The physical limits of computation inspire an open problem that concerns abstract computable sets $X \subseteq \mathbb{N}$ and cannot be formalized in the set theory $ZFC$ as it refers to our current knowledge on $X$.

Sławomir Kurpaska, Apoloniusz Tyszka

Abstract. Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. Edmund Landau’s conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite. Landau’s conjecture implies the following unproven statement $\Phi$: card(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, f(7)]$. Let $B$ denote the system of equations: \{\begin{align*} x_i! &= x_k : i, k \in \{1,\ldots,9\} \\
 x_i \cdot x_j &= x_k : i, j, k \in \{1,\ldots,9\}\end{align*}\}. We write down a system $\mathcal{U} \subseteq B$ of 9 equations which has exactly two solutions in positive integers, namely $(1,\ldots,1)$ and $(f(1),\ldots,f(9))$. Let $\Psi$ denote the statement: if a system $S \subseteq B$ has at most finitely many solutions in positive integers $x_1,\ldots,x_9$, then each such solution $(x_1,\ldots,x_9)$ satisfies $x_1,\ldots,x_9 \leq f(9)$. We write down a system $\mathcal{A} \subseteq B$ of 8 equations. Theorem 1. The statement $\Psi$ restricted to the system $\mathcal{A}$ is equivalent to the statement $\Phi$. Open Problem. Is there a set $X \subseteq \mathbb{N}$ that satisfies conditions (1)–(5)? (1) There are many elements of $X$ and it is conjectured that $X$ is infinite. (2) No known algorithm decides the finiteness/infiniteness of $X$. (3) There is a known algorithm that for every $k \in \mathbb{N}$ decides whether or not $k \in X$. (4) There is a known algorithm that computes an integer $n$ satisfying $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty,n)$. (5) There is a naturally defined condition $C$, which can be formalized in ZFC, such that for almost all $k \in \mathbb{N}$, $k$ satisfies the condition $C$ if and only if $k \in X$. The simplest known such condition $C$ defines in $\mathbb{N}$ the set $X$. Condition (5) excludes artificially defined set $X$ from the statement (i). We prove: (i) the set $X = \{k \in \mathbb{N} : (f(7) < k) \Rightarrow (f(7),k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ satisfies conditions (1)–(4); (ii) the statement $\Phi$ implies that the set $X = \{1\} \cup \mathcal{P}_{n^2+1}$ satisfies conditions (1)–(5). Proving Landau’s conjecture will disprove the statements (i) and (ii). Theorem 2. No set $X \subseteq \mathbb{N}$ will satisfy conditions (1)–(4) forever, if for every algorithm with no inputs that operates on integers, at some future day, a computer will be able to execute this algorithm in 1 second or less. Physics disproves the assumption of Theorem 2.

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1. Basic definitions and the philosophical goal of the article

Logicism is a programme in the philosophy of mathematics. It is mainly characterized by the contention that mathematics can be reduced to logic, provided that the latter includes set theory, see [3, p. 199].

**Definition 1.** Conditions (1)–(5) concern sets \( X \subseteq \mathbb{N} \).

1. There are many elements of \( X \) and it is conjectured that \( X \) is infinite.
2. No known algorithm decides the finiteness/infiniteness of \( X \).
3. There is a known algorithm that for every \( k \in \mathbb{N} \) decides whether or not \( k \in X \).
4. There is a known algorithm that computes an integer \( n \) satisfying \( \text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n] \).
5. There is a naturally defined condition \( C \), which can be formalized in ZFC, such that for almost all \( k \in \mathbb{N} \), \( k \) satisfies the condition \( C \) if and only if \( k \in X \). The simplest known such condition \( C \) defines in \( \mathbb{N} \) the set \( X \).

Condition (5) excludes artificially defined set \( X \) from Statement 2.

**Definition 2.** We say that an integer \( n \) is a threshold number of a set \( X \subseteq \mathbb{N} \), if \( \text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n] \), cf. [7] and [8].

If a set \( X \subseteq \mathbb{N} \) is empty or infinite, then any integer \( n \) is a threshold number of \( X \). If a set \( X \subseteq \mathbb{N} \) is non-empty and finite, then the all threshold numbers of \( X \) form the set \( \{\max(X), \infty\} \cap \mathbb{N} \).

Edmund Landau’s conjecture states that the set \( \mathcal{P}_{n^2+1} \) of primes of the form \( n^2 + 1 \) is infinite, see [4]–[6].

**Definition 3.** Let \( \Phi \) denote the following unproven statement:

\[
\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, ((24!)!)!)!
\]

Landau’s conjecture implies the statement \( \Phi \). In Section 4 we heuristically justify the statement \( \Phi \) without invoking Landau’s conjecture.

**Statement 1.** No known algorithm computes an integer \( k \) such that \( \text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, k] \)

Proving the statement \( \Phi \) will disprove Statement 1. Statement 1 cannot be formalized in ZFC because it refers to the current mathematical knowledge. The same is true for Statements 2–3 and Open Problem 1 in the next sections. It argues against logicism as Open Problem 1 concerns abstract computable sets \( X \subseteq \mathbb{N} \).

