Are the notions of past, present and future compatible with the General Theory of Relativity?

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Abstract

The notions of time and causality are revisited, as well as the A- and B-theory of time, in order to determine which theory of time is most compatible with relativistic spacetimes. By considering orientable spacetimes and defining a time-orientation, we formalize the concepts of a time-series in relativistic spacetimes; A-theory and B-theory are given mathematical descriptions within the formalism of General Relativity. As a result, in time-orientable spacetimes, the notions of events being in the future and in the past, which are notions of A-theory, are found to be more fundamental than the notions of events being earlier than or later than other events, which are notions of B-theory. Furthermore, we find that B-theory notions are incompatible with some structures encountered in globally hyperbolic spacetimes, namely past and future inextendible curves. Hence, GR is favorable to A-theory and the notions of past, present and future.

Keywords: General Relativity, Causality, Global hyperbolicity, Interpretation of time

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I. INTRODUCTION

Relativity changed our conception of time, it brought spatial properties to our temporal structure, and forced us to consider space and time as one object: spacetime. This mixing of space and time has led to the interpretation of statements about the current date (say, that the universe is nearly 14 billion years old) as subjective facts about our location in spacetime — akin to the statement that we are in the Milky Way — rather than describing an intrinsic feature of spacetime or the universe itself [1, 2]. The world is conceived as a block-universe in which any description of the ordering of events in time that involves the notions of future and past should be replaced by the notion of events being earlier than, or later than, other events. We call this the B-theory of time [3]. A-theorist, on the other hand, hold the opposite view: the notions of past, present and future are irreducible and fundamental [4, 5]. In this paper we will show that 1) In time-orientable spacetimes the notions of future and past are more fundamental than the notions of events being earlier than, or later than, other events, as described by the A- and B- theories of time, 2) The notions of later than and earlier than are incompatible with globally hyperbolic spacetimes. Therefore, General Relativity restricted to globally hyperbolic spacetimes is an A-theory of time.

To show this we will build, with the formalism of general relativity, notions of events and time-series (ordering of events) that are compatible with Mctaggart's usage in his seminal paper about the unreality of time [4]. We will show that some of the conclusions drawn from the theory of relativity, which historically have deemed A-theory incompatible with relativity, assume a global Minkowski spacetime (for instance, see [6]) and the notions of relative past and future. By extending these analyses to curved spacetime, i.e. general relativity on time-orientable spacetimes, the notions of relative past and future will be discarded and the A-theory will not only become relevant, but the notions of past and future will prove essential to ordering events in time.

In section II, we will describe the differences between A- and B-theory. In section III we will use Strawson's notion of *ontological prioricity* to argue that the concepts of past- and future-directedness of trajectories in General Relativity are more fundamental than the later than, earlier than relations between events in time (relations which will be mathematically defined as the $\gg$-relation). Then, in section IV, we will move from ordering events in time within one future-directed trajectory to ordering hypersurfaces of events in time; this will be done by defining the notions of chronological past and future which will eventually lead to the notion of Cauchy surfaces and global hyperbolicity. Finally in section V, we will show that global hyperbolicity is better understood within the A-theory of time, and not within the B-theory due to the crucial concept of past and future inextendibility which can be expressed in A-theory terms but cannot be translated to B-theory terms. This implies that the notions of future and past are relevant to our objective description of the universe and not a subjective statement about our location in it and that these notions should be part of our objective description of the universe at the level of spacetime theories.

II. A THEORY AND B THEORY

A-theory and B-theory are used in philosophy to distinguish two ways of describing events in time. In A-theory, we describe events as being past, present or future. Furthermore, that we are at the present, or that the present date is thus and such, are irreducible facts about events in time. For a B-theorist, events in time are to be described as being earlier than or later than other events; they may also be described as having a temporal position of t, as in a coordinate axis, but there is no further fact about the event being in the past, present or future which cannot be stated in B-theory terms.

An important difference between these types of expressions is that the description from A-theory can be true at one time and false at another time, while B-theory expressions ("later than", "earlier than") are always true. Moreover, the B-theorist states that it is always possible to recast A-expressions in terms of B-expressions. For instance, take the expression "E_1 is in the future", B-theory would rather state that the moment you are at —event E_0— is earlier than E_1.

Note that "E_0 is earlier than E_1" is an unchanging fact (true proposition) about E_1 and E_0. Also, note that while the A-theory only explicitly mentions E_1, the B-theory has to introduce a second event in its B-expression; B-expressions always involve two events (this is an important point because we will argue that General Relativity uses A-expressions to describe a situation in which we cannot find two such events). We have then reduced our A-expression, "E_1 is in the future", which is thought by the A-theory to be true now, but false in the future (when E_1 is in the past or present) to a statement about how E_1 stands in relation to E_0, which is always true.

