On the Reduction of General Relativity to Newtonian Gravitation

1. Introduction

In physics the concept of reduction is often used to describe how features of one theory can be approximated by those of another under specific circumstances. In such circumstances physicists say the former theory reduces to the latter, and often the reduction will induce a simplification of the features in question. (By contrast, the standard terminology in philosophy is to say that the less encompassing, approximating theory reduces the more encompassing theory being approximated.) Accounts of reductive relationships aspire to generality, as broader accounts provide a more systematic understanding of the relationships between theories and which of their features are relevant under which circumstances.

Thus reduction is naturally taken to be physically explanatory. Reduction can be more explanatory in other ways as well: sometimes the simpler theory is an older, predecessor of the theory being reduced. These latter "theories emeriti" are retained for their simplicity—especially regarding prediction, and sometimes understanding and explanation—and other pragmatic virtues despite being acknowledged as incorrect.[1] Under the right circumstances, one incurs sufficiently little error in using the older theory; it is empirically adequate within desired bounds and domain of application. Reduction explains why these older, admittedly false (!) theories can still be enormously successful and, indeed, explanatory themselves.

The sort of explanation that reduction involves is similar to that which Weatherall (2011) identifies in the answer to the question of why inertial and gravitational mass are the same in Newtonian gravitation. In that case, the explanandum is a successfully applied fact about Newtonian systems. The explanation is a deductive argument invoking and comparing Newtonian

[1] The concept and terminology are adapted from Belot (2005).

theory and general relativity, showing that this fact is to be expected if the latter is true. In this case, the explanandum is rather the relation of non-identity between the predictions and explanations of two successful theories. The explanation is also a deductive argument invoking both theories, but one showing that if the reducing theory is adequate, then the theory it reduces to is adequate as well, within desired bounds and domain of application.

Despite the philosophical and scientific importance of reduction, it is astounding that so few reductive relationship in physics are understood with any detail. It is often stated that relativity theory reduces to Newtonian theory, but supposed demonstrations of this fact are almost always narrowly focused, not coming close to the level of generality to which an account of reduction aspires, which is to describe the relationship between the collections of all possibilities each theories allows. Indeed, much discussion of the Newtonian limit of relativity theory (e.g., Batterman (1995)) has focused on the "low velocity limit." For example, the relativistic formula for the magnitude of the three-momentum, p, of a particle of mass m becomes the classical formula in the limit as its speed v (as measured in some fixed frame) becomes small compared with the speed of light c:

$$\begin{aligned} p &= \frac{mv}{\sqrt{1-(v/c)^2}} \\ &= mv\left(1+\frac{1}{2}(v/c)^2+\frac{3}{8}(v/c)^4+\frac{5}{16}(v/c)^6+\cdots\right) \underset{v/c\ll 1}{\approx} mv. \end{aligned} \tag{1}$$

Similar formulas may be produced for other point quantities.

Another class of narrow demonstrations, so-called "linearized gravity," concerns gravitation sufficiently far from an isolated source. In a static, spherically symmetric (and asymptotically flat) relativistic spacetime (M, g_{ab}), one can write

$$g_{ab} = \eta_{ab} + \gamma_{ab}, \tag{2}$$

where η_{ab} is the Minkowski metric. If the components of γ_{ab} are "small" relative to a fixed global reference frame, then in terms of Minkowski coordinates (t, x, y, z),

$$g_{ab} \approx (1+2\phi)(d_a t)(d_b t) - (1-2\phi)[(d_a x)(d_b x) + (d_a y)(d_b y) + (d_a z)(d_b z)] \tag{3}$$

$$T_{ab} \approx \rho(d_a t)(d_b t) \tag{4}$$

where T_{ab} is the stress-energy tensor, and the field ϕ satisfies Poisson's equation,

$$\nabla^2\phi = 4\pi\rho. \tag{5}$$

(Here ∇^2 is the spatial Laplacian.) Accordingly, the trajectories of massive test particles far from the source will in this reference frame approximately follow trajectories expected from taking ϕ as a real Newtonian gravitational potential. In practice one often has a rough and ready handle on how to apply these approximations, but it is difficult to make all the mathematical details precise.[2]

But it is not often explicitly recognized that even the collection of all point quantity formulas, such as eq. 1, or the linearized expression for the metric (eq. 3) along with trajectories of test particles far from an isolated massive body, together constitute only a small fragment of relativity theory. In particular, they say nothing about the nature of gravitation in other circumstances, or exactly how the connection between matter, energy, and spacetime geometry differs between relativistic and classical spacetimes.[3] Insofar as one is interested in the general account of the reduction of one theory to another, these particular limit relations and series expansions cannot be understood to be a reduction of relativity theory to classical physics in any strict sense. Even the operationally minded would be interested in an account of how arbitrary relativistic observables can be approximated by their Newtonian counterparts.

The development of Post-Newtonian (PN) theory, which is very general method for applying the "slow-motion" limit to a variety of formulas like equation 1 ameliorates this concern somewhat (Poisson and Will, 2014). Its goal is to facilitate complex general relativistic calculations in terms of quantities that are (in principle) measurable by experiments, and that it typically only applies as a good approximation to a small region of a relativistic spacetime. Indeed, while it seems in many circumstances to yield helpful predictions, there is no guarantee that an arbitrary PN expansion actually converges. Part of the reason for this is the theory typically relies heavily on particular coordinate systems for these small regions that may not always have felicitous properties. In a word, while enormously useful, the PN theory

[2]Wald (1984, Ch. 4.4, p. 74) is one of the few authors who makes these issues the least bit explicit.

[3]Cf. the comments of Roger Jones paraphrased by Batterman (1995, p. 198–199n1).

by itself does not constitute a general method for explaining the success of Newtonian theory.

One way to do so—to organize relatively succinctly the relationships between arbitrary relativistic observables and their classical counterparts—is to provide a sense in which the relativistic spacetimes themselves, and fields defined on them, reduce to classical spacetimes (and their counterpart fields), as all spacetime observables are defined in these terms. This geometrical way of understanding the Newtonian limit of relativity theory has been recognized virtually since the former's beginning (Minkowski, 1952), where it was observed that as the light cones of spacetime flatten out in Minkowski spacetime, hyperboloids of constant coordinate time become hyperplanes in the limit. This geometric account has since been developed further by Ehlers (1981, 1986, 1988, 1991, 1997, 1998) and others (e.g., Künzle (1976); Malament (1986a,b)) for general, curved spacetimes.[4]

But the nature and interpretation of this limit has sat uneasily with many. The image of the widening light cones seems to suggest an interpretation in which the speed of light $c \to \infty$. In one of the first discussions of this limiting type of reduction relation in the philosophical literature, Nickles (1973) considered this interpretation and pointed out some of its conceptual problems: what is the significance of letting a constant vary, and how is such variation to be interpreted physically? Rohrlich (1989) suggested that "$c \to \infty$" can only be interpreted counterfactually—really, *counterlegally*, since it corresponds literally with a sequence of relativistic worlds whose speeds of light grow without bound. Thus interpreted, the limit only serves to connect the mathematical structure of relativistic spacetime with that of classical spacetime (Rosaler, 2018). It cannot explain the success of Newtonian physics, since such an explanation "specifies what quantity is to be neglected relative to what other quantity" (Rohrlich, 1989, p. 1165) in worlds with the same laws to determine the relative accuracy of classical formulas as approximations to the outcomes of observations.

So, we are left with a conundrum: existing approaches to the reduction of general relativity to Newtonian gravitation are explanatory, or they are general; moreover, there may be some concerns about the mathematical rigor of some approaches. The received view about these three approaches is sum-

[4]It is difficult to square these detailed developments with the claim by Rosaler (2015, 2018) that geometrical approaches are vague and not mathematically well-defined.

	Single Formulas	Weak Field	Geometric
Approach	$v/c \approx 0$	$\lvert\gamma_{ab}\rvert \ll 1$	$c \to \infty$
Explanatory?	$\surd$	$\surd$ (?)	$\times$
General?	$\times$	$\times$	$\surd$
Rigorous?	$\surd$	$\times$ (?)	$\surd$

Table 1: The received view about three different approaches to the reduction of general relativity to Newtonian gravitation. The question marks under the weak field approach are supposed to indicate that its explanatory power might be questioned to the extent that (and because) its rigor might be questioned.

marized in table 1.

The purpose of this essay is to develop and provide an interpretation of the geometrical limiting account that could in fact meet the explanatory demand required of it. It is thus both perfectly general and explanatory.[5] To explicate how it works, I first present a unified framework for classical and relativistic spacetimes in §2. The models of each are instantiations of a more general "frame theory" that makes explicit the conceptual and technical continuity between the two. To define the limit of a sequence of spacetimes, however, one needs more structure than just the collection of all spacetime models. As I point out in §3, one way to obtain this structure is by putting a topology on this collection. The key interpretative move, as explicated in §4, exploits the freedom of each spacetime to represent many different physical situations through the representation of different physical magnitudes.[6]

To illustrate this move I describe three classes of examples, involving Minkowski, Schwarzschild, and cosmological (FLRW) spacetimes, that I also show have Newtonian limits in the sense defined here, with respect to a certain topology. (It still remains an open question—though I conjecture it to be true—whether every Newtonian spacetime is an appropriate limit of relativistic spacetimes.)

The topology, though, can make a significant difference to the evaluation

[5]In fact, one can use the methods developed here to describe counterlegal limits—where the speed of light does grow—and even hybrid legal/counterlegal limits of a sort, but that shall not be my focus here.

[6]The representation of different physical magnitudes by the same model is one of the ways to underwrite a general rejection of the thesis that a single model represents at most a single possible state of affairs, in general relativity in particular (Fletcher, forthcoming).

of a potential reduction. In particular, whether the Newtonian limit of a particular sequence of spacetimes exists at all can depend on it, and it is not determined automatically from the spacetime theories themselves. I indicate in §5 that certain topologies correspond with a certain the classes of observables that one demands must converge in order for a sequence of spacetimes to converge. Requiring that more observables converge in the limit leads in general to more stringent convergence criteria, thus a finer topology (i.e., with more open sets). In light of this, I argue for a slightly more stringent criterion than has been used by other authors so that observables depending on compact sets of spacetime, such as the proper times along (bounded segments of) timelike curves, also converge. Finally, in §6, I summarize what topical and methodological conclusions I think can be drawn about these results for gravitation (including PN theory) and philosophy of science, and indicate how they might be applied more generally to other reduction relations of a similar limiting type.[7]

2. Ehlers's Frame Theory

The conceptual unification of relativistic and classical spacetimes that the frame theory affords requires some preliminaries. In particular, one must recast the empirical content of Newtonian gravitation, a theory of flat spacetime with a gravitational potential, in the language of Newton-Cartan theory, which describes gravitation through a curved connection.[8] Both formulations of Newtonian gravitation share the following structure: a quintuple $(M, t_{ab}, s^{ab}, \nabla, T^{ab})$, where M is a four-dimensional smooth manifold of (pos-

[7]One might wonder whether the limit falls under the Nagelian theory of reduction (Nagel, 1961), according to which a reduction is a derivation of the laws of one theory from another. It is not my intent to engage at length with this theory, but its applicability here turns on whether one takes the existence of a mathematical limit of a theory's models to be a (component of a) derivation. Although Nagel (1970) later considered allowing a broader sense of derivation than just logical deduction, and others (e.g., Rohrlich (1988)) have suggested that limits provide a way of logically deducing one mathematical framework from another, a detailed account of limits as derivations is still not yet forthcoming.

[8]For more details, see Malament (2012, Ch. 4), whose abstract index notation I adopt throughout. Ehlers (1998) argues that it should be called the Cartan-Friedrichs theory, since Cartan (1923, 1924) and Friedrichs (1927) bear most responsibility for formulating it, but I will follow the standard terminology here.

sible) coincidence events;[9] t_{ab} and s^{ab}, which are called the *temporal* and *spatial metrics*, respectively, are smooth symmetric tensor fields on M with respective signatures $(+,0,0,0)$ and $(0,+,+,+)$;[10] ∇ is a torsion-free derivative operator compatible with t_{ab} and s^{ab}, i.e., such that $\nabla_a t_{bc} = \mathbf{0}$ and $\nabla_a s^{bc} = \mathbf{0}$; and T^{ab}, called the *stress-energy tensor*, is a smooth symmetric field on M representing the energy and momentum of various matter fields. The latter contracted with the temporal metric defines the *mass density field* $\rho = T^{ab}t_{ab}$. Moreover, one requires t_{ab} and s^{ab} to be orthogonal in the sense that $t_{ab}s^{bc} = \mathbf{0}$.

