Oversights in the Respective Theorems of von Neumann and Bell are Homologous

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We show that the respective oversights in the von Neumann's general theorem against all hidden variable theories and Bell's theorem against their local-realistic counterparts are homologous. When latter oversight is rectified, the bounds on the CHSH correlator work out to be $\pm 2\sqrt{2}$ instead of ± 2.

I. INTRODUCTION

One of the characteristic features of quantum theory is the fact that the result of any individual measurement event is not determined by the theory. If the quantum state of a physical system is not represented by an eigenvector of the observable being measured, then the theory predicts only a statistical distribution of possible results of measurements. This apparent deficiency — and it can be viewed as deficiency — has long inspired investigations into the possibility of a deeper theory underlying quantum theory, built on the so-called "dispersion-free" states [1]. Such states are supposed to be specified by "hidden variables", in addition to the usual quantum mechanical state vector, and determine the outcomes of individual measurement events. That, in turn, implies simultaneous assignment of definite values to all of the observables of a quantum mechanical system, making the resulting theory compatible with Einstein's conception of realism [2][3]. The hypothetical variables accomplishing this feat are called "hidden" because if states with prescribed values of such variables can be actually prepared, then quantum theory would be rendered observationally inadequate.

Against this background, von Neumann set out to prove in his theorem [1], based on a set of four assumptions [4][5], that no hidden variable theories consisting dispersion-free states are possible that could simultaneously assign definite values to all the observables of a quantum system consistently [6]. In other words, he set out to prove that any theory supplementing quantum theory with hidden variables to produce dispersion-free states would be logically inconsistent.

In recent decades Bell's theorem [7] has become much more familiar compared to von Neumann's theorem because of its significance for quantum information theory, quantum computing, quantum cryptography, and other quantum technologies. But unlike von Neumann's theorem, it does not purport to rule out all hidden variable theories. In fact, Bell was a supporter of Bohm's non-local hidden variable theory [8]. Indeed, the scope of Bell's theorem is limited. It claims that no locally causal and realistic theory of the kind envisaged by Einstein and formed the conceptual basis for the argument of Einstein, Podolsky, and Rosen [2] can reproduce all of the statistical predictions of quantum theory.

By now it is well known that von Neumann's theorem against hidden variable theories reproducing the predictions of quantum theory contains a rather serious oversight [9][10][11][12][13]. What is less well known and less appreciated in some quarters is that Bell's theorem against *local* hidden variable theories also contains a similar oversight, in addition to other questionable assumptions [14][15][16]. In what follows, we compare the respective oversights in the theorems of von Neumann and Bell against hidden variable theories and show that they are homologous. It is important to bear in mind, however, that while Bell's theorem is not a theorem within quantum theory itself, von Neumann's theorem is a mathematically correct theorem providing significant other results for the logical foundations of quantum theory.

II. THE OVERSIGHT IN VON NEUMANN'S THEOREM

It appears that Einstein was aware of the oversight in von Neumann's theorem before the publication of his argument against the completeness of quantum mechanics in 1935, co-authored with Podolsky and Rosen [2]. He, however, does not seem to have published his criticism anywhere. We learn about it from what has been reported by Shimony some six decades later in the collection of his own papers [6]. In the third paragraph of Section 1 of Chapter 7, Shimony writes:

> The great paper of Einstein, Podolsky and Rosen [2] of 1935 concludes that quantum mechanics is an incomplete theory, without suggesting that changes of the theory are prerequisites to the job of completion. They make no reference to von Neumann's argument, which had been published three years earlier, although they could hardly have been unaware of it. ... It is conceivable that Einstein was critical of one

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> or more of von Neumann's premises, as were Bell [11], Siegel [12], and Kochen and Specker [13] thirty years later. However, in the absence of specific evidence that this is so, I doubt it.

As noted, this quote is from the Chapter 7 in the collection of Shimony's papers, which was originally published in 1971 [6]. But by 1993 Shimony's doubt about Einstein's awareness and criticism of von Neumann's oversight were put to rest by Peter G. Bergmann, who was Einstein's collaborator. In the comment added to his original paper, Shimony writes:

> The question raised in the third paragraph of Section 1 was cleared up by a conversation with Prof. Peter G. Bergmann around 1980. He recalled a discussion with Einstein and Valentine Bargmann around 1938 at the Institute for Advanced Study, during which Einstein took von Neumann's book from the shelf and pointed to premise B′ of von Neumann's theorem (in Section 1 of Chapter IV): "If $\mathcal{R}$, $\mathcal{S}$, ... are arbitrary quantities and a, b, ... real numbers, then $\mathrm{Exp}(a\,\mathcal{R} + b\,\mathcal{S} + \dots) = a\,\mathrm{Exp}(\mathcal{R}) + b\,\mathrm{Exp}(\mathcal{S}) + \dots$." Einstein then said that there is no reason why this premise should hold in a state not acknowledged by quantum mechanics if $\mathcal{R}$, $\mathcal{S}$, etc. are not simultaneously measurable. Einstein's criticism is essentially the same as those of Siegel [12], Jauch and Piron [10], Bell [11], and Kochen and Specker [13] nearly thirty years later.

Although Shimony does not mention Grete Hermann's contribution to the debate [9], by now it is well known that she was the first to publish a criticism of von Neumann's argument in her 1935 essay on the philosophy of quantum mechanics, essentially along the line of the criticism by Einstein, Bell [11], Siegel [12], and Kochen and Specker [13].

Among these criticisms of von Neumann's assumption the one that appears to have struck the chord is that by Bell [11]. In anticipation of what we wish to demonstrate below, let us restrict his assumption to four arbitrary observables, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$, and four real numbers, a, b, c, and d. Then, within quantum theory, using normalized state vectors $|\,\psi\rangle$, his assumption is: Given the observables $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$, there exists an observable $a\,\mathcal{R} + b\,\mathcal{S} + c\,\mathcal{T} + d\,\mathcal{U}$ such that

$$a\,\langle\psi\,|\,\mathcal{R}\,|\,\psi\rangle + b\,\langle\psi\,|\,\mathcal{S}\,|\,\psi\rangle + c\,\langle\psi\,|\,\mathcal{T}\,|\,\psi\rangle + d\,\langle\psi\,|\,\mathcal{U}\,|\,\psi\rangle \;=\; \langle\psi\,|\{a\,\mathcal{R} + b\,\mathcal{S} + c\,\mathcal{T} + d\,\mathcal{U}\}|\,\psi\rangle\,, \tag{1}$$

where $\langle\psi\,|\,\mathcal{R}\,|\,\psi\rangle$ represents the quantum mechanical expectation value of the self-adjoint operator $\mathcal{R}$ in the state $|\psi\rangle$. His theorem then asserts that there exists a self-adjoint operator $\mathcal{R}$ on the corresponding Hilbert space $\mathscr{H}$ such that

$$\big\langle\psi\,\big|\,\mathcal{R}^2\,\big|\,\psi\big\rangle \neq \langle\psi\,|\,\mathcal{R}\,|\,\psi\rangle^2. \tag{2}$$

That is to say, the state defined by $\langle\psi\,|\,\mathcal{R}\,|\,\psi\rangle$ is not dispersion-free over the quantum mechanical observables $\mathcal{R}$. But as Bell points out in his exposition of von Neumann's oversight [11], the relation (1) — *i.e.*, the additivity of expectation values — is one of the peculiar properties of quantum mechanical states. It need not hold in a hidden variable theory.

