Physics in the space of quantum states

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Abstract. It has been recently shown that Newtonian dynamics is the Schrödinger dynamics of the system whose state is constrained to a submanifold in the space of states of the system. Thus defined, the submanifold can be identified with the classical phase space of the system. The classical space is then also embedded into the space of states in a physically meaningful way. The resulting unified geometric framework establishes a new connection between classical and quantum physics. The framework is rigid in the sense that the Schrödinger dynamics is a unique extension of the Newtonian one from the classical phase space submanifold to the space of states. Quantum observables in the framework are identified with vector fields on the space of states. The commutators of canonical conjugate observables are expressed through the curvature of the sphere of normalized states. The velocity and acceleration of a particle in Newtonian dynamics are components of the velocity of state under the corresponding Schrödinger evolution. The metric properties of the embedding of the classical space into the space of states result in a relationship between the normal distribution of the position of a particle and the Born rule for the probability of transition of quantum states. In this paper the implications of the obtained mathematical results to the process of measurement in quantum physics are investigated. It is argued that interaction with the environment constrains the state of a macroscopic body to the classical space. The notion of collapse of a quantum state is analyzed. The double-slit, EPR and Schrödinger cat type experiments are reviewed anew. It is shown that, despite reproducing the usual results of quantum theory, the framework is not simply a reformulation of the theory. New experiments to discover the predicted effects are proposed.

1. Questions to be addressed

The current situation in quantum theory is extremely awkward. On one hand, during more than one hundred years of its existence, the theory proved itself to be very successful and accurate in describing the observed microscopic phenomena. On the other, it represents a constellation of paradoxes and unanswered questions that kept several generations of researchers confused and led to exotic interpretations and numerous attempts to revise or complete the theory. The general issue was and still is to understand how the quantum gives rise to the classical world that we live in. The most notable question is how exactly under an observation a typical superposition of states of a particle or a system of particles produces a single observed outcome. This seemingly simple question generates a multitude of follow-up questions that indicate the level of the current confusion about the subject:

Is the quantum state a part of reality, or only the state of our knowledge of the latter? An observation takes a superposition of states to a single outcome, which is the actual value of a physical quantity, say, the position or momentum of a particle. This outcome was not present before observation. Does it mean that we create reality by making an observation? Also, if the state is real and the observed outcome is also real, then how these two realities are related? Why

do macroscopic bodies in the universe have a well defined position at all times, independently of whether we observe them or not? When is the body macroscopic and where exactly is the boundary between the quantum and the classical worlds? What happens during an observation? What process should we call an observation? Does our brain activity have anything to do with generating the outcomes? The Schrödinger dynamics is deterministic. How does the observed probability make its way into the deterministic quantum theory? Does it mean that measuring devices do not satisfy quantum dynamics and must be described differently? How can we derive the rule that determines the probabilities? How is the basis used to find the components of state determined during an observation? If several measuring devices are turned on at the same time, what determines how a particular basis is selected? What should we make of the delayed choice experiments that demonstrate that a photon is neither a particle nor a wave until observed? Is it also an act of creation by observation? Alternatively, is it a process of going back in time and fixing the nature of the photon before observation? Going beyond the non-relativistic framework, to what extent is the seemingly instantaneous transition of states compatible with the special and general relativity?

In a recent series of papers [1]-[6], summarized and further extended in [7], an important new connection between the classical and quantum dynamics was derived. The starting point was a realization of classical and quantum mechanics on an equal footing within the same Hilbert space framework and identification of observables with vector fields on the sphere of normalized states. This resulted in a physically meaningful interpretation of components of the velocity of state. Newtonian dynamics was shown to be the Schrödinger dynamics of a system whose state is constrained to the classical phase space submanifold in the Hilbert space of states. This also resulted in a formula relating the normal probability distribution and the Born rule and interpretation of quantum collapse in terms of diffusion of state on the projective space of states.

In simple words, the classical space and classical phase space of a system of particles can be identified with a submanifold of the space of states of the corresponding quantum system. When the system is constrained to the submanifold, it behaves classically. Otherwise, it behaves quantum-mechanically. The velocity of the state at any point of the classical space submanifold can be decomposed into classical (velocity, acceleration) and non-classical (phase velocity, spreading) components. The curvature of the sphere of states is determined from the canonical commutation relations. An observation creates a diffusion on the sphere of states. During the diffusion the state can reach the classical space submanifold and trigger a detector in it. The probability of reaching a particular point of the classical space submanifold is given by the Born rule.

These results suggest that there is an alternative approach to quantum mechanics that is much more appropriate for understanding and visualizing the theory and for addressing its problems and paradoxes. In this paper, such an approach is advanced. First, the earlier obtained results are reviewed, clarified and, in some cases, generalized. Then the issues of wave function collapse and transition from quantum to classical are explored. Following this, the double-slit and the EPR experiments are analyzed anew. The obtained results strongly support the point of view that the space of states is the new fundamental arena for all physical processes, classical and quantum. The classical space that serves the arena for Newtonian physics must be replaced with the space of states. By accepting this hypothesis and identifying the classical space with a submanifold in the space of states it becomes possible to understand the relationship of the classical and the quantum and to fruitfully address the difficulties and paradoxes of quantum mechanics.

2. The measurement problem

Classical mechanics is based on the notion of a *material point*. Position of a material point in the classical space is given by its *coordinates* $\mathbf{x} = \{x^i\}, i = 1, 2, 3$. The motion of a material point is

described then by the functions of time $x^i(t)$. Velocity $\mathbf{v}$ of the point is given by the derivative $\mathbf{v} = d\mathbf{x}/dt$. Acceleration $\mathbf{w}$ is given by the second derivative $\mathbf{w} = d^2\mathbf{x}/dt^2$. More complicated systems, for example, rigid bodies, are certain systems of material points. The position of a system of n material points requires in general $3n$ coordinates $\mathbf{x}_1, \ ... \ , \mathbf{x}_n$. The space of possible positions of a system is called the *configuration space* of the system. A constrained system requires fewer coordinates and the configuration space may be a non-trivial submanifold of $\mathbb{R}^{3n}$, described by *generalized* coordinates. The dynamics of a system can be derived from the *principle of stationary action*, where the *action functional* S is the integral of the *Lagrangian function* $L(x, v, t)$ of generalized positions and velocities x, v of all the particles in the system with respect to time. The Lagrangian of a single free material point of mass m in an *inertial coordinate system* can be derived from the *Galileo principle of relativity* and is equal to the kinetic energy $\frac{m\mathbf{v}^2}{2}$ of the particle. The free Lagrangian is *additive* so the Lagrangian of a system of free material points is the sum of Lagrangians of each point. The interaction is introduced by adding a function of coordinates (a *potential*). Equations of motion (the Euler-Lagrange equations) are obtained from the condition $\delta S = 0$ on variation of the action and form a system of second order differential equations for the functions $x(t)$. In particular, the Newton's equations of motion of a system of particles with interaction described by a potential $V(\mathbf{x}_1, \mathbf{x}_2, \ ... \ )$ are the Euler-Lagrange equations for the Lagrangian $L(x, v, t) = \sum_k \frac{m_k \mathbf{v}_k^2}{2} - V(\mathbf{x}_1, \mathbf{x}_2, \ ... \ )$, where m_k are masses of the particles. Alternatively, the Euler-Lagrange equations can be replaced with a system of the first order equations $\frac{\partial h}{\partial x} = -\frac{dp}{dt}$, $\frac{\partial h}{\partial p} = \frac{dx}{dt}$ (*Hamilton equations*) for the generalized coordinates and momenta $p = mv$ of the particles. Here $h = p\frac{dx}{dt} - L$ is the *Hamiltonian function* or *energy* of the system. The generalized variables x, p provide coordinates for the *phase space* and used to define the cotangent bundle and a *symplectic structure*. A point in the phase space represents the *state* of the mechanical system. Another dynamical equation that follows from differentiation of the action functional S and the Euler-Lagrange equations is the *Hamilton-Jacobi equation* $\frac{\partial S}{\partial t} + h(x, \nabla S, t) = 0$. This equation follows from the Schrödinger equation in the quasi-classical limit and is one of the points of contact of classical and quantum dynamics.

A microscopic system in quantum mechanics is maximally described by its *wave* or *state* function φ. In the *coordinate representation* the state function at time t is a function $\varphi_t(x, s)$ of coordinates x of the system and, possibly, additional quantum-mechanical degrees of freedom s such as spin. The state function is an element of a *Hilbert space* of states. The Hilbert space H is a vector space with an inner product $(\varphi, \psi)_H$ that is complete with respect to the norm $||\varphi||_H = \sqrt{(\varphi, \varphi)_H}$. Physical quantities, or *observables* are described by linear self-adjoint operators on H. The system obtains familiar physical characteristics as a result of measurement. In simple cases, a measurement on the system results in an eigenvalue of the corresponding observable. The state of the system after the measurement is the corresponding eigenvector. The initial state of the measured system can be written as a *superposition* (linear combination) of eigenvectors. The probability to find the system in a specific eigenstate of the measured observable is given by the *Born rule* (i.e., by the modulus squared of the coefficient of the corresponding normalized eigenvector in the superposition). Two observables $\widehat{a}, \widehat{b}$ may not commute: the *commutator* $[\widehat{a}, \widehat{b}] = \widehat{a}\widehat{b} - \widehat{b}\widehat{a}$ is not 0, in general. This results in the *uncertainty principle*, which puts a fundamental limit on the possible precision of measurement of two non-commuting observables. The *canonical commutation relations* between *conjugate observables*, such as position and momentum yield the algebraic structure of quantum mechanics.

The motion of the system is described by a time-dependent state function that satisfies the Schrödinger equation, which is a linear differential equation $i\hbar\frac{\partial \varphi}{dt} = \widehat{h}\varphi$. The operator $\widehat{h}$ is the *Hamiltonian*, which is an operator version of the Hamiltonian function. For a system of n interacting particles the Hamiltonian is given by $\widehat{h} = -\frac{\hbar^2}{2}\sum_k \frac{\Delta_k}{m_k} + V(\mathbf{x}_1, \ ... \ , \mathbf{x}_n)$, where Δ_k is the Laplace operator for the k-th particle and V is the operator of multiplication by

the potential of the interaction. Symmetries of the Hamitonian imply representations on the space of states. Accordingly, the theory of representations of groups becomes a foundational part of quantum mechanics. The linear nature of the Schrödinger equation signifies that the *superposition principle* holds true. Namely, a linear combination of physically possible states of a system represents another physically possible state of the system.

The linear nature of quantum mechanics poses a persistent problem when trying to reconcile the classical and quantum mechanics. The superposition principle is totally foreign to classical physics, which governs the motion of macroscopic bodies. Physical consequences of the principle feed all controversies of the theory and represent the major apparent paradoxes of modern science. All questions raised in the previous section are rooted in the superposition principle. For instance, let the wave packets $\varphi_{\mathbf{a}}$, $\varphi_{\mathbf{b}}$ be solutions to the Schrödinger equation, describing a particle near points $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$. According to the superposition principle, the sum $\varphi_{\mathbf{a}} + \varphi_{\mathbf{b}}$ must represent a physically meaningful state of the particle, which is somehow "spread over" both points. The superposition principle is confirmed by the countless experiments with microscopic particles. But how could an electron (let alone a large molecule or a microbe) go through two different holes in a plate, and yet arrive as a single particle to the screen behind the plate? This is at odds with our everyday experience as we never observe a macroscopic body in two places at once. This paradox taken in its full scope and the resulting issue of reconciliation of the classical and the quantum constitutes the *measurement problem*.

The problem can be split into several related smaller parts, of which the following three are essential. First, what is the meaning of the wave function? What does it really describe? The experiment shows its relationship to probability to find the particle in a certain state, which is one of the postulates of quantum theory. Does it go deeper than that? Does it have a physical meaning beyond the probability and information? Does it represent something physically real? This is the *problem of the reality of the state*. Second, when a certain state is obtained as a result of a measurement, what is the process that transforms the original state into the observed state? How do the observed outcomes come into being? This is the *problem of definite outcomes*. Third, there are many physical quantities that can be measured. Under each measurement the initial state transforms to a different observed state (an eigenstate of the measured observable). How could the initial state "know" what set of the final states it needs to transform to? What is the role of the measuring device in this selection? This is the *preferred basis problem*.

3. Newtonian mechanics in the Hilbert space of states

The state of a quantum system is a point in an infinite-dimensional abstract separable Hilbert space H. The Hilbert space is an infinite-dimensional version of the Euclidean space E: it is a vector space with an inner product. Geometric points of E can be identified with triples of numbers: coordinates of points in a coordinate system. The space $\mathbb{R}^3$ of all triples of numbers $\mathbf{a} = (a_1, a_2, a_3)$ with the inner product $(\mathbf{a}, \mathbf{b})_{\mathbb{R}^3} = a_1b_1 + a_2b_2 + a_3b_3$ is a *realization* of the Euclidean space E. In other words, the spaces E and $\mathbb{R}^3$ are *isomorphic*. Similarly, points of the Hilbert space can be identified with numeric-valued functions from a certain functional realization of the abstract Hilbert space. The most common realization of H is the space $L_2(\mathbb{R}^3)$ of complex-valued functions f with a finite norm given by the inner product $(f, g)_{L_2} = \int f(\mathbf{x})\overline{g}(\mathbf{x})d^3x$. All realizations of H are isomorphic.