The temporal and spatial metrics determine, respectively, the (proper) times elapsed along curves representing the possible paths of massive particles and the (proper) lengths of "spatial" paths. More precisely, call some tangent vector $\xi^a \in T_pM$ *timelike* when $t_{ab}\xi^a\xi^b > 0$ and *spacelike* when there exists a covector ξ_a at p satisfying $s^{ab}\xi_a\xi_b > 0$ and $s^{ab}\xi_a = \xi^b$.

Most treatments of spacetime geometry do not go into detail about the relations between dimensional quantities and the numerical values produced in calculations, but here it is important to be explicit about them. Given a fixed set of units, there is a freedom in choosing which numerical values for lengths and times represent one such unit in terms of which timelike and spacelike curves are parameterized. The standard convention is to parameterize them so that their tangent vectors are normalized with magnitude 1, but in what follows it will be helpful to relax this. Tangent vectors of curves remain normalized, but their magnitude may not be 1.

Specifically, parameterize every timelike curve—i.e., every curve whose tangent vector ξ^a is always timelike—so that $\tau = (t_{ab}\xi^a\xi^b)^{1/2}$ at each point of the curve is constant, hence represents a unit of time. Quantities with the dimension of time are then expressed in multiples of this magnitude. For example, suppose the chosen temporal unit is the second. If $\tau = 3$, then a timelike curve of length 6 would represent 2 seconds. Thus, given any timelike vector ζ^a at a point and a parameterization of timelike curves determining τ, the vector's *temporal magnitude* is $\|\zeta^a\| = \tau^{-1}(t_{ab}\zeta^a\zeta^b)^{1/2}$. Thus if ζ^a is the tangent vector to a timelike curve, it always has temporal magnitude 1.

[9]Throughout I shall also assume that all manifolds are connected, Hausdorff, and paracompact.

[10]This notation follows Ehlers (1997, 1998), in contrast to the common practice of using h for the spatial metric. One motivation is that it has alphabetically adjacent letters representing metrics both non-degererate (g and h) and degenerate (s and t).

Similarly, one parameterizes every spacelike curve—i.e., every curve whose tangent vector ξ^a is always spacelike—so that the quantity $\sigma = (s^{ab}\xi_a\xi_b)^{1/2}$ at each point of the curve is constant, hence represents a unit of distance. (Proposition 4.1.1 of Malament (2012, p. 255) guarantees that the choice of covector ξ_a in this equation makes no difference.) Thus, given any spacelike vector ζ^a at a point and a parameterization of spacelike curves determining σ, the vector's *spatial magnitude* is $\|\zeta^a\| = \sigma^{-1}(s^{ab}\zeta_a\zeta_b)^{1/2}$. Thus if ζ^a is the tangent vector to a spacelike curve, it always has spatial magnitude 1.

These dimensional considerations extend to the computation of proper lengths of timelike and spacelike curves. The *temporal length* of a timelike curve $\gamma : I \to M$ with tangent vector ξ^a is given by $\int_I \|\xi^a\| ds = \tau^{-1} \int_I (t_{ab}\xi^a\xi^b)^{1/2} ds$. Similarly, the *spatial length* of a spacelike curve γ is $\int_I \|\xi^a\| ds = \sigma^{-1} \int_I (s^{ab}\xi_a\xi_b)^{1/2} ds$. Thus while the integrals in the formulas for both temporal and spatial length are invariant under reparameterization of γ, a change in τ or σ does change their numerical value. Although this extra flexibility in representing dimensional quantities may seem like a gratuitous complication, it will be essential in describing the interpretation of the Newtonian limit in §4.

So far the definitions and structures described apply to models of both standard Newtonian gravitation and Newton-Cartan theory. Models of the former require in addition that ∇ be flat and postulate a further smooth scalar field, the gravitational potential ϕ, that satisfies Poisson's equation, $s^{ab}\nabla_a\nabla_b\phi = 4\pi\rho$.[11] The potential determines the gravitational force incurred by a test particle with mass m to be $ms^{ab}\nabla_b\phi$. By contrast, instead of having a gravitational potential, Newton-Cartan theory allows ∇ to be curved, and the trajectories of massive particles in a Newton-Cartan spacetime follow geodesics according to the curvature determined by

$$R_{ab} = 4\pi\rho t_{ab}, \tag{6}$$

the "geometrized" Poisson's equation.[12] There is nevertheless a systematic relationship between the former and a subset of models of the latter, captured in part by the following proposition adapted from Malament (2012, Prop. 4.2.1) and originally due to (Trautman, 1965, Sect. 5.5).

[11] Cf. eq. 3. I have chosen units for mass so that numerically Newton's gravitational constant $G = 1$.

[12] See the following footnote for the justification of this name.

Proposition 2.1. *Let* $(M, t_{ab}, s^{ab}, \nabla, T^{ab})$ *be a classical spacetime with* ∇ *flat, and let* ϕ *be a smooth scalar field satisfying Poisson's equation,* $s^{ab}\nabla_a\nabla_b\phi = 4\pi\rho$*, where* $\rho = T^{ab}t_{ab}$*. Then there is a unique derivative operator* ∇' *such that:*

1. $(M, t_{ab}, s^{ab}, \nabla', T^{ab})$ *is a classical (Newton-Cartan) spacetime;*
2. *for all timelike curves* γ *in* M*,* $\xi^n\nabla'_n\xi^a = \mathbf{0}$ *iff* $\xi^n\nabla_n\xi^a = s^{ab}\nabla_b\phi$*, where* ξ^a *is the tangent vector field to* γ*; and*
3. *the Ricci curvature* R'_{ab} *associated with* ∇' *satisfies eq. 6, the "geometrized" Poisson's equation, i.e.,* $R'_{ab} = 4\pi\rho t_{ab}$*.*[13]

The biconditional second on the list states that a timelike curve is a geodesic according to ∇'—i.e., its acceleration vanishes—just in case its acceleration according to ∇ is given by the acceleration due to gravity. Thus every classical spacetime with a gravitational potential can be "geometrized" to form an empirically equivalent Newton-Cartan spacetime. Under certain conditions the converse holds as well,[14] although this "de-geometrization" is unique only up to a gauge transformation of the potential ϕ (as one might expect). Thus there is a robust sense in which a subclass of models of Newton-Cartan theory is equivalent to the models of Newtonian gravitation (Weatherall, 2016), so in what follows it will suffice to consider the reduction of models of general relativity to models of Newton-Cartan theory.

Models of the frame theory are very similar to models of classical spacetimes. They consist too in quintuples $(M, t_{ab}, s^{ab}, \nabla, T^{ab})$, where M is a four-dimensional smooth manifold, t_{ab} and s^{ab} are the symmetric temporal and spatial metrics, ∇ is a torsion-free derivative operator compatible with the metrics, and T^{ab} is the symmetric stress-energy tensor.[15] However, one does not impose a signature on the metrics or require that they be orthogonal. Instead, one only requires that $t_{ab}s^{bc} = -\kappa\delta^c_a$ for some real constant κ, called the model's *causality constant*, and where δ^c_a is the Kronecker delta.

[13] One can show that $R'_{ab} = (s^{mn}\nabla_m\nabla_n\phi)t_{ab}$, so the "geometrized" Poisson's equation holds iff Poisson's equation holds.

[14] One can always de-geometrize a model of Newton-Cartan theory for which $R^{[a\ \ c]}{}_{(b\ \ d)} = 0$, but the resulting gravitational force may not be expressible in terms of a gravitational potential. For the latter to be possible, there must be no "global rotation" in a sense that can be made precise. (See Malament (2012, Ch. 4.5).)

[15] Ehlers also included a cosmological constant Λ, which I've assumed to vanish for simplicity since including it does not introduce any new conceptual subtleties.

Whether the causality constant is positive or zero determines the qualitative causal structure of a frame theory model. A model of Newton-Cartan theory, for example, counts as a model of the frame theory with causality constant $\kappa = 0$.[16]

Models of general relativity are models of the frame theory, too. To see why, recall that a relativistic spacetime is a pair (M, g_{ab}), where M is (again) a four-dimensional smooth manifold and g_{ab} is a smooth Lorentzian metric on M, i.e., a symmetric invertible tensor field with signature $(+,-,-,-)$. The temporal metric t_{ab} for a relativistic spacetime is just g_{ab}, while the spatial metric s^{ab} is given by $-\kappa g^{ab}$, and these define temporal and spatial lengths in the same way. The causality constant is typically fixed as $\kappa = c^{-2}$, where c is the speed of light, and ∇ is the Levi-Civita derivative operator compatible with g_{ab}. If one then defines the stress-energy tensor as

$$T^{ab} = \frac{1}{8\pi}(g^{am}g^{bn} - \frac{1}{2}g^{ab}g^{mn})R_{mn} = \frac{1}{8\pi\kappa^2}(s^{am}s^{bn} - \frac{1}{2}s^{ab}s^{mn})R_{mn}, \tag{7}$$

where R_{mn} is the Ricci curvature associated with ∇, then Einstein's equation, $R_{ab} = 8\pi(T_{ab} - \frac{1}{2}Tg_{ab})$, is satisfied, where $T_{ab} = T^{mn}t_{am}t_{bn}$ and $T = T^{mn}t_{mn}$. Although Einstein's equation and eq. 6, the "geometrized" Poisson's equation, appear to be different, the latter is actually a special case of Einstein's equation understood in a suitably generalized sense for any model of the frame theory. Note that $\rho = T$, so using the fact that for a model of Newton-Cartan theory $T^{mn}t_{am}t_{bn} = T^{mn}t_{mn}t_{ab}$,[17] one can rewrite eq. 6 as

$$\begin{aligned} R_{ab} = 4\pi\rho t_{ab} &= 8\pi(T^{mn}t_{mn}t_{ab} - \frac{1}{2}T^{mn}t_{mn}t_{ab}) \\ &= 8\pi(T^{mn}t_{am}t_{bn} - \frac{1}{2}T^{mn}t_{mn}t_{ab}) = 8\pi(T_{ab} - \frac{1}{2}Tt_{ab}). \end{aligned}$$

The last expression is formally identical to Einstein's equation when $\kappa > 0$ and the temporal metric is the Lorentz metric, i.e., when $t_{ab} = g_{ab}$. All these relationships are summarized in table 2.

Although the representation in the frame theory of classical and relativistic spacetimes requires nothing further, introducing a certain redundancy in

[16]This formulation is slightly more general than that of Ehlers, who introduces some restrictions to make the models of the general frame theory more easily interpretable in physical terms at the expense of slightly further technical complication.

[17]Locally there is always a covector field t_a such that $t_{ab} = t_a t_b$ (Malament, 2012, p. 250), so at any point, $t_{am}t_{bn} = t_a t_m t_b t_n = t_{mn}t_{ab}$.

		Specialization	
	Frame theory	Newton-Cartan	General Relativity
manifold	M	M	M
temp. metric	symmetric t_{ab}	t_{ab} has sig. $(+,0,0,0)$	$t_{ab} = g_{ab}$ has sig. $(+,-,-,-)$
spatial metric	symmetric s^{ab}	s^{ab} has sig. $(0,+,+,+)$	$s^{ab} = -\kappa g^{ab}$
derivative	torsion-free ∇	torsion-free ∇	torsion-free ∇
compatibility	$\nabla_a t_{bc} = \mathbf{0}$ $\nabla_a s^{bc} = \mathbf{0}$	$\nabla_a t_{bc} = \mathbf{0}$ $\nabla_a s^{bc} = \mathbf{0}$	$\nabla_a g_{bc} = \mathbf{0}$
causality	$t_{ab}s^{bc} = -\kappa\delta^c_a$	$\kappa = 0$	$\kappa = c^{-2} > 0$
stress-energy	T^{ab}	T^{ab}	$T^{ab} = \frac{1}{8\pi\kappa^2}(s^{am}s^{bn} - \frac{1}{2}s^{ab}s^{mn})R_{mn}$
field equation		$R_{ab} = 8\pi(T_{ab} - \frac{1}{2}Tt_{ab})$	$R_{ab} = 8\pi(T_{ab} - \frac{1}{2}Tt_{ab})$

Table 2: The components and conditions thereon of models of the frame theory, and the conditions under which these models specialize to those of Newton-Cartan theory and general relativity. The frame theory is not listed as having a field equation because it is not intended as a spacetime theory in the usual sense, but if one wished, it would do no harm to include the same field equation as for Newton-Cartan theory and general relativity.

the latter will prove enormously useful. As described so far, the causality constant for all relativistic spacetimes (M, g_{ab}) is a single fixed number. Now we let this number vary, hence consider the triples (M, g_{ab}, κ), with $\kappa \in (0, \infty)$, where all of the geometrical structure is otherwise the same. The advantage of this factor is that, combined with an appropriate choice of τ, σ, one is able to interpret a family of relativistic spacetimes $(M, \overset{\lambda}{g}_{ab}, \overset{\lambda}{\kappa})$ with a Newtonian limit as containing only models agreeing on the physical speed of light.