It is not difficult to understand why von Neumann was led to assume otherwise. For a hypothetical dispersion-free state represented by a normalized vector $|\,\psi,\,\lambda\,\rangle$ characterizing a hidden variable theory that has no statistical character by definition, the expectation value of any observable $\mathcal{R}$ would be necessarily equal to one of its eigenvalues $\mathscr{R}(\lambda_k)$:

$$\langle\,\psi,\,\lambda_k\,|\,\mathcal{R}\,|\,\psi,\,\lambda_k\,\rangle = \mathscr{R}(\lambda_k), \tag{3}$$

where $\mathscr{R}(\lambda_k)$ is a unique scalar determined by a specific hidden variable $\lambda_k \in \{\lambda_1, \dots, \lambda_n\}$ and the eigenvalue equation

$$\mathcal{R}\,|\,\psi\,\rangle = \mathscr{R}\,|\,\psi\,\rangle, \tag{4}$$

so that the measurement of $\mathcal{R}$ in the state $|\,\psi,\,\lambda\rangle$ specified by ψ and λ results with certainty in that eigenvalue $\mathscr{R}(\lambda_k)$. Consequently, such a dispersion-free state would satisfy

$$\big\langle\,\psi,\,\lambda\,|(\,\mathcal{R} - \langle\,\psi,\,\lambda\,|\,\mathcal{R}\,|\,\psi,\,\lambda\,\rangle)^2|\,\psi,\,\lambda\,\big\rangle = 0, \tag{5}$$

or, equivalently,

$$\big\langle\,\psi,\,\lambda\,|\,\mathcal{R}^2\,|\,\psi,\,\lambda\,\big\rangle = \langle\,\psi,\,\lambda\,|\,\mathcal{R}\,|\,\psi,\,\lambda\,\rangle^2, \tag{6}$$

for all observables $\mathcal{R}$, since – by hypothesis – every physical quantity would have a unique value $\mathscr{R}(\lambda_k)$. The quantum mechanical state specified by $|\psi\rangle$ is then recovered by uniform averaging over λ, which would give the expectation value

$$\langle\,\psi\,|\,\mathcal{R}\,|\,\psi\,\rangle = \lim_{n\,\gg\,1}\frac{1}{n}\sum_{k\,=\,1}^{n}\langle\,\psi,\,\lambda_k\,|\,\mathcal{R}\,|\,\psi,\,\lambda_k\,\rangle = \lim_{n\,\gg\,1}\frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{R}(\lambda_k). \tag{7}$$

In what follows we will omit the limits on summations for convenience, with the large n limits understood implicitly.

It is important to note that Eq. (7) exhibits an inter-theoretical relation between quantities pertaining to two entirely different theories. On its LHS is the standard quantum mechanical expectation value of the operator $\mathcal{R}$ in the state $|\,\psi\,\rangle$, and on its RHS is a classical average of $\mathscr{R}(\lambda_k)$ over the hidden variables λ, with no physical significance attributed to λ. To be sure, a fully developed hidden variable theory would include a detailed theory of measurement accounting for the behaviour of λ during the measurement process itself, but such a detailed theory is not required for our analysis. On the other hand, as noted in the Introduction, if states supplemented with such hidden variables can be actually prepared, then quantum theory would be rendered observationally inadequate. Therefore, λ must be traced out by uniform averaging, as done in Eq. (7). An explicit example of how that works can be found in Section 2 of Ref. [11].

Since by hypothesis the inter-theoretical relation (7) holds for all observables, we can express the LHS of Eq. (1) as

$$a\,\langle\psi\,|\,\mathcal{R}\,|\,\psi\rangle+b\,\langle\psi\,|\,\mathcal{S}\,|\,\psi\rangle+c\,\langle\psi\,|\,\mathcal{T}\,|\,\psi\rangle+d\,\langle\psi\,|\,\mathcal{U}\,|\,\psi\rangle=\frac{a}{n}\sum_{k=1}^{n}\mathscr{R}(\lambda_k)+\frac{b}{n}\sum_{k=1}^{n}\mathscr{S}(\lambda_k)+\frac{c}{n}\sum_{k=1}^{n}\mathscr{T}(\lambda_k)+\frac{d}{n}\sum_{k=1}^{n}\mathscr{U}(\lambda_k)\,, \tag{8}$$

where each of the eigenvalues $\mathscr{R}(\lambda_k)$, $\mathscr{S}(\lambda_k)$, $\mathscr{T}(\lambda_k)$, and $\mathscr{U}(\lambda_k)$ is a unique scalar number. But that makes the last equality a slippery slope. Because every school child knows that a sum of averages is equal to the average of the sum:

$$\frac{a}{n}\sum_{k=1}^{n}\mathscr{R}(\lambda_k)+\frac{b}{n}\sum_{k=1}^{n}\mathscr{S}(\lambda_k)+\frac{c}{n}\sum_{k=1}^{n}\mathscr{T}(\lambda_k)+\frac{d}{n}\sum_{k=1}^{n}\mathscr{U}(\lambda_k)=\frac{1}{n}\sum_{k=1}^{n}\big\{a\,\mathscr{R}(\lambda_k)+b\,\mathscr{S}(\lambda_k)+c\,\mathscr{T}(\lambda_k)+d\,\mathscr{U}(\lambda_k)\big\}. \tag{9}$$

Comparing the equalities (1) and (9) it is indeed tempting to conclude that (9) is a natural dispersion-free counterpart of the quantum mechanical property (1), and therefore additivity of expectation values must hold also in any hidden variable theory, as von Neumann had assumed. But the obvious disanalogy between (1) and (9) is that, while $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ are Hermitian operators in a complex Hilbert space and therefore may not commute with each other in general, $\mathscr{R}(\lambda_k)$, $\mathscr{S}(\lambda_k)$, $\mathscr{T}(\lambda_k)$, and $\mathscr{U}(\lambda_k)$ — being the specific results of the measurements of the observables $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ — are scalar numbers and therefore necessarily commute with each other. More importantly, as explained by Bell [11], eigenvalues of non-commuting observables are not additive [9]. The result of the measurement of the sum $a\,\mathcal{R}+b\,\mathcal{S}+c\,\mathcal{T}+d\,\mathcal{U}$ of non-commuting operators will not be the sum $a\,\mathscr{R}(\lambda_k)+b\,\mathscr{S}(\lambda_k)+c\,\mathscr{T}(\lambda_k)+d\,\mathscr{U}(\lambda_k)$ of the eigenvalues of those operators in a dispersion-free state. Consequently, for non-commuting operators $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$,

$$\langle\,\psi,\,\lambda_k\,|\{a\,\mathcal{R}+b\,\mathcal{S}+c\,\mathcal{T}+d\,\mathcal{U}\,\}|\,\psi,\,\lambda_k\,\rangle\neq a\,\mathscr{R}(\lambda_k)+b\,\mathscr{S}(\lambda_k)+c\,\mathscr{T}(\lambda_k)+d\,\mathscr{U}(\lambda_k), \tag{10}$$

and therefore the quantum mechanical expectation value

$$\langle\,\psi\,|\{a\,\mathcal{R}+b\,\mathcal{S}+c\,\mathcal{T}+d\,\mathcal{U}\,\}|\,\psi\,\rangle\neq\frac{1}{n}\sum_{k=1}^{n}\big\{a\,\mathscr{R}(\lambda_k)+b\,\mathscr{S}(\lambda_k)+c\,\mathscr{T}(\lambda_k)+d\,\mathscr{U}(\lambda_k)\big\}. \tag{11}$$

In the Appendix A below we prove this inequality and the fact that the eigenvalue of the sum $a\,\mathcal{R}+b\,\mathcal{S}+c\,\mathcal{T}+d\,\mathcal{U}$ of operators is not the sum $a\,\mathscr{R}+b\,\mathscr{S}+c\,\mathscr{T}+d\,\mathscr{U}$ of eigenvalues, unless $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ commute with each other. We also prove that the inequality in (11) can reduce to equality *if and only if* the operators $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ commute with each other. But that means that, for non-commuting observables, in dispersion-free states, assumption (9) is not correct:

$$\frac{a}{n}\sum_{k=1}^{n}\mathscr{R}(\lambda_k)+\frac{b}{n}\sum_{k=1}^{n}\mathscr{S}(\lambda_k)+\frac{c}{n}\sum_{k=1}^{n}\mathscr{T}(\lambda_k)+\frac{d}{n}\sum_{k=1}^{n}\mathscr{U}(\lambda_k)\neq\frac{1}{n}\sum_{k=1}^{n}\big\{a\,\mathscr{R}(\lambda_k)+b\,\mathscr{S}(\lambda_k)+c\,\mathscr{T}(\lambda_k)+d\,\mathscr{U}(\lambda_k)\big\}. \tag{12}$$

As Bell notes, it is not possible to work out the result of the measurement of a sum of non-commuting observables by trivially adding the results of separate measurements on each observable because each requires a distinct experiment.

