Empirical Underdetermination for Physical Theories in C* Algebraic Setting: Comments to an Arageorgis's Argument

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To the Memory
of Aristidis Arageorgis

Abstract.

In this paper, I reconstruct an argument of Aristidis Arageorgis against empirical underdetermination of the state of a physical system in a C*-algebraic setting and explore its soundness. The argument, aiming against algebraic imperialism, the operationalist attitude which characterized the first steps of Algebraic Quantum Field Theory, is based on two topological properties of the state space: being T1 and being first countable in the weak*-topology. The first property is possessed trivially by the state space while the latter is highly non-trivial, and it can be derived from the assumption of the algebra of observables' separability. I present some cases of classical and of quantum systems which satisfy the separability condition, and others which do not, and relate these facts to the dimension of the algebra and to whether it is a von Neumann algebra. Namely, I show that while in the case of finite-dimensional algebras of observables the argument is conclusive, in the case of infinite-dimensional von Neumann algebras it is not. In addition, there are cases of infinite-dimensional quasilocal algebras in which the argument is conclusive. Finally, I discuss Porrmann's construction of a net of local separable algebras in Minkowski spacetime which satisfies the basic postulates of Algebraic Quantum Field Theory.

Keywords: Empirical Underdetermination, Algebraic Formulation of Physical Theories, State Space, First Countable, Separability.

In this paper, I explore the possibility of *distinguishing* and *completely determining* the actual state of a physical system from its possible states on the basis of empirical evidence. In particular, I discuss an argument in favor of the view that in the limit of empirical research, when all possible empirical evidence is taken into account, the state is both distinguishable from any other state and completely determined by the evidence. The soundness of such an argument would defeat the thesis of empirical underdetermination of the state of a physical system.

The argument is due to Arageorgis (1995) and it concerns classical and quantum theories formulated in the C*-algebraic framework. It shows that the distinguishability of the actual state follows naturally from general topological assumptions related to the C*-algebraic framework. Nevertheless, the complete determination of the state of the system rests on the controversial assumptions that the state space is first countable and that the algebra of observables is separable. In an attempt to probe the nature of Arageorgis's argument, I present different cases of physical systems in which the argument either succeeds or it is inconclusive. However, these examples do not bring in anything new, they just illustrate something that Arageorgis already knew; namely, that there are physically interesting cases in which the argument is inapplicable.

The paper consists of three sections. In the first section I present a commonplace representation of state determination experiments in the C*-algebraic setting. In the second I provide a detailed reconstruction of Arageorgis's argument. Finally, the last section is devoted to the discussion of the argument in the light of examples of physical systems.

1. Algebras and Experiment

To begin with, the kinematics of any physical system is described in terms of its observables and its states. The observables may be taken to represent physical properties or magnitudes characterizing the system, which are, in principle, measurable in some spacetime locale, while a state of the system is an assignment of real values, be they interpreted as measurement outcomes, or expectation values, or probabilities, to the observables of the system.

In order to determine experimentally the state of a physical system, one performs measurements of the system's observables. Three rather commonplace assumptions govern experiments of the kind: a) the number of observables to be measured is finite; b) the number of repetitions of the measurement for any given observable is, also, finite; c) there is statistical error in the measurement of any observable which is expressed by the difference of the measured values from the experimentally determined mean value of the observable. Hence, an experiment E that determines the state of a system can be represented by a finite collection $E = \{Q_1, \dots, Q_n\}$ of observables.[1] For every observable $Q_i \in E$, one performs N_i measurements in which they obtain M_i different measurement outcomes α_{ij}; each outcome N_{ij} times. The experimentally determined mean value for each Q_i in E is,

[1] The most common interpretation of the observables is in terms of operations performed in the laboratory. However, I do not want to openly commit myself to an operationalist interpretation.

$$\langle Q_i \rangle_E = \sum_{j=1}^{M_i} \alpha_{ij} \frac{N_{ij}}{N_i}\ , \quad \text{for } i = 1, \dots, n,$$

while the corresponding variance is,

$$\Delta_E^2 Q_i = \sum_{j=1}^{M_i} {\alpha_{ij}}^2 \frac{N_{ij}}{N_i} - {\langle Q_i \rangle_E}^2, \quad \text{for } i = 1, \dots, n.$$

In the C*-algebraic framework, the observables of a physical system are self-adjoint elements of a C*- or a von Neumann algebra $\mathcal{A}$, while the physical states of the system are represented by the states of the algebra, the positive normalized complex-valued functionals on $\mathcal{A}$, $S(\mathcal{A}) = \mathcal{A}_1^{*(+)}$. Given the results yielded by a state determination experiment E, every state of the algebra of observables that assigns to each observable Q_i, a value $\omega(Q_i)$ in an open symmetric interval about $\langle Q_i \rangle_E$,

$$\omega(Q_i) \in (\langle Q_i \rangle_E - \varepsilon_i\ , \langle Q_i \rangle_E + \varepsilon_i\ )\ ,\ \ \varepsilon_i = \sqrt{\Delta_E^2 Q_i},\ \text{for } i = 1, \dots, n,$$

will count as an admissible candidate state of the system; hence, the state of the system is constrained, yet not fully determined, by experiment E. This is the thesis that one experiment underdetermines the state of a physical system, expressed in the C*-algebraic setting.

