Minimal Approximations and Norton's Dome

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1 Introduction

Two recent foci for attention to infinite idealizations in philosophy of science have been minimal models, and a distinction between idealization and approximation. Minimal models are distinguished by having their explanatory or inferential power derive from their highly idealized nature (Batterman, 2002). In conflict with more traditional accounts of idealized models in science, according to which any de-idealization making the model more referentially accurate is desirable, minimal models succeed only by omitting or simplifying unimportant details. For example, under typical conditions the description of phase transitions in materials is greatly facilitated by the so-called thermodynamic limit, an idealization of the material as an infinite collection instead of a large but finite number of interacting molecules. Understanding the generic and universal properties of materials would only be hindered by adding these details back in. In other words, for minimal models, less is more.

Sometimes, though, an attempted idealization would force a model to have a property inconsistent with its other essential properties. For example, idealizing a large sphere of radius r as having an infinite radius allows one to make precise the sense in which the ratio of a large sphere's surface area ($4\pi r^2$) to its volume ($4\pi r^3/3$) is negligibly small. Yet there is no locus of points in Euclidean space infinitely far away from any other. So, while

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there is no *model* of an infinite sphere, one can still *approximate* the surface-area-to-volume ratio of a large sphere as zero. According to a distinction proposed by Norton (2012), then, an idealization is a model (within a certain specified class) that describes a target inexactly (or incompletely), while an approximation is only an inexact description of a property of a model.

The goal of the present note is to entwine these two strands. In doing so, I have two modest goals. The first, in section 2, to is point out a way in which the strategy of minimal modeling extends beyond idealized models *per se* to approximations. A *minimal approximation*, in this sense, is an inexact description of a property of some model or class of models that is more successful than certain more exact descriptions because it omits or simplifies details that are irrelevant for the purposes towards which that model or class of models is put. Because (mere) approximations distort or abstract from properties of models rather than models themselves, they can be used even when the desired models do not exist, or when it is not as clear that the models represent concrete systems of interest. Such cases (as discussed in section 3) can arise when one uses minimal approximations to infer properties of a theory. However one uses them, though, they can be justified in similar ways as minimal models by showing that regardless of how one relaxes the approximation (i.e., how one introduces further details), the conclusions that one would draw from using the approximation still hold (as least approximately).

The second goal is to apply this idea of minimal approximations to the case of Norton's dome (2008), a simple purported classical mechanical model that violates determinism: the dome is shaped so that a particle beginning at rest on the top may spontaneously slide down at an arbitrary time. Norton's description employs a inertial reference frame associated with the (in principle) accelerating dome, which Laraudogoitia (2013) points out is not strictly achievable in classical mechanics. This raises the question of the justification of the approximation, i.e., whether the failure of determinism is only an artifact introduced by the approximation itself. In section 3, I show that the answer to this question is negative: the violation of determinism is not affected when the reference frame is in fact slightly non-inertial. Furthermore, I argue that the inertial reference frame counts as a minimal (infinite) approximation in the sense that, in giving an inexact description of a class of classical mechanical models, it shows that a wide range of particular details about the mass of the dome, or whatever massive objects it is sitting on, are not relevant to the claim about classical physics that the dome is intended to illustrate.

Before continuing, I wish to offer a couple of comments on the scope of the foregoing discussion. First, I acknowledge that there is considerable debate regarding whether minimal models provide knowledge or allow users to truly learn about their targets. Addressing these objections, as important as they are, is beyond the scope of the present note because a commitment to the justified utility of minimal approximations does not entail a similar commitment to that of minimal models. This is because minimal approximations need not describe inexactly or even refer to a concrete target system: they only describe a property of a model inexactly. Indeed, one of the advantages of Norton's dome scenario to illustrate an example of a minimal approximation is that it is intended as a means to learn about the (failure of) deterministic properties of classical mechanics, not any concrete physical system.

This brings me to my second scope comment: I also acknowledge there is considerable debate regarding whether Norton's dome scenario, regardless of whether it is described with an approximation, is a legitimate model of classical mechanics. Here I shall assume the perspective of Fletcher (2012) that there are many theories of classical mechanics with varying models, used in varying contexts, some of which are deterministic and some of which are not. The conclusions drawn about Norton's dome are thus relative to theories that admit it as a model.

