that explains [the properties of elementary particles]."[4] However, if we are to use claims about a link between physical properties and continuous symmetries in the articulation of metaphysical theories, let alone as the basis for a radical revision of metaphysics, we first need to develop a sufficiently precise understanding of the content of these claims. The purpose of this paper is to do just that.

The link between physical properties and symmetries that is of concern in this paper is distinct from a more familiar correspondence, captured by Noether's theorem, between continuous symmetries and *conserved quantities*.[5,6] For example, Noether's theorem entails that rigid rotations about a given point are a symmetry of a physical system iff the system's total angular momentum about that point is conserved, i.e. constant in time—a fact which explains why figure skaters spin more quickly when drawing in their arms. But this is not the connection physicists have in mind when they claim that a given property is linked to a continuous symmetry in the relevant sense. Whereas this link concerns "the way particles behave when you perform various *symmetry transformations*" such as rotations,[7] Noether's theorem concerns the way particles behave *under temporal evolution*, i.e. whether certain physical quantities are conserved. The correspondence captured by Noether's theorem is therefore distinct from the link between properties and continuous symmetries that is the target of this paper.

The relevant sense in which a physical property may be said to be linked to a continuous symmetry is also distinct from the claim that this property is *invariant* under that symmetry, in the sense that any two states of the system related by that symmetry agree about this property. There is something right about this proposal: every property linked to a symmetry in the relevant sense is invariant under that symmetry. But there is also something wrong about this proposal: a property can be invariant under a symmetry without being linked to that symmetry in the relevant sense. Consider a classical $n$-particle system in three-dimensional Euclidean space. At every instant of time, the system has values of various *vector* quantities, such as total linear momentum $\vec{p}$ and, for every spatial point, the total angular momentum $\vec{L}$ of the system about that point. The

[3](French, 2014, p. 109).

[4](McKenzie, 2014b, p. 21). This is echoed by other authors; e.g. Ladyman (1998); Ladyman and Ross (2007); Kantorovich (2003). Ontic structuralists also use the link between properties and symmetries to argue *against* other metaphysical views such as certain varieties of dispositional essentialism; for example, French (2014, pp. 251) and Saatsi (2017); cf. (Livanios, 2010).

[5](Noether, 1918).

[6]Physical quantities (like mass and energy) can be thought of as associated with a range of *values*, each of which is a property in its own right.

[7](Weinberg, 1987, p. 79); emphasis added.

system also has values of various *scalar* quantities, such as the magnitude of total linear momentum $|\vec{p}|$ and, for any given point, the magnitude $|\vec{L}|$ of total angular momentum about that point. Both of these magnitudes are invariant under rigid rotations about any given point.[8] And yet, for every point $x$, only the magnitude of total angular momentum about $x$ is linked to rigid rotations about $x$ in the sense identified by physicists.

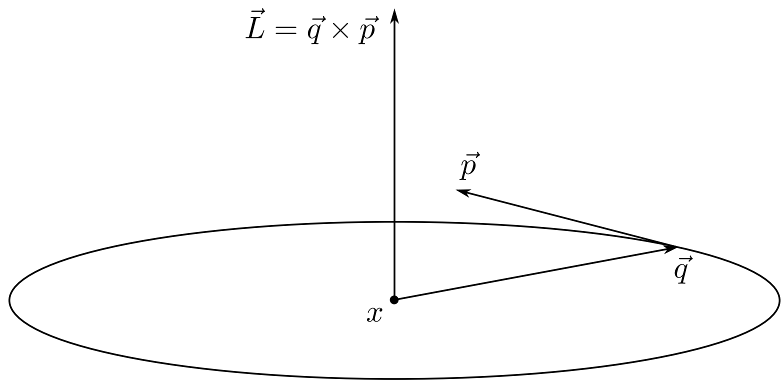


Figure 1: Illustration of a classical one-particle system in orbital motion about $x$. $\vec{q}$ represents the position of the particle in a co-ordinate system originating at $x$, $\vec{p}$ its linear momentum and $\vec{L} = \vec{q} \times \vec{p}$ its angular momentum about $x$.

The question we're asking has a standard mathematical answer—one that physicists like Weinberg presumably have in mind when making the more colourful claims cited above. This answer goes roughly as follows. The notion of a continuous symmetry is mathematically formalized by the notion of a *group*: roughly, a collection of transformations of a given type, such as rotations. And every group $G$ of continuous symmetries of a system is associated with a mathematical space capturing certain essential features of the transformations in $G$.[9] Now, there are mathematical objects (functions in the context of classical mechanics, operators in the context of quantum mechanics) whose invariance under $G$ can be derived exclusively from the essential features of transformations in $G$ captured by this space—objects which are called *Casimir invariants* of $G$ and whose number depends on $G$. The key observation is now that every property linked with a particular symmetry group is represented by one of the Casimir invariants of that group.[10] For example, the magnitude of total angular momentum about spatial point $x$ of a classical $n$-particle system is represented by a Casimir invariant of the group of rigid rotations of the system about $x$, whereas the magnitude of total linear momentum is not. Similarly, mass and spin of elementary particles are each represented by one of the two

[8]Throughout this paper, I understand the relevant transformations *actively*; that is, as transformations on physical systems, rather than merely as transformations of the mathematical objects representing those systems. See (Brading and Castellani, 2007, pp. 1342-3) for more on this distinction.

[9]This space is called the *Lie algebra* of $G$.

[10]By 'representation' I mean the way in which items of a linguistic or mathematical sort (theories, models, sentences, variables, predicates, functions, operators) are used to make claims about items of a 'metaphysical' or 'worldly' sort (possible worlds, facts, propositions, objects, properties, quantities).

Casimir invariants of the group of relativistic boosts, rotations, and translations.[11] The standard mathematical answer is thus that a physical property $P$ is linked to a continuous symmetry group $G$ in the relevant sense iff the mathematical object representing $P$ is a Casimir invariant of $G$.

A mathematical physicist or a certain sort of philosopher might think that this is all that needs to be said about the matter. According to this viewpoint, mathematical physics in itself is perfectly adequate and fully intelligible as a representation of the underlying reality, and no further understanding is to be gained by going 'beyond' physics into the murky waters of metaphysics.[12]

But for another kind of philosopher, the standard mathematical answer isn't good enough. According to this philosopher, there is something dissatisfying about characterizing features of reality by reference to the mathematical properties of their mathematical representations. If someone unfamiliar with the concept of money asks what is conveyed about a bag by saying that it contains some amount of US dollars in cash, the response 'that the bag has whatever feature is accurately represented by the dollar sign printed on its side' is likely to be dismissed as unilluminating. It is similarly inadequate to characterize the feature that a physical property has just in case it is linked with a symmetry group as whatever feature of this property is accurately represented by the fact that its mathematical representative is a Casimir invariant of that group. Therefore, once the mathematical physicist has explained the notion of a Casimir invariant, there's a further question that this philosopher would like to see addressed: what information *about a property* is conveyed by saying that its mathematical representative—the function or operator associated with it—is a Casimir invariant of some continuous symmetry group? This is the question I am going to answer in this paper.

I proceed as follows. I explain the standard mathematical answer in section 2 by considering the case study of total angular momentum about a point $x$ of a classical $n$-particle system and the group of rigid rotations of the system about $x$. Subsequently, I explain the metaphysical content of the notion of a Casimir invariant. According to the proposal I develop in section 3, what is conveyed about a property by the fact that it is represented by a Casimir invariant of a given group is that this property is a *fine-grained invariant* under that group—a notion I introduce. This notion is 'fine-grained' in the sense that not every invariant (in the ordinary sense) under a group is also a fine-grained invariant under that group: for example, it will turn out that the magnitude of total angular momentum about $x$ is a fine-grained invariant under rigid rotations about $x$, whereas the magnitude of total linear momentum is not. Finally, section 4 draws some contrasts with ontic structural realism.

[11] This group is referred to as the *Poincaré group*.

[12] One influential philosopher of physics who holds this view is David Wallace: "I'm happy to say that the way theories represent the world is inherently mathematical, so that the reason property $P$ is 'associated' with symmetry $G$ is simply because $P$ is represented by the Casimir invariant of $G$, and there's no need to say more." (Personal correspondence, quoted with permission.)

## 2 Casimir Invariance

According to the standard mathematical answer, a physical property $P$ is linked to a continuous symmetry group $G$ in the relevant sense iff the mathematical object representing $P$ is a *Casimir invariant* of $G$. The goal of this section is to introduce this notion and to characterize the theoretical and metaphysical framework I'll be using throughout this paper.

### 2.1 The Framework

To begin with, I need to tell you what notion of symmetry is operative in the link between symmetries and physical quantities. The relevant symmetries are symmetries of the mathematical space representing state space, or *state-space symmetries*, for short.