In their seminal paper on the algebraic approach to quantum fields, Haag and Kastler (1964) expressed the underdetermination thesis as follows: "Let T be the preparing operation and R_T its range (the image of $\mathfrak{A}^{*(+)}$ under T) …Then the only certain knowledge about the prepared state is that it lies in R_T. To obtain definite state we need an operation with a one-dimensional range. There are two reasons why such an ideal operation is impossible. The first has to do with the limited accuracy in the specification of T …While we are unable at the moment to give a precise analysis of the consequences of these two limitations we feel that the first one (limitation in accuracy) will result in the statement that we have no in no actual experiment a precisely defined state but rather a weak neighborhood in $\mathfrak{A}^{*(+)}$."

The neighborhood in the state space determined by an experiment E is the following:

$$N(\omega, E, \varepsilon) = \{\phi\colon |\phi(Q_i) - \omega(Q_i)| < \varepsilon\ , i = 1, \dots, n\ \},$$

where $\omega(Q_i) = \langle Q_i \rangle_E$, $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$, and it belongs in a local base at ω for the weak*-topology in the state space.

2. Arageorgis's Argument

Arageorgis in his dissertation (1995) discussed the empirical underdetermination of the state of the system as a basis of justification for algebraic imperialism, i.e. the view that "the intrinsic properties of a quantum field can be adequately described by model of the type… $\langle M, g_{ab}, O \mapsto \mathcal{A}(O), S, \alpha \rangle$" where $(M, g_{ab})$is the background spacetime, $O \mapsto \mathcal{A}(O)$ is a net of abstract C*-algebras indexed by the directed set of finite regions, S is a group and α is a realization of the group by automorphisms of the net (1995:149-150).[2] In a nutshell, the positive content of algebraic imperialism is that all physical information about a quantum field is encoded in the net of abstract C* algebras and the group of automorphisms, while the negative content is that representations of the net of abstract C*-algebras by concrete C*- or von Neumann algebras in some Hilbert space do not contain any further physical information; in this sense, all representations provide equivalent descriptions of the same physical system. Empirical underdetermination of a state as explicated in terms of the weak*-topology of the state space provides a plausible criterion of physical equivalence for two representations. Namely, if for every weak*-neighborhood of a normal state in the first representation there is a normal state in the second representation and *vice versa,* then one may consider the two representations physically equivalent: no experiment can decide whether the system in some state is best described in terms of one representation rather than the other. Hence, a conventionalist strategy emerges: the choice between any two theories defined in terms of concrete local quantum structures $\langle M, g_{ab}, \mathcal{H}, O \mapsto \mathcal{R}(O), S', U \rangle$, where $\mathcal{H}$ is a complex Hilbert space, $O \mapsto \mathcal{R}(O)$ a net of von Neumann algebras in $\mathcal{H}$, S' is a group and U a unitary representation of that group in $\mathcal{H}$, is purely conventional.

Can empirical underdetermination of states provide an adequate notion of physical equivalence and a basis for the conventionalist strategy? The objection raised at this point by Arageorgis is that physical equivalence of two representations or of two theories emerging from these representations should require much more than mere agreement with some body of available evidence. It should require, on the one hand, a stronger form of empirical equivalence which involves agreement with *all possible empirical evidence* and not just the available one (1995:160-164), on the other hand, theoretical equivalence with respect to either the description of general facts such as the description of two systems in thermodynamic equilibrium at different temperatures, or the explanation of general facts such as the explanation of the spontaneous symmetry breaking mechanism (1995:164-166).

The argument that we are interested in, in this paper, aims to support the first part of the aforementioned two-fold objection to algebraic imperialism, namely, the claim that two theories, defined in terms of (faithful) representations of abstract algebraic quantum field systems, may describe equally well all available empirical evidence, yet they may fail to provide equivalent descriptions with respect to all possible empirical evidence. The argument rests on a well-known distinction between two different underdetermination theses:

1) Inductive Empirical Underdetermination Thesis (IEU): The state of a system cannot be completely determined on the basis of available empirical evidence. In the C*-algebraic setting, the most one can expect is to confine the state of the system within a weak*-neighbourhood of the system's algebra state space, corresponding to the measurement of a finite number of observables. All states in that neighbourhood are indistinguishable with respect to the data yielded by measurements of the prescribed set of observables.

2) Strong Empirical Underdetermination Thesis (SEU): a) The state of a system cannot be completely determined on the basis of all possible empirical evidence. b) Two states indistinguishable with respect to the measurement of some finite

[2] For a detailed analysis of the concept of algebraic imperialism and its variants, consult (Ruetsche 2011).

collection of observables cannot be distinguished on the basis of any collection of observables whatsoever.

