The physical impossibility of machine computations on sufficiently large integers inspires an open problem that concerns abstract computable sets $\mathcal{X} \subseteq \mathbb{N}$ and cannot be formalized in the set theory ZFC as it refers to our current knowledge on $\mathcal{X}$

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Abstract. Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form n^2+1 is infinite. Let $\beta = (((24!)!)!)!$, and let Φ denote the implication: $\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, \beta]$. We heuristically justify the statement Φ without invoking Landau's conjecture. The set $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ satisfies conditions `(1)-(4)`. `(1)` There are a large number of elements of $\mathcal{X}$ and it is conjectured that $\mathcal{X}$ is infinite. `(2)` No known algorithm decides the finiteness/infiniteness of $\mathcal{X}$. `(3)` There is a known algorithm that for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$. `(4)` There is an explicitly known integer n such that $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$. `(5)` There is an explicitly known integer n such that $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ and some known definition of $\mathcal{X}$ is much simpler than every known definition of $\mathcal{X} \setminus (-\infty, n]$. The following problem is open: *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies conditions* `(1)-(3)` *and* `(5)`*?* The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies conditions `(1)-(3)`. Let $[\cdot]$ denote the integer part function. For every explicitly given integer $m \geqslant 1$, the set $\mathcal{X} = \left\{k \in \mathbb{N} : \left[\frac{k}{m}\right]^2 + 1 \text{ is prime}\right\}$ contains m consecutive integers and satisfies conditions `(1)-(3)`. The statement Φ implies that both sets $\mathcal{X}$ satisfy condition `(5)`.

Key words and phrases: complexity of a mathematical definition, computable set $\mathcal{X} \subseteq \mathbb{N}$, current knowledge on $\mathcal{X}$, explicitly known integer n bounds $\mathcal{X}$ from above when $\mathcal{X}$ is finite, infiniteness of $\mathcal{X}$ remains conjectured, known algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$, large number of elements of $\mathcal{X}$, mathematical statement that cannot be formalized in the set theory ZFC, no known algorithm decides the finiteness/infiniteness of $\mathcal{X}$, physical impossibility of machine computations on sufficiently large integers.

1. Basic definitions and the goal of the article

Logicism is a programme in the philosophy of mathematics. It is mainly characterized by the contention that mathematics can be reduced to logic, provided that the latter includes set theory, see [3, p. 199].

Definition 1. *Conditions* (1)-(5) *concern sets* $\mathcal{X} \subseteq \mathbb{N}$.

(1) *There are a large number of elements of* $\mathcal{X}$ *and it is conjectured that* $\mathcal{X}$ *is infinite.*

(2) *No known algorithm decides the finiteness/infiniteness of* $\mathcal{X}$.

(3) *There is a known algorithm that for every* $n \in \mathbb{N}$ *decides whether or not* $n \in \mathcal{X}$.

(4) *There is an explicitly known integer* n *such that* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.

(5) *There is an explicitly known integer* n *such that* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$ *and some known definition of* $\mathcal{X}$ *is much simpler than every known definition of* $\mathcal{X} \setminus (-\infty, n]$.

Definition 2. *We say that an integer* n *is a threshold number of a set* $\mathcal{X} \subseteq \mathbb{N}$*, if* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$*, cf.* [6] *and* [7].

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$.

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite, see [4] and [5].

Definition 3. *Let* Φ *denote the implication:*

$$\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, (((24!)!)!)!]$$

Landau's conjecture implies the statement Φ. In Section 4, we heuristically justify the statement Φ without invoking Landau's conjecture.

Statement 1. *There is no explicitly known threshold number of* $\mathcal{P}_{n^2+1}$. *It means that there is no explicitly known integer* k *such that* $\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, k]$.

Proving the statement Φ will falsify Statement 1. Statement 1 cannot be formalized in the set theory *ZFC* because it refers to the current mathematical knowledge. The same is true for Statements 2–4 and Open Problem 1 in the next sections. It argues against logicism as Open Problem 1 concerns abstract computable sets $\mathcal{X} \subseteq \mathbb{N}$.