By way of illustration, consider an $n$-particle system described in terms of the Hamiltonian formulation of classical mechanics. This theory describes the particles as each instantiating the values of certain variable quantities with a special feature: the values of these quantities, together with the masses of the particles, fix the values of all other physical quantities at that time. These quantities are *position* and *momentum*. A fact about the value of position and momentum instantiated by each particle at a given time is what's called the *instantaneous state* of the system at that time. The state space of an $n$-particle system is just the space of all possible instantaneous states of this system; that is, the space of all possible assignments of position and momentum values to particles. In Hamiltonian mechanics, this space is represented by a mathematical space called *phase space*.

Now, just like a transformation on Euclidean space is a symmetry of Euclidean space just in case it preserves what one may think of as the characteristic structural feature of Euclidean space—the *Euclidean metric*—a transformation of phase space is a symmetry of phase space just in case it preserves the characteristic structural feature of phase space: the *symplectic form* on phase space.[13] In the present context, you can think of the symplectic form as something that determines the volumes of phase space regions, and of phase space symmetries as transformations that preserve the volumes of such regions.[14]

Now, the theoretical setting in which physicsts most confidently express the claim that certain physical quantities are linked to specific symmetries is that of quantum theory, and quantum field theory in particular. Here, the mathematical spaces used to represent the state space of a given system are called Hilbert spaces. In this context, state-space symmetries are therefore Hilbert space symmetries. The relevant characteristic structural feature of Hilbert spaces—the feature that a transformation of Hilbert space must preserve to count as a Hilbert space symmetry—is the *inner product*, a function which determines the length of vectors and their projections onto one another.[15] In other words:

[13] Phase space symmetries are also referred to as *symplectomorphisms*.

[14] This is only a first approximation: although every transformation that preserves the symplectic form also preserves the volumes of phase space regions, the converse is not true.

[15] Hilbert space symmetries are also referred to as *unitary transformations*. My terminology here

Hilbert space symmetries are the direct quantum-theoretic analogues of phase space symmetries. This means that the notion of symmetry relevant to the hypothesized link between symmetries and physical quantities can be found in both the classical and the quantum setting.

It is worth noting that state-space symmetries usually have the additional feature of preserving the dynamical equations and thus also count as *dynamical* symmetries. But these two notions of symmetry—state-space symmetry and dynamical symmetry—are not logically coextensive: not every state-space symmetry is also a dynamical symmetry. Moreover, the transformations that figure in the hypothesized link between quantities and symmetries do so *qua* being state-space symmetries, even though these transformations typically have the additional feature of being dynamical symmetries.

The mathematical complexity of quantum field theory is substantial. Luckily, the above-mentioned parallels in the treatment of state-space symmetries between the classical and the quantum setting means that we don't need let ourselves be sidetracked by these complexities: we can understand the link between quantities and symmetries within the relatively simple setting of $n$-particle Hamiltonian mechanics. This is the framework I'll be using in this paper.[16]

Let me briefly introduce a few important features of $n$-particle Hamiltonian mechanics. The phase space of an $n$-particle system in three-dimensional Euclidean space is given by its configuration space—the space whose points each represent a possible instantaneous spatial configuration of the system—together with an $n$-dimensional vector space attached to every point, a space whose elements each represent a possible assignment of a specific value of linear momentum to every particle at that point.[17] We can describe any given state of the system by a $6n$-tuple of numbers with regard to some co-ordinate system of $\mathbb{R}^3$: three spatial co-ordinates as well as three momentum co-ordinates for each of the $n$ particles.

Each physical quantity of the system is associated with a smooth distribution of values over state space; that is, with a smooth assignment, to each possible state, of a value of this quantity. These distributions are specified by smooth functions from phase space to the real numbers (in the case of scalar quantities) or by smooth functions from phase space to triples of real numbers (in the case of vector quantities). For example, the distribution of total energy over the state space of the $n$-particle system is represented by a real-valued function referred to as *the* Hamiltonian and denoted by $H$.[18] For the

follows (Weinberg, 1995, pp. 50) and is based on work by Eugene Wigner (1931).

[16]To be sure: $n$-particle Hamiltonian mechanics has no hope of being the true theory of fundamental physical reality. But that's fine: this paper is not intended as a contribution to the metaphysics of classical mechanics. Instead, the point of this investigation is to draw metaphysical conclusions that can be transposed to more cutting-edge physical theories, or that (at the very least) allow us to identify general strategies for handling such cases.

[17]In mathematical terms, the phase space of an $n$-particle system is the *cotangent bundle* of its configuration space.

[18]Physical quantities associated with smooth phase space functions are generally not invariant under changes of inertial reference frames (or *Galilei frames*): for example, there are Galilei frames which disagree about linear momentum and total energy. This means that we should always be talking about these quantities *relative to a particular Galilei frame*. However, the way these quantities are talked about

sake of brevity, I will abbreviate 'the state-space distribution associated with a quantity $Q$ is represented by the phase space function $f_Q$' by '$Q$ is represented by $f_Q$'.[19]

I will take for granted that physical quantities can be distinct despite being represented by the same phase space function. More precisely: call two quantities $Q$ and $Q'$ *state-space co-extensive* just in case their associated distributions assign the same value to every state in state space, i.e. just in case the corresponding phase space functions $f_Q$, $f_{Q'}$ are mathematically co-extensive: $f_Q = f_{Q'}$. Then, the claim I take for granted is that there are distinct state-space co-extensive quantities.

One motivation for this assumption is that state-space co-extensive quantities need not be *metaphysically* co-extensive. More precisely: suppose we think of the distributions associated with physical quantities as assigning values to every *metaphysically possible state* of the system, rather than merely to the possible states represented by points in phase space. Then, for any phase space function, there will generally be distinct quantities such that the restriction to state space of their associated distributions is accurately represented by this function.

By way of example, suppose that the total energy of the $n$-particle system is in fact state-space co-extensive with the quantity whose value at a state is the result of a certain specific operation on the position and momentum facts at that state corresponding to the sum of the values of kinetic energy and Newtonian gravitational energy at that state—a fact we may express in terms of the equation $H = T + V_g$, where $T$, and $V_g$ are phase space functions representing kinetic and Newtonian gravitational energy, respectively. But total energy and the quantity represented by $T + V_g$ aren't *metaphysically* co-extensive: for example, they disagree at metaphysically possible states which contain *electrostatic energy*. This motivates the assumption that there are distinct state-space co-extensive quantities.

This paper focuses on the quantity of *total angular momentum* about a given spatial point $x$. The state-space distribution of this quantity is specified by the vector function $\vec{L} = \vec{L}^1 + ... + \vec{L}^n$, where $\vec{L}^k = \vec{q}^{\,k} \times \vec{p}^{\,k}$ represents the angular momentum of the $k$-th particle about $x$, $\vec{q}^{\,k}$ its position in a co-ordinate system centered on $x$, and $\vec{p}^{\,k}$ its linear momentum. The scalar components of $\vec{L}$ are real-valued phase space functions each of which represents the state-space distribution of the component of angular momentum along *some* axis through $x$.[20]

Certain continuous sequences of states (or *state-space curves*) capture important dynamical and modal facts about the system. Most importantly, the state-space curves that count as *possible dynamical histories* of the system allow us to answer questions such as 'what *will* the system be like in five minutes'? Another important type of state-space

in physics and philosophy often doesn't make their Galilei-dependence explicit. In this paper, I follow this practice for the sake of brevity.

[19]Not all quantities represented by smooth phase space functions are equally natural: any 'gerrymandered' function of position and momentum, such as the product $\vec{q}^{\,i} \cdot \vec{p}^{\,k}$ of the position vector of the $i$-th particle with the linear momentum vector of the $k$-th particle, is less natural than (for example) the function which represents the linear momentum of a given particle. Distinctions based on naturalness are familiar from the metaphysics of properties; cf. (Lewis, 1986, pp. 59).

[20]Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of a real vector space $V$. The *scalar components* of any $v = v_1 e_1 + v_2 e_2 + ... + v_n e_n \in V$ are the real numbers $v_1$, $v_2$,..., $v_n$.

curves allow us to answer questions such as ‘what *would* the system be like if it *were* rigidly rotated by 180° about an axis through some spatial point $x$?’ These state-space curves correspond to *rigid rotations* of the system about axes intersecting $x$.

State-space curves of a given type are specified in terms of some particular *flow* on phase space, i.e. in terms of some particular family of phase space curves such that each phase space point is intersected by exactly one curve in this family. (Think of a flow as the family of curves traced out by water molecules of a river.) Each flow has a real parameter: for example, for every real number $t$, the *dynamical* flow representing the dynamical histories of the system assigns to each phase space point its image under dynamical evolution by $t$ units of time. Similarly, for each real number $\phi$, the *rotational* flow corresponding to rigid rotations of the system about an axis through $x$ assigns to each phase space point its image under a rigid rotation by an angle of $\phi$ about this axis.