To see this, it is illuminating to compare the function of κ with constant conformal factors. Multiplying the Lorentz metric of a relativistic spacetime with such a factor is commonly interpreted as a "change of units." This can be understood in the terminology introduced so far by noting that a parameterization of timelike and spacelike curves with tangent vectors ξ^a determine $\tau = (\xi^a \xi^b t_{ab})^{1/2}$ and $\sigma = (\xi_a \xi_b s^{ab})^{1/2}$, respectively, so (keeping the parameterization of curves the same) the map $g_{ab} \mapsto \Omega^2 g_{ab}$ for some $\Omega > 0$ induces the map $\tau \mapsto \tau\Omega$ and $\sigma \mapsto \sigma/\Omega$. (Recall that $t_{ab} = g_{ab}$ and $s^{ab} = -\kappa g^{ab}$, so $g_{ab} \mapsto \Omega^2 g_{ab}$ induces the map $s^{ab} \mapsto \Omega^{-2} s^{ab}$.) If $\Omega = 1000$, then a timelike curve that previously represented 1s would, after the mapping, represent .001s = 1ms.

The role of allowing for different values of the causality constant κ is similar to that for constant conformal factors, but with more flexibility. Application of constant conformal factors transforms both τ and σ, but varying the causality constant κ only transforms σ. In particular, the map $\kappa \mapsto \omega^2 \kappa$, with $\omega > 0$, induces $\tau \mapsto \tau$ and $\sigma \mapsto \omega\sigma$. In other words, as the causality constant varies, the values of spatial magnitudes change but those of temporal magnitudes stay the same. For example, if $\omega = 1000$, then the spatial length of a spacelike curve that was originally 1m would become .001m = 1mm. With both constant conformal factors and varying κ one can thus transform τ and σ independently, although it will only be necessary to vary κ in considering the Newtonian limit.

3. The Newtonian Limit

Before developing how limits can be interpreted in this approach, one must first determine what, mathematically, a limit of a family of spacetimes is supposed to be in the first place. One way to formalize this is to put a topology on the models of the frame theory, or at least the subclass of those models that represents relativistic and classical spacetimes. In particular, if O_0 and $\{O_\lambda\}_{\lambda \in (0,1]}$ are models of the frame theory, then $\lim_{\lambda \to 0} O_\lambda = O_0$ if and only if for any open neighborhood N of O_0, there is some λ_0 such that $O_\lambda \in N$ for all $\lambda < \lambda_0$. Because these models consist in quintuples of the form $(M, t_{ab}, s^{ab}, \nabla, T^{ab})$, it is natural to consider product topologies on them, i.e., topologies whose open sets consist of the Cartesian products of the open sets of the topologies on the temporal and spatial metrics, derivative operators, and stress-energy tensors, and then restrict to the subspace of those objects that satisfy the constraints of the frame theory—namely, the causality and compatibility conditions and Einstein's equation—and its natural subspace topology.

Although Ehlers was not completely explicit about the topology for the models of the frame theory, preferring to describe the Newtonian limit directly as a kind of pointwise convergence, one can reconstruct in topological terms the particular convergence condition he (and others) used in discussing the Newtonian limit of spacetimes. To do so requires some preliminaries. First, let h_{ab} be some smooth Riemannian metric on M, and define the "distance" function between the k^{th} partial derivatives of two symmetric tensors t_{ab}, t'_{ab}, relative to h_{ab} and at each point of M, as the scalar field

$$d(t,t';h,k) = \begin{cases} [h^{ru}h^{sv}(t_{rs}-t'_{rs})(t_{uv}-t'_{uv})]^{1/2}, & k=0, \\ [h^{a_1b_1}\cdots h^{a_kb_k}h^{ru}h^{sv} \\ \quad \otimes \bar{\nabla}_{a_1}\cdots\bar{\nabla}_{a_k}(t_{rs}-t'_{rs})\bar{\nabla}_{b_1}\cdots\bar{\nabla}_{b_k}(t_{uv}-t'_{uv})]^{1/2}, & k>0, \end{cases} \tag{8}$$

where $\otimes$ is the tensor product and $\bar{\nabla}$ is the Levi-Civita derivative operator compatible with h_{ab}. (I will omit the abstract indices of tensors when they appear as arguments so as not to clutter the notation needlessly.) A particular choice of h_{ab} provides a standard of comparison for the components of the k^{th} order derivatives of t_{ab} and t'_{ab} via $d(t,t';h,k)$, which is in fact a true distance function (i.e., a metric) at each point of M.

One defines the distance function similarly for symmetric contravariant tensors, replacing h^{ab} with h_{ab}. (One can add requisite copies of h_{ab} and its inverse to extend the definition of d to any pair of tensor fields of the same index.) The case of derivative operators requires only a bit more attention. One can relate any derivative operator ∇ on M to $\bar{\nabla}$ through a symmetric connection field C^a_{bc}, denoted $\nabla = (\bar{\nabla}, C^a_{bc})$.[18] Then one can define the k^{th}-

[18] $\nabla = (\bar{\nabla}, C^a_{bc})$ means that for every smooth tensor field $\tau^{a_1\ldots a_r}_{b_1\ldots b_s}$ on M, $(\nabla_m - \bar{\nabla}_m)\tau^{a_1\ldots a_r}_{b_1\ldots b_s} = \tau^{a_1\ldots a_r}_{nb_2\ldots b_s}C^n_{mb_1} + \cdots + \tau^{a_1\ldots a_r}_{b_1\ldots b_{s-1}n}C^n_{mb_s} - \tau^{na_2\ldots a_r}_{b_1\ldots b_s}C^{a_1}_{mn} - \cdots - \tau^{a_1\ldots a_{r-1}n}_{b_1\ldots b_s}C^{a_r}_{mn}$.

order distance between two derivative operators ∇ and ∇' as $d(\nabla, \nabla'; h, k) = d(C, C'; h, k)$, the k^{th}-order distance between their connection fields, using $2k+2$ copies of h^{ab} and $k+1$ copies of h_{ab}.

Finally, consider the set S of tensor fields on M that one wishes to topologize—the cases of interest will be the temporal metrics, the spatial metrics, the stress-energy tensors, and the derivative operators (or rather their connection fields) associated with classical and relativistic spacetimes. Then sets of the form

$$B_k(t, \epsilon; h, p) = \{t' \in S : d(t, t'; h, 0)_{|p} < \epsilon, \ldots, d(t, t'; h, k)_{|p} < \epsilon\} \qquad (9)$$

constitute a basis for the C^k *point-open* topology on S, where t ranges over tensor fields in S, ϵ ranges over all positive constants,[19] and p ranges over all points of M. (One can show that any single choice of h_{ab} will do, as ranging over Riemannian metrics adds no new open sets.) One can view these basis elements as generalizations of the ϵ-balls familiar to metric spaces. And given finitely many spaces of tensor fields with their respective point-open topologies, the product point-open topology on the Cartesian product of those spaces is just the topology whose open sets are Cartesian products of the open sets of the respective topologies.

This particular description of the point-open topologies with an auxiliary Riemannian metric h_{ab} will make plain their similarities with another class of topologies, the compact-open topologies, which I consider in §5. But on its own one might wonder if introducing h_{ab} is really necessary. The following proposition shows in a sense that it is not and exhibits the connection between the point-open topologies and the notion of pointwise convergence used in the literature. (One can find an explicit formulation of this notion in Malament (1986b, p. 192).)

Proposition 3.1. *A family of tensor fields $\overset{\lambda}{\phi}{}^a_{bc}$ on a manifold M, with $\lambda \in (0, a)$ for some $a > 0$, converges to a tensor field ϕ^a_{bc} as $\lambda \to 0$ in the C^k point-open topology iff for all points $p \in M$, $\lim_{\lambda \to 0}(\overset{0}{\psi}{}^{bc}_a \overset{\lambda}{\phi}{}^a_{bc})_{|p} = (\overset{0}{\psi}{}^{bc}_a \phi^a_{bc})_{|p}$ for all tensors $\overset{0}{\psi}{}^{bc}_a$ at p and, for all positive $i \leq k$, $\lim_{\lambda \to 0}(\overset{i}{\psi}{}^{bcd_1 \ldots d_i}_a \nabla_{d_1} \cdots \nabla_{d_i} \overset{\lambda}{\phi}{}^a_{bc})_{|p} =$*

[19] One could choose a different ϵ for each derivative order, but the resulting basis generates the same topology.

$(\overset{i}{\psi}{}_a^{bcd_1 \ldots d_i} \nabla_{d_1} \cdots \nabla_{d_i} \phi^a_{bc})_{|p}$ *for all tensors* $\overset{i}{\psi}{}_a^{bcd_1 \ldots d_i}$ *at* p.[20]

(Unless otherwise stated, proofs of all propositions may be found in the appendix.) Analogous results hold for tensor fields of other index structures. The proposition states that a sequence of tensor fields $\overset{\lambda}{\phi}$ on M converges to a tensor field ϕ in the C^k point-open topology just in case all the sequences of real numbers formed by contracting the tensor field and its derivatives up to order k with any other tensor field ψ at a point converge, for all points in M. In other words, the point-open topology defines a notion of pointwise convergence.

With that definition in place, we are ready to state Ehlers's definition of a Newtonian limit of a family of relativistic spacetimes:

Newtonian Limit (Ehlers) Let $(M, \overset{\lambda}{t}_{ab}, \overset{\lambda}{s}{}^{ab}, \overset{\lambda}{\nabla}, \overset{\lambda}{T}{}^{ab})$, with $\lambda \in (0, a)$ for some $a > 0$, be a one-parameter family of models of general relativity that share the same underlying manifold M. Then $(M, t_{ab}, s^{ab}, \nabla, T^{ab})$ is said to be a Newtonian limit of the family when it is a model of Newton-Cartan theory and $\lim_{\lambda \to 0}(\overset{\lambda}{t}_{ab}, \overset{\lambda}{s}{}^{ab}, \overset{\lambda}{\nabla}, \overset{\lambda}{T}{}^{ab}) = (t_{ab}, s^{ab}, \nabla, T^{ab})$ in the C^2 point-open product topology.[21]

In practice, to prove that a family of relativistic spacetimes has a Newtonian limit, one usually just needs show that $\lim_{\lambda \to 0}(\overset{\lambda}{t}_{ab}, \overset{\lambda}{s}{}^{ab}, \overset{\lambda}{T}{}^{ab}) = (t_{ab}, s^{ab}, T^{ab})$ since a theorem of Malament (1986b, p. 194) guarantees that, if the family is at least twice differentiable in λ, convergence of the temporal and spatial metrics entails convergence of their associated derivative operators,[22]

[20]Topological spaces that are completely determined by their convergent sequences are called *sequential*. I do not know if the point-open topologies are sequential; if they are not, then there are other topologies that would also describe the convergence condition used in the literature. But in any case, the point-open topologies are clearly *sufficient* to do so, allowing one to clarify the significance of the relevant convergence condition.

[21]One actually only requires C^1 point-open convergence of derivative operators and C^0 point-open convergence of stress-energy, since the definition of the associated connection fields for the former already involves first-order derivatives and the definition of the stress-energy (eq. 7) involves the Ricci tensor, hence second-order derivatives.