The Newtonian mechanics is built in the *classical space*, which is the Euclidean space $\mathbb{R}^3$. It is based on the notion of a material point, which is a macroscopic body at a point of $\mathbb{R}^3$, whose dimensions are negligeble relative to other distances in the problem under consideration. From the everyday experience we know that macroscopic bodies possess a well-defined position in space at any moment of time. In quantum mechanics, the state of a simplest particle with a known position $\mathbf{a}$ is given by the Dirac delta function $\delta^3_{\mathbf{a}}(\mathbf{x}) = \delta^3(\mathbf{x} - \mathbf{a})$. Dirac function is a *generalized function*, equal to 0 everywhere except a single point $\mathbf{x} = \mathbf{a}$. It is understood

rigorously by means of a *delta-converging sequence* of functions [8]. The map $\omega : \mathbf{a} \longrightarrow \delta^3_{\mathbf{a}}$ provides a one-to-one correspondence between points $\mathbf{a} \in \mathbb{R}^3$ and state "functions" $\delta^3_{\mathbf{a}}$. This allows us to describe points in $\mathbb{R}^3$ in functional terms and identify the set $\mathbb{R}^3$ with the set M_3 of all delta functions in the space of state functions of the particle.

The almost obvious identification ω turns out to be central to the problem of unification of the Newtonian and Schrödinger dynamics. To explain, let us first replace ω with the similar map $\omega_\sigma : \mathbf{a} \longrightarrow \widetilde{\delta}^3_{\mathbf{a}}$, where $\widetilde{\delta}^3_{\mathbf{a}}$ is a Gaussian (bell-shaped) function whose modulus squared has variance σ^2. This will allow us to use the common Hilbert space $L_2(\mathbb{R}^3)$ in place of a more exotic Hilbert space that contains delta-functions. At the same time, other properties of the maps ω and ω_σ are the same. The manifold $M^\sigma_3 = \omega_\sigma(\mathbb{R}^3)$ of all such Gaussian functions in the space $L_2(\mathbb{R}^3)$ turns out to be mathematically identical to the Euclidean space $\mathbb{R}^3$. Figures 1 and 2 illustrate the resulting isometric embedding of the classical space into the Hilbert space. The

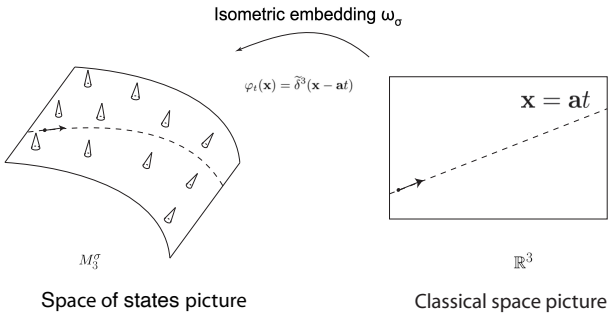

Figure 1

"spikes" on Figure 1 represent the Gaussian functions $\widetilde{\delta}^3_{\mathbf{a}}$ centered at points $\mathbf{a} \in \mathbb{R}^3$. Note that under the embedding, the manifold $M^\sigma_3 = \mathbb{R}^3$ belongs to the unit sphere in the Hilbert space $L_2(\mathbb{R}^3)$ and "spirals" through the dimensions of the Hilbert space so that there is no vector in $L_2(\mathbb{R}^3)$ that is orthogonal to the entire manifold M^σ_3. Clearly, the manifold M^σ_3 is not a vector subspace of $L_2(\mathbb{R}^3)$. However, the fact that the manifold is "curved" in $L_2(\mathbb{R}^3)$ does not preclude

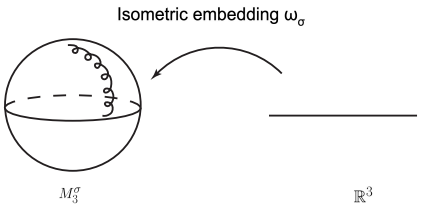

Figure 2

having a vector structure on it. In particular, a full-fledged vector structure on M^σ_3 is induced by the map ω_σ from the vector structure on $\mathbb{R}^3$.

Newtonian physics uses another important geometric construction: the phase space. For a single material point in the classical space, the phase space is the space of all possible positions $\mathbf{x} = (x_1, x_2, x_3)$ and momenta $\mathbf{p} = (p_1, p_2, p_3)$ of the particle. So it is the Euclidean space $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ of the pairs $(\mathbf{x}, \mathbf{p})$. The mechanical state of the particle is then described by a point in the phase space. In quantum mechanics, the momentum of a particle is identified by the phase factor $e^{i\mathbf{px}}$ in the state function. The one-to-one map $\Omega : (\mathbf{a}, \mathbf{p}) \longrightarrow \widetilde{\delta}^3_{\mathbf{a}} e^{i\mathbf{px}}$ identifies the classical phase space of the particle with a 6-dimensional submanifold $M^\sigma_{3,3}$ of the unit sphere S^{L_2} in the space of states of the particle.

The constant phase factor $e^{i\alpha}$ in the state function of a particle does not change the physics of the particle and is not observable in the experiments. The equivalence classes of states in $L_2(\mathbb{R}^3)$ defined up to arbitrary constant phase factors form the *physical space of states* or the *complex projective space* CP^{L_2}. The inner product of vectors in the Hilbert space $L_2(\mathbb{R}^3)$ defines a point-dependent inner product on CP^{L_2}, called the *Fubini-Study* metric. The map Ω from the classical phase space $\mathbb{R}^3 \times \mathbb{R}^3$ into the space of states CP^{L_2} with the Fubini-Study metric is an isomorphism of the classical phase space and the manifold $M^\sigma_{3,3}$.

The geometric identification of the classical space $\mathbb{R}^3$ with the manifold M^σ_3 and of the classical phase space $\mathbb{R}^3 \times \mathbb{R}^3$ of a particle with the manifold $M^\sigma_{3,3}$ is interesting in itself. However, the most important property of the identification is the relationship of Newtonian and Schrödinger dynamics that it generates. Namely, as proved in [7],

> *The Newtonian dynamics of a particle is the the Schrödinger dynamics of the particle with the state constrained to the submanifold $M^\sigma_{3,3}$. In other words, a macroscopic material point is a quantum system whose state is constrained to the manifold $M^\sigma_{3,3}$.*

This result follows, in particular, from the variational principle. The variation of the functional

$$S[\varphi] = \int \overline{\varphi}(\mathbf{x}) \left[i\hbar \frac{\partial}{\partial t} - \widehat{h} \right] \varphi(\mathbf{x}) d^3\mathbf{x} dt \tag{1}$$

with $\widehat{h} = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{x}, t)$ yields the Schrödinger equation for φ. At the same time, for the states φ constrained to the manifold $M^\sigma_{3,3}$ this functional is equal to the classical action

$$S = \int \left[\mathbf{p} \frac{d\mathbf{a}}{dt} - h(\mathbf{p}, \mathbf{a}, t) \right] dt, \tag{2}$$

where $h(\mathbf{p}, \mathbf{a}, t) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{a}, t)$ is the Hamiltonian function. So, the variation of the functional (1) with the constraint yields Newtonian equations of motion.

It is possible to show that the the extension of Newtonian to Schrödinger dynamics is unique [7]. Namely,

> *The Schrödinger evolution is the only unitary evolution on $L_2(\mathbb{R}^3)$ for which the system constrained to the classical phase space $M^\sigma_{3,3}$ satisfies Newtonian equations of motion for the particle.*

The reason behind this uniqueness is that the manifold $M^\sigma_{3,3}$ form a basis (more precisely, a *complete set of vectors*) in the Hilbert space $L_2(\mathbb{R}^3)$. As a result, there can only be one linear extension of Newtonian mechanics to the space of states.

A more detailed analysis provided in [7] reveals the details of the transition from quantum to classical dynamics. The Schrödinger equation

$$\frac{d\varphi}{dt} = -\frac{i}{\hbar}\widehat{h}\varphi \tag{3}$$

gives the velocity of the quantum state $\frac{d\varphi}{dt}$ in terms of the Hamiltonian $\widehat{h}$ of the system. The velocity is tangent to the sphere of states S^{L_2} and can be decomposed onto the components parallel and orthogonal to the phase circle (the circle $\{e^{i\alpha}\varphi\}$ obtained by changing the phase factor $e^{i\alpha}$ for a given state φ):

$$\frac{d\varphi}{dt} = -\frac{i}{\hbar}\overline{E}\varphi - \frac{i}{\hbar}\widehat{h}_\perp\varphi. \tag{4}$$

The parallel component of $\frac{d\varphi}{dt}$ (i.e., the first term in (4), or the phase velocity) is numerically the expected value of the energy. The orthogonal component of the velocity (the term $-\frac{i}{\hbar}\widehat{h}_\perp\varphi$)

is the velocity of the motion of state in the projective space CP^{L_2}. The speed of such motion turns out to be equal to the uncertainty ΔE in energy.

The orthogonal component of state φ in $M^\sigma_{3,3}$ can be further decomposed into the classical velocity $\mathbf{v} = \frac{\mathbf{p}}{m}$ and classical acceleration $\mathbf{w} = -\frac{\nabla V}{m}$ of the particle, and the velocity of spreading of the wave packet φ in $M^\sigma_{3,3}$. All components turn are orthogonal in the space of states with the classical velocity and acceleration terms tangent to $M^\sigma_{3,3}$. The norm of the total velocity $\frac{d\varphi}{dt} = \frac{i}{\hbar}\widehat{h}\varphi$ at $t = 0$ (i.e., the speed of motion of the initial state φ in $M^\sigma_{3,3}$) is given by

$$\left\| \frac{d\varphi}{dt} \right\|^2 = \frac{\overline{E}^2}{\hbar^2} + \frac{\mathbf{v}^2}{4\sigma^2} + \frac{m^2\mathbf{w}^2\sigma^2}{\hbar^2} + \frac{\hbar^2}{32\sigma^4 m^2}. \tag{5}$$

The first term is due to the phase velocity while the last one is due to the velocity of spreading. By constraining the state φ to the manifold $M^\sigma_{3,3}$, we are left with the classical velocity and acceleration of the particle (two middle terms in (5)). The Newtonian mechanics of a particle is then nothing but the quantum mechanics on the submanifold $M^\sigma_{3,3}$.

So far the mechanics of a single particle was considered. However, the situation is analogous in the case of mechanical systems consisting of any number of particles. For example, the dynamics of a system of two macroscopic particles is described in the configuration space $\mathbb{R}^3 \times \mathbb{R}^3$. The map $\omega_\sigma \otimes \omega_\sigma(\mathbf{a}, \mathbf{b}) = \widetilde{\delta}^3_{\mathbf{a}} \otimes \widetilde{\delta}^3_{\mathbf{b}}$ identifies $\mathbb{R}^3 \times \mathbb{R}^3$ with the submanifold $M^\sigma_6 = \omega_\sigma \otimes \omega_\sigma(\mathbb{R}^3 \times \mathbb{R}^3)$ of the Hilbert space $L_2(\mathbb{R}^3) \otimes L_2(\mathbb{R}^3)$ of states of the pair. Likewise, the phase space $\mathbb{R}^6 \times \mathbb{R}^6$ of the pair is realized by the manifold $M^\sigma_{6,6}$ that consists of the states $\widetilde{\delta}^3_{\mathbf{a}} e^{i\mathbf{px}} \otimes \widetilde{\delta}^3_{\mathbf{b}} e^{i\mathbf{qy}}$. The Newtonian dynamics of the pair is given by the Schrödinger equation with the state of the pair constrained to the manifold $M^\sigma_{6,6}$. The ultimate conclusion is that

> *The Newtonian dynamics of an arbitrary mechanical system is the Schrödinger dynamics of that system with the state of the system constrained to the classical phase space submanifold of the space of states of the system.*

It remains to understand, of course, what keeps the states of macroscopic bodies on the classical phase space submanifold, making Newtonian dynamics a valid description of macroscopic bodies.

The obtained embedding of the classical phase space into the space of states complemented by the obtained relationship between the Newtonian and Schrödinger evolution signifies that Newtonian dynamics found its full-fledged realization within the realm of quantum physics governed by the Schrödinger equation. This realization is valid independently of whether it is taken to mean the actual physical embedding or only as a mathematical representation.

4. Quantum observables as vector fields on the space of states

A quantum observable is a linear self-adjoint operator on a Hilbert space of states. Generally, two observables $\widehat{A}, \widehat{B}$ do not commute: the commutator $[\widehat{A}, \widehat{B}] = \widehat{A}\widehat{B} - \widehat{B}\widehat{A} \neq 0$. In simple cases the observables can be identified with matrices and multiplication of observables with the matrix multiplication. Given an observable $\widehat{A}$ on a Hilbert space H, let us define the associated linear vector field A_φ on H by

$$A_\varphi = -i\widehat{A}\varphi. \tag{6}$$

Self-adjoint operators are known to generate *unitary* (i.e., preserving the inner product) transformations. It follows that the field A_φ associated with an observable and restricted to the sphere S^H of normalized states is tangent to the sphere. In particular, the vector field $h_\varphi = -\frac{i}{\hbar}\widehat{h}\varphi$ associated with the Hamiltonian is tangent to the sphere of states. The integral curves of the vector field h_φ are solutions to the Schrödinger equation.