It is worth stressing that the relevant rotations are *non-dynamical*: whereas the parameter of the dynamical flow is interpreted as time, the parameter of a given rotational flow is interpreted as the corresponding angle of rotation; and states related by a rotation are generally not related by dynamical evolution, as illustrated in figure (2).

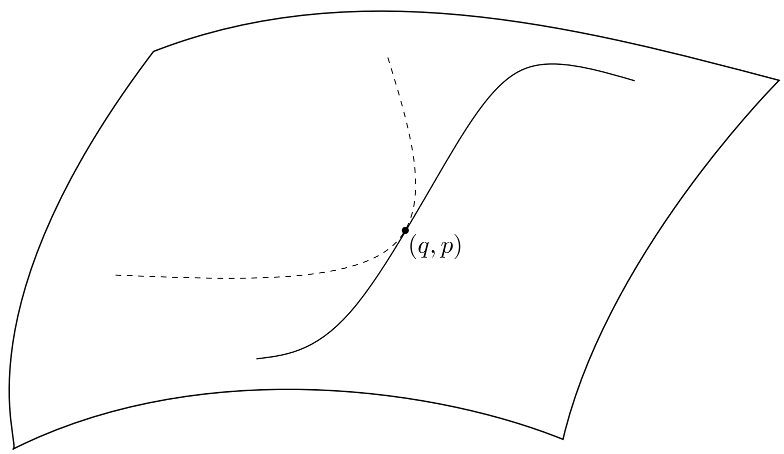


Figure 2: Illustration of phase space with a dynamical flow line and a rotational flow line (dotted) through phase space point $(q, p)$.

This is where a crucial aspect of Hamiltonian mechanics comes into play: the phase space flows representing (for example) dynamical evolution and rotations about some axis in physical space are what’s referred to as *Hamiltonian* flows. I’ll explain this notion in a moment.

But first, let me tell you why it’s important, in the present context, that these flows are Hamiltonian. The first reason is that Hamiltonian flows are *symmetries* of phase space: they preserve the symplectic form, the characteristic structural feature of state space. (Again, you can think of this as the claim that Hamiltonian flows preserve the volumes of phase space regions.[21]) Second, each of these flows has an important mathematical

[21] The qualification given in footnote (14) applies here as well.

relationship to some specific smooth phase space function: each Hamiltonian flow is *determined* by such a function in a way that *preserves* this function. The function which determines a Hamiltonian flow is called the *generator* of this flow. For example, the dynamical flow is the Hamiltonian flow generated by the total energy function $H$, and the rotational flow corresponding to rigid rotations about some axis through $x$ is generated by the scalar component of $\vec{L}$ along this axis.

The technical reason for why Hamiltonian flows preserve the symplectic form (and thus count as state-space symmetries) need not concern us here. By contrast, the determinative relationship between smooth phase space functions and the Hamiltonian flows they generate provides crucial inspiration for the metaphysical account I will propose later, so bear with me while I explain this relationship in more detail.

## 2.2 Hamiltonian Flows and their Generators

What is it for a smooth function to be a generator of its Hamiltonian flow? Here is an informal picture. We are familiar with the sense in which a *vector field* generates a flow. (Think of a vector field on some space as a function which assigns a vector to every point in this space.) For example, the flow of the velocity vector field of a river can be thought of as the family of curves traced out by the water molecules dragged along by the current of the river. But what is it for a *scalar* function to be a generator of a flow?

There's a fairly intuitive sense in which a scalar function is the generator of the flow of its *gradient*. Recall: the gradient of a smooth function $f$ on a Riemannian manifold is the vector field which assigns to each point of the manifold the vector that points in the direction of greatest increase of $f$ at that point.[22] And every smooth function $f$ on such a manifold determines its gradient in a two-step process. The first step consists in an operation that takes smooth functions as input and throws away all but two specific sorts of information: first, information about the regions of the manifold in which the input function is constant, also called *level surfaces* of this function;[23] and second, information about the differences in value of this function between each of its level surfaces. In mathematics jargon, this operation is called the *exterior derivative*. For example, the information contained in the exterior derivative of the elevation function of a piece of terrain suffices to draw an accurate map of the terrain: on the basis of this information, one can draw contour lines as well as the differences in elevation between these lines. What would be missing from the map, however—the information thrown away by the exterior derivative—is the absolute elevation of a given point of the terrain. In other words: the exterior derivative of a function is equivalent to this function up to an additive constant.

The information contained in the exterior derivative of $f$ is not yet packaged in the form of a vector field. This is the task of the second step, in which the exterior derivative

[22] A Riemannian manifold is a smooth manifold equipped with a *Riemannian metric*, a positive-definite map from pairs of vector fields to smooth functions that we can think of as a generalisation of the Euclidean dot product.

[23] Level surfaces are generalisations of the one-dimensional contour lines familiar from hiking maps, i.e. the lines of constant elevation of the relevant piece of terrain.

of $f$ is turned into the *gradient* of $f$. The specifics of this mathematical procedure are not relevant for our purposes, except for the fact that it is defined with regard to the Riemannian metric on the underlying manifold, a rule we can think of as the Euclidean 'dot product' of two vectors. Intuitively: the role of the Riemannian metric is to take something that *doesn't* point in any direction (the exterior derivative of $f$) and turn it into something that *does* point in a direction (the gradient of $f$).

It is by way of these two steps that a function determines its gradient. But as noted above, a vector field also determines its flow. Thus, $f$ also determines the flow of its gradient, which is captured by saying that $f$ is a *generator* of this flow.[24]

But what is it for a function on phase space to generate its *Hamiltonian* flow? The answer lies in a different way to package the information contained in the exterior derivative, a different procedure for turning this information into a *vector field*—a procedure peculiar to Hamiltonian mechanics and defined with regard to a mathematical device we encountered earlier as the 'characteristic structural feature of phase space': the *symplectic form* on phase space. The key difference between the procedure defined in terms of the Riemannian metric and the procedure defined in terms of the symplectic form is this: whereas the former turns the exterior derivative of a function into a vector field that everywhere points in the direction of greatest increase of the function, the latter turns the exterior derivative into a vector field that everywhere points *along the level surfaces* of this function, as illustrated in figure (3). Vector fields of this sort are referred to as *Hamiltonian vector fields* and their flows as *Hamiltonian flows.* As a result, each phase space function $f$ determines or *generates* a specific Hamiltonian flow: the flow of the Hamiltonian vector field of $f$. Moreover, since the Hamiltonian vector field of $f$ points everywhere along the level surfaces of $f$, each flow line of the Hamiltonian flow generated by $f$ is confined to some level surface of $f$. This is the reason for why smooth phase space functions are invariant under the Hamiltonian flows they generate.

We thus understand what it is for a scalar function to generate its Hamiltonian flow, and thus what it is for the component of $\vec{L}$ along some axis through spatial point $x$ to be a generator of the Hamiltonian flow corresponding to rigid rotations about this axis.

Now, the rigid rotations about arbitrary axes intersecting $x$ *taken together* have the structure of a group, namely the group $SO(3)$ of orientation-preserving isometric rotations in three-dimensional Euclidean space.[25] More precisely, rigid rotations of the $n$-particle system about a given point are implemented by an *action* of $SO(3)$ on the phase space of the system, i.e. by a rule for how the position and momentum of each constituent particle transforms under every abstract element of the group: for every spatial point, the action of $SO(3)$ as rotations about that point consists in an assignment, to each

[24] Since there is an infinity of distinct functions with the same exterior derivative—the functions that agree up to an arbitrary real constant—the flow of each gradient vector field has an infinity of generators. The same applies to Hamiltonian flows that will be introduced shortly.

[25] The statement that rotations about a point have the structure of a *group* is equivalent to the conjunction of four claims. First, the combination of any two rotations is also a rotation. Second, the result of first performing rotation $R$ followed by the combination of rotations $R'$ and $R''$ is the same as first performing the combination of $R$ and $R'$, followed by the rotation $R''$. Third, every rotation can be reversed. Fourth, some rotations are equivalent to doing nothing—e.g. rotations by 0 and 360 degrees.