[22]The same theorem ensures that $R^{[a\ \ c]}{}_{(b\ \ d)} = 0$ also holds for the Newtonian limit, which is required for the limit Newton-Cartan model to be equivalent to a model of Newtonian gravitation. (See fn. 14.)

and convergence of the stress-energy then ensures that the "geometrized" Poisson's equation (6) holds in the limit (Malament, 1986b, p. 197).[23] The convergence of each of these three fields must be checked, for the convergence of one does not in general imply the convergence of any others.

Now, since there are infinitely many topologies one can place on the collections of tensor fields on a fixed manifold and good reason to believe that there is no canonical topology on these collections (Fletcher, 2016), one may wonder why the C^2 point-open topology is appropriate here. The requirement of $k = 2$ reflects the fact that one needs the Riemann curvature tensor of a spacetime, which is defined in terms of twice repeated covariant differentiation, to converge in order to guarantee that the geometrized Poisson's equation will hold in the limit. But one may still question further why the *point-open* topology (of some flavor or other) is appropriate.

I return to this important question in §5, but for now take the Newtonian limit condition as given. Let us consider the mathematics of the Newtonian limit through the simple example of a family $\overset{\lambda}{\eta}_{ab}$ of Minkowskian spacetimes with causality constant $\overset{\lambda}{\kappa} = \lambda$ that has Galilean spacetime—a classical spacetime with flat derivative operator and vanishing stress-energy—as its Newtonian limit. Using standard global coordinate fields t, x, y, z, the temporal and spatial metrics of this family may be written as

$$\overset{\lambda}{t}_{ab} = \overset{\lambda}{\eta}_{ab} = (d_a t)(d_b t) - \lambda(d_a x)(d_b x) - \lambda(d_a y)(d_b y) - \lambda(d_a z)(d_b z), \tag{10}$$

$$\begin{aligned} \overset{\lambda}{s}{}^{ab} &= -\lambda \overset{\lambda}{\eta}{}^{ab} \\ &= -\lambda \left(\frac{\partial}{\partial t}\right)^a \left(\frac{\partial}{\partial t}\right)^b \\ &+ \left(\frac{\partial}{\partial x}\right)^a \left(\frac{\partial}{\partial x}\right)^b + \left(\frac{\partial}{\partial y}\right)^a \left(\frac{\partial}{\partial y}\right)^b + \left(\frac{\partial}{\partial z}\right)^a \left(\frac{\partial}{\partial z}\right)^b, \end{aligned} \tag{11}$$

while the temporal and spatial metrics of Galilean spacetime may be written

[23]Technically, Malament's results posit that the classical limit spacetime is temporally orientable, but this assumption can be relaxed without consequence.

as

$$t_{ab} = (d_a t)(d_b t), \tag{12}$$

$$s^{ab} = \left(\frac{\partial}{\partial x}\right)^a \left(\frac{\partial}{\partial x}\right)^b + \left(\frac{\partial}{\partial y}\right)^a \left(\frac{\partial}{\partial y}\right)^b + \left(\frac{\partial}{\partial z}\right)^a \left(\frac{\partial}{\partial z}\right)^b. \tag{13}$$

The stress-energy $\overset{\lambda}{T}{}^{ab}$ vanishes for all λ. A straightforward calculation shows that for every $\epsilon > 0$, Riemannian h, $p \in M$ and $k \in \{0, 1, 2\}$, there is a sufficiently small $\lambda > 0$ such that $d(t, \overset{\lambda}{t}; h, k) < \epsilon$, i.e., $\lim_{\lambda \to 0} \overset{\lambda}{t}_{ab} = t_{ab}$ in the C^2 point-open topology; similarly, $\lim_{\lambda \to 0} \overset{\lambda}{s}{}^{ab} = s^{ab}$ in this topology, verifying that Galilean spacetime is in fact the Newtonian limit of the Minkowskian family (as the derivative operators are flat for all λ).

This does not, of course, show that general relativity as a whole reduces to Newtonian gravitation as a whole, which would require attention to whether each Newtonian spacetime is the limit of a sequence or family of relativistic spacetimes. The point here is rather that the definition of the Newtonian Limit provides a precise sense in which the question of reduction can be answered definitively, and that there are particular examples (here and in §4.1–4.3) where that answer is affirmative.

4. Interpretation of the Newtonian Limit

As mathematical objects, the models of the frame theory are completely well-defined, and it is a mathematical matter whether a particular family, parameterized by λ, has a Newtonian limit. According to proposition 3.1, the existence of such a limit as $\lambda \to 0$ means that, given some $\epsilon > 0$ and some finite set of spacetime points, the values in any basis of the components of the temporal and spatial metrics, connection, and stress-energy (and their partial derivatives to second order) of members of the family can be approximated at those points within ϵ by those of the Newtonian limit spacetime for sufficiently small λ.

One can thus understand the Newtonian limit through its connection with the observables of the frame theory (or at least of its specialization to GR and Newton-Cartan theory). Now, there is some controversy regarding just what quantities count as observables in spacetime theories. Some of it arises in the context of the constrained Hamiltonian formalism (e.g., Rovelli (1991)) and so is exogenous to my concerns here. There is also the difficulty, by no means unique to these theories, that evenly remotely realistic representations of actual measurements with experimental apparatuses are too complicated to model in detail. Thus any tractable discussion of observables must proceed at some level of abstraction. In particular, I shall assume that one can represent observables with scalar fields that are definable from the temporal and spatial metrics, the connection, and the stress-energy (and their derivatives), along with tetrad fields associated with (the worldlines of) observers. These will be (continuous functions of) the kinds of quantities considered in proposition 3.1.[24] Elliptically, one can then speak of tensor fields as observables in the sense that their collections of components, relative to some tetrad field, are observable scalar fields. For instance, point observables, such as the velocities of particles, may be idealized representations of observations carried out at several spacetime events which nonetheless result in the measurement of a property attributable to a single event. This is a standard, if often implicit, assumption in theoretical treatments of general relativity.[25]

Note that I only take this to be a necessary condition for observability; some tensor fields so definable, e.g., very high order derivatives of the Riemann curvature, may exceed the boundaries of idealization reasonable for the inquiry at hand. But considering a slightly broad class of tensor fields will nevertheless serve the present needs, for proposition 3.1 ensures that any observables falling within such a class and defined at finitely many points will be well-approximated in the limit.

Recall now that the outcome of an observation is typically a *dimensional* quantity, i.e., one involving time, length, etc. Hence in determining the convergence of observables one must consider the parameterizations of curves, insofar as they define τ and σ, which make the numerical values of observables physically meaningful. The key is to choose them to vary with λ in such a way

[24] Note that there will be infinitely many such tetrads: their only constraint is that one of their components is timelike. There is in particular no constraint on their components' normalization. If the tetrads considered can vary only at finitely many points (as for the point-open topologies considered in section 3) or continuously across compact regions (as in the compact-open topologies introduced in section 5), whether they are normalized makes no difference to the topologies generated. However, normalization does make a difference for the open topologies (introduced in section 5), and is one reason why they have some strange properties (Fletcher, 2018).

[25] See, for instance, (Wald, 1984, Ch. 4.2) and (Malament, 2012, Ch. 2.4). That said, more could be said to justify this standard idealization, which I leave for future work; I simply adopt it here.

as to make the physical speed of light in each model of the family $(M, \overset{\lambda}{g}_{ab}, \overset{\lambda}{\kappa})$ the same. A convenient such choice is $\tau(\lambda) = 1$ and $\sigma(\lambda) = \overset{\lambda}{\kappa}^{-1/2}$.[26] In other words, one retains the same parameterization for timelike curves but linearly reparameterizes spacelike curves so that their tangent vectors ξ^a are mapped as $\xi^a \mapsto \overset{\lambda}{\kappa}\xi^a$. Parameterizing τ, σ, κ in this way is somewhat analogous to having renormalization group transformations on the relativistic spacetime models, where the time and distance scales (in general) are set only by the arbitrary choice of units, but their ratio is constrained by the speed of light. Indeed, for the above choices the speed of light in each of the models $(M, \overset{\lambda}{g}_{ab}, \overset{\lambda}{\kappa})$ will be the same, identically 1. Note that the elements of a given such family may well be isometric. All the members of the Minkowskian family defined by eqs. 10 and 11, for example, are flat complete spacetimes on $\mathbb{R}^4$, and under the above choices for $\tau(\lambda)$ and $\sigma(\lambda)$, the speed of light in each of them is the same.

Nevertheless, if $(M, \overset{\lambda}{g}_{ab}, \overset{\lambda}{\kappa})$ has a Newtonian limit as $\lambda \to 0$, then observables definable at points in M that are continuous in λ will converge in the limit as well. In other words, given some such observable and an acceptable margin of error $\epsilon > 0$, that observable will be approximated within that error by its Newtonian counterpart for sufficiently small λ. Instead of corresponding to relativistic worlds with different physical speeds of light, the members of $(M, \overset{\lambda}{g}_{ab}, \overset{\lambda}{\kappa})$ for such λ correspond to a range of physical conditions under which the Newtonian approximation is valid. Exactly what these conditions will be in specific cases will depend on the observables considered and the specification of the convergent family. But the general schema for determining them is roughly the same in most cases.[27]

First, pick a tetrad $\{\overset{i}{e}{}^a\}_{i=0}^3$ at a point and an observable definable there that converges in the limit. The former encodes the frame of an observer at that point; if the observer is represented by a timelike worldline that intersects the point, the timelike component will typically be the worldline's tangent vector. It also determines an inverse Riemannian metric $h^{ab} = \sum_{i=0}^3 \overset{i}{e}{}^a \overset{i}{e}{}^b$

[26] This is not the only choice of units that will yield the legal interpretation—any in which $\tau(\lambda)/\sigma(\lambda) = \overset{\lambda}{\kappa}^{1/2}$ will do, but for simplicity I consider just the choice described.

[27] The description given will suffice when considering an observable definable from just one of either the temporal metric, spatial metric, or stress-energy. The complications encountered for others add no new conceptual difficulty, however.

at that point.[28] Second, compute the value of the observable for the spacetime $(M, \overset{\lambda}{g}_{ab}, \overset{\lambda}{\kappa})$ as well as the limit point, using the choices $\tau(\lambda) = 1$ and $\sigma(\lambda) = \overset{\lambda}{\kappa}^{-1/2}$ for both. The absolute value of their difference will in general depend on λ. Third, pick some suitable maximum $\epsilon > 0$ for this difference, and compute the bound on λ below which the difference is below ϵ. This bound will in general depend not just on ϵ but on other quantities representing physical magnitudes that appear in the difference. Finally, consider any open neighborhood of the limit point whose intersection with the convergent family consists in exactly those members whose parameter λ is below this bound. It represents precisely those constraints on the aforementioned physical magnitudes under which the formula for the observable under consideration may be approximated, within ϵ, by its Newtonian counterpart.

Conversely, one can also work from a choice of open neighborhood of the limit point to conditions under which certain observables will be well-approximated by their values in the Newtonian limit. If one such neighborhood is a strict subset of another, its members in general correspond to better approximation by the Newtonian limit point, hence typically to more restricted physical circumstances. In many cases, the intersection of these "smaller" open sets with the convergent family will yield members with smaller values of λ.

To illustrate the schema, I will consider a variety of observables drawn from three convergent families of relativistic spacetimes: relative velocity and three-momenta of massive particles and light rays in a Minkowskian family, the acceleration of a static observer in a Schwarzschildian family, and the mass-energy and average radial acceleration of the cosmic fluid in a cosmological (FLRW) family. In most cases I will also calculate them for different choices of τ and σ to illustrate the role they play.

[28] I have assumed that the tetrads are the same for all members of the family under analysis. However, this is really just for simplicity of presentation. In the proofs of propositions 3.1 and 5.1, one can write the tetrads in terms of a fixed part plus a variable (λ-dependent) part, and then absorb the variable part into the variable (λ-dependent) tensor. As long as this maneuver is well-defined nothing about the analysis changes.