The example Bell gives to illustrate this well known fact is that of the spin components of a fermion. A measurement of σ_x can be made with a suitably oriented Stern-Gerlach magnet. But the measurement of σ_y would require a different orientation of the magnet. And the measurement of the sum $(\sigma_x+\sigma_y)$ would again require a third and quite a different orientation of the magnet from the previous two. Consequently, the result of the last measurement — *i.e.*, an eigenvalue of $(\sigma_x+\sigma_y)$ — will not be the sum of an eigenvalue of σ_x plus an eigenvalue of σ_y. The additivity of the expectation values, namely, $\langle\,\psi\,|\,\sigma_x\,|\,\psi\,\rangle+\langle\,\psi\,|\,\sigma_y\,|\,\psi\,\rangle=\langle\,\psi\,|\,\sigma_x+\sigma_y\,|\,\psi\,\rangle$, is a peculiar property of the quantum states $|\,\psi\,\rangle$. It would not hold for individual eigenvalues of non-commuting observables in a dispersion-free state of a hidden variable theory. In a dispersion-free state, every observable would have a unique value equal to one of its eigenvalues. And since there can be no linear relationship between the eigenvalues of non-commuting observables, the additivity relation (1) that holds for quantum mechanical states would not hold for dispersion-free states. Therefore, von Neumann's assumption (9) above — as innocent and inevitable it may seem mathematically — is wrong physically. And since at least one of the assumptions used in the proof of his theorem is wrong, his conclusion ruling out hidden variable theories is not valid.

III. THE OVERSIGHT IN BELL'S THEOREM

We now turn to Bell's theorem [7] and show that it harbours the same oversight as that in von Neumann's theorem. As we noted in the Introduction, Bell's theorem purports to prove that no locally causal and realistic theory in the sense envisaged by Einstein can reproduce all of the statistical predictions of quantum theory. We will follow the proof of the theorem in the manner of Clauser, Horne, Shimony, and Holt (CHSH) [17][18]. In the derivation of the bounds on the CHSH correlator, such as those in Eq. (18) below, one usually employs factorized probabilities of observing the measurement results rather than the measurement results themselves as we will do in our derivation. But employing probabilities in that manner only manages to mask the implicit assumption in the proof we intend to bring out here.

Consider the standard EPR type spin-$\frac{1}{2}$ experiment, as proposed by Bohm and later used by Bell in the proof of his theorem [7][18]. Alice is free to choose a detector direction $\mathbf{a}$ or $\mathbf{a}'$ and Bob is free to choose a detector direction $\mathbf{b}$ or $\mathbf{b}'$ to detect spins of the fermions they receive from a common source, at a space-like distance from each other. The objects of interest then are the bounds on the sum of possible averages put together in the manner of CHSH [17],

$$\mathcal{E}(\mathbf{a},\,\mathbf{b})\,+\,\mathcal{E}(\mathbf{a},\,\mathbf{b}')\,+\,\mathcal{E}(\mathbf{a}',\,\mathbf{b})\,-\,\mathcal{E}(\mathbf{a}',\,\mathbf{b}')\,, \tag{13}$$

with each average defined as

$$\mathcal{E}(\mathbf{a},\,\mathbf{b})\,=\,\lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{k\,=\,1}^{n}\,\mathscr{A}(\mathbf{a},\,\lambda_k)\;\mathscr{B}(\mathbf{b},\,\lambda_k)\right]\,\equiv\,\Big\langle\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b})\,\Big\rangle\,, \tag{14}$$

where $\mathscr{A}(\mathbf{a},\,\lambda_k)\equiv\mathscr{A}_k(\mathbf{a})=\pm1$ and $\mathscr{B}(\mathbf{b},\,\lambda_k)\equiv\mathscr{B}_k(\mathbf{b})=\pm1$ are the respective measurement results of Alice and Bob. Note that $\mathscr{A}(\mathbf{a},\,\lambda)$ and $\mathscr{B}(\mathbf{b},\,\lambda)$ are manifestly local and realistic functions [7]. Apart from the hidden variable λ, the result $\mathscr{A}=\pm1$ depends *only* on the measurement direction $\mathbf{a}$, chosen freely by Alice, regardless of Bob's actions. And, similarly, apart from the hidden variable λ, the result $\mathscr{B}=\pm1$ depends *only* on the measurement direction $\mathbf{b}$, chosen freely by Bob, regardless of Alice's actions. In particular, the function $\mathscr{A}(\mathbf{a},\,\lambda)$ *does not* depend on $\mathbf{b}$ or $\mathscr{B}$ and the function $\mathscr{B}(\mathbf{b},\,\lambda)$ *does not* depend on $\mathbf{a}$ or $\mathscr{A}$. Moreover, the hidden variable λ does not depend on $\mathbf{a}$, $\mathbf{b}$, $\mathscr{A}$, or $\mathscr{B}$.

Now, since $\mathscr{A}_k(\mathbf{a})=\pm1$ and $\mathscr{B}_k(\mathbf{b})=\pm1$, the average of their product is $-1\leqslant\Big\langle\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b})\,\Big\rangle\leqslant+1$. As a result, we can immediately read off the upper and lower bounds on the sequence of four averages considered above in (13):

$$-\,4\,\leqslant\,\Big\langle\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b})\,\Big\rangle\,+\,\Big\langle\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}')\,\Big\rangle\,+\,\Big\langle\,\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b})\,\Big\rangle\,-\,\Big\langle\,\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}')\,\Big\rangle\,\leqslant\,+4\,. \tag{15}$$

This should have been Bell's final conclusion. However, by continuing, Bell overlooked something that is physically unjustifiable. He identified the above sum of four separate averages of real numbers with the following single average:

$$\mathcal{E}(\mathbf{a},\,\mathbf{b})+\mathcal{E}(\mathbf{a},\,\mathbf{b}')+\mathcal{E}(\mathbf{a}',\,\mathbf{b})-\mathcal{E}(\mathbf{a}',\,\mathbf{b}')\,=\,\Big\langle\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b})+\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}')+\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b})-\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}')\,\Big\rangle\,. \tag{16}$$

As innocuous as this step may seem mathematically, it is in fact an illegitimate step physically, because what is being averaged on its RHS are unobservable and unphysical quantities. But it allows us to reduce the sum of four averages to

$$\Big\langle\,\mathscr{A}_k(\mathbf{a})\,\big\{\,\mathscr{B}_k(\mathbf{b})+\mathscr{B}_k(\mathbf{b}')\,\big\}\,+\,\mathscr{A}_k(\mathbf{a}')\,\big\{\,\mathscr{B}_k(\mathbf{b})-\mathscr{B}_k(\mathbf{b}')\,\big\}\,\Big\rangle\,. \tag{17}$$

And since $\mathscr{B}_k(\mathbf{b})=\pm1$, if $|\mathscr{B}_k(\mathbf{b})+\mathscr{B}_k(\mathbf{b}')|=2$, then $|\mathscr{B}_k(\mathbf{b})-\mathscr{B}_k(\mathbf{b}')|=0$, and vice versa. Consequently, using $\mathscr{A}_k(\mathbf{a})=\pm1$, it is easy to conclude that the absolute value of the above average cannot exceed 2, just as Bell concluded:

$$-\,2\,\leqslant\,\Big\langle\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b})\,+\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}')\,+\,\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b})\,-\,\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}')\,\Big\rangle\,\leqslant\,+2\,. \tag{18}$$

Let us now try to understand why the identification in (16) above is illegitimate. To begin with, Einstein's (or even Bell's own) notion of local-realism does not, by itself, demand this identification. Since this notion is captured already in the very definition of the functions $\mathscr{A}(\mathbf{a},\,\lambda_k)$ [7], the LHS of (16) satisfies the demand of local-realism perfectly well. Nor can a possible statistical independence of the four separate averages on the LHS of (16) justify their replacement with the single average on its RHS, *at the expense of what is physically possible in the actual experiments.* To be sure, mathematically there is nothing wrong with the identification of four separate averages with a single average. Indeed, every school child knows that a sum of averages is equal to the average of the sum. But this rule of thumb is not valid in the above case, because $(\mathbf{a},\,\mathbf{b})$, $(\mathbf{a},\,\mathbf{b}')$, $(\mathbf{a}',\,\mathbf{b})$, and $(\mathbf{a}',\,\mathbf{b}')$ are *mutually exclusive pairs of measurement directions*, corresponding to four *incompatible* experiments. Each pair can be used by Alice and Bob for a given experiment, for all runs 1 to n, but no two of the four pairs can be used by them simultaneously. This is because Alice and Bob do not have the ability to make measurements along counterfactually possible pairs of directions such as ($\mathbf{a}$, $\mathbf{b}$) and ($\mathbf{a}$, $\mathbf{b}'$) simultaneously. Alice, for example, can make measurements along $\mathbf{a}$ or $\mathbf{a}'$, but not along $\mathbf{a}$ *and* $\mathbf{a}'$ at the same time.

