If Bertlmann had three feet

Alexander Afriat
a.afriat-alumni@lse.ac.uk

ABSTRACT: It is argued that perfect quantum correlations cannot be due to additive conservation.

> Dr. Bertlmann likes to wear two socks of different colours. Which colour he will have on a given foot on a given day is quite unpredictable. But when you see that the first sock is pink you can already be sure that the second sock will not be pink. Observation of the first, and experience of Bertlmann, gives immediate information about the second.
>
> Bell (1981)

Most interesting features of quantum mechanics have at least something to do with interference, which will not, however, be at issue here at all. Interference is brought out by an appeal to different bases, but here the same (product) basis is adhered to throughout.

It is often claimed, and even more often suspected, that *conservation accounts for quantum correlations* (by which *perfect* quantum correlations will be meant). The underlying intuition is well expressed by Bertlmann's socks, or by the fact that the distribution of wine over two glasses can be worked out, provided one knows the total amount in both, by a measurement on one of them. Or consider a conservative classical Hamiltonian $H = T + V(q)$, where T is kinetic energy and the potential energy V depends only on position. Conservation here means that exchanges of kinetic and potential energy along a trajectory have to satisfy $H_0 = T + V$, where H_0 is the total energy of that motion. Kinetic energy will then be a function only of position, so that at any stage of the motion $T(q) = H_0 - V(q)$ can be deduced from the potential; they are perfectly correlated. Or take two free classical particles, each one subject only to the influence of the other, with initial momenta p_0 and p_0'. Irrespective of whether they collide their total momentum will remain $\pi = p_0 + p_0'$; the momentum $p' = \pi - p$ of the primed particle can always be derived from the momentum p of the other. These cases are paradigmatic for additive conservation.

Quantum correlations are similar, especially at a given instant, and with only two subsystems; but they have nothing to do with conservation. When the contrary is claimed I think *additive* conservation is meant; but that can be broken up into two logically independent parts: 1. *conservation*; and 2. *an 'additivity' condition*, presently to be defined and denoted (λ). Quantum correlations can have nothing to do with time, which has everything to do with conservation; so what is fundamentally at issue is additivity.

I will argue that an additivity condition can be constructed to account for quantum correlations with two subsystems, *but only with two*; where there are more, *quantum correlations are too strong to be explained by additivity*. An explanation that only works in a narrow special case should be viewed as no explanation at all; so quantum correlations have nothing to do with additivity.

Take three socks (on an equal number of feet) rather than two: once the pink sock is found on one foot, we know the remaining socks are on the other feet, but we cannot infer where the blue one is. With three glasses a measurement on one glass only tells us how much wine is in the other two together, not how much is in the third. Triorthogonal decompositions appear to go beyond the knowledge available in the above cases, and indeed to tell us where the blue sock is, or how much wine is in the third glass.

Consider the triorthogonal decomposition

$$|\Psi\rangle = \sum_m c_m \,|\alpha_m^1\rangle\,|\alpha_m^2\rangle\,|\alpha_m^3\rangle \in \mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2 \otimes \mathcal{H}^3, \tag{1}$$

where the Hilbert spaces $\mathcal{H}^r = \mathrm{span}\{|\alpha_1^r\rangle, |\alpha_2^r\rangle, \ldots\}$ ('span' denotes the *closed span*) have the same dimensionality, and $\langle\alpha_i^r \,|\, \alpha_j^r\rangle = \delta_{ij}$ $(r = 1, 2, 3)$. The state $|\Psi\rangle$ determines a trijective or one-to-one-to-one correspondence $|\alpha_m^1\rangle \leftrightarrow |\alpha_m^2\rangle \leftrightarrow |\alpha_m^3\rangle$ between the bases $\{|\alpha_m^1\rangle\}$, $\{|\alpha_m^2\rangle\}$ and $\{|\alpha_m^3\rangle\}$, $m = 1, 2, \ldots$. To make the correspondence observable and give rise to correlations, we can construct the self-adjoint operator

$$\mathbf{A} = \mathbf{A}^1 + \mathbf{A}^2 + \mathbf{A}^3 : \mathcal{H} \to \mathcal{H},$$

where the operators

$$\mathbf{A}^1 = A^1 \otimes I \otimes I : \mathcal{H} \to \mathcal{H}$$
$$\mathbf{A}^2 = I \otimes A^2 \otimes I : \mathcal{H} \to \mathcal{H}$$
$$\mathbf{A}^3 = I \otimes I \otimes A^3 : \mathcal{H} \to \mathcal{H},$$

and the three (maximal) operators A^r have the form

$$\sum_m \lambda_m^r \,|\alpha_m^r\rangle\langle\alpha_m^r| : \mathcal{H}^r \to \mathcal{H}^r,$$

$r = 1, 2, 3$. The operator A^r establishes a one-to-one correspondence $\lambda_1^r \leftrightarrow |\alpha_1^r\rangle$, $\lambda_2^r \leftrightarrow |\alpha_2^r\rangle, \ldots$ between eigenvalues and basis vectors, thus extending to the three spectra $\Lambda^r = \{\lambda_1^r, \lambda_2^r, \ldots\}$ the aforementioned trijective correspondence between the bases $(r = 1, 2, 3)$. The discovery of an eigenvalue therefore selects one in both of the other two factor spaces. This will be particularly surprising if we require that

$$\lambda_m^1 + \lambda_m^2 + \lambda_m^3 = \lambda \qquad (\lambda)$$

for all m (so that $\mathbf{A}|\Psi\rangle = \lambda|\Psi\rangle$); for then the entire system possesses an amount λ of the physical quantity $\mathfrak{A}$ represented by $\mathbf{A}$, whose exact distribution over all three subsystems would be determined by a measurement on any one of them. We expect this with two subsystems, maybe not with three.