It is straightforward from what we said about the representation of experiments in the C*-algebraic setting and the relevant considerations in terms of the weak*-topology in the state space that (IEU) is satisfied. However, is this the best one should expect getting from experience? Or, it would be theoretically myopic to consider only the finite data available at a given time without taking into account the full empirical potential of a theory as it reveals itself asymptotically, in the limit of research? If the state of the system remained determinate only up to a neighborhood in the weak*-topology in the light of evidence from all possible experiments, then those who profess the empirical underdetermination thesis would be vindicated.

At this point Arageorgis interferes arguing against both sub-theses of (SEU). Firstly, he argues against sub-thesis (2b) by claiming that for any two distinct states $\omega_1, \omega_2 \in S(\mathcal{A})$ of the algebra of observables $\mathcal{A}$ there is a weak*-neighborhood of ω_1, $N(\omega_1, E, \varepsilon)$, which does not contain ω_2, since $S(\mathcal{A})$ equipped with the weak*-topology forms a T_1 topological space. Now, given that each weak*-neighborhood corresponds to a possible experiment, "the T_1 property can be interpreted as guaranteeing the falsifiability of all claims asserting which is the actual state of the observed system" (1995:163).

In other words, if $\omega_1 \neq \omega_2$, there is a $Q \in \mathcal{A}$ such that $\omega_1(Q) \neq \omega_2(Q)$. Without loss of generality, we may assume that Q is a self-adjoint element; thus, it satisfies the mathematical prerequisites for being an observable.[3] Next, consider the following weak*-neighborhood of ω_1,

$$N(\omega_1, \{Q\}, \varepsilon) = \{\phi: |\phi(Q) - \omega_1(Q))| < \varepsilon\},$$

where $\varepsilon = |\omega_1(Q) - \omega_2(Q)|$, which does not contain ω_2. Since, each weak*-neighborhood corresponds to a possible experiment, the measurement of Q with experimental error - ε, verifies the assertion "the system is in state ω_1" and falsifies the assertion "the system is in state ω_2". Moreover, the crucial experiment is possible even if the two assertions agree with respect to their truth-value for some finite amount of available evidence.

To show that sub-thesis (2a) is not true either, Arageorgis makes a non-trivial assumption about the state space of the algebraic system which, to my knowledge, lacks any straightforward physical interpretation. He assumes that the weak*-topology is first countable in the state space of the physical system, i.e. for every state $\omega \in S(\mathcal{A})$ there is a countable local base of neighborhoods of ω in the weak*-topology

$$\mathcal{T}(\omega) = \{T_m(\omega), m \in \mathbb{N}\}.$$

In terms of $\mathcal{T}(\omega)$ one may define a nested local base of neighborhoods of ω,

$$\mathcal{N}(\omega) = \{N(\omega, E_n, \varepsilon_n), n \in \mathbb{N}\}$$

where,

$$N(\omega, E_n, \varepsilon_n) = T_1(\omega) \cap \ldots \cap T_n(\omega) \text{ and } N(\omega, E_{n+1}, \varepsilon_{n+1}) \subseteq N(\omega, E_n, \varepsilon_n).$$

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[3] Even if Q were not a self-adjoint element, it could be analyzed in terms of two self-adjoint elements X, Y: $Q = X + iY$. Since $\omega_1(Q) \neq \omega_2(Q)$, then either $\omega_1(X) \neq \omega_2(X)$ or $\omega_1(Y) \neq \omega_2(Y)$. Hence, there is always a self-adjoint element distinguishing to distinct states.

As previously, Arageorgis assumes that to every weak*-neighborhood in $\mathcal{N}(\omega)$ corresponds a possible experiment E_n which amounts to measuring a finite number of observables with minimum error ε_n; hence, a sequence of experiments $\{(E_n, \varepsilon_n), n \in \mathbb{N}\}$ may be associated with $\mathcal{N}(\omega)$. Since $S(\mathcal{A})$ equipped with the weak*-topology is a T_1 topological space, every $\omega \in S(\mathcal{A})$ is the intersection of all open sets containing it,

$$\{\omega\} = \bigcap_{n=1}^{\infty} N(\omega, E_n, \varepsilon_n).$$

It follows, then, that there is a sequence of possible experiments $\{(E_n, \varepsilon_n), n \in \mathbb{N}\}$ such that each subsequent member of it places more restrictions on the set of candidate states of the system, and, in this sense, it provides a better approximation of the actual state ω of the system. Asymptotically, in the limit of the empirical research, one is able to know the actual state of the system and to assign a truth-value to assertions of the type "the system is in state ω", since the sequence of experiments "converges" to the actual state.

Whether $S(\mathcal{A})$ is first-countable or not depends on the C*-algebra $\mathcal{A}$. Arageorgis suggested that a sufficient condition for $S(\mathcal{A})$ to be first-countable is for $\mathcal{A}$ to be separable i.e. to have a countable norm-dense subset. (Arveson 1976:8). Here is a justification: since $\mathcal{A}$ is a unital C*-algebra, its state space $S(\mathcal{A})$ is a weak*-compact convex subset of its dual space $\mathcal{A}^*$ (Bing-Ren 1992: 96, Prop.2.5.4). In addition, if $\mathcal{A}$ is separable, $S(\mathcal{A})$ is metrizable in the weak*-topology (Rudin: 68, Thm 3.16). Thus, $S(\mathcal{A})$ is second countable, since a separable metric space is second countable (Willard 1970:40), and, as a consequence, it is first countable. Hence, for the refutation of sub-thesis (2a), the argument rests on the separability of the algebra of observables $\mathcal{A}$.