But this inconvenient fact is rather devastating for Bell's argument, because it means that his identification (16) is illegitimate. Consider a specific run of the EPR-B experiment and the corresponding quantity being averaged in (16):

$$\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}) \,+\, \mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}') \,+\, \mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}) \,-\, \mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}')\,. \tag{19}$$

Here the index $k = 1$ now represents a specific run of the experiment. But since Alice and Bob have only two particles at their disposal for each run, only one of the four terms of the above sum is physically meaningful. In other words, the above quantity is physically meaningless, because Alice, for example, cannot align her detector along $\mathbf{a}$ and $\mathbf{a}'$ at the same time. And likewise, Bob cannot align his detector along $\mathbf{b}$ and $\mathbf{b}'$ at the same time. What is more, this will be true for all possible runs of the experiment, or equivalently for all possible pairs of particles. Which implies that all of the quantities listed below, as they appear in the average (18), are unobservable, and hence physically meaningless:

$$\begin{aligned}
&\mathscr{A}_1(\mathbf{a})\,\mathscr{B}_1(\mathbf{b}) \,+\, \mathscr{A}_1(\mathbf{a})\,\mathscr{B}_1(\mathbf{b}') \,+\, \mathscr{A}_1(\mathbf{a}')\,\mathscr{B}_1(\mathbf{b}) \,-\, \mathscr{A}_1(\mathbf{a}')\,\mathscr{B}_1(\mathbf{b}')\,,\\
&\mathscr{A}_2(\mathbf{a})\,\mathscr{B}_2(\mathbf{b}) \,+\, \mathscr{A}_2(\mathbf{a})\,\mathscr{B}_2(\mathbf{b}') \,+\, \mathscr{A}_2(\mathbf{a}')\,\mathscr{B}_2(\mathbf{b}) \,-\, \mathscr{A}_2(\mathbf{a}')\,\mathscr{B}_2(\mathbf{b}')\,,\\
&\mathscr{A}_3(\mathbf{a})\,\mathscr{B}_3(\mathbf{b}) \,+\, \mathscr{A}_3(\mathbf{a})\,\mathscr{B}_3(\mathbf{b}') \,+\, \mathscr{A}_3(\mathbf{a}')\,\mathscr{B}_3(\mathbf{b}) \,-\, \mathscr{A}_3(\mathbf{a}')\,\mathscr{B}_3(\mathbf{b}')\,,\\
&\mathscr{A}_4(\mathbf{a})\,\mathscr{B}_4(\mathbf{b}) \,+\, \mathscr{A}_4(\mathbf{a})\,\mathscr{B}_4(\mathbf{b}') \,+\, \mathscr{A}_4(\mathbf{a}')\,\mathscr{B}_4(\mathbf{b}) \,-\, \mathscr{A}_4(\mathbf{a}')\,\mathscr{B}_4(\mathbf{b}')\,,\\
&\quad\cdot\\
&\quad\cdot\\
&\quad\cdot\\
&\mathscr{A}_n(\mathbf{a})\,\mathscr{B}_n(\mathbf{b}) \,+\, \mathscr{A}_n(\mathbf{a})\,\mathscr{B}_n(\mathbf{b}') \,+\, \mathscr{A}_n(\mathbf{a}')\,\mathscr{B}_n(\mathbf{b}) \,-\, \mathscr{A}_n(\mathbf{a}')\,\mathscr{B}_n(\mathbf{b}')\,.
\end{aligned}$$

But since each of the quantities above is physically meaningless, their average appearing on the RHS of (16), namely

$$\Big\langle\, \mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}) \,+\, \mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}') \,+\, \mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}) \,-\, \mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}') \,\Big\rangle\,, \tag{20}$$

is also physically meaningless. That is to say, no physical experiment can be performed — *even in principle* — that can meaningfully allow to measure or evaluate the above average, since none of the above list of quantities could have experimentally observable values. Therefore, the innocuous looking identification (16) of Bell is, in fact, illegitimate.

In the extensive literature on Bell's theorem the identification (16) is often justified on the grounds of "counterfactual definiteness", a variant of realism, but we will see in the next section that such a justification also does not hold water.

IV. COMPARING THE OVERSIGHTS IN THE THEOREMS OF VON NEUMANN AND BELL

To compare Bell's oversight we just discussed with von Neumann's oversight discussed in Section II, let us view the measurement result $\mathscr{A}(\mathbf{a},\,\lambda_k)\,\mathscr{B}(\mathbf{b},\,\lambda_k) = +1$ or -1, simultaneously observed by Alice and Bob at remote stations, as a specific eigenvalue $\mathscr{R}(\mathbf{a},\,\mathbf{b},\,\lambda_k)$ of the observable $\mathcal{R}(\mathbf{a},\,\mathbf{b}) \equiv \boldsymbol{\sigma}_1\cdot\mathbf{a}\otimes\boldsymbol{\sigma}_2\cdot\mathbf{b}$ in the quantum mechanical singlet state

$$|\psi_{\mathbf{n}}\rangle = \frac{1}{\sqrt{2}}\Big\{|\mathbf{n},\,+\rangle_1\otimes|\mathbf{n},\,-\rangle_2 \,-\, |\mathbf{n},\,-\rangle_1\otimes|\mathbf{n},\,+\rangle_2\Big\}\,. \tag{21}$$

Here $\mathbf{n}$ is an arbitrary unit direction in space, subscripts 1 and 2 refer to particles 1 and 2, and the eigenvalue equation

$$\boldsymbol{\sigma}\cdot\mathbf{n}\,|\mathbf{n},\,\pm\rangle \,=\, \pm\,|\mathbf{n},\,\pm\rangle \tag{22}$$

determines the quantum mechanical eigenstates in which the particles have spin "up" or "down" in the units of $\hbar = 2$, with $\boldsymbol{\sigma}$ being the Pauli spin "vector" (σ_x, σ_y, σ_z). As before, our interest lies in comparing the quantum predictions of spin correlations between the constituent fermions with those derived within some locally causal dispersion-free theory [7]. For that purpose, we make the following identifications between the notations used in Sections II and III:

$$\mathscr{A}(\mathbf{a},\,\lambda_k)\,\mathscr{B}(\mathbf{b},\,\lambda_k) \equiv \mathscr{R}(\mathbf{a},\,\mathbf{b},\,\lambda_k) = \pm1 \text{ is an eigenvalue of the observable } \mathcal{R}(\mathbf{a},\,\mathbf{b}) \equiv \boldsymbol{\sigma}_1\cdot\mathbf{a}\otimes\boldsymbol{\sigma}_2\cdot\mathbf{b} \tag{23}$$

$$\mathscr{A}(\mathbf{a},\,\lambda_k)\,\mathscr{B}(\mathbf{b}',\,\lambda_k) \equiv \mathscr{S}(\mathbf{a},\,\mathbf{b}',\,\lambda_k) = \pm1 \text{ is an eigenvalue of the observable } \mathcal{S}(\mathbf{a},\,\mathbf{b}') \equiv \boldsymbol{\sigma}_1\cdot\mathbf{a}\otimes\boldsymbol{\sigma}_2\cdot\mathbf{b}' \tag{24}$$

$$\mathscr{A}(\mathbf{a}',\,\lambda_k)\,\mathscr{B}(\mathbf{b},\,\lambda_k) \equiv \mathscr{T}(\mathbf{a}',\,\mathbf{b},\,\lambda_k) = \pm1 \text{ is an eigenvalue of the observable } \mathcal{T}(\mathbf{a}',\,\mathbf{b}) \equiv \boldsymbol{\sigma}_1\cdot\mathbf{a}'\otimes\boldsymbol{\sigma}_2\cdot\mathbf{b} \tag{25}$$

$$\mathscr{A}(\mathbf{a}',\,\lambda_k)\,\mathscr{B}(\mathbf{b}',\,\lambda_k) \equiv \mathscr{U}(\mathbf{a}',\,\mathbf{b}',\,\lambda_k) = \pm1 \text{ is an eigenvalue of the observable } \mathcal{U}(\mathbf{a}',\,\mathbf{b}') \equiv \boldsymbol{\sigma}_1\cdot\mathbf{a}'\otimes\boldsymbol{\sigma}_2\cdot\mathbf{b}'. \tag{26}$$