The commutator of observables and the commutator, or *Lie bracket*, of the corresponding vector fields are related in a simple way:

$$[A_\varphi, B_\varphi] = [\widehat{A}, \widehat{B}]\varphi. \tag{7}$$

This way the algebra of observables is contained in the algebra of linear vector fields on the space of states. Moreover, as shown in [7], the commutators of observables are related to the curvature of the sphere of states. It follows that

> *Quantum mechanics can be realized as a theory of linear vector fields on the space of states. Solutions of the Schrödinger equation are integral curves of the vector field associated with the Hamiltonian. The commutator of quantum observables is equivalent to the commutator of the associated vector fields. For the canonical conjugate observables the non-vanishing commutator of the vector fields is due to the curvature of the sphere of quantum states.*

So instead of dealing with observables one can work with linear vector fields on the space of states. The algebra of observables is then ingrained into the differential geometry of the space of states. One implication of this is that the uncertainty relation becomes a statement about the geometry of the space of states [2].

5. The Born rule from the normal probability law

The embedding of the classical space M_3^σ into the space of states $L_2(\mathbb{R}^3)$ results in a relationship between distances in $\mathbb{R}^3$ and in the projective space CP^{L_2}. Let $\|\mathbf{a}-\mathbf{b}\|_{\mathbb{R}^3}$ be the distance between two points $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$. Under the embedding of the classical space into the space of states, the variable $\mathbf{a}$ is represented by the state $\tilde{\delta}^3_{\mathbf{a}}$. The set of states $\tilde{\delta}^3_{\mathbf{a}}$ form a submanifold M_3^σ in the Hilbert spaces of states $L_2(\mathbb{R}^3)$, which is "twisted" in $L_2(\mathbb{R}^3)$. It belongs to the sphere S^{L_2} and goes across the dimensions of $L_2(\mathbb{R}^3)$ (see Fig.1). The distance between the states $\tilde{\delta}^3_{\mathbf{a}}$, $\tilde{\delta}^3_{\mathbf{b}}$ on the sphere S^{L_2} or in the projective space CP^{L_2} is not equal to $\|\mathbf{a}-\mathbf{b}\|_{\mathbb{R}^3}$. In fact, the former is the length of a shortest path between the states in the space of states. The latter measures the distance between the points along the shortest path that lies in the manifold M_3^σ. The precise relation between the two distances is given by

$$e^{-\frac{(\mathbf{a}-\mathbf{b})^2}{4\sigma^2}} = \cos^2\theta(\tilde{\delta}^3_{\mathbf{a}}, \tilde{\delta}^3_{\mathbf{b}}), \tag{8}$$

where θ is the Fubini-Study distance between states in CP^{L_2}.

The relation (8) has an immediate implication onto the form of probability distributions of random variables over M_3^σ and CP^{L_2}. In particular, consider a random vector φ over the sphere S^{L_2}. Suppose φ is distributed according to some probability density function $\rho(\varphi;\psi)$ that depends only on the distance between the initial state ψ and the observed state φ. Since the classical space $M_3^\sigma = \mathbb{R}^3$ is a submanifold of the sphere, we can consider a new random variable by restricting φ to M_3^σ. Suppose that the distribution of the restricted random variable $\widetilde{\delta}^3_{\mathbf{a}}$ is normal on $\mathbb{R}^3$. That is, the conditional distribution of φ given that φ is in $M_3^\sigma = \mathbb{R}^3$ is the normal distribution. Then probability distribution $\rho(\varphi;\psi)$ satisfies the Born rule for the probability of transition between arbitrary states. The opposite is also true. In other words:

> *The normal distribution law on M_3^σ implies the Born rule on CP^{L_2}. Conversely, the Born rule on the space of states implies the normal distribution law on M_3^σ.*

So far, this relationship is purely geometrical. Because of the properties of the embedding of M_3^σ into $L_2(\mathbb{R}^3)$ the extension of the normal distribution law from $\mathbb{R}^3$ onto the space of states must satisfy the Born rule. To investigate the dynamical origin of this relationship, we will first look into a simple model of measurement of spin of a particle. This model will be considered in detail and later generalized and applied to the measurement of position and other observables.

6. The Born rule for a measurement of spin

Consider the space $\mathbb{C}^2$ of electron's spin states. The sphere S^3 of unit-normalized states in $\mathbb{C}^2$ can be identified with the group manifold $SU(2)$. For this, one identifies the space $\mathbb{C}^2$ of complex vectors $\varphi = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ with the space M of 2×2 matrices

$$\widehat{\varphi} = \begin{bmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{bmatrix}. \tag{9}$$

The map $\widehat{\omega} : \varphi \longrightarrow \widehat{\varphi}$ is an isomorphism of (real) vector spaces $\mathbb{C}^2$ and M. It identifies the sphere S^3 of unit states in $\mathbb{C}^2$ with the subset of matrices with unit determinant. Under matrix multiplication, the latter subset is the group $SU(2)$.

The differential $d\widehat{\omega}$ of the map $\widehat{\omega}$ identifies the tangent space $T_{e_1}S^3$ to the sphere S^3 at the point $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (that is, the hyperplane $\mathrm{Re} z_1 = 1$) with the Lie algebra $su(2)$ of traceless anti-Hermitian matrices

$$\widehat{A} = \begin{bmatrix} ia_2 & a_3 + ia_4 \\ -a_3 + ia_4 & -ia_2 \end{bmatrix}, \tag{10}$$

$a_2, a_3, a_4 \in R$. Under $d\widehat{\omega}$ the basis $e_2 = \begin{bmatrix} i \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $e_4 = \begin{bmatrix} 0 \\ i \end{bmatrix}$ in the tangent space $T_{e_1}S^3 = R^3$ becomes the basis $\{i\widehat{\sigma}_3, i\widehat{\sigma}_2, i\widehat{\sigma}_1\}$ in the Lie algebra $su(2)$. In particular, the real numbers a_2, a_3, a_4 acquire the meaning of coordinates of points on the tangent space $\mathrm{Re} z_1 = 1$ in the basis $\{e_2, e_3, e_4\}$.

Consider the Pauli equation for the electron interacting with a spin-measuring device. Let us assume that under the measurement the Stern-Gerlach interaction term in the equation drives the system, so that other terms can be neglected. (See the discussion in section 7.) In this case, the Hamiltonian of interaction between the electron and the device is given by $\widehat{h} = \mu\widehat{\sigma} \cdot \mathbf{B}$, where $\mathbf{B}$ is the magnetic field, $\widehat{\sigma} = (\widehat{\sigma}_x, \widehat{\sigma}_y, \widehat{\sigma}_z)$ and $\mu = e\hbar/2m$. The evolution equation for the spin states in the space $\mathbb{C}^2$ is then given by

$$\frac{d\varphi_t}{dt} = -\frac{i\mu}{\hbar}\widehat{\sigma} \cdot \mathbf{B}\varphi_t. \tag{11}$$

Using

$$(\widehat{\sigma} \cdot \mathbf{A})(\widehat{\sigma} \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} + i\widehat{\sigma} \cdot \mathbf{A} \times \mathbf{B}, \tag{12}$$

we obtain

$$(\widehat{\sigma} \cdot \mathbf{B})^2 = \mathbf{B}^2. \tag{13}$$

Since the matrix $\widehat{\sigma} \cdot \mathbf{B}$ is hermitian, we have then

$$\left\| \frac{d\varphi_t}{dt} \right\|^2_{\mathbb{C}^2} = \left(\frac{i\mu}{\hbar}\widehat{\sigma} \cdot \mathbf{B}\varphi_t, \frac{i\mu}{\hbar}\widehat{\sigma} \cdot \mathbf{B}\varphi_t \right)_{\mathbb{C}^2} = \frac{\mu^2 B^2}{\hbar^2}, \tag{14}$$

where B is the norm of $\mathbf{B}$.

Suppose the z-component of spin of an electron in a superposition of eigenstates of $\widehat{\sigma}_z$ is measured. For instance, we could insert a screen behind a Stern-Gerlach magnet and observe where the electron lands on the screen. When the electron interacts with the screen, it experiences a random magnetic field created by the molecules and atoms of the screen in their thermal motion. On the physical grounds and because of the central limit theorem, the components of the resulting magnetic field $\mathbf{B}$ can be assumed to be independent, identically distributed, normal random variables, with no correlation at different moments of time. In this

case the vector $i\widehat{\sigma}\cdot\mathbf{B}$ in the Lie algebra $su(2)$ with the Killing form is a normal random vector with an isotropic probability distribution, so that the level surfaces of the probability density are spheres. In particular, from (11) we conclude that any direction of the displacement $\delta\varphi$ of the initial spin state φ_0 in the tangent space $T_{\varphi_0}S^3$ to the sphere of states S^3 with the Killing metric is equally likely. Also, the distribution of the displacements is the same for all initial states φ_0, i.e., in all tangent spaces $T_{\varphi_0}S^3$, and is normal.

Let us look at the resulting motion of state in the projective space CP^1 of physical states. For this consider the complex lines $\{\varphi\}$ formed for each state $\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$ by the vectors $\lambda\varphi$, $\lambda\in\mathbb{C}$. Provided $\varphi_1\neq 0$, there is a unique point of intersection of the line $\{\varphi\}$ with the affine plane of vectors $\begin{bmatrix} 1 \\ \xi \end{bmatrix}, \xi\in\mathbb{C}$ in $\mathbb{C}^2$. Namely, by setting

$$\lambda\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \xi \end{bmatrix}, \tag{15}$$

we obtain

$$\xi = \frac{\varphi_2}{\varphi_1}, \tag{16}$$

and $\lambda = 1/\varphi_1$. The map $u = \{\varphi\} \longrightarrow \xi$ provides a coordinate chart on CP^1 that identifies CP^1 excluding a point (the complex line through $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$) with the set $\mathbb{C}$ of complex numbers. Under the isomorphism $\widehat{\omega}$ in (9), the vectors $\begin{bmatrix} 1 \\ \xi \end{bmatrix}$ form an affine subspace in the Lie algebra $su(2)$. The algebra $su(2)$ with the Killing form is the Euclidean space $\mathbb{R}^3$ of vectors $\mathbf{x} = \sum_k x^k i\widehat{\sigma}_k$. The stereographic projection then identifies the unit sphere S^2 at the origin of $\mathbb{R}^3$ with the above plane $\mathbb{C}$ plus a point, i.e., with CP^1 itself. Moreover, the usual metric on S^2 induced by its embedding into $\mathbb{R}^3$ is the Fubini-Study metric on CP^1. The relationship of the coordinate ξ in the plane $\mathbb{C}$ with coordinates $(x^1 = x, x^2 = y, x^3 = z)$ of the corresponding point on the sphere S^2 is given by

$$\xi = \frac{x+iy}{1-z}. \tag{17}$$

Solving this for x, y and z and using (16), one obtains:

$$x = \varphi_1\overline{\varphi}_2 + \overline{\varphi}_1\varphi_2, \tag{18}$$

$$y = i(\varphi_1\overline{\varphi}_2 - \overline{\varphi}_1\varphi_2), \tag{19}$$

$$z = \varphi_2\overline{\varphi}_2 - \varphi_1\overline{\varphi}_1. \tag{20}$$

Since the initial state φ_0 is defined only up to a phase factor, we are dealing with an ensemble of states with uniformly distributed phases. Furthermore, since the evolution equation is linear, a constant initial phase factor is preserved throughout the evolution. The random walk of the state φ on S^3 can be then described in terms of a random walk of the physical state $\{\varphi\}$ on the sphere $S^2 = CP^1$. The restriction of the volume form dV on S^3 yields the usual area form on S^2 that in the spherical coordinates (θ,ϕ) is equal to $dA = \sin\theta d\theta\wedge d\phi$. The probability of a step of a certain length is proportional to the corresponding probability density function times the area element. For short time steps, the probability density is a Gaussian function in θ,ϕ of small variance. The probability distribution is the same for all initial states φ_0 on the sphere. Because $z = \cos\theta$, we have $\sin\theta d\theta = -dz$, so that $dA = -dz\wedge d\phi$. Disregarding the change in the values of ϕ, the marginal probability of steps of an equal small increment dz is then approximately the same for each step, independent of the value of z. It follows that the process can be modeled by a simple symmetric random walk on the z-axis.

Let then $dz = \pm\Delta$ be the step of the walk with $\Delta \ll 1$ and the positive and negative values being equally likely. From (20) it follows that

$$|\varphi_1|^2 = \frac{1+z}{2}, \text{ and } |\varphi_2|^2 = \frac{1-z}{2}. \tag{21}$$

The gambler's ruin mechanism tells us now that the probability P_2 for the state φ to reach the state $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $(z = -1)$ first, as a result of the described random motion is equal to

$$P_2 \;=\; \frac{\text{number of steps from } z \text{ to } -1}{\text{number of steps from } -1 \text{ to } 1} \;=\; \frac{1-z}{2} \;=\; |\varphi_2|^2. \tag{22}$$

Similarly, the probability P_1 that φ will reach the state $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ first is given by $P_1 = |\varphi_1|^2$. This is the Born rule for transitions of spin-states.

7. The Born rule for a measurement of position

In section 5, the Born rule for transition of states was identified with an extension of the normal distribution law from the submanifold $M_3^\sigma = \mathbb{R}^3$ (the "basis") onto the space of states CP^{L_2} . The additional assumption was that the probability of transitions depends only on the Fubini-Study distance between states. In section 6, the evolution of spin-state of an electron in a random, normally distributed magnetic field was considered. The probability distribution of the displacement vector $\delta\varphi$ of the spin-state was shown to be independent of the initial state φ_0 and the direction of the displacement in the tangent space $T_{\{\varphi_0\}}CP^1$. In other words, the distribution of states driven by the field may depend only on the Fubini-Study distance between the initial and the end states. It will be now argued that a measurement of the position of a microscopic particle yields similar results. That is, the probability distribution of the state random variable may depend only on the Fubini-Study distance between states. The Born rule for transition of states of the particle then follows from section 5.