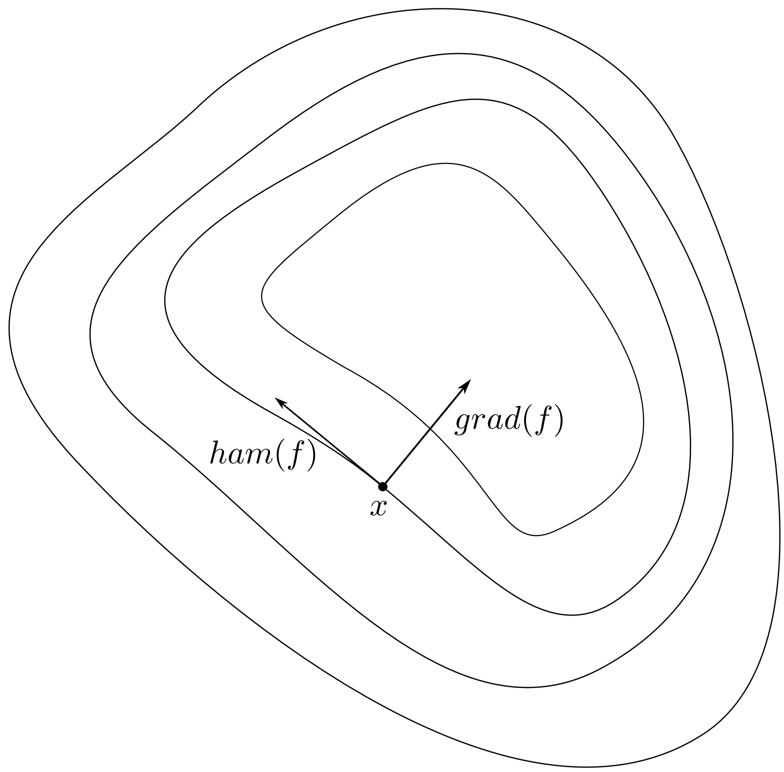


Figure 3: Illustration of the level surfaces (contour lines) of a function $f$ on a two-dimensional manifold. $grad(f)$ and $ham(f)$ denote the values at $x$ of the gradient and the Hamiltonian vector field of $f$, respectively.

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abstract element of $SO(3)$, of a rotation by *some* angle about *some* axis intersecting that point.

This has an important consequence: any two phase space points related by a rotation about spatial point $x$ are related by a rotation about *some specific axis* $i$ intersecting $x$. *A fortiori*, any two such phase space points lie on a flow line belonging to the Hamiltonian flow generated by the $i$-component of $\vec{L}$. In this way, the action of $SO(3)$ as rigid rotations about $x$ is implemented by Hamiltonian flows corresponding to rotations about axes through $x$, and so the generators of the flows about these axes—the components of $\vec{L}$ along these axes—are also generators of this action of $SO(3)$, or *$SO(3)$-generators*, for short.

Although each $SO(3)$-generator is invariant under its own Hamiltonian flow, $SO(3)$-generators are not $SO(3)$-invariant: it is not the case that any two phase space points related by a rotation about some axis through $x$ agree about every component of $\vec{L}$. Exactly how do the components of $\vec{L}$ change under rotations? This is the last question we need to answer before we can state the notion of a Casimir invariant.

## 2.3 $SO(3)$ Poisson brackets

Physics is often concerned with determining the *rates of change* of physical quantities under certain important variations of the system. For example, *velocity* is the instantaneous rate of change through time of position, and *net force* is the instantaneous rate of change through time of total linear momentum.

In Hamiltonian mechanics, facts of this sort are represented by the derivatives of phase space functions with regard to the parameter of the Hamiltonian flow that characterizes the relevant variation of the system. The instantaneous rate of change *under dynamical evolution* of the position of the $k$-th particle is given by the derivative of $\vec{q}^{\,k}$ with regard to *time*, the parameter of the Hamiltonian flow generated by the total energy function $H$. Similarly, the rate of change of the linear momentum of the $k$-th particle *under rotations* about some axis through $x$ is represented by the derivative of $\vec{p}^{\,k}$ with *with regard to the angle of rotation* about that axis, the parameter of the Hamiltonian flow generated by the component of $\vec{L}$ along that axis.

The derivative of a smooth function with regard to the parameter of a Hamiltonian flow can be expressed directly in terms of the generator of this flow. More specifically, the derivative of a function $f$ with regard to the parameter $\lambda$ of the Hamiltonian flow generated by another function $g$ is given by[26]

$$\frac{df}{d\lambda} = \{f, g\}, \tag{1}$$

where $\{\cdot\,,\cdot\}$ is the *Poisson bracket*, a product on the space of smooth functions satisfying certain conditions, the details of which need not concern us here.[27]

For present purposes, the following intuitive picture of the mathematical content of (1) shall suffice. A perfectly standard way to represent the rate of change of the elevation of a piece of terrain along a path through the terrain is by means of the *derivative* of the elevation function in the direction of this path.[28]

Suppose now that we are considering an entire family of paths through the terrain, and suppose that this family can be represented by the flow of some vector field $Y$. We then have a notion of a derivative of the elevation function in the direction of $Y$. This derivative can be thought of as the function which assigns to every point of the terrain a number that tells us how much of the greatest increase of elevation we're experiencing in the direction of $Y$—the function whose value at a point $x$ is the *projection*, with regard to the Riemannian metric, of the value of the gradient of the elevation function at $x$ onto the value of $Y$ at $x$, as illustrated in figure (4).

Equation (1) has a very similar mathematical role: $\{f, g\}$ also captures a sort of derivative of $f$ in the direction of a vector field—namely, in the direction of the Hamiltonian vector field of $g$. However, the vector field that is being projected onto the Hamiltonian

[26]Recall: we're supposing that the $f$ and $g$ are real-valued functions on phase space *only*. This entails that $\frac{\partial f}{\partial \lambda} = 0$, where $\lambda$ is the parameter of the flow of $g$.

[27]The Poisson bracket is a *Lie bracket*; that is, it is anti-symmetric, bi-linear, non-associative and satisfies the Jacobi identity: for any smooth functions $f, g, h$, $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$.

[28]In mathematical jargon, this derivative is known as the *directional derivative* of the elevation function along the path.

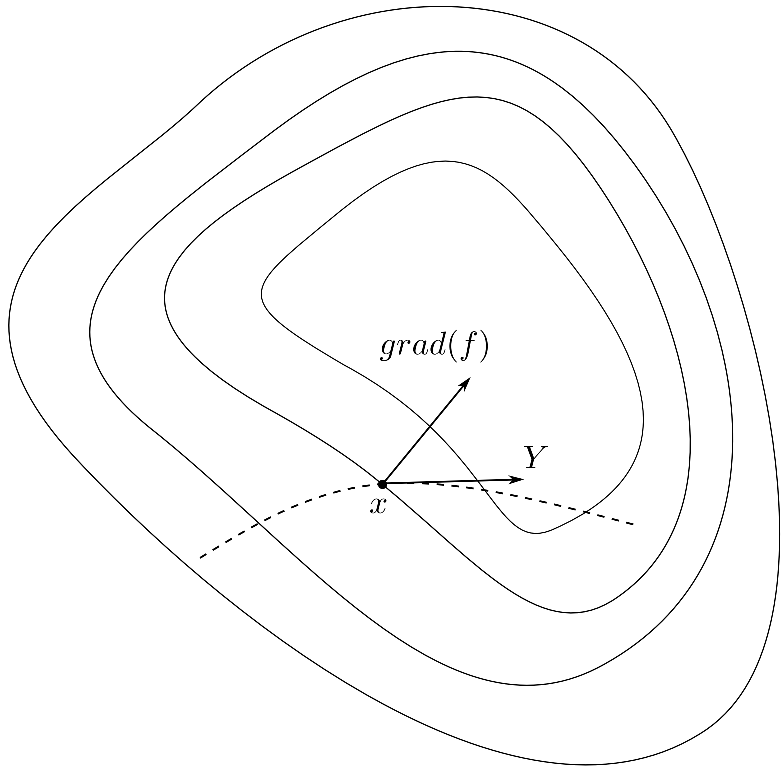


Figure 4: Illustration of a function $f$ via its contour lines, showing a flow line (dotted) of the vector field $Y$ and the value $grad(f)$ of the gradient of $f$ at $x$.

vector field of $g$ is not the gradient of $f$, but rather the Hamiltonian vector field of $f$. Moreover, the projection operation is different: whereas in the previous case the projection operation is given by the Riemannian metric, here it is given by the symplectic form. Notwithstanding these differences, we may think of the Poisson bracket $\{f, g\}$ as a derivative of $f$ *in the direction of the Hamiltonian vector field of* $g$, and thus as a phase space function that captures how $f$ *varies* along the Hamiltonian flow of $g$.