With these identifications we can now rewrite the CHSH correlator (13) for $n >> 1$ as a sum of the average values as

$$\mathcal{E}(\mathbf{a},\,\mathbf{b})\,+\,\mathcal{E}(\mathbf{a},\,\mathbf{b}')\,+\,\mathcal{E}(\mathbf{a}',\,\mathbf{b})\,-\,\mathcal{E}(\mathbf{a}',\,\mathbf{b}')\equiv\frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{R}(\lambda_k)+\frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{S}(\lambda_k)+\frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{T}(\lambda_k)-\frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{U}(\lambda_k) \tag{27}$$

by setting the scalars appearing in (1) and (9) to $a = b = c = +1$ and $d = -1$, and, similarly, the RHS of Eq. (16) as

$$\Big\langle\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b})+\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}')+\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b})-\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}')\,\Big\rangle\equiv\frac{1}{n}\sum_{k\,=\,1}^{n}\big\{\mathscr{R}(\lambda_k)+\mathscr{S}(\lambda_k)+\mathscr{T}(\lambda_k)-\mathscr{U}(\lambda_k)\big\}, \tag{28}$$

so that Bell's assumption (16) can be rewritten as

$$\frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{R}(\lambda_k)+\frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{S}(\lambda_k)+\frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{T}(\lambda_k)-\frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{U}(\lambda_k)=\frac{1}{n}\sum_{k\,=\,1}^{n}\big\{\mathscr{R}(\lambda_k)+\mathscr{S}(\lambda_k)+\mathscr{T}(\lambda_k)-\mathscr{U}(\lambda_k)\big\}. \tag{29}$$

As we discussed in the previous two sections, mathematically this assumption — without which the stringent bounds of -2 and $+2$ on the CHSH correlator cannot be derived — is a trivial identity. But is it meaningful physically? To answer this question, note that the spin observables $\mathcal{R}(\mathbf{a},\,\mathbf{b})$, $\mathcal{S}(\mathbf{a},\,\mathbf{b}')$, $\mathcal{T}(\mathbf{a}',\,\mathbf{b})$, and $\mathcal{U}(\mathbf{a}',\,\mathbf{b}')$ defined in (23) to (26) do not commute with each other [11]. Consequently, in parallel with the discussion in Section II, in the dispersion-free counterpart $|\psi_{\mathbf{n}},\,\lambda\rangle$ of the state (21) the result of the measurement of their sum $\mathcal{R}+\mathcal{S}+\mathcal{T}-\mathcal{U}$, or the eigenvalue of the observable $\mathcal{R}+\mathcal{S}+\mathcal{T}-\mathcal{U}$, will not be the sum $\mathscr{R}(\lambda_k)+\mathscr{S}(\lambda_k)+\mathscr{T}(\lambda_k)-\mathscr{U}(\lambda_k)$ of the individual eigenvalues:

$$\langle\,\psi_{\mathbf{n}},\,\lambda_k\,|\{\mathcal{R}+\mathcal{S}+\mathcal{T}-\mathcal{U}\,\}|\,\psi_{\mathbf{n}},\,\lambda_k\,\rangle\neq\mathscr{R}(\lambda_k)+\mathscr{S}(\lambda_k)+\mathscr{T}(\lambda_k)-\mathscr{U}(\lambda_k), \tag{30}$$

and therefore — as proved in Appendix A for arbitrary a, b, c, and d — the quantum mechanical expectation value

$$\langle\,\psi_{\mathbf{n}}\,|\{\mathcal{R}+\mathcal{S}+\mathcal{T}-\mathcal{U}\,\}|\,\psi_{\mathbf{n}}\,\rangle\neq\frac{1}{n}\sum_{k\,=\,1}^{n}\big\{\mathscr{R}(\lambda_k)+\mathscr{S}(\lambda_k)+\mathscr{T}(\lambda_k)-\mathscr{U}(\lambda_k)\big\}. \tag{31}$$

But in the light of (1), (27), and the identifications of the observables in (23) to (26), this lack of equivalence implies

$$\frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{R}(\lambda_k)+\frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{S}(\lambda_k)+\frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{T}(\lambda_k)-\frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{U}(\lambda_k)\neq\frac{1}{n}\sum_{k\,=\,1}^{n}\big\{\mathscr{R}(\lambda_k)+\mathscr{S}(\lambda_k)+\mathscr{T}(\lambda_k)-\mathscr{U}(\lambda_k)\big\}. \tag{32}$$

Consequently, the assumption (16), or equivalently the assumption (29), is physically wrong. As a result, the stringent bounds of ± 2 on the CHSH correlator derived on the basis of this assumption are also wrong. It is not at all surprising that they are violated in the Bell-test experiments [19]. It is also important to note that in arriving at this conclusion we have not compromised either locality or realism. We have merely brought out a flaw in the logic of Bell's argument.

In summary, since Bell's theorem, above all, is a theorem against a class of hidden variable theories, it is no different, in this regard, from von Neumann's theorem [6]. Consequently, it too pertains to simultaneous (albeit contextual [11]) assignment of definite values to *all* of the observables of a relevant quantum system, thereby making the corresponding hidden variable theory compatible with realism [2], or counterfactual definiteness [18]. But since the observables such as

$$\mathcal{X}:=\mathcal{R}\,+\,\mathcal{S}\,+\,\mathcal{T}-\,\mathcal{U}\equiv\boldsymbol{\sigma}_1\cdot\mathbf{a}\,\otimes\,\{\boldsymbol{\sigma}_2\cdot\mathbf{b}\,+\,\boldsymbol{\sigma}_2\cdot\mathbf{b}'\}\,+\,\boldsymbol{\sigma}_1\cdot\mathbf{a}'\,\otimes\,\{\boldsymbol{\sigma}_2\cdot\mathbf{b}\,-\,\boldsymbol{\sigma}_2\cdot\mathbf{b}'\} \tag{33}$$

are self-adjoint operators on the Hilbert space of the singlet system, they must also be assigned definite values, namely, one of their eigenvalues. Now Bell's theorem does assign a definite value to $\mathcal{X}$, but the value it ends up assigning is

$$\mathscr{R}(\lambda_k)+\mathscr{S}(\lambda_k)+\mathscr{T}(\lambda_k)-\mathscr{U}(\lambda_k)\equiv\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b})\,+\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}')\,+\,\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b})\,-\,\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}'), \tag{34}$$

which is not the correct eigenvalue of $\mathcal{X}$. As we have derived in the Appendix A below, the correct eigenvalue of $\mathcal{X}$ is

$$\mathscr{X}(\lambda_k)=\sqrt{\big\{\,\mathscr{R}(\lambda_k)+\mathscr{S}(\lambda_k)+\mathscr{T}(\lambda_k)-\mathscr{U}(\lambda_k)\big\}^2+\,\langle\,\psi\,|\,\mathcal{Y}\,|\,\psi\,\rangle}\,, \tag{35}$$

which is a highly nonlinear function of $\langle\,\psi\,|\,\mathcal{Y}\,|\,\psi\,\rangle$ and the linear sum $\big\{\,\mathscr{R}(\lambda_k)+\mathscr{S}(\lambda_k)+\mathscr{T}(\lambda_k)-\mathscr{U}(\lambda_k)\big\}$ of eigenvalues $\mathscr{R}(\lambda_k)$, $\mathscr{S}(\lambda_k)$, $\mathscr{T}(\lambda_k)$, and $-\mathscr{U}(\lambda_k)$, necessitating the inequality (31), as we have proved in Appendix A. Thus, the incorrect assumption Bell's theorem depends on is entirely homologous to the one von Neumann's theorem depends on. What is more, since $\langle\,\psi\,|\,\mathcal{Y}\,|\,\psi\,\rangle\neq 0$ in general, the absolute bounds on the CHSH correlator would exceed 2 in general.