2 Minimal Approximations

In this section I describe in more detail the concept of a minimal approximation and its relationship with minimal models and the approximation/idealization distinction. To begin with the approximation/idealization distinction, here is how Norton (2012, p. 209) describes it:

> An *approximation* is an inexact description of a target system. It is propositional.
>
> An *idealization* is a real or fictitious system, distinct from the target system, some of whose properties provide an inexact description of some aspects of the target system.

Note that the success conditions for both of these are not specified, for which properties one should consider, and how inexact (or abstracted) they may be, ought to be a contextual matter. What is important is rather that an idealization is a new model. So, if an idealization is intended to be within the purview of a theory, it must be a model (that is, represent a possible state of affairs) permitted by that theory. By contrast, an approximation is only a new description of (a property of) a model.

With this distinction in mind, a *minimal model* is explicitly an idealization in this sense: it is highly inexact (i.e., "highly idealized"), but that does not mean it is *ipso facto* unsuccessful. Rather it is through artful inexactness that it

> does not let a lot of [extraneous] details get in the way. In many cases, the fine details will not be needed to characterize the phenomenon of interest, and may, in fact, actually detract from an understanding of that phenomenon. ...The adding of details with the goal of "improving" the minimal model [i.e., of making the model less inexact,] is self-defeating—such improvement is illusory. (Batterman, 2002, p. 22)

For accounts of physical phenomena, a minimal model is thus one "which most economically caricatures the essential physics" (Goldenfeld, 1992, p. 33). A minimal model therefore accrues computational and explanatory advantages because the descriptions it gives can apply equally well (even though usually not exactly) to a wide variety of target systems, ones distinguished by details ignored or simplified by the minimal model.

Because a minimal model succeeds in virtue of what it does and does not describe rather than by being a model *per se*, it would seem that the same strategy of minimality could be employed for approximation. Indeed:

> An idealization can be demoted to an approximation by discarding the idealizing system and merely extracting the inexact description; however, the inverse promotion to an idealizing system will not always succeed. (Norton, 2012, p. 211)

The general failure of the "inverse promotion" is important here because it indicates that the strategy of minimal approximation may succeed when the strategy of minimal modeling cannot. Principally, by no longer requiring that the (approximated) descriptions together cohere for a model in a specified class, they allow one to implement the strategy of minimality outside of that class. For example, the target of an approximation can well be a property of a model or a collection of models, rather than a property of a concrete system. This allows one to apply minimal approximations to models of a theory, say, that are not even intended to represent concrete systems, giving inexact descriptions of their properties in order to draw conclusions about those models (hence about the theory of which they are a part), regardless of whether the approximated description could obtain exactly in any model of the theory.

I shall argue in the next section that Norton's dome scenario is such an application. Before doing so, I wish to note something about the *justification* of minimal approximations. Just as they extend, in a sense, the applicability of the minimal strategy for models, minimal approximations can be justified along the same lines. In both cases, one must show, inasmuch as possible, that the inexactness introduced is not responsible for important features of the resulting description. In other words, to show that an approximation or idealization is minimal, one must show that the details it simplifies or ignores actually don't matter to conclusions one wishes to draw. It is important that this justification is *not* achieved through the addition of *particular* descriptive details—a de-idealization or increased representational exactness—for that would defeat the purpose of the strategy of minimality. Rather, one describes the additional details generically, then shows that those details do not play any role in the conclusions derived. In this way, the justification has a form analogous to the inference rule of universal generalization from predicate logic.

3 The Case of Norton's Dome

In this section I argue that the description of the dome in Norton (2008) counts as a minimal approximation. Section 3.1 describes the dome scenario and exhibits the family of solutions to the system's equation of motion that witness the failure of determinism.[1] Section 3.2 outlines the criticism by Laraudogoitia (2013) of the dome scenario, which I argue amounts to a demand for the justification of the use of an approximation, namely that a certain reference frame is inertial. Finally, section 3.3 shows how this approximation is justified, and argues that the approximation is minimal, for it eliminates details about the mass distribution of objects in dome scenario

[1] This section is based on the analysis in Fletcher (2012, §2.2) and Malament (2008, §4–5).

that are inessential for understanding the failure of determinism that they manifest.