Another way we can describe events in A-theory is by ascribing to them the intrinsic (i.e., observer-independent) properties of pastness, presentness, and futurity. For instance, one may say that the event of us writing this paper has the property of being in the past, or pastness, while the event of you reading the paper has presentness, and the event of you being done reading this paper has futurity. These intrinsic temporal properties are called A-properties[1]. In the next section, we

[1] "Intrinsic" here is being used to describe properties that only depend on the entity that has the property in

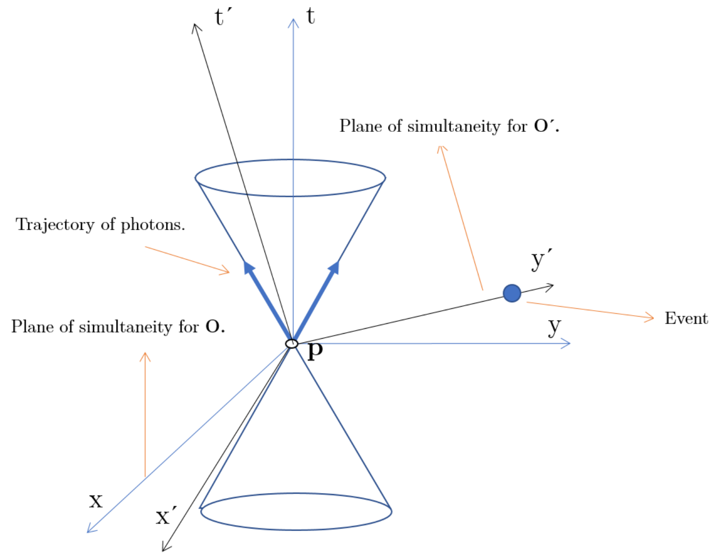

Figure 1: Minkowski diagram. The shown event in in the future for observer O and in the present for observer O'.

will see that A-properties cannot be only had by events, which will be understood as points in spacetime, but also by objects with a trajectory in spacetime. Specifically, the properties of being directed towards the future or past will be argued to be an A-property in virtue of them being intrinsic to the object at a particular time.

In this work we relate A-theory only with the existence of temporal properties (or the irreducibility of A-expressions); we will not deal with the concept of the flow of time. Following Pearson's version of A-theory [1], we maintain that the flow of time, which is normally an idea attributed to A-theory, is not directly implied the existence of A-properties (although A-expressions are friendly to the concept of the flow of time). The question we will focus on is whether events in relativity have temporal properties, or are events only to be thought of as being later or earlier than (or simultaneous with) other events[2].

question. This contrasts with the idea of extrinsic or relative properties that depend on factors exterior to the object itself. Being 10 meters away from the Eiffel tower is an extrinsic or relative property. Likewise, being later than or earlier than are extrinsic properties

[2] The growing block universe, which is thought as an A-theory, would be compatible with Pearson's take on A-theory if it is seen as defending the existence of the temporal properties of pastness and presentness, see [4, 5]

III. THE ARGUMENT FROM SPECIAL RELATIVITY

This argument stems from the notions of relative past and relative future found in Special Relativity. Observer O defines a global equal-time hypersurface given by points orthogonal to the global time coordinate axis that can be defined by the light cone in the rest frame of observer O, see Fig. 1. Depending on the velocity of a different observer O' relative to O, the light cone can have a different shape, and the equal-time hypersurfaces a different inclination, in O''s rest frame. Therefore, two observers will have two different sets of events as being present (the same goes for future and past).

For this other observer O' moving at an arbitrary velocity less than the speed of light some of the events (those outsides of the light cone) that are ordered, in our original picture, as being in the future now appear to be in the present. Moreover, some events are simultaneous in the present for O and in the past or future for O'. If this is the case then we have to accept that events cannot have presentness (or pastness, or futurity) as an intrinsic property, since for different observers the same event can be in the past or the future depending on the relative velocity between the observers O and O'.

As Adrian Bardon (a B-theorist) states, according to relativity "the temporal location of 'now' is just as much a subjective matter as the spatial location of 'here'" and that "if relativity is right, then the dynamic theory [A-theory] of time must be wrong. All events, past, present, and future, are described by some frame of reference. Without a real past, present, and future, there can be no passage of time and no dynamic change." [6].

IV. THE CONCEPTUAL SCHEME OF GENERAL RELATIVITY

Much of the reasoning behind the previous conclusion has to be modified if it is to apply to spacetimes within the framework of General Relativity, i.e. within the structure of a smooth orientable connected relativistic manifold (M, g). For this we will construct concepts that coincide with the notion of a time-seriesusing the formalism behind General Relativity. We will find that the problems for the A-theory that stem from a global Minkowski metric do not arise. First, let us briefly define a relativistic time-orientable spacetime.

> **Definition 1:** Spacetime is a D-dimensional topological connected manifold with a smooth atlas locally equipped with the Lorentzian Metric which preserves time-orientation.