V. CORRECT LOCAL-REALISTIC BOUNDS ON THE CHSH CORRELATOR

In the previous section we saw that $\mathscr{R}(\lambda_k)+\mathscr{S}(\lambda_k)+\mathscr{T}(\lambda_k)-\mathscr{U}(\lambda_k)$ is *not* one of the eigenvalues of the observable $\mathcal{X}$, and therefore it cannot be used to work out the correct local-realistic bounds on the CHSH correlator. Indeed, as we saw in Section III, the extreme values of $\mathscr{R}(\lambda_k)+\mathscr{S}(\lambda_k)+\mathscr{T}(\lambda_k)-\mathscr{U}(\lambda_k)$ are -2 and $+2$, leading to the incorrect bounds (18) on the CHSH correlator (13) because of the physical incompatibility between (1) and (9). Fortunately, the correct local-realistic bounds on the CHSH correlator can be easily worked out by working out the extrema of the expectation (or average) value of the operator $\mathcal{X}$ defined in (33), and they work out to be $-2\sqrt{2}$ and $+2\sqrt{2}$ as follows:

Let $\mathscr{X}(\lambda_k)\neq\mathscr{R}(\lambda_k)+\mathscr{S}(\lambda_k)+\mathscr{T}(\lambda_k)-\mathscr{U}(\lambda_k)$ be an eigenvalue of the observable $\mathcal{X}$ specified in (33). In Section III we saw that $\mathcal{X}$ is not observable in a typical EPR-Bohm experiment involving only two particles per run. However, as a self-adjoint operator on the Hilbert space of the composite system, it is an observable in theory. Therefore, regardless of whether $\mathcal{X}$ is actually measured in the experiment, in the dispersion-free counterpart $|\,\psi_{\mathbf{n}},\,\lambda\,\rangle$ of $|\,\psi_{\mathbf{n}}\,\rangle$ we would have

$$\langle\,\psi_{\mathbf{n}},\,\lambda_k\,|\,\mathcal{X}\,|\,\psi_{\mathbf{n}},\,\lambda_k\,\rangle=\mathscr{X}(\lambda_k). \tag{36}$$

Consequently, the quantum mechanical expectation value of $\mathcal{X}$ can be recovered by uniform averaging over λ as before:

$$\langle\,\psi_{\mathbf{n}}\,|\,\mathcal{X}\,|\,\psi_{\mathbf{n}}\,\rangle=\frac{1}{n}\sum_{k=1}^{n}\langle\,\psi_{\mathbf{n}},\,\lambda_k\,|\,\mathcal{X}\,|\,\psi_{\mathbf{n}},\,\lambda_k\,\rangle=\frac{1}{n}\sum_{k=1}^{n}\mathscr{X}(\lambda_k). \tag{37}$$

Using (37), the relation (29) can then be corrected to give the following relation that is immune to a Bell-type criticism:

$$\frac{1}{n}\sum_{k=1}^{n}\mathscr{R}(\lambda_k)+\frac{1}{n}\sum_{k=1}^{n}\mathscr{S}(\lambda_k)+\frac{1}{n}\sum_{k=1}^{n}\mathscr{T}(\lambda_k)-\frac{1}{n}\sum_{k=1}^{n}\mathscr{U}(\lambda_k)=\frac{1}{n}\sum_{k=1}^{n}\mathscr{X}(\lambda_k). \tag{38}$$

Unlike (29), this is the correct dispersion-free counterpart of the quantum mechanical relation (1) among the operators $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{U}$, and $\mathcal{X}$. Unlike that between (1) and (9) brought out by Bell, there is no physical incompatibility between the linear relationship (1) among the quantum mechanical expectation values and its dispersion-free counterpart (38).

Now, one way to derive the correct bounds on the CHSH correlator (13) is by recalling from our discussion leading to Eq. (18) that $-2\lll\mathscr{R}(\lambda_k)+\mathscr{S}(\lambda_k)+\mathscr{T}(\lambda_k)-\mathscr{U}(\lambda_k)\lll+2$ and by establishing that $0\lll\langle\,\psi\,|\,\mathcal{Y}\,|\,\psi\,\rangle\lll 4$, so that the eigenvalue $\mathscr{X}(\lambda_k)$ expressed in (35) is deduced to be bounded by $\mp 2\sqrt{2}$. Since these bounds hold for each λ_k, the CHSH correlator (13), or equivalently the LHS of Eq. (38), would also be bounded by $-2\sqrt{2}$ and $+2\sqrt{2}$. However, it is cumbersome to establish $0\lll\langle\,\psi\,|\,\mathcal{Y}\,|\,\psi\,\rangle\lll 4$. Therefore, we will employ a simpler method to derive these bounds:

Note that local commutativity of the operators acting on the spacelike separated subsystems 1 and 2 dictates that

$$[\,\boldsymbol{\sigma}_1\cdot\mathbf{a},\,\boldsymbol{\sigma}_2\cdot\mathbf{b}\,]=0\quad\forall\;\mathbf{a}\;\text{ and }\;\mathbf{b}. \tag{39}$$

Moreover, we have the identities

$$(\boldsymbol{\sigma}_1\cdot\mathbf{a})^2=(\boldsymbol{\sigma}_1\cdot\mathbf{a}')^2=(\boldsymbol{\sigma}_2\cdot\mathbf{b})^2=(\boldsymbol{\sigma}_2\cdot\mathbf{b}')^2=\mathbb{1}, \tag{40}$$

where $\mathbb{1}$ is a 2×2 identity matrix. For the next steps, it is convenient to introduce the following Tsirel'son operator,

$$\mathcal{Z}:=\frac{1}{\sqrt{2}}\left(\boldsymbol{\sigma}_1\cdot\mathbf{a}-\frac{(\boldsymbol{\sigma}_2\cdot\mathbf{b}+\boldsymbol{\sigma}_2\cdot\mathbf{b}')}{\sqrt{2}}\right)^2+\frac{1}{\sqrt{2}}\left(\boldsymbol{\sigma}_1\cdot\mathbf{a}'-\frac{(\boldsymbol{\sigma}_2\cdot\mathbf{b}-\boldsymbol{\sigma}_2\cdot\mathbf{b}')}{\sqrt{2}}\right)^2, \tag{41}$$

which is a sum of squared Hermitian operators and therefore its expectation value in any state would be non-negative:

$$\langle\,\psi_{\mathbf{n}}\,|\,\mathcal{Z}\,|\,\psi_{\mathbf{n}}\,\rangle\geqslant 0. \tag{42}$$

By expanding the RHS of Eq. (41) and using the relations (39) and (40), the Tsirel'son operator can be simplified to

$$\mathcal{Z}=2\sqrt{2}\,\mathbb{1}-\mathcal{X}. \tag{43}$$

Then, using the non-negativity (42) of $\langle\,\mathcal{Z}\,\rangle$, it is easy to see that

$$\langle\,\psi_{\mathbf{n}}\,|\,\mathcal{Z}\,|\,\psi_{\mathbf{n}}\,\rangle=2\sqrt{2}-\langle\,\psi_{\mathbf{n}}\,|\,\mathcal{X}\,|\,\psi_{\mathbf{n}}\,\rangle\geqslant 0. \tag{44}$$

Consequently, we have the desired bounds:

$$-\,2\sqrt{2}\;\leqslant\langle\,\psi_{\mathbf{n}}\,|\,\mathcal{X}\,|\,\psi_{\mathbf{n}}\,\rangle\leqslant+2\sqrt{2}. \tag{45}$$

These are the extrema of the quantum mechanical expectation value $\langle\, \psi_{\mathbf{n}} \,|\, \mathcal{X} \,|\, \psi_{\mathbf{n}} \,\rangle$ of the operator defined in (33). However, according to the relations (37) and (38), the above inequalities are equivalent to the inequalities

$$-\,2\sqrt{2}\;\leqslant\;\left\{\frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{R}(\lambda_k)+\frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{S}(\lambda_k)+\frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{T}(\lambda_k)-\frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{U}(\lambda_k)\right\}\leqslant +2\sqrt{2}, \tag{46}$$

giving the correct bounds on the CHSH correlator (13). These bounds are also derived independently in the references [15] and [16] within a manifestly local-realistic 3-sphere model. In the familiar notation of (13), they can be expressed as

$$-\,2\sqrt{2}\;\leqslant\;\mathcal{E}(\mathbf{a},\,\mathbf{b})\,+\,\mathcal{E}(\mathbf{a},\,\mathbf{b}')\,+\,\mathcal{E}(\mathbf{a}',\,\mathbf{b})\,-\,\mathcal{E}(\mathbf{a}',\,\mathbf{b}')\leqslant +2\sqrt{2}. \tag{47}$$

We have thus derived the quantum mechanical Tsirel'son bounds [20] in a purely local-realistic, dispersion-free setting. Needless to add, these corrected bounds on the CHSH correlator have never been violated in any Bell-test experiments.