4.1. Relative Velocity and Momentum in a Minkowskian Family

Consider an observer in the Minkowskian spacetimes introduced above (eqs. 10 and 11) whose worldline has tangent vector

$$\mu^a = \left(\frac{\partial}{\partial t}\right)^a$$

at a point q and who will measure at that point the three-momentum p of a particle of known mass m with tangent vector

$$\overset{\lambda}{\xi}{}^a = \frac{1}{\sqrt{1-\lambda v^2}}\left[\left(\frac{\partial}{\partial t}\right)^a + v\left(\frac{\partial}{\partial x}\right)^a\right], \tag{14}$$

where $0 \leq v < 1$. The coefficients for both tangent vectors have been chosen so that they each have magnitude 1, i.e., so that $\tau(\lambda) = 1$. Further, parameterize spacelike curves so that $\sigma(\lambda) = \lambda^{-1/2}$ so to make the physical speed of light constant throughout the family. To calculate the speed and three-momentum of the particle relative to the observer, one can decompose $\overset{\lambda}{\xi}{}^a$ into its components collinear and orthogonal to μ^a:

$$\overset{\lambda}{\xi}{}^a = \left(\frac{\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c}{\overset{\lambda}{\eta}_{bc}\mu^b\mu^c}\right)\mu^a + \left(\overset{\lambda}{\xi}{}^a - \left(\frac{\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c}{\overset{\lambda}{\eta}_{bc}\mu^b\mu^c}\right)\mu^a\right) = (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c)\mu^a + (\overset{\lambda}{\xi}{}^a - (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c)\mu^a).$$

The relative speed is given by the ratio of the magnitude of the spatial component to that of the temporal component, which are given respectively by

$$\begin{aligned}
\|\overset{\lambda}{\xi}{}^a - (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c)\mu^a\| &= (\sigma(\lambda))^{-1}[\overset{\lambda}{s}{}^{ad}(\overset{\lambda}{\xi}_a - (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c)\overset{\lambda}{\mu}_a)(\overset{\lambda}{\xi}_d - (\overset{\lambda}{\eta}_{ef}\mu^e\overset{\lambda}{\xi}{}^f)\overset{\lambda}{\mu}_d)]^{1/2} \\
&= \lambda^{1/2}[-\lambda^{-1}\overset{\lambda}{\eta}_{ad}(\overset{\lambda}{\xi}{}^a - (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c)\mu^a)(\overset{\lambda}{\xi}{}^d - (\overset{\lambda}{\eta}_{ef}\mu^e\overset{\lambda}{\xi}{}^f)\mu^d)]^{1/2} \\
&= ((\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c)^2 - 1)^{1/2}, \qquad (15)\\
\|(\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c)\mu^a\| &= (\tau(\lambda))^{-1}[\overset{\lambda}{t}_{ad}(\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c)\mu^a(\overset{\lambda}{\eta}_{ef}\mu^e\overset{\lambda}{\xi}{}^f)\mu^d]^{1/2} \\
&= |\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c|. \qquad (16)
\end{aligned}$$

Thus the speed of the particle is given by

$$\frac{((\overset{\lambda}{\eta}_{ab}\mu^a\overset{\lambda}{\xi}{}^b)^2 - 1)^{1/2}}{|\overset{\lambda}{\eta}_{ab}\mu^a\overset{\lambda}{\xi}{}^b|} = \left(1 - \frac{1}{(\eta_{ab}\xi^a\mu^b)^2}\right)^{1/2} = \sqrt{\lambda}v$$

and the magnitude of its three-momentum by

$$p = m((\overset{\lambda}{\eta}_{ab}\mu^a\overset{\lambda}{\xi}^b)^2 - 1)^{1/2} = \frac{m\sqrt{\lambda}v}{\sqrt{1-\lambda v^2}}.$$

As remarked in the introduction, one can approximate this momentum by the classical formula of the product of the mass with the speed, $p = m\sqrt{\lambda}v$, which is indeed the measured momentum in the Newtonian limit, if λv^2—hence just λ for a fixed v—is sufficiently small. Thus choosing some allowed error $\epsilon > 0$ entails there is some $\delta > 0$, depending on v, such that $|p - m\sqrt{\lambda}v| < \epsilon$ when $\lambda < \delta$. This condition will be satisfied for the members of the Minkowskian family that lie in a certain neighborhood of the limiting Galilean spacetime. For simplicity, let

$$h^{ab} = \left(\frac{\partial}{\partial t}\right)^a\left(\frac{\partial}{\partial t}\right)^b + \left(\frac{\partial}{\partial x}\right)^a\left(\frac{\partial}{\partial x}\right)^b + \left(\frac{\partial}{\partial y}\right)^a\left(\frac{\partial}{\partial y}\right)^b + \left(\frac{\partial}{\partial z}\right)^a\left(\frac{\partial}{\partial z}\right)^b$$

be the (inverse) Riemannian metric constructed from the tetrad compatible with the standard global coordinates t, x, y, z on $(\mathbb{R}^4, \overset{\lambda}{\eta}_{ab}, \lambda)$. (An analogous calculation can be performed for Riemannian metrics constructed from other tetrads.) Then the intersection of the open neighborhood $B_0(s, \delta; h, q)$ of the Galilean spatial metric s^{ab} with the spatial metrics of the Minkowskian family yields $\{\overset{\lambda}{s}^{ab} : 0 < \lambda < \delta\}$, exactly those spatial metrics for which measurements of the particle's three-momentum will be within the allowed error of the classical formula. One can show the analogous neighborhood for the temporal metrics will be $B_0(t, 3\delta; h, q)$—the factor of 3 comes from the dimension of space.

One can also calculate the speed that any observer will measure of a light ray at q. The value will differ depending on the choices of $\tau(\lambda)$ and $\sigma(\lambda)$, which I shall demonstrate below. I shall also perform the calculation of this value for the Minkowskian family, but it generalizes straightforwardly to any spacetime. To begin, consider a family of $\overset{\lambda}{\eta}_{ab}$-null vectors $\overset{\lambda}{\nu}^a$ representing the trajectories of light rays at q. These will not in general be collinear because they must lie along the light cone, which is (eventually) widening as $\lambda \to 0$. The speed that the observer with tangent vector μ^a at q will assign to the light ray will be the ratio of the spatial magnitude of the component of $\overset{\lambda}{\nu}^a$ orthogonal to μ^a to the temporal magnitude of the component of $\overset{\lambda}{\nu}^a$ collinear to μ^a, just like with the speed of the massive particle. Now suppose we set $\tau(\lambda) = \sigma(\lambda) = 1$. So after decomposing $\overset{\lambda}{\nu}{}^a$ into these components via $\overset{\lambda}{\nu}{}^a = (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\nu}{}^c)\mu^a + (\overset{\lambda}{\nu}{}^a - (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\nu}{}^c)\mu^a)$, one can calculate the spatial (eq. 15) and temporal (eq. 16) magnitudes

$$\begin{aligned}
\|\overset{\lambda}{\nu}{}^a - (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\nu}{}^c)\mu^a\| &= (\sigma(\lambda))^{-1}[\overset{\lambda}{s}{}^{ad}(\overset{\lambda}{\nu}_a - (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\nu}{}^c)\overset{\lambda}{\mu}_a)(\overset{\lambda}{\nu}_d - (\overset{\lambda}{\eta}_{ef}\mu^e\overset{\lambda}{\nu}{}^f)\overset{\lambda}{\mu}_d)]^{1/2} \\
&= [-\lambda^{-1}\overset{\lambda}{\eta}_{ad}(\overset{\lambda}{\nu}{}^a - (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\nu}{}^c)\mu^a)(\overset{\lambda}{\nu}{}^d - (\overset{\lambda}{\eta}_{ef}\mu^e\overset{\lambda}{\nu}{}^f)\mu^d)]^{1/2} \\
&= \lambda^{-1/2}|\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\nu}{}^c|^{1/2}, \\
\|(\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\nu}{}^c)\mu^a\| &= (\tau(\lambda))^{-1}[\overset{\lambda}{t}_{ad}(\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\nu}{}^c)\mu^a(\overset{\lambda}{\eta}_{ef}\mu^e\overset{\lambda}{\nu}{}^f)\mu^d]^{1/2} = |\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\nu}{}^c|.
\end{aligned}$$

Thus the observer will determine the speed of a light ray with tangent vector $\overset{\lambda}{\nu}{}^a$ to be $\lambda^{-1/2}$, the ratio of the magnitude of the spatial component to that of the temporal component of $\overset{\lambda}{\nu}{}^a$ (relative to μ^a). Since the choice of this observer was arbitrary, arbitrary observers in the spacetimes $(M, \overset{\lambda}{\eta}_{ab}, \lambda)$ will measure larger and larger speeds of light as $\lambda \to 0$. On the other hand, if one uses instead $\sigma(\lambda) = \overset{\lambda}{\kappa}{}^{-1/2} = \lambda^{-1/2}$ while retaining $\tau(\lambda) = 1$, the above calculation yields a speed of 1 for the speed of light independently of λ. In other words, in this interpretation all observers will always measure the speed of light to be the same fixed value.[29]

To see the significance of these calculations, it may be illuminating to focus on the tangent space at q and consider the relative velocity that a certain observer passing through that point would attribute to particles with various other tangent vectors. (See figure 1 for an illustration.) When $\tau(\lambda) = \sigma(\lambda) = 1$, as the light cone widens, the velocity she would attribute to particles with a given tangent vector (up to normalization) would remain the same, but she would count more and more vectors as timelike. She would still attribute a fixed speed to a particle whose tangent vector was initially null for $\lambda = 1$ but became timelike as the cone widened.

By contrast, she would judge the speed of a light ray whose tangent vector must lie along the widening cone to be larger than that of a light ray for $\lambda = 1$. By contrast, when $\tau(\lambda) = 1$ but $\sigma(\lambda) = \lambda^{-1/2}$, as the light

[29]Note that, for Newton-Cartan theory, one does not usually countenance particles (massive or otherwise) whose trajectories are not timelike. Nevertheless one can still consider the behavior of the "relative speed" observable in the limit regardless of the choices of τ and σ. (Cf. Weatherall (2011, p. 430, fn. 16).)

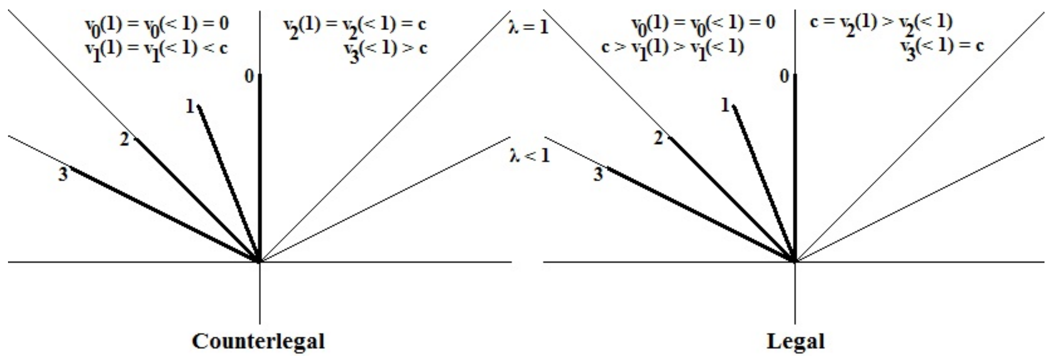

Figure 1: Depicted are four vectors (labeled 0, 1, 2, and 3) at some point q, along with the light cones associated with the $\lambda = 1$ member and some $\lambda < 1$ member from a family of relativistic spacetimes with $\kappa = \lambda$. The $v_i(\lambda)$ are the speeds that an observer at q with tangent vector proportional to vector 0 would measure of a particle whose worldline at q has (co-directed) tangent vector proportional to vector i. For the figure labeled "counterlegal," $\tau(\lambda) = \sigma(\lambda) = 1$, and the relative speed corresponding to each vector does not change, but as $\lambda \to 0$ the speed of a light ray increases, and more vectors become timelike. For the figure labeled "legal," on the other hand, $\tau(\lambda) = 1$ and $\sigma(\lambda) = \lambda^{-1/2}$, and as $\lambda \to 0$ the relative speed of each vector decreases, but the speed of a light ray stays the same. More vectors become timelike for these choices of τ and σ, too, but as λ becomes sufficiently small their speeds relative to vector 0 also decrease.

cone widens, more and more vectors count as timelike, but particles with null tangent vectors are always measured with the same speed. Accordingly, a particle with a fixed tangent vector at q not comoving with the observer will be attributed smaller and smaller relative velocities as the light cone widens. (Only particles comoving with the observer maintain the same (vanishing) measured velocity as the cone widens.) Heuristically, one can interpret v/c becoming small as $\lambda \to 0$ as v being fixed but $c \to \infty$, or as c being fixed but $v \to 0$. Of course, these remarks bear only on relative velocity observables at individual points. Other observables deserve their own treatment, to some of which I turn presently.