3. Discussion

To explore the range of application of Arageorgis's refutation of (SEU), I focus on the refutation of sub-thesis (2a), since (2b) is false for every C*-algebra, given that the weak*-topology on the state space $S(\mathcal{A})$ is always T_1. Recalling that the refutation of (2a) rests on the first countability of the state space of the algebra of observables which, in turn, is derived from the separability of the C*-algebra $\mathcal{A}$, let me make some introductory remarks that may improve the understanding of the import of this assumption.

It has been suggested[4] that if time is assumed to be real-valued, and, if measurements can be performed at each time instant, then the class of possible state determination experiments as well as the corresponding class of weak*-neighborhoods of the actual state of the system are, *in principle*, uncountable. Thus, it is quite puzzling to restrict our attention to a countable set of experiments as it seems to suggest the supposition that the state space is first countable. Wouldn't it be more reasonable to stipulate that local bases in the state space had the cardinality of the set of real numbers?

[4] I thank Ben Feintzeig for stirring up this part of the discussion.

Firstly, let me stress that the first countability of the state space does not imply that all local bases are countable neither that only countably many state determination experiments can be performed. It just claims that there is at least one countable local base for every point of the state space. Arageorgis's argument exploits the existence of such a local base to show that a countable set of experiments, suitably devised, *suffice*, in principle, to determine completely the actual state of the system.

Secondly, it is vital for Arageorgis's argument to be able to determine a sequence of possible experiments, corresponding to a nested local base of weak*-neighborhoods of the state of the system, in order to show that in the light of all possible empirical evidence the state can be determined. Yet, on the assumption that all local bases were of the cardinality of the set of real numbers, one could not follow a *similar* strategy as before to generate a decreasing *net*, and not a sequence, now, of local neighborhoods of a state. To illustrate this fact, assume that for every state $\omega \in S(\mathcal{A})$ there is a local base of neighborhoods of ω in the weak*-topology

$$\mathcal{T}(\omega) = \{T_x(\omega), x \in \mathbb{R}\}$$

Since the class of compact subsets of $\mathbb{R}$ equipped with the usual set-theoretic inclusion is a directed set. Next, one may define a decreasing net of subsets of the state space as follows:

$$X_1 \subseteq X_2, \quad N_{X_2}(\omega) \subseteq N_{X_1}(\omega),$$

where

$$N_X(\omega) = \bigcap_{x \in X} T_x(\omega).$$

However, $N_X(\omega)$ is not necessarily a neighborhood, since it is not constructed as a *finite* intersection of open sets and, respectively, the decreasing net does not define a local base of neighborhoods of ω. Just by repeating the construction used for countable local bases one cannot yield a net of possible experiments that determine the state of the system. Hence, no matter how difficult it might be to provide a physical understanding of the stipulation that the state space of the system is first countable, it seems essential for Arageorgis's argument to conclude.

Arageorgis has already admitted that "many C*-algebras of physical interest will not satisfy these conditions, and, so, one cannot invariably assume the first countability of the weak*-topology on the corresponding state spaces." (1995:164) A similar judgement has been expressed by Porrmann (2004) motivating his attempt to provide countable and separable versions of the fundamental assumptions of relativistic algebraic quantum field theory: "While being concerned with a separable Hilbert space is common from a physicist's point of view, the corresponding requirement on the C*-algebra… is too restrictive to be encountered in physically reasonable theories from the outset." The remaining section is devoted to the illustration of these claims.

To begin with, consider an example from classical mechanics. The phase space of a system of N classical particles is a subset Γ of $\mathbb{R}^{2N}$. Assuming that the particles' motion is restricted to a bounded part of the physical space and that the range of admissible energy values for the system is also bounded, one may justifiably accept that Γ is a compact subset of $\mathbb{R}^{2N}$ (Strocchi 2005:10). The observables of the system are, in general, real polynomials of the position and the momentum of the particles. Since the topological algebra of real polynomials with domain of definition a compact subset Γ of $\mathbb{R}^{2N}$, equipped with the sup-norm, is dense in the algebra of real-valued continuous functions in Γ, by the Stone-Weierstrass theorem, one may consider the algebra of real-valued continuous functions to be the algebra of observables for the system the observables. Or, equivalently, the observables can be represented by the self-adjoint elements of the C*-algebra of complex-valued continuous functions of a compact domain of definition Γ, $C(\Gamma)$.

It is easy to justify that $C(\Gamma)$ is separable, for every compact subset Γ of $\mathbb{R}^{2N}$. It can be proven that $C(\Gamma)$, is separable if and only if Γ is second countable (Chou 2012: Thm. 2.4). However, every compact subspace Γ of $\mathbb{R}^{2N}$ is second countable, since $\mathbb{R}^{2N}$ is a second countable space.[5] Hence, $C(\Gamma)$ is separable. An alternative justification invokes the Stone-Weierstrass theorem to show that the set of all polynomials in $x \in \Gamma$ with coefficients with rational real and imaginary part, form a countable dense subset of $C(\Gamma)$; hence, by definition, $C(\Gamma)$ is separable.