3.1 Setting the Ground

Norton (2008) considers a cylindrically symmetric dome on which a point particle of mass m slides under the influence of a uniform gravitational force directed parallel to the dome's axis of symmetry. The dome's surface may be described as the surface of revolution of the curve $\boldsymbol{\gamma} : [0, r_0] \to \mathbb{R}^2$ described in x-y-coordinates (i.e., with the z-coordinates suppressed):

$$\boldsymbol{\gamma}(r) = (\gamma_x(r), \gamma_y(r)) = \left(\frac{2g^2}{3b^4} - \frac{2g^2}{3b^4} \left(1 - \frac{b^4 r}{g^2}\right)^{3/2}, -\frac{2b^2}{3g} r^{3/2} \right), \tag{1}$$

where r is the arc length of this generatrix, g is the constant free acceleration due to gravity, b is a dimensional constant, and x and y are, respectively, horizontal and vertical coordinates (the latter parallel to the dome's axis of symmetry). If the particle is constrained to move on the surface, then in general the force propelling the particle down the slope is the component of the total force $\boldsymbol{F}$ tangential to the surface:

$$\boldsymbol{F}_{||} = \left(\frac{d\boldsymbol{\gamma}}{dr} \cdot \boldsymbol{F} \right) \frac{d\boldsymbol{\gamma}}{dr}. \tag{2}$$

Its tangential acceleration, meanwhile, is in general

$$\boldsymbol{a}_{||} = \left(\frac{d^2 r}{dt^2} \right) \frac{d\boldsymbol{\gamma}}{dr}. \tag{3}$$

Combining equations 2 and 3 with the identity $\boldsymbol{F}_{||} = m\boldsymbol{a}_{||}$ yields the scalar equation of motion for the distance traversed r as

$$\frac{d^2 r}{dt^2} = \frac{d\boldsymbol{\gamma}}{dr} \cdot \frac{\boldsymbol{F}}{m}, \tag{4}$$

where

$$\frac{d\boldsymbol{\gamma}}{dr} = \left(\left(1 - \frac{b^4 r}{g^2}\right)^{1/2}, -\frac{b^2}{g} r^{1/2} \right). \tag{5}$$

Applying equation 5 and the fact that $\boldsymbol{F} = (0, -mg)$ to equation 4 then yields

$$\frac{d^2 r}{dt^2} = -g\frac{d\gamma_y}{dr} = b^2\sqrt{r}. \tag{6}$$

Supposing that the particle is placed at rest the top of the dome at time $t = 0$, one can show via a straightforward calculation that

$$r(t) = \begin{cases} 0, & \text{if } t \leq T, \\ \frac{1}{144}(b[t-T])^4, & \text{if } t > T, \end{cases} \tag{7}$$

is a solution to its equation of motion (6) where T is any nonnegative real number whatsoever. Consequently, the dome-and-particle system with a uniform gravitational force is not deterministic since there is a non-denumerable number of distinct solutions to the particle's equation of motion.

Note a couple of assumptions crucial to this analysis and central to the following discussion: first, the application of equation 6 assumes that a reference frame in which the dome is at rest is inertial; second, neither the total mass nor the mass distribution within the rigid dome (and whatever it may be attached to) need be invoked. The first figures in the criticism of the analysis by Laraudogoitia (2013), while the second figures in the claim that the approximation yielding an inertial reference frame is in fact minimal.