Poisson brackets are therefore the tool we need to determine how the components of $\vec{L}$ vary under rotations about some spatial point $x$. Denote by $L_i$ the scalar component of $\vec{L}$ along axis $i$ through $x$, and let 1, 2, and 3 be the labels of pairwise perpendicular axes through $x$. Then, $\{L_1, L_2\}$ is the derivative of $L_1$ along the Hamiltonian flow generated by $L_2$. And as it turns out, the derivative of any given $SO(3)$-generator along the flow of another $SO(3)$-generator is also an $SO(3)$-generator: $\{L_1, L_2\} = L_3$. Equations of this sort are called *SO(3)-Poisson bracket equations* and summarized as

$$\{L_i, L_j\} = \epsilon_{ijk} L_k, \tag{2}$$

where $i, j, k$ take values in the labels of any given triple of pairwise perpendicular axes intersecting $x$ and $\epsilon_{ijk}$ is 1 for even permutations of $ijk$, $-1$ for uneven permutations and 0 otherwise.[29]

We need to take note of a related, important use of Poisson brackets. It might be remembered from high school calculus that a function of a single variable is constant just in case its derivative vanishes everywhere. Similarly, a function $f$ is constant (or invariant) along the Hamiltonian flow of another function $g$ iff the derivative of $f$ along this flow vanishes everywhere in phase space; that is, iff $\{f, g\} = 0$. Facts about the invariance of functions along Hamiltonian flows are thus facts about vanishing Poisson brackets.

We can use this to say what it is for a function to be invariant under rotations about some spatial point $x$: a function $f$ has this property just in case, for every axis $i$ through $x$, its derivative along the Hamiltonian flow corresponding to rigid rotations about $i$ vanishes: $\{f, L_i\} = 0$. I refer to functions which have this property as *SO(3)-constant* functions.

Note that the physical quantities represented by $SO(3)$-constants are such that any two states related by an $SO(3)$-transformation agree about these quantities—which is to say that these quantities are $SO(3)$-invariant. Whenever there is a risk of confusion, I refer to the quantities represented by $SO(3)$-constant functions as *ordinary* $SO(3)$-invariants.

The goal of this section was to explain how the $SO(3)$-generators change under rotations. As promised, this was the last thing we needed to understand before articulating the notion of a Casimir invariant. To this I now turn.

### 2.4 Casimir Invariants

I follow the standard definition in physics texts.[30] For a smooth function $C$ on phase space to be a *Casimir invariant* of the action of $SO(3)$ as rigid rotations about $x$ (or an $SO(3)$-*Casimir*, for short) is for $C$ to satisfy two conditions. First, $C$ must be an $SO(3)$-constant: $\{C, L_i\} = 0$ for every axis $i$ through $x$. Second, $C$ must be a function exclusively of the $SO(3)$-generators.

One phase space function which satisfies these conditions is $L^2 = L_1^2 + L_2^2 + L_3^2$, the Euclidean dot product of the angular momentum vector function with itself. First, $L^2$ is an $SO(3)$-constant; that is, $\{L^2, L_i\} = 0$ for every axis $i$ through $x$.[31] Second, $L^2$ is a function only of $SO(3)$-generators, as can be seen by noting that no two states can differ about $L^2$ without differing about at least one component of $\vec{L}$. Importantly, $L^2$ is the unique 'independent' $SO(3)$-Casimir, in the sense that every other $SO(3)$-Casimir is a function of $L^2$—for example, the function $cos(L^2)$. For this reason, we may refer to $L^2$ as *the* $SO(3)$-Casimir.

$L^2$ has a systematic relationship with the function $|\vec{L}|$ representing the magnitude of angular momentum $M$: $|\vec{L}|$ is the square root of $L^2$. In the remainder of this paper, I follow an informal convention in physics by treating $L^2$ and $|\vec{L}|$ as having the same representational content, so that $M$ may be said to be represented by $L^2$ as much as by

[29] In mathematical jargon, equation (2) means that the set of $SO(3)$-generators forms a *Lie algebra* with regard to the Poisson bracket, called the *SO(3)-Poisson Lie algebra*.

[30] (Penrose, 2004, pp. 553,568), (Fuchs and Schweigert, 2003, pp. 254).

[31] This is derived in (Goldstein et al., 2001, p. 418).

$|\vec{L}|$. In so doing, it is important to keep in mind that any given value of $M$ is represented by the square root of the corresponding numerical value of $L^2$.

The functional dependence of $SO(3)$-Casimirs on $SO(3)$-generators is responsible for an important feature of Casimirs: a function $C$ is an $SO(3)$-Casimir iff the fact that $C$ is $SO(3)$-constant can be derived from the $SO(3)$-Poisson brackets alone. In particular, the $SO(3)$-Poisson brackets $\{L_i, L_j\} = \epsilon_{ijk} L_k$ are sufficient to derive the equations $\{L^2, L_i\} = 0$. This feature of $SO(3)$-Casimirs will be important later, so keep it in mind.

Not every $SO(3)$-constant is also an $SO(3)$-Casimir. For example, the function $p^2$ representing the magnitude of total linear momentum is an $SO(3)$-constant: it obeys $\{p^2, L_i\} = 0$ for every relevant axis $i$ and thus satisfies the first condition of being an $SO(3)$-Casimir. But $p^2$ is not a function of the $SO(3)$-generators: the phase space points $(\vec{q}, \vec{p})$ and $(2\vec{q}, \frac{1}{2}\vec{p})$ representing states of a one-particle system agree about $\vec{L} = \vec{q} \times \vec{p}$ and thus about all components thereof, but disagree about $p^2$. It follows that $p^2$ is not a function of the $SO(3)$-generators, and thus not an $SO(3)$-Casimir. This is confirmed by the fact that the $SO(3)$-constancy of $p^2$ cannot be derived from the $SO(3)$-Poisson brackets alone.

According to the standard mathematical account, the magnitude of total angular momentum about $x$ is linked to the group $SO(3)$ of rigid rotations about $x$ (in the relevant sense) just in case the mathematical representative of this quantity is an $SO(3)$-Casimir. We now understand what this means: to be an $SO(3)$-Casimir is to be an $SO(3)$-constant function of the $SO(3)$-generators. I will now explain what is conveyed *about the magnitude of total angular momentum about $x$* by saying that its mathematical representative is an $SO(3)$-Casimir invariant.

# 3 Fine-Grained Invariance

Let me dispense with a tempting proposal immediately. As we just saw, one feature that distinguishes $SO(3)$-Casimirs from mere $SO(3)$-constants is that the former, but not the latter, are functions only of the $SO(3)$-generators. This is reflected in the fact that, for any triple of co-ordinate axes through $x$, $L^2$ can be written as the sum of squares of components of $\vec{L}$ along these axes, whereas the mere $SO(3)$-constant $p^2$ cannot. It may now be tempting to think that it is a very similar feature that distinguishes the $SO(3)$-invariant *quantity* represented by the $SO(3)$-Casimir from other $SO(3)$-invariant quantities. Observe that the magnitude of total angular momentum $M$ can be thought of as having the following sort of functional dependence on the components of angular momentum: $M$ is the quantity such that, for any triple of pairwise perpendicular axes through $x$, the value of $M$ at a state is the result of whatever specific operation on the position and momentum facts at that state corresponds to taking the square root of the sum of squares of the values of the angular momentum components along these axes at that state. According to the tempting proposal, what is conveyed about $M$ by saying that its mathematical representative is an $SO(3)$-Casimir is the conjunction of two claims: first, $M$ is an ordinary $SO(3)$-invariant; and second, $M$ has just this sort of functional dependence on the components of angular momentum.

The problem with this proposal is that it cannot be applied in quantum mechanics. For example, actions of $SO(3)$ as rigid spatial rotations of a non-relativistic $n$-particle quantum system about a spatial point are generated by self-adjoint operators $\hat{L}_i$ that represent components of the total *orbital* angular momentum about that point. In this setting, the $SO(3)$-Poisson bracket equations (2) become the *commutator bracket* equations $[\hat{L}_i, \hat{L}_j] = \epsilon_{ijk}\hat{L}_k$, and the Casimir invariant associated with this action of $SO(3)$ is the operator that can be written as $\hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2$, interpreted as the magnitude of total orbital angular momentum.

The crux is now that it is impossible for a quantum system to simultaneously possess determinate values of quantities represented by non-commuting operators.[32] This means that a quantum system cannot simultaneously have values for every component of total orbital angular momentum, and so no physical quantity can functionally depend on these components in the way fleshed out above.

The simple idea therefore does not carry over to quantum theory. But it is precisely the context of quantum theory in which the link between properties and symmetries is of greatest physical significance, and so any plausible account of this link ought to be applicable to this context. Since the simple idea does not meet this requirement, it should be rejected.