The time-orientation is a vector field $\mathcal{T}$ defined everywhere in the manifold such that at the tangent space at each point $g(\mathcal{T}, \mathcal{T}) > 0$. Given a notion of co-orientation defined for any two vectors as $g(\mu, \nu) > 0$, we find two equivalence classes of vectors co-oriented with the time-orientation along the manifold corresponding to the two half-light cones of the Lorentzian metric. In general it may not be possible to consistently relate all the light cones in a spacetime with a single equivalence class of vectors. That is, there may not exist a smooth vector field $\mathcal{T}$ such that at every point it is co-oriented with a timelike or null vector, but when we say that the spacetime is time-orientable, we are stating that there exists such a smooth vector field $\mathcal{T}$.

> **Definition 2:** A vector μ is future-oriented if and only if it is co-oriented with the time-orientation $\mathcal{T}$.

That is, when:

$$g(\mu, \mathcal{T}) > 0. \tag{1}$$

Likewise, we say a vector ν is past-oriented if and only if:

$$g(\nu, \mathcal{T}) < 0. \tag{2}$$

Any vector must be co-oriented with $\mathcal{T}$ in order to be pointing to the future, the direction of $\mathcal{T}$ representing one of the two half-light cones. Defining a time-orientation implies assigning the vectors within one of the cones to be pointing to the future, and assigning the vectors in the other cone to be pointing to the past (hence we introduce the asymmetry of time into the theory).

The inclusion of a time-orientation in the definition of spacetime is not required by Einstein's equations. Dennis Lehmkuhl notes that conformal structure only allows for the possibility of choosing which half cone in the double-conic structure is directed towards the future or the past, but it does not force such a choice. Einstein's equations are compatible with a spacetime with no time-orientation, although spacetime itself must be orientable. This means, the vector-field $\mathcal{T}$ must be definable but the spacetime does not require the addition of this structure for Einstein's equations to apply [7].

Lehmkuhl also notes that Friedmann-Lemaitre-Robertson-Walker spacetimes, which corresponds to spacetimes that can have a global metric at large scales in which we can define a global time-coordinate [7], may best represent our universe. Spacetimes where such global time-coordinates can be defined are called globally hyperbolic. In this section we will not consider the solutions of the Einstein equations in particular, we will only need spacetime to be temporally orientable with time-orientation $\mathcal{T}$. Stronger time-related conditions on the manifold, based on causality, will be considered in section V. Now we ask: how is time-orientability a reasonable assumption?

If we were to permeate spacetime with material particles the velocities of their curves would define a time-orientation. In fact, this is precisely what is done in Friedmann spacetimes; the energy-momentum tensor $T^{\mu\nu}$ is defined to represent an ideal fluid, isotropic and homogenous, defined on all points of the manifold M. The solutions to these equations are metrics that indeed define a time-orientable universe. Imposing such an energy-momentum tensor together with Einstein equations are much stronger restriction than just time-orientability. Given the above considerations about causality and cosmology however, time-orientation is a reasonable assumption. Now that a time-orientation is assumed, we can create a time-series by considering a timelike curve in spacetime:

Definition 3: A curve γ is said to be future-directed if the velocity field defined by its trajectory satisfies $g(v_\gamma, \mathcal{T}) > 0$ at the tangent space T_pM of every point p, where p is an image of the curve.

Note that the expression "$v \in T_pM$ at point p is a future-directed vector" is an A-expression given that it only involves one event at p, and it serves to order events in time. The time-series constructed by following a future-oriented trajectory would be an A-series. Nevertheless, this A-expression can be translated to a B-expression as well, and hence we can likewise order the events from earlier to later. We can define a formal relation that stems from timelike trajectories and can produce the time-series above, but this relation will still require the notions of future and past orientation.

Definition 4: Consider a timelike future-oriented curve $\gamma : [a, b] \to M$. These statements are equivalent:

- $\gamma(a) = p$ and $\gamma(b) = q$ for $b > a$.
- $q \gg p$.[3]
- q is later than p.

[3] if the curve γ is null, then the relation between q and p is written as $\geq$.

Points related by the $\gg$-relation form a B-series, the $\gg$-relation expressing a time order between the two events, and hence making the relation a B-property (being later than or earlier than). Moreover, events are ordered in this way for any observer, any chart, and hence they form a family of objective B-series. Nevertheless, while both series are compatible with our ordering of events within a timelike curves, A-properties (being in the future, being in the past) will prove more fundamental than B-properties. This will ultimately be the case due to the fundamentality, or *ontological prioricity* of the future-directed velocity of curves (understood as directional derivatives) in the scheme of general relativity.