Consider the Cartesian product $\Lambda = \Lambda^1 \times \Lambda^2 \times \Lambda^3 = \{(\lambda_{m^1}^1, \lambda_{m^2}^2, \lambda_{m^3}^3)\}$ of the spectra, and the subset

$$\Lambda_{(\lambda)} = \{(\lambda_{m^1}^1, \lambda_{m^2}^2, \lambda_{m^3}^3) : \lambda_{m^1}^1 + \lambda_{m^2}^2 + \lambda_{m^3}^3 = \lambda\} \subset \Lambda$$

satisfying condition (λ). The discovery of an eigenvalue λ_n^s (the value $s = 1, 2$ *or* 3 of the superscript is chosen by the experimenter, that of the subscript by nature) will determine a subset

$$\Lambda_{(\lambda; m^s = n)} = \{(\lambda_{m^1}^1, \lambda_{m^2}^2, \lambda_{m^3}^3) : \lambda_{m^1}^1 + \lambda_{m^2}^2 + \lambda_{m^3}^3 = \lambda; m^s = n\} \subset \Lambda_{(\lambda)},$$

which would be a singleton if there were only two subsystems.

The triorthogonal decomposition (1) determines another subset of $\Lambda_{(\lambda)}$, namely $\Lambda_{(1)} = \{(\lambda_m^1, \lambda_m^2, \lambda_m^3)\} \subset \Lambda_{(\lambda)}$. Here the discovery of the same eigenvalue λ_n^s would select the triple

$$\Lambda_{(1; m^s = n)} = (\lambda_n^1, \lambda_n^2, \lambda_n^3) \subset \Lambda_{(\lambda; m^s = n)}.$$

We would have $\Lambda_{(1)} = \Lambda_{(\lambda)}$ and $\Lambda_{(1; m^s = n)} = \Lambda_{(\lambda; m^s = n)}$ with two subsystems, *and only then*. This means that the correlations due to the triorthogonal decomposition, being stronger than those due to (λ), cannot be attributed to such an additivity condition.

The matter can also be seen as follows. A vector $|\Phi\rangle$ belonging to eigenspaces corresponding to eigenvalues $\lambda_{m^r}^r$ that add up to λ will be an eigenvalue of $\mathbf{A}$ corresponding to λ; in other words conditions

$$\left(|\alpha_{m^1}^1\rangle\langle\alpha_{m^1}^1| \otimes |\alpha_{m^2}^2\rangle\langle\alpha_{m^2}^2| \otimes |\alpha_{m^3}^3\rangle\langle\alpha_{m^3}^3|\right)|\Phi\rangle = |\Phi\rangle \text{ and } \lambda_{m^1}^1 + \lambda_{m^2}^2 + \lambda_{m^3}^3 = \lambda$$

together imply that $\mathbf{A}|\Phi\rangle = \lambda|\Phi\rangle$. So

$$\Omega_{(\lambda)} = \mathrm{span}\{|\alpha_{m^1}^1\rangle|\alpha_{m^2}^2\rangle|\alpha_{m^3}^3\rangle : \lambda_{m^1}^1 + \lambda_{m^2}^2 + \lambda_{m^3}^3 = \lambda\}$$

is the eigenspace belonging to λ.

The commuting set $\{\mathbf{A}, \mathbf{A}^s\}$ (again, $s = 1, 2$ *or* 3), representing the amount of $\mathfrak{A}$ respectively possessed by the whole system and by subsystem s, *will not be complete*, since $\dim \Omega_{(\lambda; m^s = n)} > 1$ for the subspace

$$\Omega_{(\lambda;m^s=n)} = \text{span}\{|\alpha^1_{m^1}\rangle|\alpha^2_{m^2}\rangle|\alpha^3_{m^3}\rangle : \lambda^1_{m^1} + \lambda^2_{m^2} + \lambda^3_{m^3} = \lambda; m^s = n\}$$

determined by λ^s_n. But with a triorthogonal expansion, the two measurements $\mathbf{A}$ and $\mathbf{A}^s$ determine a single product $|\alpha^1_m\rangle|\alpha^2_m\rangle|\alpha^3_m\rangle$. So the correlations contained in a triorthogonal expansion go beyond those due to the set $\{\mathbf{A}, \mathbf{A}^s\}$, which would otherwise only have selected the larger subspace $\Omega_{(\lambda;m^s=n)}$.

If there were only two subsystems, $\{\mathbf{A}, \mathbf{A}^s\}$ would be a *complete* commuting set, since a single product would be determined by measurement of $\mathbf{A}$ and $\mathbf{A}^s$; $\{\mathbf{A}, \mathbf{A}^s\}$ would represent neither more nor less correlation than what is contained in a triorthogonal decomposition.

The question has so far concerned a single instant; but one can also wonder about evolution. The triorthogonal decomposition is preserved if the vectors $|\alpha^1_m\rangle|\alpha^2_m\rangle|\alpha^3_m\rangle$ are energy eigenvectors, for then the time evolution operator does not change their directions. We can then speak of *conservation*, and say that the correlations in question *cannot be attributed to an additive conservation law*.

Reference

J.S. Bell (1981) “Bertlmann’s socks and the nature of reality” in J.S. Bell, *Speakable and unspeakable in quantum mechanics*, Cambridge University Press, 1987, pp.139-58