Since the separability of the C*- algebra is a sufficient condition for the first countability of its state space, which, in turn, guarantees the refutation of sub-thesis (2a) of (SEU), one may justifiably infer that (SEU) is false for all classical particle systems of bounded energy moving in bounded regions of the physical space. Namely, there are observables that distinguish any two different states of the system, as represented by normalized positive functionals on $C(\Gamma)$, and in the limit of empirical research, when one takes into consideration data from all possible state determination experiments, they can determine unambiguously the state of the particle system, for any given preparation.

A second physically interesting case in which (SEU) can be proved false as long as one determines the local state of the system or considers the physical state of the system to be represented by a state of the quasi-local algebra, is provided by the Heisenberg model for ferromagnetism (Bagarello and Trapani 1996)[6]. Roughly, the model consists of a finite number of atoms of spin ½ placed at fixed lattice sites and it describes the spin-spin interactions between the nearest neighbors. For every finite region V of the d-dimensional lattice ($d = 1,2,3$), the local algebra of observables of the spin system is represented by a finite non-commutative von Neumann algebra. To see that consider any finite region V of the lattice. At each lattice point $p \in V$, the algebra of observables $\mathcal{A}_p = M_2(\mathbb{C})$ is generated by the Pauli matrices $\sigma_p^1, \sigma_p^2, \sigma_p^3$,

$$\sigma_p^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_p^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_p^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the unit 2×2 matrix I_p. Hence, the projection of the spin, or the magnetic moment of a particle, at p along a given direction $n = (n_1, n_2, n_3)$ in $\mathbb{R}^3$ is represented by the matrix,

[5] The Euclidean space $\mathbb{R}^n$ is second countable since the set $\mathcal{B} = \{B_r(x): x \in \mathbb{Q}^n, r \in \mathbb{Q}, r > 0\}$ of open balls centered at points having rational coordinates is a countable base for the topology.

[6] For a philosophical discussion of symmetry breaking in quantum spin systems, consult (Ruetsche 2006).

$$n_1\sigma_p^1 + n_2\sigma_p^2 + n_3\sigma_p^3,$$

with spectrum ± 1, corresponding to the possible up and down orientations of the spin.

The region V has $|V|$-many lattice sites and the algebra of observables associated with V is the tensor product of $\mathcal{A}_p$ for every $p \in V$,

$$\mathcal{A}_V = \otimes_{p=1}^{|V|} \mathcal{A}_p = \overbrace{M_2(\mathbb{C}) \otimes \ldots \otimes M_2(\mathbb{C})}^{|V|\ times} \simeq M_{2^{|V|}}(\mathbb{C}),$$

which is isomorphic to the algebra of $2^{|V|} \times 2^{|V|}$ complex matrices. This algebra contains all observables that can be measured in the finite region V.

The local algebra of observables associated with a region V, is a separable von Neumann algebra, since it is finite-dimensional. This implication can be justified as follows: every finite-dimensional topological vector space over $\mathbb{C}$ is topologically isomorphic to a finite-dimensional topological space over $\mathbb{R}$ with double dimension (Koethe 1969: 151). But for every finite-dimensional topological vector space over the reals, one can pick out a base of finite cardinality, and the rational linear combinations of that base form a countable norm-dense subset of the topological vector space; hence the topological vector space is separable over $\mathbb{R}$ is separable as well as its topologically isomorphic over $\mathbb{C}$. The converse implication, from the separability of the von Neumann algebra to its finite-dimensionality, is a fact stated as an exercise in (Takesaki 1979: 139). Thus, a von Neumann algebra $\mathcal{M}$ is separable if and only if it is finite dimensional.

As in the case of the classical particle system, one may argue along Arageorgis's line of reasoning for the claim that in the limit of empirical research, if all possible local measurements in a given region were performed and the relevant data were taken into account, the local state of the system would be fully determined; hence, sub-thesis (2a) of (SEU) is false.

What about the entire infinite d-dimensional lattice? Is it possible to infer by means of the same argument whether one can fully determine the state of the spin system by possible local measurements performed in its entire volume? To answer the question, one needs to produce the quasi-local algebra of observables for the lattice and examine whether it is separable. It is well-known that the quasi-local algebra of a quantum spin system is a UHF algebra constructed over the finite subsets of the lattice (Bratteli-Robinson II:240). A unital C*-algebra $\mathcal{A}$ is a uniformly hyperfinite (UHF) algebra if there exists an increasing sequence of C*-subalgebras, $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$, $\mathcal{A}_n \subset \mathcal{A}_m$, for $n < m$, each containing the unit of $\mathcal{A}$, such that $\mathcal{A}$ is the uniform closure of $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ and $\mathcal{A}_n$ is *-isomorphic to the algebra of bounded operators on a finite-dimensional Hilbert space, $\mathcal{A}_n \simeq \mathcal{B}(\mathcal{H}_n)$, $\dim \mathcal{H}_n < \infty$, $\forall n \in \mathbb{N}$. In the case of a quantum spin system, the increasing sequence of subalgebras is obtained by considering the algebras of an increasing sequences of lattice regions $\{V_n\}_{n\in\mathbb{N}}$,