Let us justify the fundamentality of velocity in General Relativity. To do so we will use Strawson's notion of *ontological prioricity.* He invites us to imagine a language and "to suppose, for instance, it should turn out that there is a type of particulars, β, such that particulars of type β cannot be identified without reference to particulars of another type, α, whereas particulars of the type α can be identified without reference to particulars of type β" [8]. Strawson claims that under the conceptual scheme that underlies our ability to identify particulars (given a language), α-particulars will be more fundamental than, or ontologically prior to, β-particular. If such a dependence were to be found in our language, Strawson argues, will have "some significance for an inquiry into the general structure of the conceptual scheme in terms of which we think about particulars" [8]. That there is such a conceptual structure is then revealed by the fact that we find this dependence when identifying particulars. Strawson's objective is not to create an ontology in which α –particulars and β-particular are related in this way, but rather just to describe the hierarchy of entities already present in the conceptual scheme we use when speaking and thinking about these particulars.

Consider the conceptual scheme of General Relativity. We can translate this definition to reveal the fundamentality of concepts in GR. Imagine a function that is defined in terms of certain objects which can, in turn, be defined without the aid of the function. That is, we cannot identify such functions without knowing the values or characters of the objects that define it, but the objects themselves can be identified without the function. This situation is quite analogous to Strawson's but instead of speaking of particulars using English we are using mathematical objects. The objects we are considering are precisely the velocities and lengths of trajectories.Given the structures of dependencies, velocity is more fundamental than the length in the conceptual scheme of General

Relativity. The velocity of the curve γ at point p is a linear map:

$$v_{\gamma,p} : \mathcal{C}^{\infty}(M) \to \mathbb{R}. \tag{3}$$

$$f \to v_{\gamma,p} := (f \circ \gamma)'(\lambda_0), \tag{4}$$

where $\mathcal{C}^{\infty}(M)$ is the set of all smooth functions defined in the manifold and the prime ($\prime$) signifies differentiation with respect to the parameter λ.

This definition of velocities amounts to saying that the directional derivative or the rate of change of a function defined in spacetime over a trajectory (with respect to a parameter λ) is the velocity. The value of the operation is not what defines the velocity, but rather the operator: the velocity is the directional derivative itself, in the direction of the trajectory, at a point p, independently of the function f on which it acts. Unlike in Newtonian mechanics, in General Relativity we define the velocity without any reference to position or lengths: and this has to be so, given the framework of differentiable manifolds.

One could protest that the concept of velocity demands a speed, and hence it has to include its magnitude, which requires a metric to be defined. To this it will suffice to say that we only need this coarser definition of velocity as a directional derivative to develop our argument. It is the fact that this notion is allowed prior to defining a chronological order of events that will make the notions of future-directed and past-directed vectors more fundamental than proper time and other temporal notions that require the velocity in their definitions. As an example we take the proper time or length of a trajectory, which is given by choosing a metric and defined by the length functional:

$$\tau = \int_{\gamma} \sqrt{g_{\mu\nu} dx^{\mu} dx^{\nu}}. \tag{5}$$

By our notion of *ontological prioricity* we then find that velocities (understood as directional derivatives of trajectories) are prior to lengths. We know that mathematically this is so, and the further claim is that this mathematical structure reveals an ontological structure of dependencies in the theory. In the same way that the identification of particulars reveals the ontological structure present in the scheme we use to describe time. Now we are in a position to prove the fundamentality of A-properties:

> **Proposition 1:** A future directed curve is more fundamental than (*ontologically prior* to) the $\gg$-relation.

Proof. We defined a future directed curve γ as one where $g(v_\gamma, \mathcal{T}) > 0$ at every point p, where $\mathcal{T}$ is the time-orientation. Nevertheless, we require the notion of such a curve to define the $\gg$-relation. Moreover, within relativistic spacetimes with a time-orientation $(M, g, \mathcal{T})$ the only structure available are the metric and the time-orientation, so only in terms of these can points be ordered in time. [4] $\square$

This proves the fundamentality of a future-directed curve within this conceptual scheme. Moreover, given that future-orientation is an A-property, and that the $\gg$-relation is a B-property, we have a case where A-properties are more fundamental than B-properties. This is the hierarchy of entities suggested by the conceptual scheme of General Relativity. In this hierarchy, A-properties come out on top of B-properties, but they are both still compatible with General Relativity on time-orientable spacetimes. To see when the problem arises from the B-theory we will have to consider the ordering of an entire subset on M in a time-series and globally hyperbolic spacetimes. The following is a definition related to this section that will be used later:

Definition 5: Let $\gamma[a, b]$ be a segment of a null future-directed curve.If $a < b$ and $\gamma[a] = p$, $\gamma[b] = q$, then we say that $q \geq p$.

V. TIME-SERIES AND GENERAL RELATIVITY.

To reach globally hyperbolic spacetimes we will expand the notions of a time-series beyond the notions of a timelike-curve and consider the most general form of a time-series that GR can offer. Then we will answer the question: are the notions of past, present or future necessary for these descriptions or do the notions of later than, earlier than suffice? As we apply the constraints required for a globally hyperbolic spacetime.[5] the shortfall of B-expressions will become evident. We start be defining the chronological future of a subset of point.