2. The physical impossibility of machine computations on sufficiently large integers inspires Open Problem 1

Definition 4. *Let* $\beta = (((24!)!)!)!$.

Lemma 1. $\beta \approx 10^{10^{10^{10^{25.16114896940657}}}}$.

Proof. We ask Wolfram Alpha at `http://wolframalpha.com`. □

Statement 2. *The set* $\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ *satisfies conditions* `(1)-(4)`.

Proof. Condition `(1)` holds as $\mathcal{X} \supseteq \{0, \ldots, \beta\}$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than β, see [2]. Thus condition `(2)` holds. Condition `(3)` holds trivially. Since the set

$$\{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

is empty or infinite, the integer β is a threshold number of $\mathcal{X}$. Thus condition `(4)` holds. □

In Statement 2,

$$\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, \beta]$$

and the sets

$$\mathcal{X} = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

and

$$\mathcal{X} \setminus (-\infty, \beta] = \{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

have definitions of similar complexity. The following problem arises:

Open Problem 1. *Is there a set* $\mathcal{X} \subseteq \mathbb{N}$ *that satisfies conditions* `(1)-(3)` *and* `(5)`*?*

Theorem 1. *Assume that for every positive integers b and s, at some future day, machine computations will be possible on every integers from the interval* $[-b, b]$ *and this will be possible with the speed of s FLOPS. These assumptions contradict the current paradigm of physics, although they alone have no consequences in mathematics formalized in ZFC. We claim that our assumptions alone imply that no set* $\mathcal{X} \subseteq \mathbb{N}$ *will satisfy conditions* `(1)-(4)` *forever.*

Proof. The proof goes by contradiction. Since conditons `(2)-(4)` will hold forever, the algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$n+1 \notin \mathcal{X},\ n+2 \notin \mathcal{X},\ n+3 \notin \mathcal{X},\ \ldots \tag{T}$$

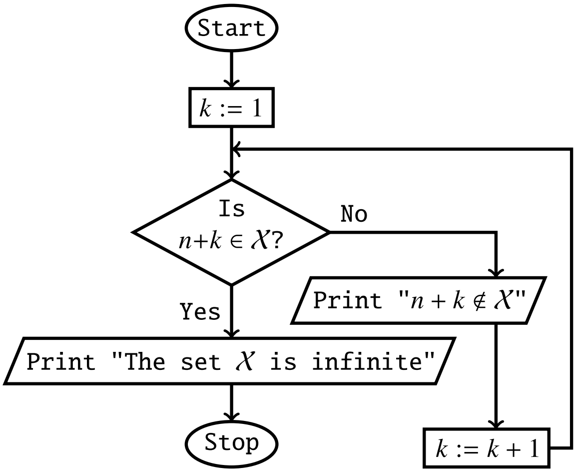

Fig. 1 Algorithm whose execution never terminates if the set $\mathcal{X}$ is finite

The sentences from the sequence `(T)` and our assumptions alone imply that for every explicitly given integer $m > n$, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap \mathcal{X} = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $\mathcal{X}$ is finite, contrary to the conjecture in condition `(1)`. □

3. Number-theoretic statements Ψ_n

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geqslant 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} \qquad\qquad\qquad x_1! &= x_1 \\ \qquad\qquad\quad x_1 \cdot x_1 &= x_2 \\ \forall i \in \{2, \ldots, n-1\}\ x_i! &= x_{i+1} \end{cases}$$

The diagram in Figure 2 illustrates the construction of the system $\mathcal{U}_n$.

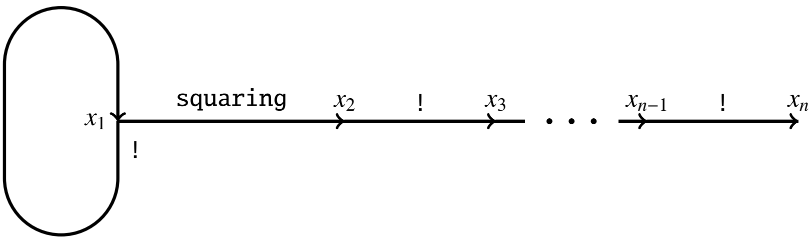

Fig. 2 Construction of the system $\mathcal{U}_n$

Lemma 2. *For every positive integer n, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely* $(1, \ldots, 1)$ *and* $\big(f(1), \ldots, f(n)\big)$.