What, then, is the right account? The simple idea we just considered hypostatizes one mathematical feature of Casimir invariants: the functional dependence of Casimir invariants on the relevant generators. I propose that we focus on *another* distinctive feature of Casimir invariants: the fact that the $SO(3)$-constancy of $SO(3)$-Casimirs is derivable exclusively from the $SO(3)$-Poisson bracket equations. Let the *SO(3)-variation facts* be the collection of facts consisting, for every triple of pairwise perpendicular axes through $x$, of the facts stated by the $SO(3)$-Poisson bracket equations $\{L_i, L_j\} = \epsilon_{ijk}L_k$; where $i, j, k$ take values in the labels of these axes. According to my proposal, what is conveyed about a property by saying that it is represented by an $SO(3)$-Casimir invariant is that the ordinary $SO(3)$-invariance of this property is *determined* by the $SO(3)$-variation facts; a determinative relationship which corresponds to the derivability of one collection of Poisson bracket equations from another collection of Poisson bracket equations. This proposal is illustrated in figure (5).

To implement this proposal, we need to do two things: first, we need to explain the nature of $SO(3)$-variation facts; and second, we need to understand the nature of the determinative relationship whose mathematical proxy is the derivability between the relevant mathematical propositions.

[32] Note that this claim does not depend on the controversial *eigenstate-eigenvalue link*. In other words: it is not presupposed that (1) *every* physical quantity is represented by a Hermitian operator, nor that (2) if a quantity $Q$ is represented by a Hermitian operator, the *only* way of determinately having $Q$ is being in an eigenstate of $Q$.

$$
\begin{tikzcd}
\{f_Q, L_i\} = 0 & Q \text{ is an ordinary } SO(3)\text{-invariant} \\
\{L_i, L_j\} = \epsilon_{ijk} L_k \arrow[u, "derivation"'] & SO(3)\text{-variation facts} \arrow[u, "determination"']
\end{tikzcd}
$$

Figure 5: Illustration of the proposal, according to which the mathematical feature of the $SO(3)$-Casimir invariant function $f_Q$ shown on the left is reflected in a similar feature, shown on the right, of the quantity $Q$ represented by $f_Q$.

### 3.1 Modality, Variation, and Determination

What is the content of Poisson bracket equations such as $\{L_1, L_2\} = L_3$? For every axis $i$ through spatial point $x$, denote by $M_i$ the $i$-component of total angular momentum about $x$. Then our question seems to have the following, straightforward answer. The Poisson bracket $\{L_1, L_2\}$ expresses the definite description 'the quantity whose value at every state equals the rate of change of $M_1$ under rotations about the 2-axis'; and the equation $\{L_1, L_2\} = L_3$ expresses the non-trivial identity of the satisfier of this definite description with $M_3$.

But there is a problem with this idea. The Poisson bracket $\{L_1, L_2\}$ prominently mentions $L_2$, the mathematical representative of $M_2$; and it does so for an important mathematical reason: $L_2$ is the generator of the flow in the direction of which we're taking the derivative of $L_1$. By contrast, the proposed definite description mentions only $M_1$, but not $M_2$. This suggests that an important part of the content of $\{L_1, L_2\}$ is missed by the straightforward proposal.

A natural thought is to make the definite description expressed by $\{L_1, L_2\}$ reflect the mathematical content of $\{L_1, L_2\}$ more closely. Mathematically, $\{L_1, L_2\}$ can be paraphrased as 'the function whose value at every phase space point equals the derivative of $L_1$ along the Hamiltonian flow generated by $L_2$.' This suggests that the definite description stated by $\{L_1, L_2\}$ is 'the quantity whose value at every state equals the rate of change of $M_1$ along the family state-space curves generated by $M_2$.' As it stands, however, this is obscure: there is no obvious sense in which a *quantity* counts as a generator of a family of state-space curves. The key challenge is thus to give content to this sort of definite description.

The mathematical account reviewed in section (2.2) is suggestive of the following proposal. What is conveyed about a quantity $Q$ by the fact that its mathematical surrogate $f_Q$ is a generator of its associated phase-space flow is that certain facts about the state-space distribution of $Q$ to *determine* facts about the state-space curves represented by the Hamiltonian flow of $f_Q$ in a way that mirrors the mathematical determination of this flow by $f_Q$, for some appropriate notion of determination.

To implement this proposal, we need to do three things. We need to understand the nature of the facts determined by certain facts about the quantities characterized by generators. We need to explain the nature of facts about those quantities that determine these facts. And we need to say more about the nature of the determination relation that obtains between the relevant facts.

What is the nature of the facts determined by certain facts about quantities characterized by generators? I said earlier that Hamiltonian flows on the phase space of an $n$-particle system capture certain *modal* features of the system. For example, the Hamiltonian flow of $L_1$ can be used to answer questions such as 'what *would* the system be like if it *were* rigidly rotated by 180° about the 1-axis?' Think of the facts determined by the relevant facts about $M_1$ as answers to questions of this kind.

More precisely, let's say that state $s'$ is *rotationally accessible* from state $s$ just in case there is an angle $\phi$ such that $s$ is related to $s'$ by a rotation about the 1-axis by $\phi$. Second, stipulate that a proposition $p$ is *1-possible* at a state $s$ iff $p$ holds at some state rotationally accessible from $s$, and that $p$ is *1-necessary* at $s$ if $p$ holds at every state rotationally accessible from $s$. Third, for any two states $s$ and $s'$ related by a rotation about the 1-axis, let the *modal distance* between $s$ and $s'$ be the smallest number $|\phi|$ such that $s$ and $s'$ are related by a rotation by an angle of $\pm\phi$.[33] Finally, let the *1-modal facts* at a state $s$ be the collection of facts about what is 1-possible and 1-necessary at $s$, together with facts about the modal distances between $s$ and states rotationally accessible from $s$. The facts determined by the relevant facts about $M_1$ are the 1-modal facts at every state.

Next, consider a physical quantity characterized by the generator of some phase-space flow. What is the nature of the facts about this quantity that are supposed to determine the relevant modal facts? Start by observing that the 1-modal facts at every state are characterized by the Hamiltonian flow of $L_1$: a state $s$ is rotationally accessible from another state $s'$ just in case the phase space points representing $s$ and $s'$ lie on the same Hamiltonian flow line of $L_1$. This suggests that we can use the mathematical account of the facts that determine a particular Hamiltonian flow line of $L_1$ as a blueprint for an account of the nature of the facts that determine the 1-modal facts at every state.

According to the mathematical account reviewed in section (2.2), the Hamiltonian flow of a smooth phase space function is determined by the exterior derivative of this function, i.e. by facts of two sorts: first, facts about which regions of phase space are level surfaces of the function; and second, facts about the differences in value of the function between these level surfaces. And a particular flow line is selected by additionally specifying a particular phase space point—the 'initial value' of this flow line. For example: the exterior derivative of $L_1$, together with a specific point $(q, p)$, determines which phase space points are related to $(q, p)$ by a rotation about the 1-axis.

The facts about a function $f_Q$ captured by its exterior derivative correspond to facts

[33] To obtain a reparametrisation-invariant measure of the modal distance—a measure invariant under a change from *degrees* to *radians*, for example—an appropriate normalisation factor needs to be added: $\frac{1}{2\pi}|\phi|$ in radians and $\frac{1}{360°}|\phi|$ in degrees. In mathematics jargon, this measure is referred to as the *Haar measure* on the underlying group; cf. (Fuchs and Schweigert, 2003, pp. 375-8).

about the state-space distribution of the *physical quantity* $Q$ represented by $f_Q$, facts I'll refer to as the *level surface facts* about $Q$: first, facts about the regions of state space in which the quantity has a given constant value; and second, facts about the differences in value of this quantity between those regions. And in close analogy to the mathematical story just described, to infer which states are rotationally accessible from actuality—to infer the actual 1-modal facts—one also needs to know which state one is in; i.e. one needs to know the actual position and momentum facts. This suggests that the range of facts which determine the 1-modal facts are the level surface facts about $M_1$, together with the position and momentum facts.

Finally, we need to say more about the nature of the determination relation. To begin with, determination in the relevant sense is a relation between *facts*: namely, the relation that obtains between (say) the 1-modal facts at some state and the level surface facts about $M_1$, together with the position and momentum facts at that state. Moreover, this relation is not identity: the conjunction of the relevant modal facts is clearly distinct from the conjunction of the corresponding level surface facts. It is also asymmetric: the relevant notion of determination is supposed to give metaphysical content to a piece of asymmetric mathematical language, namely the mathematical sense in which a smooth phase-space function is the *generator* of some specific phase-space flow. Thus, the target notion of determination is such that the 1-modal facts at some state are determined by, but do not determine, the level surface facts about $M_1$, together with the position and momentum facts at that state. The determination relation is also clearly not a causal relation: there is no obvious sense in which the 1-modal facts are caused by the level surface facts about $M_1$, together with the position and momentum facts.