Definition 6: The chronological future of $p \in M$, $I^+(p)$ is defined as the set that can be reached by future-directed timelike curves going through p.

$$I^+(p) = \{q \in M | \exists \, \text{a future directed timelike curve} \, \lambda(t) \, \text{with} \, \lambda(0) = p \, \text{and} \, \lambda(t') = q\}\,.$$

[4] A possible alternative would be to not start with spaces $(M, g, \mathcal{T})$ but rather pre-endow a set with relations $\gg$, $\geq$, and then attempt to recover the metric and the time-orientation. These objects, $(M, \gg \geq)$ are called causal spaces and while they are distinct from relativistic spacetime, in section VI we will consider the possibility of starting with a causal space and deriving a time-orientation and future-directed curves. We will see that not all features can be recovered without additional requirements.

[5] See Refs. [9, 10] for an extended review about globally hyperbolic spacetime.

By drawing a family of timelike, future-directed curves, we are collecting a series of events that we know are, in the framework of General Relativity, ordered from earlier to later absolutely, within each trajectory. This relation between the events cannot be contested, it is chart-independent. Each timelike or lightlike future-directed curve by itself can be considered a time-series of ordered points. Nevertheless, there is still a further question of how are the points between the different curves in $I^+(p)$ ordered with respect to each other. To do this we have to talk about the chronological future of subsets. We can define the chronological future of a subset $S \subset M$, denoted $I^+(S)$ as:

$$I^+(S) = \bigcup_{p \in S} I^+(p). \tag{6}$$

Here we are drawing all possible timelike curves from all points in S and stating that the points within these trajectories form a set which is in the future of the set S. Nevertheless, how the points in $I^+(S)$ relate temporally within themselves is not yet clear. To do this we have to introduce achronal sets.

> **Definition 7:** A subset S is called achronal if there are no points $p, q \in S : q \in I^+(p)$ [11], that is, points within the subset that are not communicated by timelike curves.

This subset S is not necessarily a hypersurface on M or any subset of M. Nevertheless, there is a way to guarantee that the points in an achronal subset form a three-dimensional subset. The next theorem (theorem 8.1.3 in [12]) states that the boundary of any subset S of M defines an achronal subset that forms a three-dimensional surface embedded on M.

Theorem 1. Let $(M, g_{\mu\nu})$ be a time-orientable spacetime, and let $S \subset M$. Then the boundary of $I^+(S)$ is an achronal, three-dimensional, embedded C^0- submanifold of M.

Since these points do not form a time-series, but they do form a three-dimensional surface reminiscing of a plane of simultaneity, one may be tempted to say that they are all at the same time. Nevertheless, there is not enough structure to define such a notion clearly. Only when we have the much more stringent structure of a partial Cauchy surface can we say that the formalism allows us to define an achronal surface to be at a specific time t.

From the velocity fields defined by our many future-oriented trajectories emanating from S we could choose ξ^a, a smooth, future-directed, unit timelike vector field that is twist-free (that is, it satisfies the condition $\xi_{[a} \nabla_b \xi_{c]} = 0$). If this choice is possible, then we can locally foliate a subspace of M with a one-parameter family of hypersurfaces. This will be equivalent to defining a series of anachronal surfaces ordered from earlier to later in time by defining a twist-free velocity field in $I^+(S)$. This cannot be done in general, it requires a locally well behaved-causal structure. (Check section 2.8 in [13]).

This will not order the events outside these achronal surfaces S however, since they have an *edge* ($edge(S) \neq \emptyset$). That is to say, there may exist a point $r \in I^+(S)$ and a point $q \in I^-(S)$ in the neighborhood U of a point p in the boundary of S that are connected by a timelike or null curve that does not intersect with S. The $edge(S)$ is defined to be the set of all points in U that can be connected in this way. This would mean that an observer in the chronological past of S can get to the chronological future by skipping the time t that the surface S would be representing through points that are not chronologically related to S.

Any stricter chronological ordering will require more structure to be added. Particularly, to further order all events in spacetime we will need these achronal surfaces to have no edge and for their chronological past and future to contain the manifold. When a spacetime admits this we say it is globally hyperbolic, which will be defined more precisely in the next section. Without considering global hyperbolicity however, general relativity leaves us with many series of events localized in space that are ordered from the past to the future within each series, which forms an A-series. [6]

We can then say that events are in the past, present or future relative to a point p by saying that a future-directed trajectory going through p would reach them, or that the point p could have been reached by timelike trajectories from the past. In the case of global hyperbolicity, we can speak of entire Cauchy surfaces being in the past, present, or future. In the next section we will argue that, not only that we are in a better position to talk about A-properties than we were before (when we only considered SR), but that the notion of time-orientation unavoidably introduced A-expressions in our conception of time. The following definitions are related to this section and it will be used in the next section:

> **Definition 8:** The *edge(S)* of an achronal surface S is the set of all points p in the boundary of S such that, in a neighborhood U of p there are points $r \in I^+(S)$ and $q \in I^-(S)$ that are connected by a timelike or null curve that does not intersect with S.