3.2 The Inertial Approximation

Laraudogoitia (2013) criticizes the use of a uniform gravitational field in the dome system. His primary complaint derives from the fact that the frame of reference in which the motion of the particle is described cannot be inertial—that is, Newton's second law would not hold—unless it assumes that the particle's acceleration is due to gravitational interaction with an infinite mass:

> [Norton] applies Newton's second law in a frame of reference R linked to [the dome] D and he does it in such a way as to presuppose that the frame is inertial. But D interacts with [the particle] P despite being at rest in the inertial system R (P's movements do not affect D). This is only possible if the mass of D is infinite, or if the mass of D+M is (M being some material body rigidly joined to D). (Laraudogoitia, 2013, p. 2930)

The argument runs in more detail as follows: according to Newton's third law, two objects interacting under mutual gravitational influence exert equal and opposite forces on one another. The particle in the dome system experiences a gravitational force in falling down the dome, so the massive objects generating that force—whether just the dome itself or the dome along with some other object(s)—experience an equal and oppositely directed force. Yet the reference frame used to describe the motion of the particle, in which the dome is at rest, is assumed to be inertial. The reference frame of an object experiencing a net force can only be inertial if that object has an infinite mass (for otherwise the object would be accelerating).[2] But if the object has infinite mass, then the gravitational acceleration it induces for the particle is infinite, not finite, as is assumed in the description of the dome and particle system.[3] Thus the dome scenario is not idealized, since as it cannot be described as a consistent model within classical mechanics with Newtonian gravitation, in contrast to what Norton (2008, p. 795) claims.[4] But it can be said to use an approximation, since treating the reference frame of the dome as inertial is an inexact description of its actual, non-inertial reference frame in a consistent model of classical mechanics.

It's important to emphasize that the use of this approximation does not make the description of dome scenario illegitimate for Laraudogoitia, as he

[2]This assumes some delicate details about how classical mechanics ought to treat systems of infinite mass. However, even if one takes infinitely massive systems as falling outside the scope of classical mechanics, the conclusion of the argument would still hold, for in that case no reference frame of an object experiencing a net force can be inertial.

[3]Laraudogoitia (2013, p. 2930) draws the further conclusion that the "infinite mass must be distributed across an infinite region of space because infinite mass distributed in a finite region of space has no physical meaning, at least if one does not 'decouple' the gravitational interaction," i.e., no longer require that gravitational and inertial mass be equal to each other. This conclusion is correct, but not for quite the reasons Laraudogoitia gives. There are perfectly mundane distributions of total infinite mass in a finite region of space—consider, for example, the mass density $\rho(r) \propto 1/r$ for $0 < r < 1$, and $\rho(r) = 0$ otherwise—but the mass concerned must nevertheless be distributed in an infinite region of space in order to generate a uniform gravitational attraction. In any case, this conclusion is not essential for the present discussion.

[4]In the terminology of Norton (2008), this is called a strong failure of internal idealization. The dome scenario has also been criticized as involving illegitimate idealizations, e.g. of infinite rigidity (Korolev, 2007a,b), but Norton (2008, p. 795) correctly points out that there can be no failure of idealization—what he calls in that paper "external idealization"—if the model in question is not intended to represent an independently specified system or phenomenon.

recognizes that, commonly in classical mechanics,

> an assumption of infinite mass is introduced to simplify calculation when we know that it will not affect the qualitative characteristics of the resulting evolution. This is where the difference from Norton's dome lies: in this last case we do not know if the indeterminism will be maintained when the mass of the dome is finite, (Laraudogoitia, 2013, p. 2930)

that is, whether the infinite approximation is justified in the sense described in the previous subsection. Until the question of justification of the approximation has been discharged, the analysis of the dome as manifesting a failure of determinism remains inconclusive. He ends with the charge that

> the burden of proof falls on anyone arguing that indeterminism is maintained in Norton's model even with total finite mass. I have no proof that with total finite mass there is no indeterminism and resolving the question looks complicated to say the least. (Laraudogoitia, 2013, p. 2931)

In the following subsection I resolve this question. Readers uninterested in the technical details may skip to the last two paragraphs of that subsection for a summary and discussion.

3.3 Justifying the Inertial Approximation

My goal in this section is to show that even when the particle and dome are both allowed to accelerate under their mutual (finite) gravitational force, the system remains indeterministic. To do so, I show that the particle can still fall down in the reference frame of the dome without assuming that this reference frame is inertial.