Moreover, the relevant notion of determination is also distinct from necessitation and supervenience. I'll here focus on the case of supervenience with metaphysical necessity, according to which facts of type $Y$ supervene on facts of type $X$ iff it is metaphysically impossible for there to be a difference in the $Y$-facts without a difference in the $X$-facts. (The case against metaphysical necessitation is analogous.) The idea is as follows: for the 1-modal facts to be determined by the level surface facts about $M_1$, together with the position and momentum facts, is for the 1-modal facts to supervene on the level surface facts about $M_1$, together with the position and momentum facts. We have a clear grip on the supervenience of the 1-modal facts on the position and momentum facts: there can be no difference in which states are rotationally accessible without a difference in which possible state is actual. By contrast, if we hold fixed the actual state of the system, i.e. the actual position and momentum facts, then the 1-modal facts are *metaphysically necessary:* given that the actual state of $n$-particle system is thus-and-such, and given what it is for the system to be 'rigidly rotated' about the 1-axis, facts about which states are rotationally accessible from actuality could not have failed to obtain. This implies that, given the position and momentum facts, the 1-modal facts supervene with metaphysical necessity on any fact whatsoever. But it is not the case that, given the position and momentum facts, the 1-modal facts are determined by any fact whatsoever. The relevant notion of determination is therefore more fine-grained than metaphysical necessitation and supervenience: although determination implies necessitation and supervenience, it is not tantamount to these notions.[34]

This feature of the determination relation is consistent with the following observation about its fineness of grain. Whereas the mathematical proxy of the determination relation is *extensional* in the sense that every phase space function co-extensive with $f$ counts as a generator of the Hamiltonian flow of $f$, we have reason to deny that, if the level surface facts about some quantity $Q$ (together with the position and momentum facts) determine the relevant modal facts, so do the level surface facts about every quantity state-space coextensive with $Q$. Even if the total energy of the $n$-particle system is in fact state-space co-extensive with the quantity equivalent to the sum of kinetic and Newtonian gravitational energy, someone who thinks that facts about the *latter* determine the dynamical state-space curves would derive wrong claims about the dynamical evolution of the system in the metaphysical possibility in which the system contains electrostatic energy in addition to kinetic and gravitational energy. The relevant notion of determination is more fine-grained than its mathematical surrogate.

Finally, the 'generating' idiom suggests that determination is an explanatory relation. For example: when we say that a device is a *generator* of electricity, we literally mean that it *produces* electricity in the sense that there is an underlying physical process in virtue of which the generator is *causally responsible for* the electricity produced by it. This suggests that determination is an explanatory relation of a similarly productive (albeit non-causal) kind. According to this conception, when $p$ determines $q$, then $q$ may be thought of as 'produced' by $p$ in a sense that may be paraphrased as '$p$ is *responsible* for $q$' or as '*$p$ makes it the case that $q$*'.

So, the relevant notion of determination is that of an asymmetric non-causal explanatory relation between facts that implies (but is not implied by) metaphysical necessitation and supervenience.[35] These characteristics are compatible with various more specific metaphysical notions that have been discussed in the literature.[36] The project of this paper, however, does not require a commitment to any particular such notion. For our purposes, the characterization of the fairly general features of the determination relation just given is sufficient.

This account of what is conveyed about a property by the fact that it is represented by a generator might seem abstract and unintuitive. But the general kind of explanatory move should be familiar from everyday modal claims. According to the account just

[34]Note that this reasoning applies *a fortiori* to proposals which deploy narrower necessities, such as nomic necessity.

[35]Given the holding of the non-trivial identities stated by the Poisson bracket equations $\{L_i, L_j\} = \epsilon_{ijk}L_k$, there are different, equally correct ways to *state* the relevant determination claims. For example, the claim that the 1-modal facts are determined by the level surface facts about $M_1$, together with the position and momentum facts, can be stated just as accurately as the claim that the 1-modal facts are deterrmined by the level surface facts about *the quantity whose value at every state is equal to the rate of change of $M_2$ along state-space curves generated by $M_3$*, together with the position and momentum facts. In other words: the relata of the determination relation involve the relevant generating quantities, but not the definite descriptions by which these quantities can be picked out.

[36]The fineness of grain of the determination relation is suggestive of resources from hyperintensional metaphysics, such as metaphysical grounding (Fine, 2001, 2012; Rosen, 2010), essence (Fine, 1994a,b; Rosen, 2015), ontological dependence (Fine, 1995; Correia, 2008; Koslicki, 2012) or metaphysically impossible worlds (Nolan, 2013; Jago, 2014).

presented, the fact that some proposition $p$ is 1-possible is determined by the level surface facts about $M_1$, together the position and momentum facts. Now, the level surface facts about $M_1$, together with the position and momentum facts, include the fact that all rotationally accessible states agree about $M_1$, and *a fortiori* they include the fact about the *actual* value of $M_1$. Therefore, the fact that $p$ is 1-possible is partially determined, and thus partially explained by, the fact about the actual value of $M_1$. In other words, the proposed account entails that an invariant feature of actuality partially explains what is possible, in the relevant sense of 'possible'.

This is reminiscent of more familiar types of modality. When we make claims about what is possible, we always hold certain contextually relevant features of actuality fixed. For example, if you want to know whether it's possible to fit eight people in my car, you are presumably holding fixed the actual size of my car. And the features of actuality we hold fixed in making these claims often seem explanatory of the latter: the fact that it's impossible to fit eight people in my car is explained by the fact that my car has room for no more than five people.

We are now in a position to understand the definite descriptions that are expressed by Poisson brackets. The Poisson bracket $\{L_1, L_2\}$ expresses the definite description 'the quantity whose value at every state is equal to the rate of change of $M_1$ along the state-space curves determined by the level surface facts about $M_2$'; and $\{L_1, L_2\} = L_3$ expresses the non-trivial identity of the satisfier of this definite description and $M_3$.

Moreover, we can now sharpen our understanding of ordinary $SO(3)$-invariance. Let $Q$ be the quantity represented by the function $f_Q$. The fact that $Q$ is an ordinary $SO(3)$-invariant consists in the conjunction, for every axis $i$ through spatial point $x$, of the variation facts stated by $\{Q, L_i\} = 0$.

Let me pause briefly to review what we have accomplished so far. Our goal is to understand what is conveyed about a physical quantity by the fact that it is represented by a Casimir invariant under some relevant group—say, $SO(3)$. According to our strategy, the answer is that the ordinary $SO(3)$-invariance of that quantity is determined by the $SO(3)$-variation facts. The central result of this section is that we now have a precise account of the $SO(3)$-variation facts. The remaining task is to characterize the sense in which these facts *determine* the ordinary $SO(3)$-invariance of the relevant quantity. This is what I turn to now.

### 3.2 Fine-Grained Invariance

There is reason to think that the relevant notion of determination relation is the same as the notion discussed in the previous section. Like before, determination is a relation between facts that is distinct from the identity relation: the conjunction of the $SO(3)$-variation facts is clearly not identical to the ordinary $SO(3)$-invariance of the magnitude of angular momentum. The relation is arguably asymmetric: the $SO(3)$-variation facts determine, but are not determined by, the fact that the magnitude of angular momentum is an ordinary $SO(3)$-invariant. It is also non-causal: there is no obvious sense in which facts of the former type *cause* facts of the latter type. Moreover, by familiar reasoning, determination is not necessitation or supervenience. Given what it is for the $n$-particle system to be 'rigidly rotated' about a point, it is *metaphysically necessary* that magnitudes of vector quantities are $SO(3)$-invariant. Intuitively, that the magnitudes of vectors don't change under rotations follows from *what it is* to rotate a vector. This means that the ordinary $SO(3)$-invariance of any vector quantity supervenes with metaphysical necessity on, and is metaphysically necessitated by, any fact whatsoever; but the ordinary $SO(3)$-invariance of this quantity is not determined by any fact whatsoever.

The relevant determination relation also has explanatory characteristics. Intuitively, facts about how every *component* of a vector changes under a given type of transformation (such as rotations) explain facts about whether the *magnitude* of this vector is invariant under these transformations. In the present context, the fact that the *function* $L^2$ is an $SO(3)$-constant, i.e. the fact that $\{L^2, L_i\} = 0$ for every axis $i$ through $x$, is explained by the fact that equations $\{L_i, L_j\} = \epsilon_{ijk} L_k$ hold—i.e. by facts about how the components of $\vec{L}$ change under rotations. This suggests that the ordinary $SO(3)$-invariance of the magnitude of total angular momentum is explained by facts about how total angular momentum components change under rotations—i.e. by the $SO(3)$-variation facts. Determination is an explanatory relation.