[6] In the next section we will see that the B-theorist can use the »-relation to order the same events from earlier to later and hence form a B-series as well.

Definition 9: An achronal set is said to be a partial Cauchy surface if it is a three-dimensional, embedded C^0-submanifold with $edge(S) = \emptyset$.

VI. A-THEORY AND GENERAL RELATIVITY

Both A-series and B-series were defined using time-orientation and timelike or lightlike trajectories. Nevertheless, the B-theorist, while recognizing the importance of the A-notions of future- and past-directness, can still define the $\gg$-relation to indicate that there is a future-directed or past-directed timelike curve.

Definition 10: We say a curve $\gamma : [a, b] \to M$ is future-directed when for points $\gamma(b) = q$ and $\gamma(a) = p$ we have $q \gg p$. Similarly for past directed curves.

Points related in this way will also form a B-series, $\gg$ expressing the relation between the two events, and hence constituting a B-expression. The notions of chronological future and past can be similarly defined. Nevertheless, given the dependency between future- and past-direction and this two-event relation ($\gg$), we find that A-expressions are more fundamental than B-expressions in relativistic spacetimes. Nevertheless, this has not shown the stronger claim that there are A-expressions in relativistic spacetimes that cannot be translated into a B-expression as we can translate, for instance, "the curve is future-directed at point p" into "there are some points q, such that $p \ll q$". This last step is important since there are spaces $(M, \geq, \gg)$, called causal spaces, which are sets pre-endowed with the $\gg$ and $\geq$ relations that can recover a lot of the structure of relativistic spacetimes (including future and past-directed curves) without appealing to a metric or a time-orientation [14, 15]. Therefore we are to show that there are properties of curves in relativistic spacetimes that cannot be derived with respect to the $\gg$ or $\geq$ relations and, moreover, that these properties are A-properties.

At least two of these properties is that of the past and future-inextendibility of curves, which we will prove are implied by globally hyperbolic spacetimes within GR. Moreover, we will show that the inextendibility of curves constitutes an A-property in virtue of curves being inextendible at one parameter value λ_0, which in turns involves only one event. To define past(future)-inextendibility we first have to define the past (future) endpoints of a curve γ.

Definition 11: Let γ be a past(future)-directed curve. A point p is a past (future) endpoint if for every neighborhood U of p (open set in M containing p) there is a parameter value λ_0 such that $\gamma(\lambda) \in U \; \forall \, \lambda > \lambda_0$. γ is past (future)-inextendible if there is no such past (future) endpoint.

Note that by definition, there are no two events p and q to B-relate in order to define inextendible curves in terms the $\gg$-relation (that is, in terms of B-theory). It does constitute, however, an A-property since a curve is inextendible at a point p. This is because A future(past) inextendible curve cannot be extended to take any parameter value $\lambda$and hence there is a point p that is imaged by the biggest(lowest) affine parameter value λ_0. Now, let γ be a curve, then of γ we can say:

- γ a future-directed at $\gamma(\lambda_0) = p$.
- γ is inextendible to the future at $\gamma(\lambda_0) = p$.

Given the inexistence of events in γ required for it to be not extendible, neither of these claims can be translated into B-expressions. The B-theorist may accept the prioricity of A-theory concepts but may complain that we are being too stringent and that a B-expression can suffice to describe events in certain spacetimes. The Friedmann-Robertson-Walker metric, for instance, can estimate an age to the Universe, and order event via a global time coordinate. This characteristic (called global hyperbolicity), however, requires past inextendibility to be defined. We define the domain of dependence of S, $D^+(S)$, as

$$D^+(S) = \{q \in M | \, \text{Every past inextendible causal curve through } q \text{ intersects } S\}$$

When

$$D(S) = D^+(S) \cup D^-(S) = M. \tag{7}$$

Then S is a Cauchy surface (see Fig. 2), and a spacetime with a Cauchy surface, by Geroch's theorem, is globally hyperbolic [16]. Moreover, note that $edge(S) = \emptyset$.

Theorem 2. Let (M, g) be globally hyperbolic, and S a spacelike Cauchy hypersurface. Then, there exists a Cauchy temporal function $\mathcal{T} : M \to \mathbb{R} \times S$ such that $S = \mathcal{T}^{-1}(0)$.