VI. CONCLUDING REMARKS

We have compared the mistaken assumptions in the respective theorems of von Neumann and Bell against hidden variable theories and found them to be homologous. While Bell's theorem concerns a rather restricted class of locally causal theories, both theorems claim to rule out theories based on dispersion-free states by assuming the relationship

$$\frac{a}{n}\sum_{k\,=\,1}^{n}\mathscr{R}(\lambda_k)+\frac{b}{n}\sum_{k\,=\,1}^{n}\mathscr{S}(\lambda_k)+\frac{c}{n}\sum_{k\,=\,1}^{n}\mathscr{T}(\lambda_k)+\frac{d}{n}\sum_{k\,=\,1}^{n}\mathscr{U}(\lambda_k)=\frac{1}{n}\sum_{k\,=\,1}^{n}\big\{a\,\mathscr{R}(\lambda_k)+b\,\mathscr{S}(\lambda_k)+c\,\mathscr{T}(\lambda_k)+d\,\mathscr{U}(\lambda_k)\big\} \tag{9}$$

between expected values of observables to arrive at their respective conclusions. But, rather ironically, it is precisely the requirement of hypothetical dispersion-free states that forces the above assumption to fail for hidden variable theories. It is the very notion of realism — the idea that the outcomes of individual measurement events are predetermined, with simultaneous assignment of definite values to all observables of a physical system — that forces the assumption (9) to fail, as Bell himself has argued in his criticism of von Neumann's theorem [11]. In Section 3 of his paper Bell writes:

> Thus the formal proof of von Neumann does not justify his informal conclusion. ... It was not the objective measurable predictions of quantum mechanics which ruled out hidden variables. It was the arbitrary assumption of a particular (and impossible) relation between the results of incompatible measurements either of which might be made on a given occasion but only one of which can in fact be made [11].

But that is precisely the mistake Bell himself has made in his own famous theorem [7][17]. Bell's theorem can be proven only by considering three or four incompatible physical experiments involving mutually exclusive detector directions. Since experiments along mutually exclusive detector directions cannot be performed simultaneously, they amount to observing non-commuting observables. And for non-commuting observables of a system in a dispersion-free state, the result of the measurement of a sum of the observables is not the same as the sum of the results of the measurements of the individual observables in that sum [11]. Therefore, Bell's assumption (29) does not hold for dispersion-free states, rendering the informal conclusion of the formal proof of his theorem invalid. Paraphrasing Bell from the above quote, we can therefore conclude: It is not the objectively measurable predictions of quantum mechanics that rule out the possibility of a local-realistic theory. It is the *ad hoc* assumption of three or four physically incompatible experiments, any one of which might be performed on a given occasion, but only one of which can, in fact, be performed in practice.

Appendix A: Eigenvalue of a Sum of Non-Commuting Operators and the Proof of Eq. (11)

In this appendix we prove that the eigenvalue of the sum $a\,\mathcal{R}+b\,\mathcal{S}+c\,\mathcal{T}+d\,\mathcal{U}$ of operators is not equal to the sum $a\,\mathscr{R}+b\,\mathscr{S}+c\,\mathscr{T}+d\,\mathscr{U}$ of the individual eigenvalues of the operators $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$, unless they commute with each other. It is not difficult to prove this by first evaluating the square of the operator $\{a\,\mathcal{R}+b\,\mathcal{S}+c\,\mathcal{T}+d\,\mathcal{U}\}$ as follows:

$$\begin{aligned}\{a\,\mathcal{R}+b\,\mathcal{S}+c\,\mathcal{T}+d\,\mathcal{U}\}\{a\,\mathcal{R}+b\,\mathcal{S}+c\,\mathcal{T}+d\,\mathcal{U}\} = a^2\mathcal{R}^2 &+ ab\,\mathcal{R}\mathcal{S}+ac\,\mathcal{R}\mathcal{T}+ad\,\mathcal{R}\mathcal{U}\\ &+ ba\,\mathcal{S}\mathcal{R}+b^2\mathcal{S}^2+bc\,\mathcal{S}\mathcal{T}+bd\,\mathcal{S}\mathcal{U}\\ &+ ca\,\mathcal{T}\mathcal{R}+cb\,\mathcal{T}\mathcal{S}+c^2\mathcal{T}^2+cd\,\mathcal{T}\mathcal{U}\\ &+ da\,\mathcal{U}\mathcal{R}+db\,\mathcal{U}\mathcal{S}+dc\,\mathcal{U}\mathcal{T}+d^2\mathcal{U}^2.\end{aligned} \tag{A1}$$

Now, assuming that the operators $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ do not commute in general, let us define the following operators:

$$\mathcal{L} := \mathcal{S}\mathcal{R} - \mathcal{R}\mathcal{S} \iff \mathcal{S}\mathcal{R} = \mathcal{R}\mathcal{S} + \mathcal{L}, \tag{A2}$$

$$\mathcal{M} := \mathcal{T}\mathcal{R} - \mathcal{R}\mathcal{T} \iff \mathcal{T}\mathcal{R} = \mathcal{R}\mathcal{T} + \mathcal{M}, \tag{A3}$$

$$\mathcal{N} := \mathcal{T}\mathcal{S} - \mathcal{S}\mathcal{T} \iff \mathcal{T}\mathcal{S} = \mathcal{S}\mathcal{T} + \mathcal{N}, \tag{A4}$$

$$\mathcal{O} := \mathcal{U}\mathcal{R} - \mathcal{R}\mathcal{U} \iff \mathcal{U}\mathcal{R} = \mathcal{R}\mathcal{U} + \mathcal{O}, \tag{A5}$$

$$\mathcal{P} := \mathcal{U}\mathcal{T} - \mathcal{T}\mathcal{U} \iff \mathcal{U}\mathcal{T} = \mathcal{T}\mathcal{U} + \mathcal{P}, \tag{A6}$$

$$\text{and} \quad \mathcal{Q} := \mathcal{U}\mathcal{S} - \mathcal{S}\mathcal{U} \iff \mathcal{U}\mathcal{S} = \mathcal{S}\mathcal{U} + \mathcal{Q}. \tag{A7}$$

These operators would be null operators with vanishing eigenvalues if the operators $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ did commute with each other. Using the above relations for the operators $\mathcal{S}\mathcal{R}$, $\mathcal{T}\mathcal{R}$, $\mathcal{T}\mathcal{S}$, $\mathcal{U}\mathcal{R}$, $\mathcal{U}\mathcal{T}$ and $\mathcal{U}\mathcal{S}$, Eq. (A1) can be simplified to

$$\begin{aligned}\{a\,\mathcal{R} + b\,\mathcal{S} + c\,\mathcal{T} + d\,\mathcal{U}\}\{a\,\mathcal{R} + b\,\mathcal{S} + c\,\mathcal{T} + d\,\mathcal{U}\} = a^2\mathcal{R}^2 &+ 2ab\,\mathcal{R}\mathcal{S} + 2ac\,\mathcal{R}\mathcal{T} + 2ad\,\mathcal{R}\mathcal{U} \\ &+ ab\,\mathcal{L} + b^2\mathcal{S}^2 + 2bc\,\mathcal{S}\mathcal{T} + 2bd\,\mathcal{S}\mathcal{U} \\ &+ ac\,\mathcal{M} + bc\,\mathcal{N} + c^2\mathcal{T}^2 + 2cd\,\mathcal{T}\mathcal{U} \\ &+ ad\,\mathcal{O} + bd\,\mathcal{Q} + cd\,\mathcal{P} + d^2\mathcal{U}^2 \qquad \text{(A8)} \\ = \{a\,\mathcal{R} + b\,\mathcal{S} + c\,\mathcal{T} &+ d\,\mathcal{U}\}^2_{\mathbf{c}} + \mathcal{Y}, \qquad \text{(A9)}\end{aligned}$$

where

$$\mathcal{Y} := ab\,\mathcal{L} + ac\,\mathcal{M} + bc\,\mathcal{N} + ad\,\mathcal{O} + cd\,\mathcal{P} + bd\,\mathcal{Q}\,. \tag{A10}$$