So, consider again the system with the point particle of mass m sliding under the influence of Newtonian gravitation on the dome, which is possibly attached rigidly to some other object. To show that the inertial approximation for the reference frame of the dome is justified, one can work in that reference frame using, for the particle, the modified version of Newton's second law adapted to non-inertial reference frames (Thornton and Marion, 2004, p. 392):

$$\boldsymbol{F} = \boldsymbol{F_{net}} - m\dot{\boldsymbol{\omega}} \times \boldsymbol{\gamma} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\gamma}) - 2m\boldsymbol{\omega} \times \dot{\boldsymbol{\gamma}}, \tag{8}$$

where $\boldsymbol{F_{net}}$ is the net force on the particle in any inertial reference frame, $\boldsymbol{\gamma}$ is the location of the particle on the dome given by equation 1, $\boldsymbol{\omega}$ is the angular velocity of the dome's reference frame with respect to any inertial reference frame, and the overdot represents differentiation with respect to time. The second term arises from the angular acceleration of the reference frame, while the third and fourth terms are the ("fictional") centrifugal and Coriolis forces, respectively.

It will be helpful to write the first term as $\boldsymbol{F_{net}} = \boldsymbol{F}_{||} + \boldsymbol{F_{cor}}$, where $\boldsymbol{F}_{||}$ is the force that would arise in the original dome scenario analyzed in section 3.1, with an unmoving dome and a uniform gravitational force, and $\boldsymbol{F_{cor}}$ is the "correction" to it arising from non-uniformities in the gravitational force and corresponding differences in the normal force of the dome keeping the particle passing through its surface. Then define

$$\boldsymbol{f} = \boldsymbol{F} - \boldsymbol{F}_{||} = \boldsymbol{F_{cor}} - m\dot{\boldsymbol{\omega}} \times \boldsymbol{\gamma} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\gamma}) - 2m\boldsymbol{\omega} \times \dot{\boldsymbol{\gamma}}, \tag{9}$$

which represents the net apparent force on the particle departing from $\boldsymbol{F}_{||}$, whether due to the "correction" force $\boldsymbol{F_{cor}}$ or any "fictitious" forces arising from describing the motion of the particle in a non-inertial reference frame.

Combining the analysis of section 3.1 (equations 2–4) with equation 9 yields that[5]

$$\ddot{r}(r) = -g\frac{d\gamma_y(r)}{dr} + \frac{d\boldsymbol{\gamma}(r)}{dr} \cdot \frac{\boldsymbol{f}(r)}{m}.$$

Next, applying equation 6 and the fact that $\ddot{r} = [d(\dot{r}^2)/dr]/2$ allows one to solve for $d(\dot{r}^2)/dr$:

$$\frac{d(\dot{r}^2)}{dr}(r) = 2\left(b^2\sqrt{r} + \frac{d\boldsymbol{\gamma}(r)}{dr} \cdot \frac{\boldsymbol{f}(r)}{m}\right).$$

Integrating this ordinary differential equation using the dummy variable r', from $r' = 0$ to $r' = r$, and assuming the same initial conditions as before, i.e., that the particle begins at rest at the top of the dome ($\dot{r}(0) = 0$), then returns that

$$\dot{r}(r) = \sqrt{\frac{4b^2r^{3/2}}{3} + \frac{2W(r)}{m}}, \tag{10}$$

where

$$W(r) = \int_0^r \frac{d\boldsymbol{\gamma}(r')}{dr'} \cdot \boldsymbol{f}(r')\, dr' \tag{11}$$

[5] The general strategy for the following calculations is found in Norton (2008, Appendix) and Laraudogoitia (2013, fn. 3).

is a kind of (effective) work done by the net apparent "correction" force $\boldsymbol{f}$ on the particle in the dome's reference frame. Inverting equation 10 to solve for t in terms of r results in

$$t(r) = \int_0^r \left(\frac{4b^2 (r')^{3/2}}{3} + \frac{2W(r')}{m} \right)^{-1/2} dr'. \tag{12}$$

This describes the time t needed for the particle to traverse a distance r along the surface of the dome. If determinism were to hold, then the particle would remain at the top of the dome ($r = 0$) forever, so that the integral would be infinite (or undefined). Thus, if this integral exists and is finite (at least for sufficiently small r), then determinism fails.