A specific example of a measurement of the position that will be kept in mind here is the one where the particle is exposed to a stream of photons of sufficiently high frequency and number density. The scattered photons are then observed to determine the position of the particle. The field of photons in the analysis of this experiment will be treated classically, as a fluctuating potential in a region surrounding the source. Despite the classical treatment of the field and other assumptions to be made about the potential, a more general proof in section 9 will confirm that the result derived here is general.

Let us partition the space $\mathbb{R}^3$ into the cubical cells of edge $\gamma > 0$ centered at the lattice points $\mathbf{a} - \gamma\mathbf{n}$ for some vector $\mathbf{a}$ and $\mathbf{n}$ in the lattice of integers $\mathbb{Z}^3$. Let $\chi_{\mathbf{n}}$ be the unit-normalized indicator function of the $\mathbf{n}$th cell. By choosing γ sufficiently small, one can approximate an arbitrary state ψ in $L_2(\mathbb{R}^3)$ as well as necessary by a finite superposition

$$\psi \approx \sum_{\mathbf{n}} C_{\mathbf{n}} \chi_{\mathbf{n}}. \tag{23}$$

For the same partition, the potential associated with the measurement can be approximated by the sum $\sum_{\mathbf{n}} V_{\mathbf{n}} \widehat{P}_{\mathbf{n}}$, where $\widehat{P}_{\mathbf{n}}$ is the projector onto the subspace of functions with support in the $\mathbf{n}$th cell. The components $V_{\mathbf{n}}$ of the potential are random variables. In the case of position measurement by scattering photons off the particle, $V_{\mathbf{n}}$ can be associated with a photon in the $\mathbf{n}$th cell at time t.

Let $\widehat{V}_\perp = \widehat{V} - \overline{V}I$, as before. From the definition of $\overline{V}$, and within the approximation (23), we have

$$\overline{V} = \sum_{\mathbf{n}} V_{\mathbf{n}} |C_{\mathbf{n}}|^2. \tag{24}$$

Because $\sum |C_{\mathbf{n}}|^2 = 1$, the mean value of the random variable $V_{\perp m}$ is zero:

$$E(V_{\mathbf{m}} - \overline{V}) = E(V_{\mathbf{m}}) - E(V_{\mathbf{m}}) \sum_{\mathbf{n}} |C_{\mathbf{n}}|^2 = 0. \tag{25}$$

On the physical grounds and because of the central limit theorem, the components $V_{\perp \mathbf{n}}$ for different values of $\mathbf{n}$ will be assumed to be independent, identically distributed, normal random variables with the zero mean and no correlation in time.

Let us neglect the kinetic energy term in the Hamiltonian $\widehat{h}$. The discussion of when such an approximation is valid will be given in section 8. The Schrödinger equation for the motion of state in the projective space CP^{L_2} is then given by

$$\frac{d\psi(t)}{dt} = -\frac{i}{\hbar} \widehat{V}_\perp \psi(t). \tag{26}$$

Realistically, the potential $\widehat{V}_\perp$ can be assumed to act only on a compact subset D of $\mathbb{R}^3$. In this case, only the projection of the initial state $\psi = \psi(0)$ onto D will be relevant for the outcomes of the position measurement in D. In particular, we can assume that the support of ψ is in D.

The set up is now very similar to the one considered in section 6. The random magnetic field $\mathbf{B}$ in $\mathbb{R}^3$ is replaced here with the n-component random vector $\mathbf{V}_\perp$ with components $V_{\perp \mathbf{n}} = V_{\mathbf{n}} - \overline{V}$. The generators $i\widehat{\sigma} \cdot \mathbf{B}$ of the Lie algebra $su(2)$ are replaced with the operators A_V defined by $iA_V = \sum_{\mathbf{n}} (V_{\mathbf{n}} - \overline{V}I)\widehat{P}_{\mathbf{n}}$. These operators are elements of the Lie algebra of the unitary group $U(N)$, where N is the number of cells in D. Although the operators A_V do not span the Lie algebra of the group, the one-parameter subgroups of $U(N)$ generated by these operators sweep out the symmetric spaces S^{2N-1} and CP^{N-1}, that is, the sphere of states and the projective space of states. Furthermore, the Killing metric on $U(N)$ yields the Fubini-Study metric on CP^{N-1}. Under such a realization the identity of the group is identified with a particular initial state $\{\psi\}$ in CP^{N-1}. The subspace of the Lie algebra formed by the generators A_V is identified with the tangent space $T_{\{\psi\}}CP^{N-1}$.

From the distribution of the potential, it follows that all directions of the velocity vector $\frac{d\psi}{dt} = -\frac{i}{\hbar}\widehat{V}_\perp \psi$ in the tangent space $T_{\{\psi\}}CP^{N-1}$ are equally likely. Furthermore, the action of the unitary group on CP^{N-1} is transitive. By moving the initial state $\{\psi\}$ and the tangent space $T_{\{\psi\}}CP^{N-1}$ around, we conclude that the distribution of the velocity vector is also independent of the initial state $\{\psi\}$. Therefore, the same is true for the increments $\delta\psi = \frac{d\psi}{dt}\tau$ for a small time step τ.

The motion of state under the evolution (26) is a sequence of random increments $\delta\psi$. By the above, the distribution of the increments is homogeneous (identical for all tangent spaces), isotropic (direction-independent in the tangent space), with no correlation in time. It follows that the distribution of the end states ψ_t at any time t may depend only on the Fubini-Study distance between the initial and the end states.

On the other hand, the distances between two neighboring states $\tilde{\delta}^3_{\mathbf{a}}$, $\tilde{\delta}^3_{\mathbf{b}}$ in M^σ_3 and the corresponding points $\mathbf{a}$, $\mathbf{b}$ in $\mathbb{R}^3$ are equal. Also, the distribution of the increments $\delta\psi$ is normal. It follows that the state constrained to M^σ_3 will undergo a Gaussian random walk (a random walk with the step size distributed normally). Therefore, the position of a particle in $\mathbb{R}^3$ undergoing such a walk is distributed normally. From this and the result of section 5, it follows that the distribution of the end states for the unconstrained evolution (26) must satisfy the Born rule.

8. The validity of assumptions in the models

The models used in sections 6 and 7 are based on the Schrödinger equation and the following two assumptions. First, that the kinetic energy term in the Hamiltonian of the particle under a measurement can be neglected. Second, that the components of the random potential in the models are independent, identically distributed normal random variables with no correlation in time. In light of the central limit theorem and its functional extension (the Donsker's theorem), the latter assumption is typical in the theory of stochastic processes based on a random walk. However, the validity of the former assumption requires a discussion.

Let us begin with the model in section 7, and suppose that the position of an electron is measured by subjecting it to a stream of photons. As discussed, the photons will be treated classically as a potential acting on the particle. Consider first the case when the initial state of the electron belongs to the classical phase space submanifold $M^{\sigma}_{3,3}$ of the space of states. The wave length of the photons will be taken initially to be $1nm = 10^{-9}m$ (x-rays). A simple estimate [7] of the terms in the decomposition (5) yields for the classical velocity component of $\frac{d\varphi}{dt}$, given by the second term in (5):

$$\frac{v}{2\sigma} \sim 10^{14} s^{-1}. \tag{27}$$

For the classical acceleration component, given by the third term in (5), we have

$$\frac{mw\sigma}{\hbar} = 10^{17} s^{-1}. \tag{28}$$

For the spreading component, given by the last term in (5), we obtain

$$\frac{\hbar}{4\sqrt{2}\sigma^2 m} \sim 10^{13} s^{-1}. \tag{29}$$

In the estimate, the acceleration term is the largest of the three. Also, the wave length of the photons in the non-relativistic position measurement experiments is typically much larger than the used value of $1nm$. With the increase in λ, the velocity term decreases as $\lambda^{-\frac{3}{2}}$, the acceleration term decreases as $\lambda^{-\frac{1}{2}}$ while the spreading term decreases as λ^{-2}. In particular, for the scattering in visible light $\lambda \sim 10^{-5}m$ we obtain the velocity term of the order of $10^8 s^{-1}$, the acceleration term $\sim 10^{15} s^{-1}$ and the spreading term $\sim 10^5 s^{-1}$. Furthermore, if the mass m increases, the value of the velocity term further decreases as $m^{-\frac{1}{2}}$, the value of the acceleration term increases as $m^{\frac{1}{2}}$, while the spreading term decreases as m^{-1}, showing that the acceleration terms is by far the dominant term under these conditions [7].

These results were obtained for the initial state in $M^{\sigma}_{3,3}$. Let us now write an arbitrary initial state ψ as a superposition of states in $M^{\sigma}_{3,3}$. Then the variation $\delta\psi = \frac{d\psi}{dt}\tau$ can be also written as a series in functions from $M^{\sigma}_{3,3}$, so that each term of the series is a constant times a function in $M^{\sigma}_{3,3}$. In this case for each term in the series the acceleration term will be dominant. Given the near-orthogonality of the terms of the series, we can neglect the velocity and spreading parts in each term. This amounts to keeping only the potential term in the Hamiltonian. In particular, the motion of the state in these conditions amounts to a jiggling of the wave packet without much spreading or displacement.

To check that the kinetic energy term in the Hamiltonian can be dropped, we also need to check that the periods of a free evolution of the electron state during the measurement can be neglected. That is, interaction with the electromagnetic field is happening continuously in time. Only in this case the discussion in the previous paragraphs is applicable. From the number density of photons, we can estimate the number of photons in one cubic meter of space by

$N \approx 2.02 \times 10^7 T^3$ and the average energy of a photon by $2.7 k_B T$, where k_B is the Boltzmann constant and T is temperature. For instance, taking $T \sim 500K$, we obtain $N \sim 10^{15}$. The photon with the average energy at this temperature has the wave length $\lambda \sim 10^{-5}m$. Under these conditions, at any time t there is about one photon per cube of the volume λ^3. So, at any t, each $M^{\sigma}_{3,3}$ component of ψ experiences the potential of a photon passing by. Given these conditions, neglecting the free evolution of the electron state is a reasonable approximation.

The model in section 6 is different in its use of the magnetic field, needed for the random walk of the spin state. The full dynamics is described by the Pauli equation

$$\frac{d\Psi_t}{dt} = -\frac{i}{\hbar}\left[\frac{(\widehat{\mathbf{p}} + e\mathbf{A})^2}{2m} + \mu\widehat{\sigma}\cdot\mathbf{B} - e\phi\right]\Psi_t. \tag{30}$$

Here Ψ_t is the two-component state function of the electron, $\mathbf{A}$ and ϕ are magnetic vector and electric scalar potentials, $\widehat{\mathbf{p}}$ is the momentum operator and the rest of the terms are as in the section 6. Consider the case of a product state $\Psi_t(\mathbf{x}) = \psi_t(\mathbf{x})\varphi_t$, where ψ_t is a scalar function of $\mathbf{x}$ and φ_t is the spin state function used in section 6. In this case the variables separate and we obtain two equations:

$$\frac{d\psi_t}{dt} = -\frac{i}{\hbar}\left[\frac{(\widehat{\mathbf{p}} + e\mathbf{A})^2}{2m} - e\phi\right]\psi_t \tag{31}$$

and

$$\frac{d\varphi_t}{dt} = -\frac{i}{\hbar}\mu\widehat{\sigma}\cdot\mathbf{B}\varphi_t. \tag{32}$$

For the first equation, we can use the Coulomb gauge $\nabla\cdot\mathbf{A} = 0$ and apply the estimates in section 7 to conclude that the term that determines the evolution of the state under the same conditions is equal to $\widehat{V} = \frac{ie}{\hbar}\phi$. Under the conditions of section 7, the potential $\widehat{V}$ ensures the distribution of states on the space of states that is consistent with the Born rule. The second equation is the one already discussed in section 6. The general case of a superposition of the product states in the random electromagnetic field satisfying the stated conditions is treated similarly. In this case we obtain a simultaneous motion of the position and the spin states. The Born rules still applies to the total state Ψ. The details related to the motion of non-separable states are discussed in section 14.

These estimates support the assumption that during a measurement considered in this section, the potential term is the main term in the Hamiltonian responsible for the Schrödinger dynamics of the particle. A more general approach to the motion of state of a measured system will be discussed in the following section.

9. The motion of state under measurement

In the non-relativistic quantum mechanics, the unitary property of evolution means that the state of a particle cannot disappear or get created. For a spinless particle, the state under the evolution can only move along the unit sphere in the space of states $L_2(\mathbb{R}^3)$. To express this conservation of states in the case of observation of position of the particle, let us introduce the density of states functional $\rho_t[\varphi;\psi]$. Recall that under a position measurement considered in section 7, the state is driven by the potential term in the Schrödinger equation. The space is partitioned into cubical cells of the size comparable to the wavelength of the photons scattering off the particle. The state and the potential are written in terms of their values in the cells. The resulting space of states H is finite dimensional, say, of dimension N, and the state under the evolution is confined to this space. In particular, the usual volume form on the space of states is defined.