We have thus separated out the commuting part $\{a\,\mathcal{R} + b\,\mathcal{S} + c\,\mathcal{T} + d\,\mathcal{U}\}_{\mathbf{c}}$ and the non-commuting part $\mathcal{Y}$ of the operator $\{a\,\mathcal{R} + b\,\mathcal{S} + c\,\mathcal{T} + d\,\mathcal{U}\}$. Note that the operators $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{O}$, $\mathcal{P}$, and $\mathcal{Q}$ defined in (A2) to (A7) will not commute with each other in general unless their constituents $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ themselves are commuting. Next, we work out the eigenvalue $\mathscr{X}$ of the operator $\mathcal{X} := \{a\,\mathcal{R} + b\,\mathcal{S} + c\,\mathcal{T} + d\,\mathcal{U}\}$ in a state $|\,\psi\,\rangle$ using the eigenvalue equations

$$\mathcal{X}\,|\,\psi\,\rangle = \mathscr{X}\,|\,\psi\,\rangle \tag{A11}$$

and

$$\mathcal{X}\,\mathcal{X}\,|\,\psi\,\rangle = \mathcal{X}\{\mathcal{X}\,|\,\psi\,\rangle\} = \mathcal{X}\,\{\mathscr{X}\,|\,\psi\,\rangle\} = \mathscr{X}\,\{\mathcal{X}\,|\,\psi\,\rangle\} = \mathscr{X}^2\,|\,\psi\,\rangle, \tag{A12}$$

in terms of the eigenvalues $\mathscr{R}$, $\mathscr{S}$, $\mathscr{T}$, and $\mathscr{U}$ of the operators $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ and the expectation value $\langle\,\psi\,|\,\mathcal{Y}\,|\,\psi\,\rangle$:

$$\mathscr{X} = \sqrt{\langle\,\psi\,|\,\mathcal{X}\,\mathcal{X}\,|\,\psi\,\rangle} = \sqrt{\langle\,\psi\,|\{a\,\mathcal{R} + b\,\mathcal{S} + c\,\mathcal{T} + d\,\mathcal{U}\}^2_{\mathbf{c}}\,|\,\psi\,\rangle + \langle\,\psi\,|\,\mathcal{Y}\,|\,\psi\,\rangle}\,, \tag{A13}$$

where we have used Eq. (A9). But the eigenvalue of the commuting part $\{a\,\mathcal{R} + b\,\mathcal{S} + c\,\mathcal{T} + d\,\mathcal{U}\}_{\mathbf{c}}$ is simply the linear sum $a\,\mathscr{R} + b\,\mathscr{S} + c\,\mathscr{T} + d\,\mathscr{U}$ of the eigenvalues of the operators $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$. Consequently, using the equation analogous to (A12) for the square of the operator $\{a\,\mathcal{R} + b\,\mathcal{S} + c\,\mathcal{T} + d\,\mathcal{U}\}_{\mathbf{c}}$ we can express the eigenvalue $\mathscr{X}$ of $\mathcal{X}$ as

$$\mathscr{X} = \sqrt{\left\{a\,\mathscr{R} + b\,\mathscr{S} + c\,\mathscr{T} + d\,\mathscr{U}\right\}^2 + \langle\,\psi\,|\,\mathcal{Y}\,|\,\psi\,\rangle}\,. \tag{A14}$$

Now, because the operators $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{O}$, $\mathcal{P}$, and $\mathcal{Q}$ defined in Eqs. (A2) to (A7) will not commute with each other in general if their constituent operators $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ are non-commuting, the eigenvalue $\mathscr{Y}$ of the operator $\mathcal{Y}$ defined in (A10) will not be equal to the linear sum of the corresponding eigenvalues $\mathscr{L}$, $\mathscr{M}$, $\mathscr{N}$, $\mathscr{O}$, $\mathscr{P}$, and $\mathscr{Q}$ in general,

$$\mathscr{Y} \neq ab\,\mathscr{L} + ac\,\mathscr{M} + bc\,\mathscr{N} + ad\,\mathscr{O} + cd\,\mathscr{P} + bd\,\mathscr{Q}\,, \tag{A15}$$

even if we assume that the operators $\mathcal{X}$ and $\mathcal{Y}$ commute with each other so that $\langle\,\psi\,|\,\mathcal{Y}\,|\,\psi\,\rangle = \mathscr{Y}$ is an eigenvalue of $\mathcal{Y}$. That is to say, just like the eigenvalue $\mathscr{X}$ of $\mathcal{X}$, the eigenvalue $\mathscr{Y}$ of $\mathcal{Y}$ is also a nonlinear function in general. On the other hand, because we wish to prove that the eigenvalue of the sum $a\,\mathcal{R} + b\,\mathcal{S} + c\,\mathcal{T} + d\,\mathcal{U}$ of the operators $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ is not equal to the sum $a\,\mathscr{R} + b\,\mathscr{S} + c\,\mathscr{T} + d\,\mathscr{U}$ of the individual eigenvalues of the operators $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ unless they commute with each other, we must make sure that the eigenvalue $\mathscr{Y}$ does not vanish for the unlikely case in which the operators $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{O}$, $\mathcal{P}$, and $\mathcal{Q}$ commute with each other. But even in that unlikely case we would have

$$\mathscr{Y} = ab\,\mathscr{L} + ac\,\mathscr{M} + bc\,\mathscr{N} + ad\,\mathscr{O} + cd\,\mathscr{P} + bd\,\mathscr{Q} \tag{A16}$$

as eigenvalue of the operator $\mathcal{Y}$ defined in (A10), and consequently the eigenvalue $\mathscr{X}$ in (A14) will at best reduce to

$$\mathscr{X} = \sqrt{\left\{a\,\mathscr{R} + b\,\mathscr{S} + c\,\mathscr{T} + d\,\mathscr{U}\right\}^2 + ab\,\mathscr{L} + ac\,\mathscr{M} + bc\,\mathscr{N} + ad\,\mathscr{O} + cd\,\mathscr{P} + bd\,\mathscr{Q}}\,. \tag{A17}$$

In other words, even in such an unlikely case $\mathscr{Y}$ will not vanish, and consequently the eigenvalue $\mathscr{X}$ will not reduce to

$$\mathscr{X} = a\,\mathscr{R} + b\,\mathscr{S} + c\,\mathscr{T} + d\,\mathscr{U}. \tag{A18}$$

As a result, we can now prove Eq. (11) of Section II: Unless $\langle\,\psi\,|\,\mathcal{Y}\,|\,\psi\,\rangle \equiv 0$, the average of the eigenvalue $\mathscr{X}$ will be

$$\langle\,\psi\,|\,\mathcal{X}\,|\,\psi\,\rangle = \frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{X}(\lambda_k) = \frac{1}{n}\sum_{k\,=\,1}^{n}\sqrt{\big\{a\,\mathscr{R}(\lambda_k) + b\,\mathscr{S}(\lambda_k) + c\,\mathscr{T}(\lambda_k) + d\,\mathscr{U}(\lambda_k)\big\}^2 + \langle\,\psi,\,\lambda_k\,|\,\mathcal{Y}\,|\,\psi,\,\lambda_k\,\rangle} \tag{A19}$$

$$\neq \frac{1}{n}\sum_{k\,=\,1}^{n}\sqrt{\big\{a\,\mathscr{R}(\lambda_k) + b\,\mathscr{S}(\lambda_k) + c\,\mathscr{T}(\lambda_k) + d\,\mathscr{U}(\lambda_k)\big\}^2 + \mathscr{Y}(\lambda_k)}\;\;\text{if}\;[\,\mathcal{X},\,\mathcal{Y}\,] \neq 0 \tag{A20}$$

$$\neq \frac{1}{n}\sum_{k\,=\,1}^{n}\big\{a\,\mathscr{R}(\lambda_k) + b\,\mathscr{S}(\lambda_k) + c\,\mathscr{T}(\lambda_k) + d\,\mathscr{U}(\lambda_k)\big\}\;\text{if}\;\mathscr{L},\,\mathscr{M},\,\mathscr{N},\,\mathscr{O},\,\mathscr{P},\,\mathscr{Q} \neq 0. \tag{A21}$$

This proves Eq. (11) discussed in Section II. Note that, because $\mathscr{X}(\lambda_k)$ and $\langle\,\psi\,|\,\mathcal{Y}\,|\,\psi\,\rangle$ are highly *nonlinear* functions in general (recall, *e.g.*, that $\sqrt{x^2 \pm y^2} \neq \sqrt{x^2} \pm \sqrt{y^2}$), the inequality in (A21) can reduce to equality *if and only if* the operators $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ commute with each other. In that case, the operators $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{O}$, $\mathcal{P}$, and $\mathcal{Q}$ defined in (A2) to (A7) will also commute with each other, as well as being null operators, with each of the eigenvalues $\mathscr{L}$, $\mathscr{M}$, $\mathscr{N}$, $\mathscr{O}$, $\mathscr{P}$, and $\mathscr{Q}$ reducing to zero. Consequently, in that case $\langle\,\psi\,|\,\mathcal{Y}\,|\,\psi\,\rangle$ will vanish identically and (A14) will reduce to (A18). In particular, for the CHSH correlator (27) for which $a = b = c = +1$ and $d = -1$, (A18) will simplify to

$$\mathscr{X}(\lambda_k) = \mathscr{R}(\lambda_k) + \mathscr{S}(\lambda_k) + \mathscr{T}(\lambda_k) - \mathscr{U}(\lambda_k). \tag{A22}$$

It is *this* eigenvalue $\mathscr{X}(\lambda_k)$ in (A22) that has been implicitly and unjustifiably assumed in the proof of Bell's theorem.

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