To show that the integral of equation 12 is indeed finite, it suffices to show that there are positive constants C and ϵ such that

$$b^2 \sqrt{r} + \frac{d\boldsymbol{\gamma}(r)}{dr} \cdot \frac{\boldsymbol{f}(r)}{m} > C r^{1-\epsilon} \tag{13}$$

for sufficiently small $r > 0$. For, in this case, the integral of both sides (from 0 to r) would yield

$$\frac{4b^2 r^{3/2}}{3} + \frac{2W(r)}{m} > \frac{2C}{1-\epsilon} r^{2-\epsilon} \tag{14}$$

after multiplying by two and using both equation 11 and the fact that both sides of the inequality are non-negative. Rearranging inequality 14 after taking the square root of both sides, then integrating again, reveals that

$$t(r) < \int_0^r \sqrt{\frac{1-\epsilon}{2C(r')^{2-\epsilon}}} dr' = \sqrt{\frac{2(1-\epsilon) r^{\epsilon}}{C \epsilon^2}}, \tag{15}$$

hence $t(r)$ would be finite.

So, to prove inequality 13, first note that for any C and $\epsilon < 1/2$, $b^2 \sqrt{r} > C r^{1-\epsilon}$ for sufficiently small $r > 0$. Thus it suffices to find conditions under which the remaining terms do not alter this inequality too much. Calculating that

$$\frac{d\boldsymbol{\gamma}}{dr} \cdot \frac{\boldsymbol{f}}{m} = \frac{d\boldsymbol{\gamma}}{dr} \cdot \left(\frac{\boldsymbol{F_{cor}}}{m} - \dot{\boldsymbol{\omega}} \times \boldsymbol{\gamma} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\gamma}) - 2\boldsymbol{\omega} \times \dot{\boldsymbol{\gamma}} \right) \tag{16}$$

using equation 9 (and dropping the argument r to reduce clutter), one can analyze first the term containing the "correction" force $\boldsymbol{F_{cor}}$: it is always non-negative so long as the magnitude of the component of the gravitational force on the particle tangential to the dome is non-decreasing as the particle slides down. This will be the case for sufficiently small r when the dome system is sufficiently far away from other gravitational sources besides the dome (and whatever it might be attached to).

Second, the last term, involving the Coriolis force, vanishes, as $\dot{\boldsymbol{\gamma}} = (d\boldsymbol{\gamma}/dr)(dr/dt)$, so in fact $\boldsymbol{\omega} \times \dot{\boldsymbol{\gamma}}$ and $d\boldsymbol{\gamma}/dr$ are orthogonal. Since the remaining terms are negative but vanish at $r = 0$, to prove inequality 13 it will suffice to show that their magnitude grows no faster than $b^2\sqrt{r}$ for small r. To do this, it will be helpful to bound their magnitudes using the fact that $|\boldsymbol{a} \cdot \boldsymbol{b}| \leq |\boldsymbol{a}||\boldsymbol{b}|$ and $|\boldsymbol{a} \times \boldsymbol{b}| \leq |\boldsymbol{a}||\boldsymbol{b}|$ for any vectors $\boldsymbol{a}, \boldsymbol{b}$, and that $|d\boldsymbol{\gamma}/dr| = 1$ by direct calculation from equation 5:

$$\begin{aligned} \left| \frac{d\boldsymbol{\gamma}}{dr} \cdot (\dot{\boldsymbol{\omega}} \times \boldsymbol{\gamma}) \right| &\leq |\dot{\boldsymbol{\omega}}||\boldsymbol{\gamma}|, \\ \left| \frac{d\boldsymbol{\gamma}}{dr} \cdot (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\gamma})) \right| &\leq |\boldsymbol{\omega}|^2|\boldsymbol{\gamma}|. \end{aligned}$$

Inspection of equation 1 indicates that $|\boldsymbol{\gamma}|$ grows no faster than $\sqrt{r}$ for small r,[6] while $|\boldsymbol{\omega}|$ and $|\dot{\boldsymbol{\omega}}|$ are both non-negative and continuous functions of r, vanishing at $r = 0$. Hence for small r, both terms are in fact bounded above in magnitude by $|\boldsymbol{\gamma}|$. This shows that both of the middle two negative terms of equation 16 are dominated by $b^2\sqrt{r}$ for small r, proving that inequality 13 holds.