So we begin with an ensemble of particles whose initial state belongs to a neighborhood of the state ψ on the sphere of states S^{2N-1} in H or on the projective space CP^{N-1}. The functional

$\rho_t[\varphi;\psi]$ measures the number of states that by the time t belong to a neighborhood of a state φ in S^{2N-1}. It is approximately equal to the number of states in a small region around φ in S^{2N-1} divided by the volume of the region. Under the identification of the classical space with the manifold M^σ_3, the position $\mathbf{a}$ of a particle is identified with the state $\tilde{\delta}^3_{\mathbf{a}}$. So the density of states functional $\rho_t[\varphi;\psi]$ must be related to the usual density of particles $\rho_t(\mathbf{a};\mathbf{b})$ with initial position $\mathbf{b}$. Namely, when the volume form is constrained to M^σ_3, we must have $\rho_t(\mathbf{a};\mathbf{b}) = \rho_t[\tilde{\delta}^3_{\mathbf{a}};\tilde{\delta}^3_{\mathbf{b}}]$.

The conservation of a large number of small macroscopic particles is expressed in differential form by the continuity equation. For instance, if $\rho_t(\mathbf{a};\mathbf{b})$ is the density at a point $\mathbf{a}\in\mathbb{R}^3$ of an ensemble of Brownian particles with initial position near $\mathbf{b}$ and $\mathbf{j}_t(\mathbf{a};\mathbf{b})$ is the current density of the particles at $\mathbf{a}$, then

$$\frac{\partial\rho_t(\mathbf{a};\mathbf{b})}{\partial t} + \nabla\mathbf{j}_t(\mathbf{a};\mathbf{b}) = 0. \tag{33}$$

We will assume that $\rho_t(\mathbf{a};\mathbf{b})$ and $\mathbf{j}_t(\mathbf{a};\mathbf{b})$ are normalized per one particle, i.e., the densities are divided by the number of particles. In this case, the particle density and the probability density can be identified.

The conservation of states of an ensemble of microscopic particles is expressed by the continuity equation that follows from the Schrödinger dynamics and has the same form (33) with

$$\rho_t = |\psi|^2, \quad \text{and} \quad \mathbf{j}_t = \frac{i\hbar}{2m}(\psi\nabla\overline{\psi} - \overline{\psi}\nabla\psi). \tag{34}$$

For the states $\psi\in M^\sigma_{3,3}$ we obtain

$$\mathbf{j}_t = \frac{\mathbf{p}}{m}|\psi|^2 = \mathbf{v}\rho_t. \tag{35}$$

Because the restriction of Schrödinger evolution to $M^\sigma_{3,3}$ is the corresponding Newtonian evolution, the function ρ_t in (35) must be proportional to the density of particles, denoted earlier by $\rho_t(\mathbf{a};\mathbf{b})$. The relation $\rho_t(\mathbf{a};\mathbf{b}) = \rho_t[\tilde{\delta}^3_{\mathbf{a}};\tilde{\delta}^3_{\mathbf{b}}]$ tells us that ρ_t in (34) must be then the density of states $\rho_t[\tilde{\delta}^3_{\mathbf{a}};\psi]$. It gives the number of particles initially in a state near ψ found under the measurement at time t in the state near $\tilde{\delta}^3_{\mathbf{a}}$.

The obtained relationship

$$\rho_t[\tilde{\delta}^3_{\mathbf{a}};\psi] = |\psi_t(\mathbf{a})|^2 \tag{36}$$

explains the identification of $|\psi_t(\mathbf{a})|^2$ with the probability density, postulated in quantum theory. Indeed, the probability density to find the system in a state for an ensemble of states is proportional to the value of the density of states functional on that state, which for the states in M^σ_3 is given by (36). So $|\psi_t(\mathbf{a})|^2$ is the probability density to find the particle near $\mathbf{a}$ simply because this quantity is the density of quantum states near the point $\tilde{\delta}^3_{\mathbf{a}}$. If there are more states near $\tilde{\delta}^3_{\mathbf{a}}$, it becomes more likely to find the state under an observation near that point.

The relationship (36) and the Born rule that follows from it were derived solely from the Schrödinger equation and the fact that the Schrödinger dynamics constrained to $M^\sigma_{3,3}$ is equivalent to the Newtonian one. The continuity equation (33) in quantum mechanics used to obtain (36) follows from the Schrödinger equation with *any* potential. Suppose we begin with an arbitrary random potential V that under the Newtonian dynamics yields the normal distribution of the position random variable for a macroscopic particle. As we know from section 3, there exists a unique extension of the Newtonian to Schrödinger dynamics. The formula (36) asserts then the validity of the Born rule for the resulting distribution of states undergoing the Schrödinger evolution with the potential V.

This conclusion generalizes the results of section 7, obtained under more stringent constraints on the potential. In addition, a purely geometric derivation of the Born rule in section 5 acquires here its dynamical validation. Note also that the isotropy of the probability distribution that

needed to be assumed in the derivation of section 5 now follows directly from the Schrödinger dynamics and its reduction to the Newtonian one.

10. The classical behavior of macroscopic bodies

The motion of state in quantum mechanics satisfies the deterministic Schrödinger equation with a given potential. However, as advocated here, under the conditions typically associated with a measurement, the state evolves by the Schrödinger equation with a random potential. The potential initiates a random motion of the state on the space of states and the resulting change in the density of states functional. The difference between these two types of evolution is analogous to the difference between the usual Newtonian motion of a macroscopic particle in a given potential and the Brownian motion of the particle. In light of the discussed relationship of Newtonian and Schrödinger dynamics, this analogy is not surprising. Note that the typical process of measurement must be sufficiently fast to be able to neglect the deterministic Schrödinger evolution during the measurement. The corresponding estimates for the measurement by the light scattering were given in section 7. In the opposite case, the motion of the state will consist of the deterministic drift and a random motion about the moving mean. The analogy with the measurement on a macroscopic particle can serve here a guiding principle.

The conservation of states in $L_2(\mathbb{R}^3)$ during measurement can be written in the following integral form

$$\rho_{t+\tau}[\varphi;\psi] = \int \rho_t[\varphi+\eta;\psi]\gamma[\eta]D\eta. \tag{37}$$

Here ρ_t is the density of states functional, $\gamma[\eta]$ is the probability functional of the variation η in the state φ and integration goes over all variations η such that $\varphi+\eta$ is in S^{L_2}. The functional $\gamma[\eta]$ is assumed to be independent of the direction of η. The equation (37) relates the density ρ_t at the time $t+\tau$ at a point φ to the density at the time t and the flow of states from the neighboring points $\varphi+\eta$ to the point φ during the time interval τ due to the random motion.

When the state of the particle is constrained to $M_3^\sigma = \mathbb{R}^3$, we have $\varphi = \tilde{\delta}^3_{\mathbf{a}}$ and $\eta = \tilde{\delta}^3_{\mathbf{a}+\epsilon} - \tilde{\delta}^3_{\mathbf{a}}$, where ϵ is a displacement vector in $\mathbb{R}^3$. By section 9, the function $\rho_t(\mathbf{a};\mathbf{b}) = \rho_t[\tilde{\delta}^3_{\mathbf{a}};\tilde{\delta}^3_{\mathbf{b}}]$ is the usual density of particles in space. Also, from the properties of $\gamma[\eta]$ it follows that the function $\gamma(\epsilon) = \gamma[\tilde{\delta}^3_{\mathbf{a}+\epsilon} - \tilde{\delta}^3_{\mathbf{a}}]$ is independent of the direction of ϵ in $\mathbb{R}^3$. The integral in (37) becomes the usual integral over $\mathbb{R}^3$. As known since Einstein's work on the Brownian motion, the differential form of the resulting integral equation is the diffusion equation

$$\frac{\partial \rho_t(\mathbf{a};\mathbf{b})}{\partial t} = K\Delta\rho_t(\mathbf{a};\mathbf{b}), \tag{38}$$

where $K = \frac{1}{2\tau}\int \epsilon^2\gamma(\epsilon)d\epsilon$.

The diffusion equation (38) describes the dynamics of an ensemble of particles in the classical space $M_3^\sigma = \mathbb{R}^3$. If initially all particles in the ensemble are at the origin, then the density of the particles at a point $\mathbf{a} \in \mathbb{R}^3$ at time t is given by

$$\rho_t(\mathbf{a};0) = \left(\frac{1}{4\pi kt}\right)^{\frac{3}{2}} e^{-\frac{\mathbf{a}^2}{4Kt}}. \tag{39}$$

In particular, for the mean-squared displacement of the Brownian particle we obtain

$$\frac{d\overline{\mathbf{a}^2}}{dt} = 2nK, \tag{40}$$

where $n = 3$ for $\mathbb{R}^3$.

As we know from section 3, the metric on the submanifold M_3^σ is the metric on the space of states CP^{L_2}, constrained to M_3^σ. In other words, the embedding of M_3^σ into CP^{L_2} is isometric. In particular, if θ is the distance between the states $\tilde{\delta}^3_{\mathbf{a}}$ and $\tilde{\delta}^3_{\mathbf{o}}$ centered at $\mathbf{a}$ and the origin in $\mathbb{R}^3$ in the Fubini-Study metric, we have $\mathbf{a}^2 = \theta^2$ for small values of $\mathbf{a}^2$. This can be also seen directly from (8) by the series expansion. We also know that the density of particles is equal to the density of states functional constrained to M_3^σ.

Suppose that position of a microscopic particle is measured by subjecting the state to a random potential, as investigated in section 7. We know that the state will undergo a random motion on the space of states, that the probability of reaching a particular point is given by the Born rule, and that the motion observed on M_3^σ is the usual Brownian motion. In particular, for the points on the x-axis in $\mathbb{R}^3$ and the corresponding states in M_3^σ, we have for small t

$$\frac{d\overline{x^2}}{dt} = \frac{d\overline{\theta^2}}{dt} = 2K, \tag{41}$$

where as before θ is the Fubini-Study distance between the states. Since the probability distribution is homogeneous and isotropic, the equality

$$\frac{d\overline{\theta^2}}{dt} = 2K \tag{42}$$

must be true for all initial states ψ and along arbitrary directions in the tangent space to the space of states at any ψ, for small t. From (42), we can conclude that, when K is vanishingly small, the state of the system in the space of states does not change. In particular, if the initial state is in M_3^σ, the position of the particle is defined and remains constant.

Suppose the initial state of a macroscopic body is given by a point $\tilde{\delta}^3_{\mathbf{b}}$ in M_3^σ. (It is more accurate to identify the state of the body at rest with a tensor product of the states of cells in the body, but this would not change the main outcome; see the discussion at the end of the section.) It is a well established and experimentally confirmed fact that macroscopic bodies experience an unavoidable interaction with the surroundings. Their "cells" are pushed in all possible directions by the surrounding particles. For instance, a typical Brownian particle of radius between $10^{-9}m$ and $10^{-7}m$ experiences about 10^{12} random collisions per second with surrounding atoms in a liquid. The number of collisions of a solid of radius $10^{-3}m$ in the same environment is then about 10^{19} per second. Collisions with photons and other surrounding particles must be also added. Even empty space has on average about 450 photons per cm^3 of space.

Let us estimate the value of the diffusion coefficient for a macroscopic body. As known after the works of Stokes and Einstein, the diffusion coefficient for a spherical particle is well described by the expression

$$K = \frac{k_B T}{6\pi\eta r}, \tag{43}$$

where r is the radius of the particle and η is the dynamic viscosity. In particular, for a macroscopic particle of radius $r \sim 1mm$ in the air, $\eta \sim 10^{-5} N \cdot s/m^2$, at room temperature, we get $K \sim 10^{-12} m^2/s$. The variance of the x-coordinate of position of the particle is given by $\overline{x^2} = 2Kt$. In particular, it would take about $10^6 s$ or more than 10 days for the standard deviation of $1mm$ in the distribution of the displacement of the particle to occur.

The actual time of observation of position of particles in experiments is much shorter. For instance, if we scatter visible light off the particle to determine its position, the time interval of observation could be as short as $10^{-13}s$, which for a $1mm$ of radius particle in the air would amount to the displacement of the order of $10^{-21}m$. This quantity is much less than the accuracy of measurement, limited by the wavelength $\lambda \sim 10^{-5}m$, and cannot be observed in the measurement. Note that the Fubini-Study distance between Gaussian states that are

$10^{-21}m$ apart in M_3^σ with $\sigma \sim 10^{-5}m$ can be calculated via (8) and is about $10^{-16}rad$. So the state is hardly moving away from its original position and cannot realistically reach points in the space of states that are away from that position. In particular, it becomes impossible to find the state positioned initially in the classical space M_3^σ at a different point of that space.

The situation is surprisingly similar to that of a pollen grain and a ship initially at rest in still water. While under the kicks from the molecules of water, the pollen grain experiences a Brownian motion, the ship in still water will not move at all. Because of the established relation of Newtonian and Schrödinger dynamics, this is more than an analogy. In fact, when the state is constrained to the classical phase space submanifold, the "pushes" experienced by the state become the classical kicks in the space that could lead to the Brownian motion of the body.

Suppose now an external potential V is applied to the macroscopic system. According to (5), this will "push" the state that belongs to the classical phase space submanifold in the direction tangent to the submanifold. Therefore, the external potential applied to a macroscopic body will not affect the motion of state in the directions orthogonal to the classical phase space submanifold. That means that the state will remain constrained to the submanifold. On the other hand, as we know from the same section, the constrained state will evolve in accord with Newtonian dynamics in the total potential $V + V_S$, where V_S is the potential created by the surroundings. However, since at any time t the total force $-\nabla V_S$ exerted on the macroscopic body by the particles of the surroundings can typically be neglected, the body will evolve according to Newtonian equations with the force term $-\nabla V$. To be sure, the particles of the surroundings are responsible for the friction. In the Hamiltonian description of interaction of the body with the surroundings (as in the Ullersma model [25]), the friction comes from a contribution to the total potential in the Hamiltonian. However, whenever the friction can be neglected, the dynamics of the solid is determined by the force $-\nabla V$.