Let

$$B_n = \Big\{x_i! = x_k :\ i,k \in \{1,\ldots,n\}\Big\} \cup \Big\{x_i \cdot x_j = x_k :\ i,j,k \in \{1,\ldots,n\}\Big\}$$

For a positive integer n, let Ψ_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1,\ldots,x_n$, then each such solution $(x_1,\ldots,x_n)$ satisfies $x_1,\ldots,x_n \leqslant f(n)$.* The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The statements Ψ_1 and Ψ_2 hold trivially. There is no reason to assume the validity of the statement Ψ_9, cf. Conjecture 1 in Section 4.

Theorem 2. *For every statement Ψ_n, the bound $f(n)$ cannot be decreased.*

Proof. It follows from Lemma 2 because $\mathcal{U}_n \subseteq B_n$. □

Theorem 3. *For every integer $n \geqslant 2$, the statement Ψ_{n+1} implies the statement Ψ_n.*

Proof. If a system $\mathcal{S} \subseteq B_n$ has at most finitely many solutions in positive integers $x_1,\ldots,x_n$, then for every integer $i \in \{1,\ldots,n\}$ the system $\mathcal{S} \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers $x_1,\ldots,x_{n+1}$. The statement Ψ_{n+1} implies that $x_i! = x_{n+1} \leqslant f(n+1) = f(n)!$. Hence, $x_i \leqslant f(n)$. □

Theorem 4. *Every statement Ψ_n is true with an unknown integer bound that depends on n.*

Proof. For every positive integer n, the system B_n has a finite number of subsystems. □

4. A conjectural solution to Open Problem 1

Lemma 3. *For every positive integers x and y, $x! \cdot y = y!$ if and only if*

$$(x+1=y) \lor (x=y=1)$$

Lemma 4. *(Wilson's theorem,* [1, p. 89]*). For every integer $x \geqslant 2$, x is prime if and only if x divides $(x-1)!+1$.*

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_2! &= x_3 \\ x_3! &= x_4 \\ x_5! &= x_6 \\ x_8! &= x_9 \\ x_1 \cdot x_1 &= x_2 \\ x_3 \cdot x_5 &= x_6 \\ x_4 \cdot x_8 &= x_9 \\ x_5 \cdot x_7 &= x_8 \end{cases}$$

Lemma 3 and the diagram in Figure 3 explain the construction of the system $\mathcal{A}$.

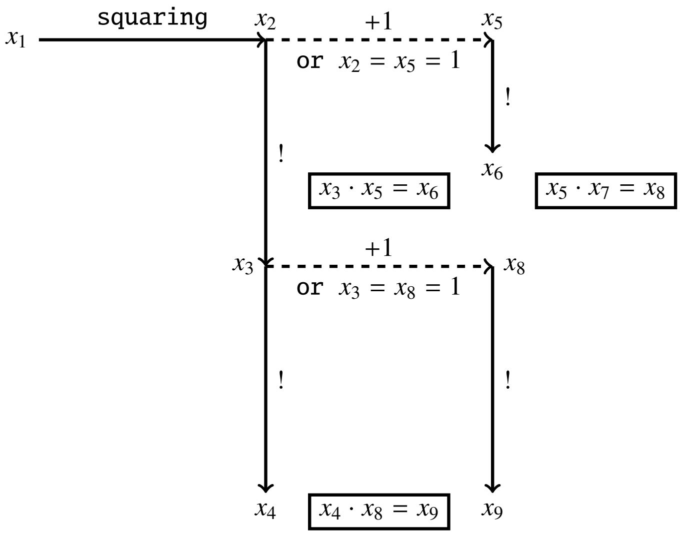

Fig. 3 Construction of the system $\mathcal{A}$

Lemma 5. *For every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$
\begin{array}{lcl}
x_2 & = & x_1^2 \\
x_3 & = & (x_1^2)! \\
x_4 & = & ((x_1^2)!)! \\
x_5 & = & x_1^2 + 1 \\
x_6 & = & (x_1^2 + 1)! \\
x_7 & = & \dfrac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 & = & (x_1^2)! + 1 \\
x_9 & = & ((x_1^2)! + 1)!
\end{array}
$$