4.2. Acceleration in a Schwarzschildian Family

Consider a family of Schwarzschildian spacetimes, whose temporal (hence Lorentz) metrics are given in the standard spherical coordinates t, r, θ, ϕ as

$$\overset{\lambda}{t}_{ab} = \overset{\lambda}{g}_{ab} = (1 - 2M\lambda/r)(d_a t)(d_b t) - \left(\frac{\lambda}{1 - 2M\lambda/r}\right)(d_a r)(d_b r) - \lambda r^2[(d_a\theta)(d_b\theta) + \sin^2\theta(d_a\phi)(d_b\phi)],$$

where M denotes the mass of the black hole. For present purposes I will be concerned with the "external" region of the spacetime, i.e., that for which $r > 2M\lambda$. When restricted to this region, the family $(M, \overset{\lambda}{t}_{ab}, \lambda)$ has as a Newtonian limit: the spacetime associated with a point mass (Ehlers, 1981, 1991, 1997). (Note that the Schwarzschild radius, $2M\lambda$, vanishes in the limit.) Thus one can evaluate in this family the conditions under which various observables may be approximated by their Newtonian counterparts.

In particular, consider a static observer, i.e., one whose tangent vector at some point p is given by the unit vector

$$\overset{\lambda}{\mu}{}^a = \frac{1}{\sqrt{1 - 2M\lambda/r}} \left(\frac{\partial}{\partial t}\right)^a. \tag{17}$$

Such an observer is always accelerating, for she maintains her position (with respect to the static coordinates) despite the gravitational "attraction" of the black hole. When will the acceleration she experiences be approximated by that she would experience if she were in the gravitational field of a Newtonian point mass? Let $\overset{\lambda}{\nabla}$ be the covariant derivative operator associated with the spacetime $(M, \overset{\lambda}{t}_{ab}, \lambda)$, let ∇ be that associated with the Newtonian limit spacetime, and let $\bar{\nabla}$ be that compatible with the (flat) Riemannian metric

$$h_{ab} = (d_a t)(d_b t) + (d_a r)(d_b r) + r^2[(d_a\theta)(d_b\theta) + \sin^2(d_a\phi)(d_b\phi)]$$

arising from the standard coordinates and according to which the observer's worldline is a geodesic. (Again, one can perform analogous calculations with respect to Riemannian metrics constructed from other observers.) In both the Schwarzschildian family and its Newtonian limit spacetime, the observer's acceleration is completely radial, i.e., proportional to $(\partial/\partial r)^a$. Thus the quantity of interest will be the difference in magnitude of these accelerations. The acceleration in the family is given by

$$\overset{\lambda}{\mu}{}^b \overset{\lambda}{\nabla}_b \overset{\lambda}{\mu}{}^a = \overset{\lambda}{\mu}{}^b \bar{\nabla}_b \overset{\lambda}{\mu}{}^a - \overset{\lambda}{\mu}{}^b \overset{\lambda}{\mu}{}^c \overset{\lambda}{C}{}^a{}_{bc} = -\overset{\lambda}{\mu}{}^b \overset{\lambda}{\mu}{}^c \overset{\lambda}{C}{}^a{}_{bc},$$

where $\overset{\lambda}{\nabla} = (\bar{\nabla}, \overset{\lambda}{C}{}^a{}_{bc})$. Using the standard result (see, e.g., Malament (2012, p. 78)) that

$$\overset{\lambda}{C}{}^a{}_{bc} = \frac{1}{2} \overset{\lambda}{g}{}^{ad} (\bar{\nabla}_d \overset{\lambda}{t}_{bc} - \bar{\nabla}_b \overset{\lambda}{t}_{dc} - \bar{\nabla}_c \overset{\lambda}{t}_{db}),$$

one can compute that

$$-\overset{\lambda}{\mu}^b\overset{\lambda}{\mu}^c\overset{\lambda}{C}^a{}_{bc} = \frac{M}{r^2}\left(\frac{\partial}{\partial r}\right)^a,$$

the spatial magnitude of which is then given by

$$\begin{aligned}\overset{\lambda}{a} = \| - \overset{\lambda}{\mu}^b\overset{\lambda}{\mu}^c\overset{\lambda}{C}^a{}_{bc}\| &= (\sigma(\lambda))^{-1}[\overset{\lambda}{s}^{ad}(-\overset{\lambda}{\mu}^b\overset{\lambda}{\mu}^c\overset{\lambda}{C}_{abc})(-\overset{\lambda}{\mu}^b\overset{\lambda}{\mu}^c\overset{\lambda}{C}_{dbc})]^{1/2}\\ &= (\sigma(\lambda))^{-1}[-\lambda^{-1}\overset{\lambda}{g}_{ab}(-\overset{\lambda}{\mu}^b\overset{\lambda}{\mu}^c\overset{\lambda}{C}^a{}_{bc})(-\overset{\lambda}{\mu}^b\overset{\lambda}{\mu}^c\overset{\lambda}{C}^d{}_{bc})]^{1/2}\\ &= \frac{M/r^2}{\sigma(\lambda)\sqrt{1-2M\lambda/r}}. \end{aligned} \tag{18}$$

Because these observables are continuous in λ and are functions of tensors that converge in the Newtonian limit, their limiting values must match their corresponding values in the limit. Indeed, a similar calculation for the Newtonian limit spacetime yields that

$$a = \| - \mu^b\mu^c C^a_{bc}\| = (\sigma(\lambda))^{-1}M/r^2. \tag{19}$$

When $\sigma(\lambda) = 1$, the magnitude of the acceleration in the Schwarzschildian family approaches that of its Newtonian limit spacetime as $\lambda \to 0$. When $\sigma(\lambda) = \lambda^{-1/2}$, eq. 19 still approximates eq. 18 when $2M\lambda/r$, i.e., the ratio of the Schwarzschild radius to the radial distance of the observer, is sufficiently small. For fixed M, one can interpret this as a sufficiently large r, or for a fixed r, as a sufficiently small M. In other words, given some allowed error $\epsilon > 0$, there is some $\delta > 0$, such that when the ratio of the Schwarzschild radius to the radial distance of the observer is less than it, the magnitude of the acceleration experienced by the observer can be approximated within ϵ by the formula for its Newtonian counterpart. (The calculations showing this are identical to the ones for relative velocity in §4.1.)

4.3. Mass-Energy and Average Radial Acceleration in a Cosmological (FLRW) Family

Consider a family of spatially flat cosmological (FLRW) spacetimes, whose temporal (hence Lorentz) metrics are given in Cartesian coordinates as

$$\overset{\lambda}{t}_{ab} = \overset{\lambda}{g}_{ab} = (d_a t)(d_b t) - \lambda a^2[(d_a x)(d_b x) + (d_a y)(d_b y) + (d_a z)(d_b z)], \tag{20}$$

where the cosmological scale factor $a > 0$ depends only on t and is normalized so that $a_{|t=0} = 1$. The stress-energy tensor has the form of that of a perfect fluid with density ρ and pressure p, both of which also only depend on t:[30]

$$\overset{\lambda}{T}{}^{ab} = (\rho + \lambda p) \left(\frac{\partial}{\partial t}\right)^a \left(\frac{\partial}{\partial t}\right)^b + p\overset{\lambda}{s}{}^{ab}. \tag{21}$$

The family $(\mathbb{R}^4, \overset{\lambda}{t}_{ab}, \lambda)$ has, as $\lambda \to 0$, a Newtonian limit representing a homogeneous universe (Ehlers, 1988, 1997). Thus observables that are continuous functions in λ will converge to their Newtonian counterparts as well.

Consider, for example, an observer whose tangent vector at some point is given by

$$\overset{\lambda}{\xi}{}^a = \frac{1}{\sqrt{1 - \lambda a^2 v^2}} \left[\left(\frac{\partial}{\partial t}\right)^a + v \left(\frac{\partial}{\partial x}\right)^a \right], \tag{22}$$

with $0 \leq v < a^{-1}$. What mass-energy density $\overset{\lambda}{\rho}$ would the observer measure? Under what circumstances can it be approximated by its corresponding Newtonian observable, the mass density ρ? For the relativistic family, one can calculate

$$\overset{\lambda}{\rho} = \overset{\lambda}{t}_{ac}\overset{\lambda}{t}_{bd}\overset{\lambda}{T}{}^{ab}\overset{\lambda}{\xi}{}^c\overset{\lambda}{\xi}{}^d = \frac{\rho + \lambda p}{1 - \lambda a^2 v^2} - \lambda p = \frac{\rho + \lambda a^2 v^2}{1 - \lambda a^2 v^2}, \tag{23}$$

which yields $\lim_{\lambda \to 0} \overset{\lambda}{\rho} = \rho$ as expected. Clearly there will only be a discrepancy from ρ when the observer is not comoving with the fluid, i.e., when $v > 0$, but for *any* $\epsilon > 0$ one can find a sufficiently small λ such that $|\overset{\lambda}{\rho} - \rho| < \epsilon$. The bound for this will depend on $a^2 v^2$, so under the legal interpretation, one can interpret the smallness of λ to be the smallness of a relative to v, or vice versa.

One can also examine the limiting behavior of another observable, sometimes called the *average radial acceleration* (*ARA*), which measures (in a sense) the average tidal forces that a small cluster of massive test particles undergoing geodesic motion would experience. To calculate it, one must first invert eq. 7 to yield the Ricci tensor

$$\overset{\lambda}{R}_{ab} = 8\pi(\overset{\lambda}{t}_{am}\overset{\lambda}{t}_{bn} - \frac{1}{2}\overset{\lambda}{t}_{ab}\overset{\lambda}{t}_{mn})\overset{\lambda}{T}{}^{mn} = 8\pi(\rho + \lambda p)(d_a t)(d_b t) - 4\pi(\rho - \lambda p)\overset{\lambda}{t}_{ab}, \tag{24}$$

[30] One may also require them to satisfy various energy condition or equations of state, but these play no role in the calculations below.

where I have substituted in eq. 21. Now, suppose that the observer with tangent vector $\overset{\lambda}{\xi}{}^a$ at some point is undergoing geodesic motion in a neighborhood of that point, and pick a smooth tetrad field whose timelike component is the observer's tangent vector field and whose spacelike components vanish when Lie differentiated by that field. The ARA is then defined as the average of the magnitudes of the relative acceleration between the observer and "infinitesimally close" observers "connected" by a spacelike component, for each of the three components. It turns out that the ARA is independent of the choice of these spacelike components and can be determined from the observer's tangent vector and the Ricci tensor (Malament, 2012, p. 165–6):[31]

$$\begin{aligned} ARA = -\frac{1}{3\sigma(\lambda)}\overset{\lambda}{R}_{ab}\overset{\lambda}{\xi}{}^a\overset{\lambda}{\xi}{}^b &= -\frac{1}{3\sigma(\lambda)}\left(\frac{8\pi(\rho+\lambda p)}{1-\lambda a^2v^2} - 4\pi(\rho-\lambda p)\right) \\ &= -\frac{4\pi}{3\sigma(\lambda)}\left(\frac{\rho+3\lambda p+\lambda a^2v^2(\rho-\lambda p)}{1-\lambda a^2v^2}\right). \end{aligned} \tag{25}$$

Because the ARA is composed from tensorial fields that have a Newtonian limit and is continuous in λ, its limiting value as $\lambda \to 0$ must be the value it takes on in the Newtonian limit model. Indeed, when $\sigma(\lambda) = 1$, this value is $-4\pi\rho/3$, just as expected (Malament, 2012, p. 281). Under the legal interpretation, the conditions under which eq. 25 may be approximated by its Newtonian formula are somewhat complicated, but roughly, this will be when $\lambda a^2 v^2$ is much less than 1 and p is much less than ρ, i.e., when the relative velocity of the observer to the integral curves of the cosmic fluid is sufficiently small and the pressure is small compared to the mass density.