So, the origin of the classical behavior of macroscopic bodies in the theory is three-fold. First of all, the initial state of a macroscopic material point is a point of $M_{3,3}^\sigma$. That is, a macro body is *created* at a point of the submanifold $M_{3,3}^\sigma$. More generally, the state of an extended macroscopic body is a point of the classical phase space manifold $M_{3n,3n}^\sigma$ in the tensor product space of states $L_2(\mathbb{R}^3) \otimes \ ... \ \otimes L_2(\mathbb{R}^3)$ of the "parts" of the body. To "build" the body, we either use the parts that are already in the classical space, or we measure position of each part with a sufficient precision in the process of building, and thus prepare a state in $M_{3n,3n}^\sigma$.

Second, because of the interaction of the body with the surroundings (radiation, atoms and molecules of the media), its state experiences a fluctuating potential. The values of the potential in different "cells" of the space are represented by independent and identically distributed random variables, having no correlation in time. As a result, the components of the state over different cells evolve independently, the coherence is lost and the state undergoes a diffusion process rather than a free Schrödinger evolution. The state's position on the space of states is described by the probability density function.

Finally, because of the macroscopic character of the body, the coefficient of diffusion for the process is extremely small. The density of states functional for the body is nearly constant in time. We don't see a quantum evolution of the state, but rather a negligible "jiggling". Under a position measurement of the body this jiggling results in a Gaussian distribution of the position random variable.

Note that the diffusion equation (38) was derived under the assumption that the state of the system is constrained to the manifold $M_{3,3}^\sigma$. As discussed earlier in this section, a spatially-extended solid is more accurately represented by the state with values in the manifold $M_{3n,3n}^\sigma$. However, the linear displacement of the solid is given by the displacement of its center of mass. For the states constrained to $M_{3n,3n}^\sigma$ under the usual assumptions, the equation (37) yields the diffusion equation (38) for a material point positioned at the center of mass of the solid. In particular, the previous consideration applies.

From this analysis, it becomes clear that the transition of macroscopic to microscopic happens for the bodies whose Brownian motion under the influence of the atoms and molecules of the media is observable. If a macroscopic body is sufficiently small so that the Brownian motion of the body in the surrounding media can be observed in an experiment, then the superposition of states of different positions of the body becomes observable as well. In fact, it was demonstrated that under the conditions typical for the Brownian motion, the state of the system has equal probability of any direction of displacement in the space of states. In particular, the state may become a superposition of states of a given position in $\mathbb{R}^3$. Interference effects on such states can be then observed.

11. Collapse of quantum state

In the previous sections 6-10, we considered a diffusion on the space of states, described by the Schrödinger equation with a random potential. Although the probability of transition between states under the diffusion was shown to satisfy the Born rule, the resulting process is very different from what is usually understood by the collapse. The fact that a random potential may lead to a random fluctuation of state is rather trivial and goes against of what one normally tries to achieve when explaining or modeling the collapse. The existing collapse models utilize various ad hoc additions to the Schrödinger equation with the goal of explaining why the state under the resulting stochastic process "concentrates" to an eigenstate of the measured observable (usually, position or energy) [10]-[21]. Instead, it is argued here that under a generic measurement, an ensemble of states with an initial position near ψ "diffuses" isotropically into the space of states by a unitary Schrödinger evolution. Whenever a particular state in the ensemble reaches a neighborhood of an eigenstate of the measured observable, we say that the "collapse" has occurred. In this case, the measuring device can record the value of the measured physical quantity.

According to this scenario, the measuring device has two separate functions. On one hand, it initiates a diffusion by creating a "noise". On the other, it registers a particular location of the diffused state. For instance, the "noise" in the position measuring device could be due to a stream of photons. The device then registers the state reaching a point in M_3^σ. In a similar way, a momentum measuring device registers the states that under the diffusion reach the eigen-manifold of the momentum operator (the image of M_3^σ under the Fourier transformation). Note the similarity in the role of measuring devices in quantum and classical mechanics: in both cases the devices are designed to measure a particular physical quantity and inadvertently create a "noise", which contributes to a distribution of values of the measured quantity.

According to this, the measuring device in quantum mechanics is not responsible for creating a basis into which the state is to be expanded. If several measuring devices are present, they are not "fighting" for the basis. When the eigen-manifolds (the collection of eigenfunctions) of the corresponding observables don't overlap, only the device measuring one of the observables can "click" for the measured particle as the state can reach only one of the eigen-manifolds at a time.

What does it all say about measurement of position of macroscopic and microscopic particles? During the period of observation of position of a macroscopic particle, the position is a random variable whose probability density function satisfies the diffusion equation. Normally, observation happens during a short enough interval of time and the variance of the probability distribution is sufficiently small. A particular value of position variable during the observation is simply a realization of one of the possible outcomes. The change in observed position of the particle can be equivalently thought of as either a stochastic process $\mathbf{b}_t$ with values in $\mathbb{R}^3$ or a process $\tilde{\delta}^3_{\mathbf{b},t}$ with values in M_3^σ. The advantage of the latter representation is that the position random variable gives both the position of the particle in $M_3^\sigma = \mathbb{R}^3$ and, under a proper choice of σ, the probability density to find it in a different location $\mathbf{a}$ (in the state $\tilde{\delta}^3_{\mathbf{a}}$), due to

uncontrollable interactions with the measuring device under the observation.

Measuring position of a microscopic particle has, in essence, a very similar nature. Under observation, the state ψ is a multivariate random variable with values in the space of states CP^{L_2}. To measure position is to observe the state on the submanifold M_3^σ (or $M_{3,3}^\sigma$) in CP^{L_2}. In this case, the random variable ψ assumes one of the values $\tilde{\delta}_{\mathbf{a}}^3$, with the uniquely defined probability density compatible with the normal density on $\mathbb{R}^3$. As we know from the previous sections, this probability density signifies the validity of the Born rule. Here too, the random variable ψ gives both the position of the state of the particle in CP^{L_2} and the probability density to find the particle in a different state $\tilde{\delta}_{\mathbf{a}}^3$.

So the difference between the measurements is two-fold. First, under a measurement, the state ψ of a microscopic particle is a multivariate random variable with values in the entire space of states CP^{L_2} and not just in the submanifold M_3^σ. Second, unless ψ is already constrained to M_3^σ (the case that would mimic the measurement of position of a macroscopic particle), to measure position is to observe the state that "diffused" enough to reach the submanifold M_3^σ. To put it differently, the measuring device is not where the initial state is. Assuming the state has reached M_3^σ, the probability density of reaching a particular point in M_3^σ is given, as we saw, by the Born rule.

We don't use the term collapse of position random variable when measuring position of a macroscopic particle. Likewise, there seems to be no physics in the term collapse of the state of a microscopic particle. Instead, due to the diffusion of state, there is a probability distribution to find the state of the particle in various locations on CP^{L_2}. In particular, the state may reach the space submanifold $M_3^\sigma = \mathbb{R}^3$. If that happens and we have detectors spread over the space, then one of them clicks. If the detector at a point $\mathbf{a} \in \mathbb{R}^3$ clicks, that means the state is at the point $\tilde{\delta}_{\mathbf{a}}^3 \in CP^{L_2}$ (that is, the state *is* $\tilde{\delta}_{\mathbf{a}}^3$). The number of clicks at different points $\mathbf{a}$ when experiment is repeated is given by the Born rule. The state is not a "cloud" in $\mathbb{R}^3$ that shrinks to a point under observation. Rather, the state is a point in CP^{L_2} which may or may not be on $\mathbb{R}^3 = M_3^\sigma$. When the detector clicks, we know that the state is on M_3^σ.

Note once again that there is no need for any new mechanism of "collapse" in the model. An observation is not about a "concentration" of state and the stochastic process initiated by the observation is in agreement with the conventional Schrödinger equation with a randomly fluctuating potential. The origin of the potential depends on the type of measuring device or properties of the environment capable of "measuring" the system. Fluctuation of the potential can be traced back to thermal motion of molecules, atomic vibrations in solids, vibrational and rotational molecular motion, and the surrounding fields.

12. Entanglement and decoherence

So far, the state of the measured system was considered independently of the measuring device and the environment. This is possible as long as the influence of the environment can be modeled, at least approximately, by a potential. However, in many cases the state of the measured particle, whose position is measured, and the surroundings cannot be described independently. The state of the total system is then a linear combination of the terms

$$\tilde{\delta}_{\mathbf{a}}^3 \otimes E_{\mathbf{a}}, \tag{44}$$

where $E_{\mathbf{a}}$ represents the state of the surroundings when the particle is in the state $\tilde{\delta}_{\mathbf{a}}^3$. At the same time, the result of measurement is always a single term like (44). How could it be?

As we know from section 3, the Newtonian dynamics of an arbitrary mechanical system is the Schrödinger dynamics of the system with the state constrained to the classical phase submanifold of the space of states. To understand the entanglement, let us begin then with a classical mechanical analogue of an entangled state. Consider a pair of macroscopic particles,

for simplicity in one dimension, connected by a weightless rigid rod that keeps the particles at a fixed distance d from each other. Suppose the position of one of the particles is measured. As before, the diffusion mechanism can be used to describe the resulting normal distribution of the position random variable of the particle. If position of the particle is found to be a, then position of the second is guaranteed to be $a + d$. If position of the second particle is measured and found to be b instead, then position of the first is guaranteed to be $b - d$.

In the language of quantum states, the state of the pair at any time is $\tilde{\delta}_a \otimes \tilde{\delta}_{a+d}$ for some a. This state belongs to a one-dimensional submanifold N of the two-dimensional manifold $M_1^\sigma \otimes M_1^\sigma$, which itself is a submanifold of the Hilbert space $L_2(\mathbb{R}) \otimes L_2(\mathbb{R})$. Here M_1^σ is the submanifold of $L_2(\mathbb{R})$ made of the normalized Gaussian functions $\tilde{\delta}_a$ of width σ. The difference between this state and the state of a pair of microscopic particles entangled in a "similar" manner is that in the latter case the state belongs to the linear envelop $L(N)$ of N, which is a subspace of $L_2(\mathbb{R}) \otimes L_2(\mathbb{R})$. So, a normalized entangled state lives on the unit sphere $S^{L(N)}$ in $L(N)$ and not only on the submanifold N of thereof. A system of two particles connected by a rod is a classical mechanical analogue of the entangled state of a pair of particles and is, at least mathematically, a special case of the entangled state.

We can now approach the measurement of the position of the pair in an arbitrary entangled state in a way similar to the case of a single particle. As in section 7, we conclude that the distribution of the displacements of the initial state ψ in $CP^{L(N)}$ under a typical measurement is homogeneous and isotropic. From the normal distribution of the position random variable of one of the particles when the state of the pair is constrained to $M_1^\sigma \otimes M_1^\sigma$, it then follows that the Born rule for an arbitrary initial state ψ is satisfied. That is, under the measurement the state ψ undergoes a random motion on the space of states and can potentially reach the submanifold N in $CP^{L(N)}$. The probability of reaching the neighborhood of a particular point $\tilde{\delta}_a \otimes \tilde{\delta}_{a+d}$ in $CP^{L(N)}$ is given by the Born rule. To find the pair in the state $\tilde{\delta}_a \otimes \tilde{\delta}_{a+d}$ is the same as to find the first particle at a (and, therefore, the second particle at $a + d$).

The number of "parts" in a quantum system makes no difference for the proposed mechanism to work. For instance, a macroscopic crystalline solid with the lattice constant Δ (in one dimension) can be described by the state

$$\tilde{\delta}_a \otimes \tilde{\delta}_{a+\Delta} \otimes \ \ldots \ \otimes \tilde{\delta}_{a+n\Delta}. \tag{45}$$

According to section 10, a macroscopic solid interacts with the surroundings, which results in a trivial diffusion of the state. In particular, an initial state of the form (45) cannot evolve into a superposition of such states. If, however, the number n of cells in the body is small, then superposition of states of different positions of the system becomes possible. In this case the process of measurement for the system is similar to the one described for the pair of particles.

The nature of the entanglement between the particle and the surrounding (whatever it may be) raised at the beginning of this section is analogous. Such an entanglement usually comes with a significant increase in the dimension of the Hilbert space of possible states of the total system. The measurement of only one part of the entangled system results in a random motion of the total state on the space of states. The measurement of position of a particle is successful if under this motion the state reaches the submanifold formed by the product states $\tilde{\delta}^3_{\mathbf{a}} \otimes E_{\mathbf{a}}$. Analogously to the two-particle system, the relative probabilities of finding the state near a specific point of the submanifold is given by the Born rule. Whenever a part of the surroundings, say, the measuring apparatus, is macroscopic, its state is maintained in the classical phase space submanifold of the space of states. The mechanism of this constraint was explained in section 10. The consequence of this is that the state of a macroscopic measuring apparatus and the measured system is always a product state.

Because the Newtonian dynamics was shown to be a constrained Schrödinger dynamics, the issue of where to place the "cut" in the surroundings, to avoid considering the entire

universe in an entangled state with the measured particle is analogous to the same issue in Newtonian mechanics. There too, in principle, the entire universe influences the motion of the measured particle. However, a good approximation can be obtained by assuming that the particle represents a closed system, by reducing the role of the environment to a potential, or by considering a system of finitely many particles, or else, by yet another mechanism that effectively reduces the degrees of freedom of the system and makes the problem solvable.