We can define a global time function $\mathcal{T} : M \to \mathbb{R}$, such that it is constant through each Cauchy surface S. $\mathcal{T}$ would be the time-coordinate and since M can be foliated then the topology of M is diffeomorphic to $\mathbb{R} \times S$, see proof in [17, 18]. We have constructed a time-series relating the events of the entire manifold. Can two observers differ on the time at each surface? Yes, the global function is not unique and can be defined differently in different charts. Will they disagree with

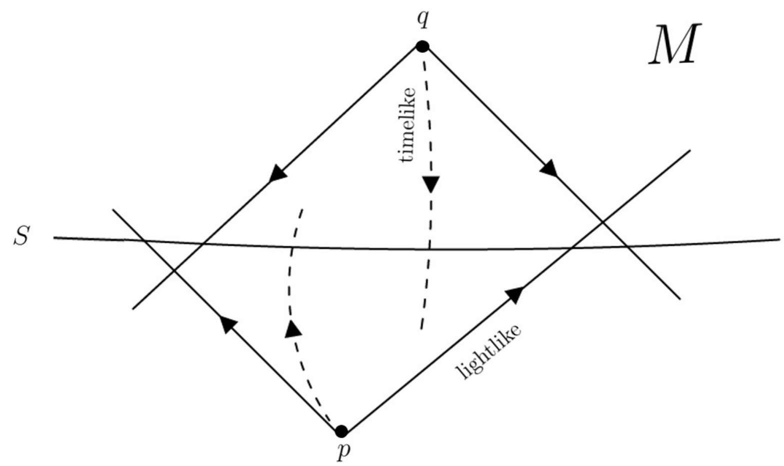

Figure 2: S is a Cauchy surface on M at time t. For any point $p \in D^{+}(S)$ ($q \in D^{-}(S)$) all the future(past)-directed non-spacelike curves only intersect S once.

respect to the order of events? No, Because we have ordered the events strictly by them being possible-causally accessible, through the definition of the domain of dependence.

Nevertheless, global hyperbolicity can be defined in an alternative way: namely using causal constraints and requiring that for any p and q the subset $J^{+}(p) \bigcap J^{-}(p)$ is compact [9, 10]. In fact, this definition is one that can be specified in causal spaces (Lorentz Length-Spaces) without any reference to time-inextendible curves [15]. Nevertheless, globally hyperbolic spacetimes imply the existence of Cauchy surfaces which implies the existence of past(future)-inextendible curves. Therefore, even with an alternative definition, there will be past(future)-inextendible curves of spacetimes that cannot be described within B-theory. We explicitly show this in proposition 2.

> **Proposition 2:** Let (M, g) be a spacetime with partial Cauchy surface S such that $edge(S) = \emptyset$. Then there exists a curve λ that has no past endpoint in M.

Proof. Let S be a partial Cauchy surface. Consider a point p in the boundary of S, and let p be an endpoint of the null curve $\gamma \in M$, $\gamma(t) = p$. Then, there is a $t_1 \in \mathbb{R}$ and a neighborhood $U \subset M$ such that if $t_1 < t$, $\gamma(t_1) \in U$ (from the definition of a past endpoint). Consider extending the curve γ into the past from p to a point q in the neighborhood of p. Thus, $q \in I^{-}(p)$. Then from q one can generate a timelike curve that, by the virtue of being timelike, can get arbitrary close to γ, and hence remain contained in U, but fail to intercept with p and S, however, it can join with a point $r \in U$ such that $r \in I^{+}(p)$. Therefore, the endpoint $p \in edge(S)$ with $edge(S) \neq \emptyset$. But S is a partial Cauchy surface and hence $edge(S) = \emptyset$. Consequently, there can be no endpoint p of the curve γ. ☐

Corollary 1: γ is past-inextendible since it has no endpoint, a similar proof follows for future-inextendible curves.

The proposition 2 also follows as a corollary of proposition 6.5.3 in Ref. [9].

Proposition 3: Let S be a closed achronal set. Then, $H^{-}(S)$ (Cauchy horizon) is generated by null geodesic segments which either have no past endpoints or have past endpoints at $edge(S)$.[7]

Corollary 2: if $edge(S) = 0$. Then, the points in $H^{-}(S)$ are generated by null geodesic segments that are inextendible to the past.

Therefore, given a globally hyperbolic spacetime M, there is a smooth spacelike Cauchy hypersurface S, thus, a global diffeomorphism between M and $\mathbb{R} \times S$ that requires future- and past-inextendible curves to exist (A-notions) which cannot be described by B-theory terms. To see how the B-notions fall short, consider definition 10 for future directed curves in terms of $\gg$, $\geq$. Now consider the affine parameter value λ_0 from which an inextendible curve γ cannot be extended. Since $g(v_\gamma(\lambda_0), \mathcal{T}) > 0$ we can say that the curve is future-oriented at λ_0. We cannot, however, express this using the $\gg$ relation since there is no point q later than $\gamma(\lambda_0)$ in the curve.