$$V_n \subset V_m \Rightarrow |V_n| < |V_m| \Rightarrow M_{2^{|V_n|}}(\mathbb{C}) \subset M_{2^{|V_m|}}(\mathbb{C}) \Rightarrow \mathcal{A}_n \subset \mathcal{A}_m \text{ , for } n < m,$$

where the third implication is established via the unital, injective and trace preserving *-homomorphism, $\phi_k: M_{2^k}(\mathbb{C}) \subset M_{2^{k+1}}(\mathbb{C})$,

$$\phi_k : a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

$\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ is a normed *-algebra and its uniform closure $\overline{\bigcup_{n\in\mathbb{N}} \mathcal{A}_n}^{\|\cdot\|}$ is the UHF algebra $\mathcal{A}$, the quasi-local algebra of the spin system. All UHF algebras are separable (Bratteli 1972: 202, 1.10), thus, $\mathcal{A}$ is separable and the argument entails that it is possible to infer the state of the infinite lattice by means of state determination experiments employing local measurements.

To complete the discussion of the quantum spin system, I consider a well-known representation of the quasi-local algebra in terms of a tracial state. A positive linear functional τ on the quasi-local algebra $\mathcal{A}$ is said to be tracial if $\tau(a^*a) = \tau(aa^*),\ a \in \mathcal{A}$. Von Neumann algebras $M_{2^k}(\mathbb{C})$ have a unique tracial state ($\|\tau_k\| = 1$), $\tau_k(a) = \frac{1}{2^k}\sum_{i=1}^{2^k} a_{ii}$ where $a = (a_{ij}) \in M_{2^k}(\mathbb{C})$ (Murphy 1990:179). Since the $\phi_k$is trace preserving, $\tau_{k+1} \circ \phi_k = \tau_k$, one can define a unique tracial state on $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$, $\tau: \bigcup_{n\in\mathbb{N}} \mathcal{A}_n \to \mathbb{C}$, $\tau(a) = \tau_k(a)$, for $a \in \mathcal{A}_k$ which can be extended uniquely to a tracial state on its uniform closure $\overline{\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n}^{\|\cdot\|}$. Next, consider the GNS representation $(\mathcal{H}_\tau, \pi_\tau, \psi_\tau)$ of the quasi-local algebra induced by the tracial state τ. The von Neumann algebra $\mathcal{M} = (\pi_\tau(\mathcal{A}))'' = \left((\pi_\tau(\mathcal{A}))'\right)'$, where $(\pi_\tau(\mathcal{A}))' = \{A \in \mathcal{B}(\mathcal{H}_\tau): AB = BA, B \in \pi_\tau(\mathcal{A})\ \}$, generated by π_τ is a hyperfinite II_1 factor.

I argued previously that a von Neumann algebra $\mathcal{M}$ is separable if and only if they are finite-dimensional. Moreover, if a von Neumann algebra $\mathcal{M}$ is finite-dimensional, it has minimal projections. Minimal projections are abelian (Takesaki 1979:296), therefore, if $\mathcal{M}$ is finite-dimensional then it has abelian projections or, equivalently, if a von Neumann algebra $\mathcal{M}$ does not have abelian projections, then it is infinite-dimensional. Von Neumann algebras of type II (hyperfinite II_1 factors, included) and of type III do not have abelian projections; hence, von Neumann algebras of type II, III are infinite-dimensional, therefore, they are non-separable.

In the case of the quantum spin system, the hyperfinite II_1 factor $\mathcal{M} = (\pi_\tau(\mathcal{A}))''$ is non-separable. Since the representation $(\mathcal{H}_\tau, \pi_\tau, \psi_\tau)$ is faithful (Murphy 1990:182) one has

$$\mathcal{A} \simeq \pi_\tau(\mathcal{A}) \subseteq (\pi_\tau(\mathcal{A}))'' = \mathcal{R}.$$

This relation shows that a separable C*-subalgebra may be contained in a non-separable von Neumann algebra, for a given representation. Hence, separability is not bequeathed from a C*-subalgebra to the von Neumann algebra obtained in the representation. This is not surprising, since one may show that, in general, for every von Neumann algebra $\mathcal{M}$ acting on a separable Hilbert space there is a separable C*-algebra $\mathcal{A}$, $\mathcal{A} \subset \mathcal{M}$ which is weakly dense in $\mathcal{M}$ (Arveson 1976:9, Prop.1.2.3).[7] Hence, $\bar{\mathcal{A}}^{WOT} = \mathcal{M}$ or by the double commutant theorem $\mathcal{A}'' = \mathcal{M}$.

[7] Porrmann (2004, *Appendix A*) delivers a proof of a slightly more general fact about the existence a separable C*-subalgebra $\mathcal{A}_{sep}$ of *any* unital subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space, which is dense in $\mathcal{A}$ in the strong operator topology: $\overline{\mathcal{A}_{sep}}^{SOT} = \mathcal{A}$.