So far, decoherence was not present in the discussion. Formally, decoherence is a mathematical expression of the fact that a quantum system interacting with the environment behaves like a probabilistic mixture and needs to be described by the probability and not by the state itself. The dynamical part of the process that leads to decoherence consists in the entanglement between the state of the particle and the environment. This part is due to the usual unitary evolution of the total system. The decoherence itself consists in a fast decrease in the interference effects between terms of the total state of the system, due to the near-orthogonality of the states of the environment $E_{\mathbf{a}}(t)$, as the system evolves in time. The theory is centered around, and does not go beyond the issue of entanglement and the resulting loss of coherence. It does not describe the way in which specific measurement results are obtained and does not derive the Born rule. At the same time, decoherence theory uses an array of very useful models that provide physical content for the theory. These models testify to the universal character of the loss of coherence and transition to classical probability resulting from interaction with the environment.

Most importantly, the interaction with the surroundings, particularly, with the incident particles in the model of spatial decoherence by scattering is what also triggers the diffusion of state, under the discussion here. In this context, decoherence may be considered a superficial expression of the underlying physical process of diffusion of state. In fact, the diffusion of state under a measurement signifies that the system is described by the density of states functional. When applied to measurement of position, the functional yields a probabilistic distribution of the position random variable, which is the end-result of the (spatial) decoherence. Note that the diffusion is a dynamical process, unlike the "pure entanglement form" of decoherence itself. Despite being a "fake" or "microscopically unitary" decoherence [23], the diffusion of state seems to provide a universal dynamical mechanism for the loss of coherence *and* collapse of state.

13. The double-slit experiment

The derivation of Newtonian from Schrödinger dynamics, the relationship of the Born rule to the normal probability distribution, an explanation of the classical behavior of macroscopic bodies and a simple picture of collapse all suggest that the isomorphism between the classical space $\mathbb{R}^3$ and the manifold M_3^σ must be considered a physical and not just a mathematical identification. Let us accept this hypothesis and explore its consequences.

Based on the hypothesis, the way to picture the correspondence between the classical and the quantum is the following. All physical processes happen in the space of states of the system, and not just in the classical space. The classical space and classical phase space are submanifolds in the space of states. The classical dynamics is the Schrödinger dynamics of the system constrained to the classical phase space submanifold. Interaction with the environment is what constrains the state of a macroscopic body to the classical space. Some process with microscopic particles begin on the classical space submanifold and continue into the Hilbert space of states. Some processes begin in the Hilbert space and end-up on the classical space submanifold. All dynamical processes (classical and quantum) are described by paths in the space of states. Any such path satisfies the Schrödinger equation. When the path takes values in the classical phase space, we see it as a classical motion, satisfying Newtonian equations of motion. The state of a particle that belongs to the classical phase space submanifold represents a classical material point. This correspondence makes the states physical or "real" and results

in an identification of particles and states.

Let us use the obtained correspondence between the classical and the quantum to revisit the essential paradoxes of quantum mechanics. As discussed, the superposition principle in quantum mechanics represents the main obstacle to reconciliation of the quantum and the classical. Let us therefore begin with the simplest manifestation of the superposition principle: the double-slit experiment. Different forms of the experiment are well known and don't need to be reviewed here. We are going to discuss the simplest set-up of the experiment, involving an electron gun, a plate with a pair of parallel slits, and a scintillating screen or a photographic plate to observe the interference pattern. Our first task is to identify the Hilbert space of the system and the submanifold of the corresponding classical system. We will deal with a single electron. Also, since the origin of the electron will not be important, the electron gun will be left out of the picture. For now we will also leave out the screen registering the outgoing particles and the surroundings.

The Hilbert space of the system is the tensor product of spaces $L_2(\mathbb{R}^3)$, one for each particle in the system. However, the state of the macroscopic plate with the slits has the form (45) in section 12. That is, the plate is given by a point ψ_P on the submanifold $M^\sigma_{3n} = M^\sigma_3 \otimes ... \otimes M^\sigma_3$ in $CP^{L_{2,n}}$. Here $L_{2,n}$ is the tensor product of Hilbert spaces $L_2(\mathbb{R}^3)$ for all particles of the plate. The isomorphism $\omega_n : \mathbb{R}^3 \times ... \times \mathbb{R}^3 \longrightarrow M^\sigma_{3n}$, $\omega_n(\mathbf{a}_1, ..., \mathbf{a}_n) = \tilde{\delta}^3_{\mathbf{a}_1} \otimes ... \otimes \tilde{\delta}^3_{\mathbf{a}_n}$ allows us to view the states in M^σ_{3n} as points in the classical configuration space $\mathbb{R}^{3n}$ or positions of n particles in the single classical space $\mathbb{R}^3$. That is how our usual view of the plate becomes possible and how the state ψ_P gets identified with a set of material points that represent the particles of the plate in $\mathbb{R}^3$. As discussed in sections 10 and 12, the interaction between the plate and the environment prevents an entanglement between the states of the macroscopic plate and the electron. Moreover, since the plate is practically unaffected by the electron, its state during the experiment remains ψ_P so that the state of the total system belongs to the manifold $L_2(\mathbb{R}^3) \otimes \psi_P$.

We can now proceed with the analysis of the experiment. First, the wave packet of the electron propagates toward the plate. If the electron is sufficiently fast, the spreading of the packet on the approach to the plate can be neglected. During this time interval, the propagation of the initial packet ψ is happening essentially by a displacement $\psi_t(\mathbf{x}) = \psi(\mathbf{a} - \mathbf{v}t)$. The electron state moves along (parallel to) the classical space submanifold M^σ_3 in the projective space CP^{L_2} for the Hilbert space $L_2(\mathbb{R}^3)$. The state $\Psi_t = \psi_t \otimes \psi_P$ of the total system moves along the submanifold $M^\sigma_3 \otimes \psi_P$ in $CP^{L_2} \otimes \psi_P$, diffeomorphic, i.e., identical to the manifold $M^\sigma_3 = \mathbb{R}^3$. The motion can be thought of in the classical terms; we have a material point propagating towards the plate. If desirable, we can add photons to this picture, to ensure that we can "see" the plate. In fact, if refraction in the media is neglected, the photon wave packet always propagates along the classical space M^σ_3 without spreading. The isomorphism ω_n can then be used to interpret the entire process in terms of the electron, the plate and the photon, all in the same classical space $\mathbb{R}^3$.

During the second stage of the experiment, the electron goes "through" the slits in the plate. Although it is useful to "visualize" the plate by the state ψ_P, or by the corresponding set of points in $\mathbb{R}^3$, the effect of the plate on the electron can be described by a potential $\widehat{V}$, which is infinite on the plate and zero at the slits and away from the plate. The potential acts non-trivially on all state functions in $L_2(\mathbb{R}^3)$, not equal to zero on the plate. The Schrödinger evolution of the electron is still described by a path ψ_t in the Hilbert space. However, at this time, the shape of the function ψ_t is different. After interaction with the potential the state function is a superposition $c_1\psi_1 + c_2\psi_2$, where the packets ψ_1, ψ_2 represent the state of the electron passing though one of the slits with the second slit closed. The resulting superposition continues propagating in the same direction, forming a path ψ_t.

What happens at this step is very important. Let us describe the motion of the state in terms of the Schrödinger evolution on the space of states $H = L_2(\mathbb{R}^3) \otimes L_2$ of the electron-plate

system. The state $\Psi_t = \psi_t \otimes \psi_P$ of the system propagates along the classical space submanifold $M_3^\sigma \otimes \psi_P = \mathbb{R}^3$ in the space H or, equivalently, the projective space CP^H with the induced Fubini-Study metric. On interaction with the potential representing the plate, the state ψ_t evolves into a superposition $c_1\psi_{1t} + c_2\psi_{2t}$. In terms of the geometry on the space of states, the path ψ_t is no longer valued in the classical space submanifold M_3^σ in $L_2(\mathbb{R}^3)$. In fact, the classical space submanifold is formed by the Gaussian states. Those states have a single "hump", while ψ_t behind the plate is a "double-humped" state function. As the state interacts with the plate, the distance from the state to the the classical space submanifold $\mathbb{R}^3$ increases.

Using the identification of the state ψ_P with the set P in M_3^σ of states of all particles of the plate, we can also view the entire process of interaction with the plate in a single space $L_2(\mathbb{R}^3)$. The set P is the image of the plate in $\mathbb{R}^3$ under the isomorphism ω_σ of section 3. The classical part of the experiment can be formulated within the submanifold M_3^σ alone and consists of the electron state (a point on, or near M_3^σ) approaching the plate P. Under interaction with the plate, the state of the electron becomes $\psi_t = c_1\psi_{1t} + c_2\psi_{2t}$. Even when the states ψ_{1t} and ψ_{2t} are Gaussian states, representing the points in the classical space M_3^σ, the superposition is not a Gaussian state. Therefore, the superposition at time t is *not* a point in the classical space. As the result of interaction with the plate, the path ψ_t moves away from the classical space M_3^σ and, therefore, passes *over* the plate with the slits (which is a subset P of M_3^σ, in this representation).

The origin of the paradox of the double-slit experiment is now clear. When trying to view the dynamics of the electron in the experiment within the classical space $M_3^\sigma = \mathbb{R}^3$, we are facing the dilemma: which slit did the electron go through? When formulated in these terms, the only correct answer seems to be that it went "through both" or to admit that position is not defined. This violently clashes with everything we know about the world around us and contradicts Newtonian mechanics. It forces us to think of the electron in terms of some kind of "electron cloud" that can "assemble" back to the particle (collapse) when measured. Alternatively, that the answer to Einstein's question - "is the moon there, when nobody looks?", - must be negative, at least for the electrons.

Under the Schrödinger dynamics, the evolution of the electron is a path ψ_t in the Hilbert space. It is a path in the usual sense; a continuous and single-valued function of time with values in $L_2(\mathbb{R}^3)$. When the state is constrained to M_3^σ, ψ_t is the usual path of a macroscopic particle in Newtonian dynamics. When the electron interacts with the plate, the path continues into the Hilbert space. Because the path can be written now as a sum $\psi_t = c_1\psi_{1t} + c_2\psi_{2t}$, we tend to think that both parts, ψ_{1t} and ψ_{2t} are real, so that the path of the electron splits into the paths that go through slits 1 and 2. This is paradoxical and contradictory. In fact, if the same wave function is written as a superposition of eigenstates of a different observable, then, by the same logic, the new components must be real as well. Since there are many observables, the notion of reality becomes ill-defined. The way out is to accept that the adequate way to describe the reality is by the vector ψ_t and not by its components ψ_{1t} and ψ_{2t}, that depend on the choice of a basis. When the state function belongs to the submanifold M_3^σ, the electron behaves classically. In general, however, the state of the electron is not confined to M_3^σ and satisfies the Schrödinger equation.

The issue of reality of the components ψ_{1t} and ψ_{2t} is similar to the following question in classical physics. When a physical vector (say, a velocity vector) is written in terms of its components in a certain basis, should we count the components as real? The answer is obvious: the physical vector itself is real because it is basis independent. However, the components of the vector are just shadows of the real thing as they change with the change of basis, similar to the way a shadow changes when the source of light is moved around. Our problem with the superposition principle is rooted in the desire to attach to the classical components like ψ_{1t} and ψ_{2t} the status of a "real thing". The paradox of the superposition is resolved by accepting the total state ψ_t as an adequate description of reality, while considering ψ_{1t} and ψ_{2t} for what

they really are: representation dependent components of the vector ψ_t. To answer Einstein's question: The moon and the electron *are* there, when nobody looks. Their existence is described by the state, at any time and not just when the object is measured. Whenever the state belongs to the classical space $\mathbb{R}^3 = M_3^\sigma$, it describes the usual classical existence in the Newtonian sense. But unlike the classical position, the state also catches the quantum origin of nature.

Suppose that position of the electron is measured by the screen behind the plate. As discussed in sections 9 and 11, a measurement of position induces a diffusion on the projective space of states. If the initial state of the electron was ψ, the density of states functional at the point $\tilde{\delta}^3_{\mathbf{a}}$ was shown to be $|\psi(\mathbf{a})|^2$. Because the state is a superposition of two states that describe the electron passing through one of the slits, the density of states functional contains the cross term. This term in the density results in an alternating probability of reaching different parts of the screen, producing a typical interference picture on the screen.

What happens when we place a source of light between the plate with the slits and the screen? In this case, the diffusion of the electron state begins earlier. After passing through the plate, the electron state is "two-humped". In particular, this initial state of the electron is positioned away from M_3^σ. Suppose that on interaction with the photons of the source of light, the electron is observed near one of the slits. That means, in particular, that the diffused electron state is on the classical space submanifold M_3^σ. So the state function of the electron observed near one of the slits must be "single-humped". The electron in such a Gaussian-like state is later observed on the screen. Clearly, no interference picture would appear on the screen.

What about a delayed-choice version of the experiment when we decide to determine which slit the electron went through *after* the electron has passed the plate with the slits? For instance, we could turn the light on after the electron went through the slits. The paradox is that the electron seems to "decide" retroactively to behave as a particle or a wave, and, accordingly, to go through one slit, or both, depending on our decision to turn the light on. However, the previous analysis is not altered by this change in the experiment. Whether or not the light source is present, the state of the electron after the slits is "two-humped". In particular, inserting a screen between the plate and the light source will show the interference pattern. When the light source is turned on and the electron is observed near one of the slits, the "two-humped" state is moved by the Brownian motion to a "single-humped, Gaussian-like position. As a result, the screen behind the light source will not show interference picture. As we know, the probability of finding the state by one slit or the other in the process of diffusion is given by the Born rule.