5. Topology and Observables

We can now return to the question I posed after the definition of the Newtonian limit: why use the point-open topology? Since there is no canonical topology for the spacetime metrics (Fletcher, 2016), it must be justified relative to the nature of the investigation. In light of the foregoing discussion of the legal interpretation of the limit, it is clear that a topology is equivalent with a set of relevant observables that one requires be well-approximated by the Newtonian limit spacetime. In the case of the C^2 point-open topology,

[31] The factor of σ arises because one is averaging the magnitudes of components of acceleration vectors, which are spacelike.

the observables are scalar point quantities at finitely many points arising from contraction of the temporal and spatial metrics, the stress-energy tensor, and their derivatives to second order.

At least this much seems warranted, since we do measure, at least in some idealized sense, relativistic point observables that we can approximate with their classical counterparts.[32] But our experiments and observations are not confined to points: there are many observables corresponding more generally to *extended compact* regions for whose classical approximations one should account as well. These may include continuous measurements of point quantities, such as the momentum flux over time, integrated observables, such as the proper time along a worldline, and analogous quantities over areas and volumes in spacetime.

To see why any point-open topology is insufficient to take these kinds of observables into account, consider the following family of relativistic temporal metrics on $\mathbb{R}^4$ restricted to temporal coordinates in the range $[0,1]$:

$$\begin{aligned}(\overset{\lambda}{t}_{ab})_{|t^{-1}[0,1]} = {} & (1+\lambda^{-3}t(1-t)^{1/\lambda})(d_a t)(d_b t) \\ & - \lambda(d_a x)(d_b x) - \lambda(d_a y)(d_b y) - \lambda(d_a z)(d_b z), \quad (26)\end{aligned}$$

where $0 < \lambda \leq 1$. Restricted to the strip $[0,1] \times \mathbb{R}^3$, the relativistic family $(\mathbb{R}^4, \overset{\lambda}{t}_{ab}, \lambda)$ has the "Galilean" spacetime $([0,1]\times\mathbb{R}^3, t_{ab}, s^{ab}, \nabla, \mathbf{0})$ (cf. eqs. 12,13) as its Newtonian limit in the C^2 point-open topology. (The members of this family have many smooth extensions to the whole spacetime, but which extension one chooses is immaterial for my purposes here as long as the extensions for the whole family also have a Newtonian limit.) The limit exists because, for every point p with t-coordinate within $[0,1]$ and any $\epsilon > 0$, one can find a sufficiently small λ such that $d(t, \overset{\lambda}{t}; h, i)_{|p} < \epsilon$ for $i = 0, 1, 2$, and similarly for the spatial metric. (The exponential term $(1-t)^{1/\lambda}$ dominates as $\lambda \to 0$.) But one cannot find any λ such that the distance function will be *uniformly* bounded by ϵ for all $t \in [0,1]$. This is because as $\lambda \to 0$, the "bump" in the metric moves toward $t = 0$, becoming more localized but also taller. (See fig. 2.) Consequently there will be observables, depending on the

[32]A topology's system of open sets must be closed under arbitrary union and finite intersection, so any topology characterized by the sufficient approximation of a class of observables is sensitive as well to finite conjunctions and arbitrary disjunctions of well-approximations of those observables.

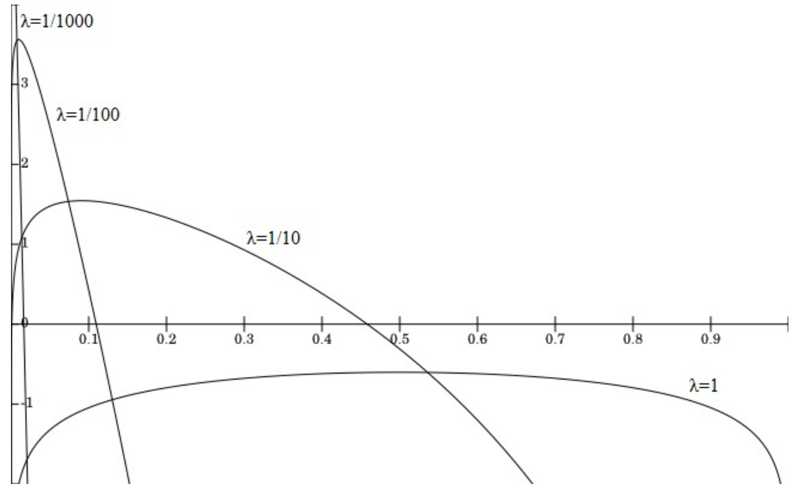

Figure 2: Log-plot of the conformal "bump" in eq. 26 as a function of t-coordinate. At $\lambda = 1/1000$, it is sharply peaked near $t = 0$ and nearly vanishing elsewhere.

values the metric takes on intervals containing a point with $t = 0$, that do not converge in the Newtonian limit.

For example, consider a timelike curve passing through the region with t-coordinate in $[0, 1]$ and with tangent vector everywhere proportional to $(\frac{\partial}{\partial t})^a$. Since the lengths of timelike curves are invariant under reparameterization, the proper time elapsed along the curve between temporal coordinates $t = 0$ and $t = 1$ according to $\overset{\lambda}{t}_{ab}$ is given by

$$\int_0^1 \left[\overset{\lambda}{t}_{ab} \left(\frac{\partial}{\partial t} \right)^a \left(\frac{\partial}{\partial t} \right)^b \right]^{1/2} dt = 1 + \frac{1}{\lambda(2\lambda^2 + 3\lambda + 1)}, \tag{27}$$

whereas the same quantity according to the Galilean temporal metric $(d_a t)(d_b t)$ is just 1. The former diverges as $\lambda \to 0$, meaning that there are *no* experimental contexts in which observations adhere to their classical formulas within some ranges of error. Because these observations include measurable quantities, such as proper time, the point-open topology is evidently too coarse to capture the demand that the Newtonian limit point of a family of relativistic spacetimes approximate standard observables that depend on extended (but compact) regions.

Requiring the convergence of more kinds of observables corresponds with introducing more open sets into the topology on temporal and spatial metrics, i.e., requiring convergence in a *finer* topology. There is a natural such choice to supplant the point-open topology, one that is defined quite similarly but controls convergence uniformly on compacta instead of pointwise. A basis for the C^k *compact-open* topology on the smooth tensor fields in some set S may be written as sets of the form

$$B_k(t, \epsilon; h, C) = \{t' \in S : \sup_C d(t, t'; h, 0) < \epsilon, \ldots, \sup_C d(t, t'; h, k) < \epsilon\}, \quad (28)$$

where, much as with the point-open topologies, t ranges over all tensor fields in S, ϵ ranges over all positive constants, and C ranges over all compact subsets of M. The compact-open topologies have many nice properties, the most important of which for this context is that the sequence given by eq. 26 does *not* converge—it is sensitive to the fact that the lengths of some timelike curves diverge in the $\lambda \to 0$ limit. Besides what's already been said, one can see this as a consequence of the following proposition that gives convergence conditions for the compact-open topologies analogous to those for the point-open topologies.

Proposition 5.1. *A family of tensor fields* $\overset{\lambda}{\phi}{}^a_{bc}$ *on* M*, with* $\lambda \in (0, a)$ *for some* $a > 0$*, converges to a tensor field* ϕ^a_{bc} *as* $\lambda \to 0$ *in the* C^k *compact-open topology iff for all compacta* $C \subseteq M$*,* $\lim_{\lambda \to 0} \sup_C(\overset{0}{\psi}{}^{bc}_a \overset{\lambda}{\phi}{}^a_{bc}) = \sup_C(\overset{0}{\psi}{}^{bc}_a \phi^a_{bc})$ *for all tensor fields* $\overset{0}{\psi}{}^{bc}_a$ *on* C *and, for each positive* $i \leq k$*,* $\lim_{\lambda \to 0} \sup_C(\overset{i}{\psi}{}^{bcd_1 \ldots d_i}_a \nabla_{d_1} \cdots \nabla_{d_i} \overset{\lambda}{\phi}{}^a_{bc}) = \sup_C(\overset{i}{\psi}{}^{bcd_1 \ldots d_i}_a \nabla_{d_1} \cdots \nabla_{d_i} \phi^a_{bc})$ *for all tensor fields* $\overset{i}{\psi}{}^{bcd_1 \ldots d_i}_a$ *on* C*. Moreover, the* C^k *compact-open topology is the unique topology with this property.*

Analogous results hold for tensor fields of other index structures. One can interpret the proposition as showing just how the compact-open topologies formalize a notion of uniform convergence of observables defined on compacta.[33]

Although the sequence given by eq. 26 does not converge in any compact-open topology, the Minkowskian family given by eqs. 10 and 11 does still have

[33] There is a slightly coarser topology than the compact-open that prevents eq. 27 from converging, in which one replaces the distance function (eq. 8) with the L^2 norm. The essential point I want to make, which would still stand under this proposal, is that one needs to use a topology that is sensitive (in some appropriate way) to observables defined on extended regions.

Galilean spacetime as its Newtonian limit. In light of these considerations, I propose modifying Ehlers's definition of the Newtonian limit to require convergence in the C^2 compact-open topology:[34]

Newtonian Limit (Revised) Let $(M, \overset{\lambda}{t}_{ab}, \overset{\lambda}{s}^{ab}, \overset{\lambda}{\nabla}, \overset{\lambda}{T}^{ab})$, with $\lambda \in (0, a)$ for some $a > 0$, be a one-parameter family of models of general relativity that share the same underlying manifold M. Then $(M, t_{ab}, s^{ab}, \nabla, T^{ab})$ is said to be a Newtonian limit of the family when it is a model of Newton-Cartan theory and $\lim_{\lambda \to 0}(\overset{\lambda}{t}_{ab}, \overset{\lambda}{s}^{ab}, \overset{\lambda}{\nabla}, \overset{\lambda}{T}^{ab}) = (t_{ab}, s^{ab}, \nabla, T^{ab})$ in the C^2 compact-open product topology.

This characterization depends, of course, on considering as relevant all and only observables subsisting on compact regions of spacetime. Thus one need not be too insistent that this is the sole "right" topology to characterize the Newtonian limit, for this class of observables may very well be too expansive or too meager for certain contexts. For example, one might want to consider observables associated not just with compact regions of spacetime, but also with non-compact curves with finite proper length, as arise in singular spacetimes. (Consider starting a stopwatch and then throwing it into a black hole.)

Another case of interest in this regard is whether one should include some global (non-compact) observables, especially in the context of cosmological models. This case is less clear because even in cosmology, it is not obvious that one can have experimental access to observables depending on (data on) non-compact sets, without which it seems one cannot determine the global structure of virtually all spacetimes of interest.[35] In any case, it is harder to find an even plausible topology to encode such global observables. The most common topologies chosen in the literature to control the global behavior of smooth tensor fields in a collection S are the C^k open topologies, a basis for

[34]This is not a significant departure from Ehlers. Although he effectively stated the definition of the Newtonian limit using the C^2 point-open topology (i.e., "pointwise convergence"), he did on occasion discuss compact observables, such as proper time, in which case he referred to locally uniform convergence. This mode of convergence is equivalent to compact convergence for locally compact spaces. Since every finite-dimensional manifold is locally compact, the corresponding topology is the compact-open topology.