If the kinematics of the quantum spin system is described in terms of the hyperfinite II_1 factor $\mathcal{M}$, then Arageorgis's argument cannot be used to infer whether the global state of the infinite lattice can be determined by data yielded by possible experiments, since the implication from the separability of the algebra to the first countability of the state space is blocked. Thus, unless other reasons are given, one should suspend judgement regarding the truth of sub-thesis (2a) of (SEU).

My last, but of no less importance, comment on Arageorgis's argument is related to the Haag-Araki formulation of relativistic algebraic quantum field theories in Minkowski spacetime. A Haag-Araki theory is defined as any 6-tuple,

$$\langle \mathbb{R}^4, \eta_{ab}, \mathcal{H}, O \mapsto \mathcal{R}(O), G, U \rangle,$$

where $(\mathbb{R}^4, \eta_{ab})$is Minkowski spacetime, $\mathcal{H}$ is a complex Hilbert space, $O \mapsto \mathcal{R}(O)$ a net of von Neumann algebras in $\mathcal{H}$, G is a group and U a (strongly continuous) unitary representation of that group in $\mathcal{H}$, which satisfies the usual axioms of (a) isometry; (b)weak additivity; (c) locality; (d) primitive causality; (e) Poincaré covariance; (f) spectrum condition (see, Horuzhy 1990:12ff).

Several results indicate that the local algebras of observables, $\mathcal{R}(O)$, where O is a finite region of Minkowski spacetime, in a Haag-Araki theory are type III_1 factors. Moreover, concrete models of relativistic fields, such as free Bose field, support the same result.[8] Type III_1 factors, as mentioned previously, are infinite-dimensional and $\mathcal{R}(O)$ are not separable; thus, one cannot infer on the basis of the algebra's separability whether the local state space is first countable and the argument for the refutation of (2a) of (SEU) is blocked. Hence, it cannot be established whether possible evidence coming from a finite region suffice to determine the local state of the physical system in that region.

Moreover, to express mathematically the absence of superselection rules and the condition that the system is described in a single coherent superselection sector, one may assume that the global algebra of the system, $(\bigcup_{o\in\mathbb{R}^4} \mathcal{R}(O))'' = \mathcal{R}$ is a type I_∞ von Neumann factor (Horuzhy 1990:24). This fact can also be inferred, in a Haag-Araki theory, by the cyclicity of the vacuum subspace for the complement of the centre of $\mathcal{R}$, Z' (Horuzhy 1990:106, Prop.1.3.46). Thus, if the global algebra $\mathcal{R}$ is a type I_∞von Neumann factor then it is non-separable, and nothing can be said about the first countability of global state space of the system; once more, Arageorgis's argument is inconclusive.

I close this section with Porrmann's suggestion of a *separable* reformulation of the fundamental assumptions of relativistic algebraic quantum field theory in Minkowski spacetime. I briefly sketch this approach, since it provides a general account of an algebraic quantum field theory that satisfies the prerequisites of Arageorgis's argument against sub-thesis (2a) of (SEU).

Porrmann considers a denumerable dense subgroup $P_{+\,count}^{\uparrow}$ of the proper orthochronous group of Poincaré transformations $P_{+}^{\uparrow}$ constructed as a semi-direct

[8] For a recent review of the relevant facts, consult (Halvorson and Mueger 2007:749-752).

product of the corresponding countable subgroups of Lorentz transformations, $L^{\uparrow}_{+count}$, and of the group of translations in $\mathbb{R}^4$, T_{count} :

$$P^{\uparrow}_{+count} = L^{\uparrow}_{+count} \ltimes T_{count} \ .$$

By applying these transformations to symmetric double cones[9] with rational radii, centered at the origin, i.e.

$$D = \{x = (x^0, \vec{x}) \in \mathbb{R}^4 : |x^0| + |\vec{x}| < r, r \in \mathbb{Q}\},$$

one obtains a countable family of open bounded regions $\mathfrak{R}$, which has the following desirable features: (a) it is a countable base for the Euclidean topology of $\mathbb{R}^4$; (b) it is invariant under the geometric transformations thus constructed. The first feature entails that the elements of the family can be as small as one wishes and that they cover $\mathbb{R}^4$. The second feature says that by applying an element of $P^{\uparrow}_{+count}$ to some element of $\mathfrak{R}$, one obtains a region of the same family.