As before, we see that the paradox is due to our assumption that the electron must be on the classical space manifold M_3^σ at any time. In this case, the observed interference pattern signifies that the electron somehow "spreads out" over both slits and behaves like a wave. On the other hand, if the light source is on, then the electron visibly goes through one of the slits only and behaves like a particle. The paradox is resolved by accepting that evolution of the electron is described by a path ψ_t in the space of states CP^{L_2}. When the electron interacts with the plate, the path abandons the classical space submanifold M_3^σ in CP^{L_2}, the state function is "two-humped" and the interference picture is observable. When the source of light is turned on and the electron is observed by one of the slits, the path returns to the classical space, the state function is "single-humped" and the interference is not present. The moment when the light source is turned on is irrelevant. The nature of the electron does not change. In particular, the electron does not go back in time to "adjust" its nature depending on our decision to turn the light source on. The electron does not spread over the slits. Moreover, the electron *does not go through* the slits. If anything, it goes *over* the slits into the large dimensions of the space of states and *comes back* whenever its position is measured. This resolves the paradox of the double-slit experiment.

14. EPR experiment

The state of a pair of microscopic particles is an element of the tensor product Hilbert space $H = L_2(\mathbb{R}^3) \otimes L_2(\mathbb{R}^3)$. When positions of both particles are known, the state belongs to the submanifold $M_3^\sigma \otimes M_3^\sigma$ in CP^H. In section 12, a classical-mechanical version of an entangled state was discussed. It consists of two macroscopic particles connected by a weightless rigid rod, considered for simplicity in one dimension. If position of one particle in such a system is measured to be a, then position of the second particle is automatically known to be $a+d$, where d is the length of the rod. The state of the pair is then $\tilde{\delta}_a \otimes \tilde{\delta}_{a+d}$. A pair of microscopic particles in a superposition of such states is an example of an EPR pair. If momentum of the first particle in an EPR pair is found to be p, then the momentum of the second will be $-p$.

There are essentially two paradoxes associated with EPR-pairs. The first one consists of the non-local character of "communication" between the particles of the pair. Namely, how could a measurement performed on one particle instantaneously affect the other particle, no matter how far away? The other paradox is related to our ability to influence the reality of position or momentum of the second particle by choosing to measure either position or momentum of the first. This calls into question the notion of physical reality as well as completeness of quantum theory.

Similarly to the single particle case, the evolution of the pair is a path in the space of states CP^H. Whenever the path takes values in the submanifold $M_3^\sigma \otimes M_3^\sigma$, the position of both particles is known. Moreover, if the state is constrained to $M_3^\sigma \otimes M_3^\sigma$, then the Schrödinger dynamics of the pair is equivalent to the Newtonian one. As before, the constructed isomorphisms ω_n allow us to identify the state of the pair in $M_3^\sigma \otimes M_3^\sigma$ with a point in the configuration space $\mathbb{R}^3 \times \mathbb{R}^3$ of the system of two point-particles or positions of both particles in the classical space $\mathbb{R}^3$.

Suppose the state of the pair is a point on CP^H away from the submanifold $M_3^\sigma \otimes M_3^\sigma$. Suppose that position of one of the particles is measured. As discussed in section 12, the state of the system will undergo a diffusion in CP^H and the probability for the state of reaching a particular point in $M_3^\sigma \otimes M_3^\sigma$ is given by the Born rule. Note that position of only one of the particles needs to be measured for the state to be able to reach the manifold $M_3^\sigma \otimes M_3^\sigma$. Under the measurement, the state of the pair will undergo a random motion while following a continuous path ψ_t from the initial state to a point in $M_3^\sigma \otimes M_3^\sigma$.

It is important that the distance d between the points a and $a+d$ has nothing to do with the motion of the state ψ to an observed position state $\tilde{\delta}_a \otimes \tilde{\delta}_{a+d}$. The observed properties of one particle are not communicated to the other one by any signal or a field in space. Moreover, there are no particles in the sense of objects on $M_3^\sigma \otimes M_3^\sigma$, or on $M_3^\sigma = \mathbb{R}^3$. Rather, there is a state ψ_t representing the pair. When the state is constrained to $M_3^\sigma \otimes M_3^\sigma$, the particles are described by the classical Newtonian dynamics. So we can think of them in purely classical terms, as indeed, material points. However, the state in CP^H, not constrained to the classical space or phase space submanifolds describes the pair as a quantum object that embraces and supersedes the material point of Newtonian mechanics.

We see that the paradoxical "spooky action at a distance" is not present anymore. The state of the pair takes over the individual reality of the particles. There is no instantaneous collapse that somehow makes the pair "real". The pair is always real and exists in a form described by the state. For the state constrained to the classical phase space $M_6^\sigma \otimes M_6^\sigma$, we recover the usual classical-mechanical description of a pair of material points in $\mathbb{R}^3$. But in all cases, the state is an appropriate entity to describe physical characteristics of the observed world. The paradox of "creation" of reality of position or momentum of one particle by measuring the corresponding quantity of the second clears up as well. These physical characteristics only make sense for the state constrained to the manifold $M_6^\sigma \otimes M_6^\sigma$ and alike. In that particular case, their relation to the motion of state was derived in section 3. Otherwise, these physical characteristics are only

"shadows" of the deeper physics described by the state. The space of states is the new physical arena that extends the classical space. The state offers a more complete way of identifying characteristics of physical bodies. It generalizes the notion of position, momentum and other observed quantities and reproduces these quantities when constrained to an appropriate classical submanifold.

15. Schrödinger's cat paradox

The issue here is that the existence of entangled states of microscopic systems results in a contradiction when applied to macroscopic objects. In particular, in the famous Schrödinger thought experiment we get superpositions of states of a cat being alive and dead. As discussed in section 10, a macroscopic system is subjected to interaction with the environment, or, to put it differently, is "measured by the environment". From the same section we know that the state of the "measured" macroscopic system undergoes a trivial diffusion and so it does not change at all. The macroscopic object is therefore constrained to the classical space M^{σ}_{3n}. Assume that an entangled state of a cat and a decaying atom is somehow created. To measure the system is to measure one component of the system. For instance, when we check if the cat is alive, we also know that the atom has not decayed. The measured system is then in the product state. However, because the cat is always "measured" the state of the total atom-cat system is maintained in the product form. So, under the normal conditions the state of a microscopic and a macroscopic system cannot be entangled. There cannot be Schrödinger cats running around.

Note that the inconsistent view of reality by different observers in the Wigner's friend type of experiment, discovered by Frauchiger and Renner [24], is only present when an entanglement of microscopic and macroscopic objects is possible. As discussed, such an entanglement is not possible. At the same time, there is much more to be investigated now that the physical arena became the space of states. In particular, the notion of reality is altered for the objects not constrained to the classical space submanifold. We need to understand what it means in detail. Further, if the space is now a submanifold in the space of states, then what is an appropriate extension of the space-time manifold? How does it fit into the scheme? Would the results of relativity theory need to be changed? This requires further investigation.

16. Summary and experimental verification

The classical space and classical phase space of an arbitrary system of particles have been realized as submanifolds in the space of states of the system. This has been done in such a way that the Newtonian dynamics is the Schrödinger dynamics of the system whose state is constrained to the classical phase space submanifold. All physical processes happen in the space of states of the system, and not just in the classical space. Some process with microscopic particles begin on the classical space submanifold and continue into the Hilbert space of states. Some processes begin in the Hilbert space and end-up on the classical space submanifold. All dynamical processes (classical and quantum) are described by paths in the space of states. Any such path satisfies the Schrödinger equation. When the path takes values in the classical phase space, we see it as a classical motion, satisfying Newtonian equations of motion. The state of a particle that belongs to the classical phase space submanifold represents a classical material point. This correspondence makes the states physical or "real" and results in an identification of particles and states.

Interaction with the environment is capable of constraining the state of a macroscopic body to the classical space. The interaction is responsible for the normal distribution of the position of the body in the classical space. For a microscopic particle, the same interaction results in the Born rule for the probability of transition of the particle's state in the space of states. Physical collapse is not a process of "shrinking" of the "probability cloud" or the state itself in the classical space. Rather, it is a diffusive motion of the state on the space of states from the initial

to the end point. In particular, the state in a position measurement experiment undergoes a random walk from the initial state in the space of states to a point on the submanifold M_3^σ. The measuring device does not constrain the motion of state under measurement. It simply records the states that reach the set of eigenstates of the measured observable accessible to the measuring device. The measuring device is also inadvertently responsible for the random potential or the noise that results in a random walk of the state of a microscopic particle.

This new understanding of physical processes clarify the nature of quantum reality and addresses many of the paradoxes and conceptual difficulties that we have with the quantum. The "splitting" of the electron in the double-slit experiment is now represented by a usual, single-valued path of the electron in the space of states. The initial electron propagates along the classical space submanifold M_3^σ. Interaction with the screen makes the path of the electron's state bend. The path leaves then the submanifold and continues into the space of states. When we shine photons onto the electron, the path of the electron is described by a random walk. Under the walk, the path may return to the submanifold M_3^σ. Likewise, the state of an EPR pair of particles at any time is a point in the space of states of the pair. When the state is in the classical space or phase space submanifold, it represent a pair of classical particles at a certain fixed distance from each other. A superposition of such states of a determined position of the particles is a point in the space of states of the pair. The evolution of the pair is a path in the space of states. Under a measurement of position of one of the particles, the point representing the state of the pair undergoes a random walk and can reach the classical space submanifold. If this happens, the position of both particles in the classical space becomes known, without any "communication" between the particles.

The proposed realization of the Newtonian and Schrödinger dynamics in the space of states is more than a reformulation of the theory. In addition to reproducing the results of these theories, the realization also predicts new phenomena. In particular, the state under a measurement is predicted to undergo a random walk on the space of states. The state does not "converge" to the eigenstates of any particular observable but rather propagates isotropically into the space of states. In particular, if several devices measuring different observables are turned on at the same time, the state should "collapse" equally frequently to the eigenstates of these observables as long as they are at the same Fubini-Study distance from the initial state. This result goes against the existing models of collapse and can be checked, for instance, by measuring different components of the spin of a particle. A different experiment to check the validity of the proposed realization is related to testing the boundary between the classical and the quantum. According to the theory, this boundary is determined by the largest particles whose Brownian motion in an appropriate media is observable. In fact, as long as the Brownian motion for the particle is observable, the state of the particle will diffuse into the space of states. In particular, superpositions of the position eigenstates become possible and can be observed.

The obtained results strongly suggest that the macroscopic and the microscopic particles have the same nature. The macroscopic particles live in the classical space submanifold of the space of states and are maintained in the submanifold by their interaction with the environment. The microscopic particles are not constrained to the classical space submanifold and are free to move around the space of states, until their position or other observable is measured. Under a measurement, the state of a microscopic particle is capable of reaching the classical space submanifold so that its position at that time is known. All in all, this point of view signifies that we have to give up our attachment to the classical space and recognize that the space of quantum states is the actual arena for physical processes, both classical and quantum.

References

[1] Kryukov A 2005 *Int. J. Math. Math. Sci.* **2005** 2241

[2] Kryukov A 2007 *Phys. Lett.* A **370** 419

[3] Kryukov A 2017 *J. Math. Phys.* **58** 082103
[4] Kryukov A 2018 *J. Math. Phys.* **59** 052103
[5] Kryukov A 2019 *J. Phys.: Conf. Ser.* **1239** 012022
[6] Kryukov A 2019 *J. Phys.: Conf. Ser.* **1275** 012050
[7] Kryukov A 2020 *J. Math. Phys.* **61** 082101
[8] Gelfand I, Shilov G, Saletan E, Vilenkin N and Graev M 1964 *Generalized functions* (Academic Press, New York)
[9] Klauder J 2012 *J. Phys. A: Math. Theor.* **45** 285304
[10] Pearle P 1976 *Phys. Rev.* D **13** 857
[11] Pearle P 1999 *Collapse Models* in: Lecture Notes in Physics vol 526 (Berlin: Springer) p 195
[12] Ghirardi G, Rimini A and Weber T 1986 *Phys. Rev.* D **34** 470
[13] Ghirardi G, Pearle P and Rimini A 1990 *Phys. Rev.* A **42** 78
[14] Diósi L 1989 *Phys. Rev.* D **40** 1165
[15] Adler S and Horwitz L 2000 *J. Math. Phys.* **41** 2485
[16] Adler S, Brody D, Brun T and Hughston L 2001 *J. Phys.* A **34** 8795
[17] Adler S and Brun T 2001 *J. Phys.* A **34** 4797
[18] Adler S 2002 *J. Phys.* A **35** 841
[19] Adler S 2004 *Quantum theory as an emergent phenomenon* (Cambridge: Cambridge University Press)
[20] Hughston L 1996 *Proc. Roy. Soc. London* A 953
[21] Bassi A, Lochan K, Satin S, Singh T and Ulbricht H 2013 *Rev. Mod. Phys.* **85** 471
[22] Ullersma P 1966 *Physica* **32** 27
[23] Joos E, Giulini D, Kiefer C, Kupsch J and Stamatescu I 2003 *Decoherence and the appearance of a classical world in quantum theory* (Berlin: Springer)
[24] Frauchiger D and Renner R 2018 *Nature communications* **9** 3711
[25] Ullersma P 1966 *Physica* **32** 90