[35]In particular, if a spacetime is not causally bizarre, then it is observationally indistinguishable from a spacetime that has holes and is extendible, anisotropic, and not globally hyperbolic. See Manchak (2011) and references cited therein.

which may be given as sets of the form

$$B_k(t, \epsilon; h) = \{t' \in S : \sup_M d(t, t'; h, 0) < \epsilon, \ldots, \sup_M d(t, t'; h, k) < \epsilon\}, \quad (29)$$

where t ranges over all tensor fields in S, ϵ ranges over all positive constants, and (importantly) h ranges over all Riemannian metrics. One can show that, when M is compact, the open topologies are identical to the compact-open topologies. But when M is non-compact, in contrast to the point-open and compact-open topologies, different choices of h in general generate different collections of open sets. One is thus obliged to consider all possible choices of h because, as an arbitrary smoothly varying choice of basis at each point, no particular choice corresponds with anything physically meaningful in a spacetime model. And in this case, the convergence condition for the open topologies is *not* as similar to those for the compact-open and point-open topologies as one might have expected:[36]

Proposition 5.2. *Let $g, \{\overset{n}{g}\}_{n \in \mathbb{N}}$ be tensor fields on a non-compact manifold M. Then $\overset{n}{g} \to g$ in the open C^k topology iff there is a compact $C \subset M$ such that:*

1. *for sufficiently large n, $\overset{n}{g}_{|M-C} = g_{|M-C}$; and*
2. *$\overset{n}{g}_{|C} \to g_{|C}$ in the compact-open C^k topology.*

Thus the mere fact that the temporal and spatial metrics of a relativistic spacetimes differ in signature from those of a Newton-Cartan spacetime is sufficient to entail the following negative result:

Corollary 5.3. *No family of relativistic spacetimes on a non-compact manifold has a Newtonian limit in any open topology.*

6. Conclusions

One can draw a number of topical and methodological conclusions from the above discussion. In §2, I described how one can give a unified description of both general relativity and Newton-Cartan theory under the banner of Ehlers's frame theory. It is only superficially an example of a unification in

[36] For a proof sketch, see Golubitsky and Guillemin (1973, p. 43–44).

the traditional sense in philosophy of science, for the theories thereby "unified" do not have different domains but in fact concern the same phenomena: one (general relativity) is a successor to the other (Newtonian gravitation) and improves upon it. Thus the frame theory should perhaps more properly be called a frame*work* theory,[37] since it provides a common terminology for and exhibits the conceptual continuity between Newtonian and relativistic gravitation (Ehlers, 1981, 1986, 1988, 1998).

Even considered as a framework theory, it is not without philosophical interest (Lehmkuhl, 2017). It reveals that, if only in retrospective rational reconstruction, the transition to relativity theory from Newtonian physics involves much more conceptual continuity than is usually emphasized. This kind of claim is of interest for structural realists, who are keen to find the structural continuity between old and new theories (Worrall, 1989; Redhead, 2001; Votsis, 2009). Framework theories like Ehlers's frame theory provide just the technical apparatus needed to draw a comparison. Certain critics of structural realism, both realists and otherwise, would also find the frame theory of interest, for they can point to the rather minimal structure that relativity and Newtonian theory share, skeptical that something so meager commits one to much at all, ontologically. More generally, the methodology used here, in which the models of two theories of the same (or at least overlapping) subject matter are united in a common framework (that is to be equipped with the relevant topology), does not require in any essential way that the theories under consideration be physical theories—any sufficiently mathematized theories will do. Thus it could also prospectively apply to certain theories of, say, economics, climate science, or population ecology.

Coming back to concerns more internal to the conceptual structure of physics, once the frame theory is constructed, a topology on its space of models defines a convergence condition for them, thereby making precise what it means to take a limit within that space. In §3, I proved how the convergence condition used in the literature can be understood topologically and used this reconstruction to define the Newtonian limit of a family of models of general relativity. The reduction relation (under the legal interpretation) may then be understood as supporting an explanation of why Newtonian gravitation was successful by providing the conditions under which its mod-

[37] Ehlers (1981) used the term *Rahmentheorie*, which is ambiguously translated as either "frame theory" or "framework theory."

els successfully approximate those of general relativity. Because the pattern of providing a unifying framework for models of the two theories and then topologizing those models does not appear in principle to depend on anything topically specific to gravitation, it might be applied fruitfully to other reduction relations. It may even serve as a general pattern for how one can explain the success of one (sufficiently mathematized) theory from another. Of course, further case studies must be pursued to test the viability of this hint.

A longstanding obstacle to physically interpreting the geometrical methods used to define the Newtonian limit is the ambiguous conceptual status of letting the limiting parameter be the value of a fixed physical constant, namely the speed of light. I attempted to clarify the nature of the limit in §4 by showing through explication and example how it may be interpreted so that the physical value of the speed of light is in fact the same in all models. This makes the geometric limit, already completely general in its capability to capture properties and observable of interest, appropriate for showing how models of classical spacetime can approximate relativistic spacetimes in specified observational contexts. In some ways, the geometric Newtonian limit subsumes the older, more limited local approximation methods (like the "low velocity limit") more commonly used, although I do not expect this way of approaching approximations to supplant traditional methods. Indeed, there has already been some work investigating how these geometrical methods relate to and clarify the nature of the PN theory (Rendall, 1992; Tichy and Flanagan, 2011), described briefly in the introductory section of this essay. Rather, I see one of the primary upshots of this section to be a general way to understand reductive limiting relationships between theories that are usually described in terms of the elimination of a constant of nature.

In §5 I returned to a question broached in §3 about the definition of the Newtonian limit: why use the C^2 point-open topology? Indicating how the topology corresponds with a class of observables one demands be well-approximated in the Newtonian limit, I argued that the point-open topology is not fine enough—it allows too many convergent sequences of spacetimes, including ones in which observables one would think should converge do not, such as the proper time measured along a compact timelike worldline. These observables depend on data on compact sets, so I tentatively suggest using the C^2 compact-open topology instead. This allows one to recover the convergence of observables depending on data on compact sets. An even finer class of topologies seemingly attuned to global features, the open topologies, however allows for no sequences of relativistic spacetimes with a Newtonian limit. This exhibits the fact that whether a particular sequence converges (or is "singular") depends on the topology on their models, hence how we understand the relationships theories have with each other depends on what features we take to be important in judging the similarities of their models. A topology implicitly picks out such features, but the significance of these choices in understanding limiting relationships has been unduly neglected.

Appendix A. Proofs of Propositions

Proposition (3.1)**.** *A family of tensor fields* $\overset{\lambda}{\phi}{}^a_{bc}$ *on* M*, with* $\lambda \in (0, a)$ *for some* $a > 0$*, converges to a tensor field* ϕ^a_{bc} *as* $\lambda \to 0$ *in the* C^k *point-open topology iff for all points* $p \in M$*,* $\lim_{\lambda \to 0}(\overset{0}{\psi}{}^{bc}_a \overset{\lambda}{\phi}{}^a_{bc})_{|p} = (\overset{0}{\psi}{}^{bc}_a \phi^a_{bc})_{|p}$ *for all tensors* $\overset{0}{\psi}{}^{bc}_a$ *at* p *and, for all positive* $i \leq k$*,* $\lim_{\lambda \to 0}(\overset{i}{\psi}{}^{bcd_1 \ldots d_i}_a \nabla_{d_1} \cdots \nabla_{d_i} \overset{\lambda}{\phi}{}^a_{bc})_{|p} = (\overset{i}{\psi}{}^{bcd_1 \ldots d_i}_a \nabla_{d_1} \cdots \nabla_{d_i} \phi^a_{bc})_{|p}$ *for all tensors* $\overset{i}{\psi}{}^{bcd_1 \ldots d_i}_a$ *at* p*.*

Proof. Consider the case $k = 0$, as the others can be treated similarly, and fix some Riemannian metric h_{ab} on M. First suppose that $\lim_{\lambda \to 0} \overset{\lambda}{\phi}{}^a_{bc} = \phi^a_{bc}$ in the C^0 point-open topology, and consider some tensor ψ^{bc}_a at an arbitrary $p \in M$. Letting Greek indices indicate tensor components in the basis for the tangent and cotangent spaces at p that makes h_{ab} and h^{ab} the identity matrices,

$$\begin{aligned}
|\psi^{bc}_a(\overset{\lambda}{\phi}{}^a_{bc} - \phi^a_{bc})|_{|p} &= \left| \sum_{\alpha,\beta,\gamma=0}^{3} \psi^{\beta\gamma}_\alpha (\overset{\lambda}{\phi}{}^\alpha_{\beta\gamma} - \phi^\alpha_{\beta\gamma}) \right| \\
&\leq \left| \sum_{\alpha,\beta,\gamma=0}^{3} (\psi^{\beta\gamma}_\alpha)^2 \right|^{1/2} \left| \sum_{\alpha,\beta,\gamma=0}^{3} (\overset{\lambda}{\phi}{}^\alpha_{\beta\gamma} - \phi^\alpha_{\beta\gamma})^2 \right|^{1/2} \\
&= d(\psi, 2\psi; h, 0)_{|p} d(\phi, \overset{\lambda}{\phi}; h, 0)_{|p},
\end{aligned}$$

using the Cauchy-Schwarz inequality. Since $d(\psi, 2\psi; h, 0)_{|p}$ is constant with respect to λ, by hypothesis we have that $\lim_{\lambda \to 0} |\psi^{bc}_a(\overset{\lambda}{\phi}{}^a_{bc} - \phi^a_{bc})|_{|p} = 0$.

For the reverse direction, assume instead that for all $p \in M$ and all tensors of the form ψ^{bc}_a at p, $\lim_{\lambda \to 0} |\psi^{bc}_a(\overset{\lambda}{\phi}{}^a_{bc} - \phi^a_{bc})|_{|p} = 0$. In particular choosing ψ^{bc}_a

to vanish in all but one component shows that that each component of $\overset{\lambda}{\phi}{}^a_{bc}$ converges to each component of ϕ^a_{bc}. Since

$$d(\phi, \overset{\lambda}{\phi}; h, 0)_{|p} = \left| \sum_{\alpha,\beta,\gamma=0}^{3} (\overset{\lambda}{\phi}{}^\alpha_{\beta\gamma} - \phi^\alpha_{\beta\gamma})^2 \right|^{1/2}$$

is a continuous function in each $\overset{\lambda}{\phi}{}^\alpha_{\beta\gamma} - \phi^\alpha_{\beta\gamma}$ and $d(\phi, \phi; h, 0)_{|p} = 0$, by definition $\lim_{\lambda\to 0} \overset{\lambda}{\phi}{}^a_{bc} = \phi^a_{bc}$ in the C^0 point-open topology. □

Proposition (5.2)**.** *A family of tensor fields* $\overset{\lambda}{\phi}{}^a_{bc}$ *on* M*, with* $\lambda \in (0, a)$ *for some* $a > 0$*, converges to a tensor field* ϕ^a_{bc} *as* $\lambda \to 0$ *in the* C^k *compact-open topology iff for all compacta* $C \subseteq M$, $\lim_{\lambda\to 0} \sup_C (\overset{0}{\psi}{}^{bc}_a \overset{\lambda}{\phi}{}^a_{bc}) = \sup_C (\overset{0}{\psi}{}^{bc}_a \phi^a_{bc})$ *for all tensor fields* $\overset{0}{\psi}{}^{bc}_a$ *on* C *and, for each positive* $i \leq k$, $\lim_{\lambda\to 0} \sup_C (\overset{i}{\psi}{}^{bcd_1\ldots d_i}_a \nabla_{d_1} \cdots \nabla_{d_i} \overset{\lambda}{\phi}{}^a_{bc}) = \sup_C (\overset{i}{\psi}{}^{bcd_1\ldots d_i}_a \nabla_{d_1} \cdots \nabla_{d_i} \phi^a_{bc})$ *for all tensor fields* $\overset{i}{\psi}{}^{bcd_1\ldots d_i}_a$ *on* C*. Moreover, the* C^k *compact-open topology is the unique topology with this property.*

Proof. The proof of the biconditional is analogous to that of proposition 3.1. To prove uniqueness, it suffices to prove that the C^k compact-open topology is first-countable, i.e., it has a countable local neighborhood base (Willard, 1970, Corollary 10.5, p. 71). Fix a Riemannian h, and let $C_i \subset M$ for $1 \leq i < \infty$ be a sequence of compacta such that $\bigcup_{i=1}^{\infty} C_i = M$. I claim that, for each tensor field χ, $B_k(\chi, 1/n; h, \bigcup_{i=1}^{n} C_i)$ for $1 \leq n < \infty$ is a countable local basis at χ if M is non-compact. For consider some open neighborhood of χ, which by definition must contain a set of the form $B_k(\chi, \epsilon; h, C)$ for some $\epsilon > 0$ and compact $C \subset M$. Let $m = \max\{\arg\min_n 1/n < \epsilon, \arg\min_n C \subseteq \bigcup_{i=1}^{n} C_i\}$. Then clearly $B_k(\chi, 1/m; h, \bigcup_{i=1}^{m} C_i) \subseteq B_k(\chi, \epsilon; h, C)$. If M is compact, then similarly for any set of the form $B_k(\chi, \epsilon; h, C)$, $B_k(\chi, 1/m; h, M) \subseteq B_k(\chi, \epsilon; h, C)$ with $m = \arg\min_n 1/n < \epsilon$, so $B_k(\chi, 1/n; h, M)$ for $1 \leq n < \infty$ is a countable local basis at χ. □

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