Proof. By Lemma 3, for every integer $x_1 \geqslant 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 5 follows from Lemma 4. □

Lemma 6. *There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$. This is true as every such tuple $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \in \{1, 2\}$.*

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leqslant 2$. □

Conjecture 1. *The statement Ψ_9 is true when is restricted to the system $\mathcal{A}$.*

Theorem 5. *Conjecture 1 proves the following implication: if there exists an integer* $x_1 \geqslant 2$ *such that* $x_1^2 + 1$ *is prime and greater than* $f(7)$*, then the set* $\mathcal{P}_{n^2+1}$ *is infinite.*

Proof. Suppose that the antecedent holds. By Lemma 5, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geqslant f(7)$. Hence, $(x_1^2)! \geqslant f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geqslant (f(8) + 1)! > f(8)! = f(9)$$

Conjecture 1 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 5 and 6, the set $\mathcal{P}_{n^2+1}$ is infinite. □

Theorem 6. *Conjecture 1 implies the statement* Φ.

Proof. It follows from Theorem 5 and the equality $f(7) = (((24!)!)!)!$. □

Theorem 7. *The statement* Φ *implies Conjecture 1.*

Proof. By Lemmas 5 and 6, if positive integers $x_1, \ldots, x_9$ solve the system $\mathcal{A}$, then

$$(x_1 \geqslant 2) \wedge (x_5 = x_1^2 + 1) \wedge (x_5 \text{ is prime})$$

or $x_1, \ldots, x_9 \in \{1, 2\}$. In the first case, Lemma 5 and the statement Φ imply that the inequality $x_5 \leqslant (((24!)!)!)! = f(7)$ holds when the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_9$. Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every x_i does not exceed $f(9)$. □

Statement 3. *The set* $\mathcal{X} = \mathcal{P}_{n^2+1}$ *satisfies conditions* `(1)-(3)`. *The statement* Φ *implies that the set* $\mathcal{X}$ *satisfies condition* `(5)`.

Proof. Since the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite, condition `(1)` holds for $\mathcal{X}$. Condition `(3)` holds trivially. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$, see [2]. Thus condition `(2)` holds for $\mathcal{X}$. Suppose that the statement Φ holds. This implies that β is a threshold number of $\mathcal{X} = \mathcal{P}_{n^2+1}$. Thus condition `(4)` holds for $\mathcal{X}$. The definition of $\mathcal{P}_{n^2+1}$ is much simpler than the definition of $\mathcal{P}_{n^2+1} \setminus (-\infty, \beta]$. The last two sentences imply that condition `(5)` holds for $\mathcal{X}$. □

Let $[\cdot]$ denote the integer part function.

Statement 4. *For every explicitly given integer* $m \geqslant 1$*, the set* $\mathcal{X} = \left\{k \in \mathbb{N} : \left[\frac{k}{m}\right]^2 + 1 \text{ is prime}\right\}$ *contains* m *consecutive integers and satisfies conditions* `(1)-(3)`. *The statement* Φ *implies that the set* $\mathcal{X}$ *satisfies condition* `(5)`.

Proof. The set $\mathcal{X}$ contains m consecutive integers because the number 2 is prime and the equality $\left[\frac{k}{m}\right]^2 + 1 = 2$ holds for every integer $k \in \{m, \ldots, 2m - 1\}$. The rest of the proof goes as in the proof of Statement 3, although the statement Φ allows us to compute a threshold number of $\mathcal{X}$ that depends on m. □

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