Secondly, Porrmann constructs a net of separable C*-algebras over the aforementioned countable family of open bounded regions of $\mathbb{R}^4$. He begins with a net of concrete local C*- algebras $O \mapsto \mathcal{A}(O)$, on a separable Hilbert space $\mathcal{H}$ and focuses on those algebras that correspond to regions O_k in $\mathfrak{R}$. Each local algebra $\mathcal{A}(O_k)$ contains a separable C*-subalgebra $\mathcal{A}_{sep}(O_k)$ (see note 6). Porrmann takes the separable algebra $\mathcal{A}_{sep}(O_k)$ to be over the field $\mathbb{Q} + i\mathbb{Q}$. Next, he constructs a separable algebra over $\mathbb{C}$, $\mathcal{A}^{\blacksquare}_{sep}(O_k)$ as the algebra generated by the union of all transforms $U(\Lambda, s)\mathcal{A}_{sep}(O_i)U^{-1}(\Lambda, s)$, where $(\Lambda, s) \in P^{\uparrow}_{+count}$; U is the unitary representation of $P^{\uparrow}_{+count}$ in $\mathcal{H}$; and $\Lambda O_i + s \subseteq O_k$. Thus, $\mathcal{A}^{\blacksquare}_{sep}(O_k)$ is a separable C*-algebra, strongly dense to $\mathcal{A}(O_k)$ such that

$$\mathcal{A}_{sep}(O_k) \subseteq \mathcal{A}^{\blacksquare}_{sep}(O_k) \subseteq \mathcal{A}(O_k),$$

By construction the net $O_k \mapsto \mathcal{A}^{\blacksquare}_{sep}(O_k)$ satisfies the conditions of isotony, locality and covariance, with respect to $\mathfrak{R}$ and $P^{\uparrow}_{+count}$.[10] The countable algebra $\mathcal{A}_{sep}$ over $\mathbb{Q} + i\mathbb{Q}$ generated by the union of $\mathcal{A}_{sep}(O_k)$, $O_k \in \mathfrak{R}$, is uniformly dense in the C*-inductive limit $\mathcal{A}^{\blacksquare}_{sep}$ of the net $O_k \mapsto \mathcal{A}^{\blacksquare}_{sep}(O_k)$, which in turn is strongly dense in the quasi-local algebra $\mathcal{A}$,

$$\mathcal{A}_{sep} \subseteq \mathcal{A}^{\blacksquare}_{sep} \subseteq \mathcal{A}.$$

Porrmann's suggestion seems to integrate nicely the principal postulates of relativistic algebraic quantum field theory with the assumption of countability of spacetime regions and spacetime symmetries and the separability of the local algebras. It has been developed in the process of exploring a new conception of particles in terms of

[9] Porrmann mentions "standard diamonds" but I believe he refers to double cones. A diamond is any (open) region of Minkowski space time, bounded or unbounded, that satisfies the relation: $O = O''$, where O' is the subset of $\mathbb{R}^4$ which contains all points at spacelike distance with every point of O (Horuzhy 1990:24). A double cone is a well-known case of a bounded diamond region defined as the interior of the intersection of the forward and the backward light cones of two timelike distant spacetime points. Alexandrov has observed that double cones provide a base for the topology of Minkowski spacetime, $\mathbb{R}^4$.(Borchers and Sen 2006:5)

[10] For more details, especially about relativistic covariance, consult (Porrmann 1999:55).

particle weights. Moreover, Porrmann (2004) explicitly claims that under reasonable physical assumptions about the state space[11] of the system "no information on a physical system described by a normal state of bounded energy… gets lost" when considerations about particle weights are made in the context of the above constructions.

4. Some open questions

Let us make clear that what is discussed in this paper is just one argument against (SEU). In particular, I focused on a refutation of sub-thesis (2a) of (SEU) which rests on the first-countability of the state space. It remains an open question whether there are different arguments against (2a), not based on this topological property.

Moreover, the argument infers the first-countability of the state space from the separability of the algebra of observables. The discussion above has indicated that separability of the algebra of observables is related to two factors: (a) the dimension of the algebra, considered as a vector space; (b) whether it is a C*- or von Neumann algebra. I argued that finite-dimensional C*-algebras and von Neumann algebras are separable while infinite-dimensional von Neumann algebras are non-separable and infinite-dimensional C*-algebras can be separable. These considerations make this argument inconclusive in all cases in which the algebras are non-separable.

Regarding Porrmann's suggestion it is not at all clear *to me* whether one can start from a countable family of Minkowski spacetime regions and the corresponding separable local algebras and subsequently recover, under reasonable conditions, the full physical content of a Haag-Kastler theory. Although it may concern my limited understanding of the theory, yet I consider it an open issue.

Nevertheless, separability is just one sufficient condition. One may find a different sufficient condition to infer the desired first-countability of the state space. This is a third possibility that remains open. Actually, in the process of writing this paper I asked for the help of the "web-next-door" mathematician on this issue, admittedly, without any remarkable results[12].

Finally, a fourth possibility that has not been explored so far, has to do with the restriction of the physically admissible states of an (infinite-dimensional) von Neumann algebra to some subset of the state space, for instance the set of normal states. Is the topological space defined on the subset of normal states of a von Neumann algebra equipped with a weak*-topology, first countable?

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[11] Porrmann refers to the Fredenhagen-Hertel compactness condition which restricts the number of states of finite total energy on a given local algebra.

[12] See, https://mathoverflow.net/questions/311534/topology-of-state-space-in-von-neumann-algebras/311653?noredirect=1#comment777613_311653 and https://math.stackexchange.com/questions/2929380/sufficient-conditions-for-a-c-algebra-to-be-separable I would like to thank especially Robert Furber and Martin Argerami for their contribution to this discussion.

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