The foregoing have therefore established the justification for the inertial approximation used in Norton's original dome scenario, for, under the right conditions, the conclusions drawn from the approximation about the failure of determinism still hold when the reference frame of the dome is more exactly

[6] In fact, one can show that $|\boldsymbol{\gamma}(r)|$ grows no faster than Ar for some $A > 0$ when r is small, as its derivative at $r = 0$ is positive and finite. To show this, without loss of generality select units in which $b^2 = g$. Then $|\boldsymbol{\gamma}(r)| = \frac{2}{3}[(1 - (1 - r)^{3/2})^2 + r^3]^{1/2} = \frac{2}{3}[2 - 3r - 2(1 - r)^{3/2} + 3r^2]^{1/2}$. Direct calculation shows that

$$\frac{d|\boldsymbol{\gamma}(r)|}{dr} = \frac{(1 - r)^{1/2} + 2r - 1}{[2 - 3r - 2(1 - r)^{3/2} + 3r^2]^{1/2}}.$$

Although the limit $r \to 0$ yields an indeterminate form, one can perform the substitution $z = (1 - r)^{1/2}$, rewrite the resulting fraction as a square root with which one commutes the limit $z \to 1$, then apply l'Hôpital's rule.

treated as being slightly non-inertial. These conditions are fairly simple: the mass of the dome system must be distributed so that, at least for locations close to the apex of the dome (i.e., for small r), the net gravitational force on the particle into the dome is not less than that which would arise from a uniform gravitational force, as treated in section 3.1 and considered originally by Norton (2008).[7] This will be case if the dome system is sufficiently far away from other large gravitational sources.

Moreover, this all shows that the approximation is a minimal one, for it reveals that details about the construction of the dome (outside of a neighborhood of its apex) and that of any object to which it is attached do not matter for the conclusions about the failure of determinism. Adding in those details would only complicate the dome scenario, distracting one from the essential aspects of the physics. Finally, the justification of the minimal approximation did not require this detail either, for it described the system without the inertial approximation generically, to which universal generalization was applied.

4 Conclusions and Future Work

In this note, I've suggested how the strategy of minimality employed for models—idealize as much as possible to simplify or abstract away details extraneous from the essential features—can be extended to approximations, too, and justified in much the same way. This expands the scope of the strategy because approximations are only inexact descriptions of a model: they do not require a new model or even a particular concrete system to describe. So, while the question of whether, and if so, how minimal models inform us of their target systems is important, this question needn't arise for minimal approximations, which make possible broader and more insightful inferences and calculations about properties of a model or models one already has.[8]

In my application of these ideas to Norton's dome scenario, I have argued

[7] This requirement implies, of course, that the particle must not lose contact with the dome, at least initially. See Malament (2008) for a discussion of this issue in the original dome scenario.

[8] In this sense, minimal approximations fit with the perspective of Knuuttila and Boon (2011), who wish to reduce the representational role for models in favor of their use as tools to forge more general representational relationships.

that it uses a minimal approximation of an inertial reference frame, showing that the details of the distribution of mass in the dome (and whatever it may be attached to) are not important for the demonstration of the failure of determinism. It is intended as a vehicle for inference about classical mechanics, not a model for a real-world system, so the question of the exactness of its description does not arise. This answers Laraudogoitia's concern that the scenario must assign an infinite mass to the dome.

I suspect that one can fruitfully identify minimal approximations in many other circumstances in the foundations of science. To take some examples from physics, Norton (2012, §4.2) himself suggests that many techniques from statistical mechanics, as well as reversible processes in thermodynamics (Norton, 2014) are approximations rather than idealizations. Further, Fletcher (2017, p. 190) raises the question of whether point test particles in spacetime theories count as idealizations, but perhaps they are better understood as approximations.[9] Showing that these are minimal, and justified, would go a long way towards a better understanding of the conceptual basis of those theories.

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