As a final stand the B-theorist may hold that, while causal spaces $(M, \geq, \gg)$ cannot reproduce the structure of relativistic spacetimes solely with the notions of $(\geq, \gg)$ [15], they can reproduce notions of global hyperbolicity and ultimately recover the Lorentzian metric by recovering the null-cone structure, under the condition that the space is strongly causal [14]. Nevertheless, these spaces *are not equivalent* to strongly causal relativistic spacetimes or general relativistic spacetimes for that matter: not only they fail to recover all structure from relativistic spacetimes [14], but further undesired causal behaviour arise from these spaces (see section 5 in ref. [15]). Moreover, the Lorentzian metric formulation of relativistic spacetime (M, g) is the preferable framework since Einstein's equations are formulated in terms of the metric, and it is conceptually simpler. Ultimately, the limitations found in causal spaces and in the $\gg$ and $\geq$ relations when describing structure in relativistic spacetimes (M, g) are what shows the limitations of B-theory when describing the ordering of events in the framework of General Relativity.[8]

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[7] Proved in [9].

[8] Recently, Grant et al. [19] defined notions of extendibility and inextendibility of curves within the framework of causal spaces. A null (timelike) curve $\lambda : [a, b) \to M$ is extendible as a null (timelike) curve if $\lambda[a, b] \in M$. Note that this is a much weaker notion than the definition above, since an extendible curve in relativistic spacetimes

It seems our result favors the A-theory of time in GR since the notions of past and future, via future-directedness and past-directness, are to be found written in most results regarding time in GR textbooks. In general, we cannot order events outside our future and past domain of dependence. These events may not form a time-series: they can be achronal. Causal notions, like the trajectory of timelike particles, are what allowed us to order the events in time, and causality cannot be relative, so the ordering of events is not observer-dependent. This allows us to avoid the problem of relative pastness and relative futurity found in special relativity.

Time-orientation is needed in order to make progress in formulating our concept of time and causality. First we talk about a particular direction in the manifold as being towards the future, then we define chronological past and future, future- and past-directed curves, future- and past-inextendible curves, the $\gg$-relation, the global time function, and the Cauchy Surface. The B-theorist could suggest that we should refer to events later than or earlier than, instead of vectors being towards later times (future-directed), but these expressions do not fully capture the nature of a time-oriented spacetime. In terms of which expressions pick out fundamental aspects of spacetime, the A-theorist has an upper hand in General Relativity.

VII. CONCLUSION

We have constructed notions of future and past chronicity that allow us to define time-series, as akin as possible to the ones found in McTaggart's work and the philosophy of time. The ordering of events came fundamentally from the definition of future- and past-directness, and the time-orientation; further structures were defined with the Cauchy surface, and the global time function, to end up ordering all the events on the manifold.

Every such structure is always connected to the existence of timelike (or lightlike) curves communicating causally one point to the next. Curves will allow us to order events in time, but we need to tell them where to go, they have to know where the future is first, in order to allow us to define any chronical ordering on the manifold M. First we need to define future-directed velocities at a point p, then we can locate the appropriate future events and relate them via the B-relations $\gg$ and $\geq$.

In this view, unless we require global hyperbolicity, we cannot refer to a "one and only" present

(M, g) can be extended to all values of parameter space (and still lie in the Manifold while keeping its timelike or null character).

(a present that is common to all points in space); each trajectory has its own present. If we can define a twist-free vector field ξ^a to locally foliate a subspace of M into achronal surfaces, then each collection of achronal surfaces will have its own present. That is not to say that whether an event is in the present can be relative; if an event is in the present for me (it is in my vicinity up to a partial Cauchy surface) then another observer cannot say that according to them then the event is in the future or in the past (because they can only talk about the future or the past in their vicinity). [9]

To conclude, let us not forget we are basing ourselves on an account of time derived from the notion of the velocity (directional derivative) of a trajectory in spacetime. It may be that different accounts of time can be extracted from the formalism of General Relativity, or that one may have good reasons to prefer a particular solution of Einstein's Equations. Nevertheless, when dealing with a general, time-orientable solution to Einstein's equations, we see no other way of ordering events in time and, when doing so, notions related to A-theory appear as fundamental. Moreover, this result extends beyond GR since more fundamental theories at low energy (effective theories) should recover the same time features of general relativity, hence, in this limit the ordering of events in time should be characterized by the A-theory.

Until now, we know of no other spacetime theory that is so friendly to the A-theorist. For lifting any ordering of events off the ground, one needs the notions of time-orientation, future-orientation, and past-orientation, which we argued are A-notions. Are the notions of objective past, present, and future found in GR? Yes, given that 1) any ordering of events in time will use them and 2) they are based on notions of causality, that are chart-independent. We find that, in the theory of General Relativity, events are ordered from past, to present, to future, agreeing with the A-theory of time.

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[9] This notion fulfills some of the desire description formulated by Oliver Pooley about what a relativistic notion of the flow of time can be. It has A-theory notions and it goes away with a global *now* [4]. Nevertheless, the topic of the flow of time requires a separate study.

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