In sum, the operative notion of determination is, again, that of an asymmetric non-causal explanatory relation between facts that implies (but is not implied by) metaphysical necessitation and supervenience.[37]

We are now in control of all notions involved in our proposal for the feature that is attributed to a quantity by saying that its mathematical representative is an $SO(3)$-Casimir invariant. I call this feature of the *fine-grained SO(3)-invariance* of this quantity:

> **Fine-Grained SO(3)-Invariance.** For a quantity $Q$ to be a *fine-grained SO(3)-invariant* is for the fact that $Q$ is an ordinary $SO(3)$-invariant to be determined by the $SO(3)$-variation facts.[38]

Fine-grained invariance is the notion we've been looking for. I said earlier that the ordinary invariance of a physical quantity under a symmetry group is too coarse-grained to account for the link between properties and symmetries: a quantity can be invariant under a group without being linked to that group in the relevant sense. By contrast, fine-grained invariance has exactly the right fineness of grain: the fine-grained $SO(3)$-invariants are all and only the quantities linked to $SO(3)$ in the sense identified by physicists.

[37] As before, for present purposes we can leave open which more specific metaphysical notion is best suited to play this role; see footnote 36 for some options.

[38] Note that the $SO(3)$-variation facts are closed under substitutions of co-denoting definite descriptions. For example, the variation fact stated by the equation obtained from $\{L_1, L_2\} = L_3$ by replacing $L_1$ with $\{L_2, L_3\}$, $L_2$ with $\{L_3, L_1\}$, and $L_3$ with $\{L_1, L_2\}$ is also among the $SO(3)$-variation facts. It follows that the relevant determination claims are preserved under substitutions of co-denoting definite descriptions of the sort expressed by $SO(3)$-Poisson bracket equations.

## 4 Ontic Structural Realism

Earlier, I noted that the phenomenon that motivates this paper—assertions of a link between symmetries and properties encoded in the notion of a Casimir invariant—has inspired ontic structural realists to draw substantive metaphysical conclusions about the relative metaphysical priority of structures on the one hand and physical objects and their properties on the other. The core idea is that, if the essential propeties of elementary particles can be exhaustively characterized in terms of mathematical objects that are defined in terms of symmetry groups—such as Casimir invariants—this suggests that symmetry structure is more fundamental than particles and their properties. To what extent does my account differ from ontic structuralist approaches?

To begin with, my ambitions in this paper are more modest. The goal has been merely to explain what is conveyed about a physical quantity by the fact that it is mathematically characterized by a Casimir invariant, rather than to defend a claim about the relative fundamentality of such quantities and their bearers *vis-à-vis* symmetry structure. Moreover, the present investigation is not predicated on the goal of vindicating a structuralist metaphysics. The question as to whether the notion of a fine-grained invariant is consistent with ontic structuralism strikes me as open.

I take it that this is an advantage of the present investigation. One persistent criticism of ontic structural realism has been that the operative notion of 'structure' has not been sufficiently clarified.[39] It therefore seems worth examining whether the metaphysical content of Casimir invariance can be identified without an antecedent commitment to ontic structural realism. This is what I have done in this paper.

Another major difference between my account and ontic structural realist approaches concerns the status of mathematical objects in the specification of the relevant metaphysical claims. Structural realists have been forthright in their use of explicitly metaphysical notions to capture the relationship between symmetry structure and elementary particles: extant accounts deploy the language of necessitation,[40] supervenience,[41] determinables and determinates,[42] constitution,[43] ontological dependence,[44] fundamentality,[45] metaphysical grounding,[46] and essence.[47] However, the entities to which these notions are applied are often straightforwardly mathematical. A few examples: French (2014, p. 283) proposes that "the Poincaré group is a determinable". Roberts (2011, p. 50) argues for the claim that "the existing entities described by quantum theory are organized into a hierarchy, in which a particular symmetry group occupies the top, most fundamental position." And McKenzie (2014a, pp. 21-2) entertains the hypothesis that "the Poincaré

[39] For example, (Psillos, 2001; Dorr, 2010; Wolff, 2011; Arenhart and Bueno, 2015); cf. (McKenzie, 2016).

[40] (Berenstain and Ladyman, 2012).

[41] (Ladyman and Ross, 2007; Wolff, 2011; Nounou, 2012).

[42] (French, 2014, Section 10.7-8).

[43] (Castellani, 1998).

[44] (Wolff, 2011; McKenzie, 2014b).

[45] (Ladyman, 1998; Ladyman and Ross, 2007; Roberts, 2011; McKenzie, 2014b,a; French, 2014).

[46] (French, 2014, pp. 169,283).

[47] (McKenzie, 2014b).

group [...] forms part of [an elementary particle's] essence" which implies "the dependence of relativistic particles on the Poincaré group". What these claims have in common is that the subjects of the respective metaphysical attributes are mathematical entities introduced as part of the mathematical language used to state claims about physical reality.

As I explained in the introduction, it is far from obvious what such claims amount to. At face value, the above structuralist theses have the implication that symmetry groups—mathematical objects introduced as part of the mathematical apparatus of physics—are themselves among the constituents of physical reality. If this is not the claim these authors intend, we are owed a more transparent account of the ontological content of claims involving symmetry groups, an account which clearly and rigorously distinguishes between the underlying physical claims and the mathematical language used to express them.

One important goal of this paper has been to respect this constraint, and the advantages of this approach should now be clear. We elucidated the physical content of what is perhaps one of the most important mathematical notions in elementary particle physics and thereby provided a transparent ontological basis for a symmetry-based metaphysics of modern physics. Moreover, our investigation demonstrates what we can achieve when we look beyond the mathematical apparatus in our ontological theorizing. For example, the fine-grained determinative relationship between the modal facts captured in terms of symmetry groups and the level surface facts of physical quantities is something we would likely have overlooked had we settled for the mathematical formalism of symmetry groups and their generators.

Finally, the account presented in this paper creates conceptual space for a view which makes precise sense of the suggestion that the essential properties of particles are definable in terms of symmetries.[48] According to this view, for every essential property $P$ of elementary particles, there is a continuous symmetry group $G$ such that 'being a fine-grained $G$-invariant' defines $P$.[49] Applied to the classical-mechanical context of this paper, the idea is that 'being a fine-grained $SO(3)$-invariant' defines the magnitude of angular momentum.[50] The notion of a fine-grained invariant developed in this paper therefore promises a symmetry-based metaphysics of properties. Of course, the structuralist credentials such a view can only be fully assessed once it has been developed in detail. This is a task for another day.

[48]This suggestion has been made, for example, by Kantorovich (2003), Roberts (2011), McKenzie (2014a), French (2014), and Weinberg (1987, 1993). The original idea is based on a paper by Wigner (1939).

[49]To be sure: the sense of 'definition' operative in this view must be *metaphysical* rather than nominal: that is, the relevant definitions are definitions of *properties*, not of the predicates or concepts that refer to these properties. There are various accounts of metaphysical definition: see (Fine, 1994a; Rosen, 2015) for hyperintensional accounts and (Dorr, 2016) for an account in terms of higher-order identifications.

[50]Modifications to this strategy have to be made for group actions which, unlike $SO(3)$, are linked to more than one fine-grained invariant in the relevant sense. I take up this issue in future work.

## 5 Conclusion

In this paper, I have done two things. First, I reviewed the standard mathematical account of the link between properties and symmetries by introducing the notion of a Casimir invariant. Second, I explained what is conveyed *about a property* by saying that it is represented by a Casimir invariant under some group $G$. According to the proposal developed in this paper, the answer consists in the notion of a fine-grained $G$-invariant: that is, the notion of a property whose ordinary $G$-invariance is determined by the $G$-variation facts.

I said earlier that the notion of symmetry operative in the hypothesized link between quantities and symmetries is the notion of a state-space symmetry. I also noted that this type of symmetry is not confined to the classical setting: a crucial feature of symmetries in quantum theory is that they are symmetries of Hilbert space, the mathematical space used to characterize the state spaces of quantum systems. Moreover, the theoretical framework of quantum theory contains exact analogues of the mathematical features of Hamiltonian mechanics that inspired the notion of fine-grained invariance. For example, in quantum mechanics, *self-adjoint operators* on Hilbert space play the role of Hamiltonian vector fields and *unitary* flows on Hilbert space play the role of Hamiltonian flows on phase space. The work of the symplectic form is done by a theorem due to Marshall Stone (1932) which ensures that every self-adjoint operator generates a unitary flow which preserves this operator. Finally, the role of the Poisson bracket is played by the commutator bracket on the space of self-adjoint operators. Quantum theory thus contains all structural features we need to articulate the notion of a fine-grained invariant in this framework. A detailed study of the notion of fine-grained invariance in the quantum setting should be the subject of future work.

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