2. The physical limits of computation inspire Open Problem 1

**Definition 4.** Let \( \beta = ((24!)!)! \).

**Lemma 1.** \( \log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta))))) \approx 1.42298. \)

**Proof.** We ask Wolfram Alpha at [http://wolframalpha.com](http://wolframalpha.com)
**Statement 2.** The set \( X = \{ k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset \} \) satisfies conditions (1)–(4).

**Proof.** Condition (1) holds as \( X \supseteq \{0, \ldots, \beta\} \) and the set \( \mathcal{P}_{n^2+1} \) is conjecturally infinite. By Lemma[1] due to known physics we are not able to confirm by a direct computation that some element of \( \mathcal{P}_{n^2+1} \) is greater than \( \beta \), see [2]. Thus condition (2) holds. Condition (3) holds trivially. Since the set

\[ \{ k \in \mathbb{N} : (\beta < k) \land (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset \} \]

is empty or infinite, the integer \( \beta \) is a threshold number of \( X \). Thus condition (4) holds.

\[ \square \]

Proving Landau’s conjecture will disprove Statement[2]

**Open Problem 1.** Is there a set \( X \subseteq \mathbb{N} \) that satisfies conditions (1)–(5)?

**Theorem 1.** No set \( X \subseteq \mathbb{N} \) will satisfy conditions (1)–(4) forever, if for every algorithm with no inputs that operates on integers, at some future day, a computer will be able to execute this algorithm in 1 second or less.

**Proof.** The proof goes by contradiction. Since conditions (2)–(4) will hold forever, the algorithm in Figure 1 never terminates and sequentially prints the following sentences:

\[ n + 1 \notin X, \ n + 2 \notin X, \ n + 3 \notin X, \ldots \] \( (T) \)

Fig. 1 An algorithm whose execution never terminates if the set \( X \) is finite

The sentences from the sequence \((T)\) and our assumption imply that for every integer \( m > n \) computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that \((n, m) \cap X = \emptyset \). Thus, at some future day, numerical evidence will support the conjecture that the set \( X \) is finite, contrary to the conjecture in condition (1).

\[ \square \]

Physics disproves the assumption of Theorem[1]
3. Number-theoretic statements $\Psi_n$

Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geq 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases}
x_1! = x_1 \\
x_1 \cdot x_1 = x_2 \\
\forall i \in \{2, \ldots, n-1\} \ x_i! = x_{i+1}
\end{cases}$$

The diagram in Figure 2 illustrates the construction of the system $\mathcal{U}_n$.

![Fig. 2 Construction of the system $\mathcal{U}_n$](image)

**Lemma 2.** For every positive integer $n$, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(n))$.

Let $B_n$ denote the following system of equations:

$$\{x_i! = x_k : i, k \in \{1, \ldots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\}$$

For a positive integer $n$, let $\Psi_n$ denote the following statement: if a system of equations $S \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leq f(n)$. The statement $\Psi_n$ says that for subsystems of $B_n$ with a finite number of solutions, the largest known solution is indeed the largest possible. The statements $\Psi_1$ and $\Psi_2$ hold trivially. There is no reason to assume the validity of the statement $\forall n \in \mathbb{N} \setminus \{0\} \ \Psi_n$.

**Theorem 2.** For every statement $\Psi_n$, the bound $f(n)$ cannot be decreased.

**Proof.** It follows from Lemma 2 because $\mathcal{U}_n \subseteq B_n$. □

**Theorem 3.** For every integer $n \geq 2$, the statement $\Psi_{n+1}$ implies the statement $\Psi_n$.

**Proof.** If a system $S \subseteq B_n$ has at most finitely many solutions in positive integers $x_1, \ldots, x_n$, then for every integer $i \in \{1, \ldots, n\}$ the system $S \cup \{x_i! = x_{n+1}\}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_{n+1}$. The statement $\Psi_{n+1}$ implies that $x_i! = x_{n+1} \leq f(n + 1) = f(n)!$. Hence, $x_i \leq f(n)$. □

**Theorem 4.** Every statement $\Psi_n$ is true with an unknown integer bound that depends on $n$.

**Proof.** For every positive integer $n$, the system $B_n$ has a finite number of subsystems. □
4. A conjectural solution to Open Problem 1

Lemma 3. For every positive integers $x$ and $y$, $x! \cdot y = y!$ if and only if 
\[(x + 1 = y) \lor (x = y = 1)\]

Lemma 4. (Wilson’s theorem, [1, p. 89]). For every integer $x \geq 2$, $x$ is prime if and only if $x$ divides $(x - 1)! + 1$.

Let $\mathcal{A}$ denote the following system of equations:

\[
\begin{align*}
    x_2! & = x_3 \\
    x_3! & = x_4 \\
    x_5! & = x_6 \\
    x_8! & = x_9 \\
    x_1 \cdot x_1 & = x_2 \\
    x_3 \cdot x_5 & = x_6 \\
    x_4 \cdot x_8 & = x_9 \\
    x_5 \cdot x_7 & = x_8 \\
\end{align*}
\]

Lemma [3] and the diagram in Figure 3 explain the construction of the system $\mathcal{A}$.

Fig. 3 Construction of the system $\mathcal{A}$
**Lemma 5.** For every integer $x_1 \geq 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:

\[
\begin{align*}
x_2 &= x_1^2 \\
x_3 &= (x_1!)! \\
x_4 &= ((x_1!)!)! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)! \\
x_7 &= \frac{x_1^3}{x_1^2 + 1} \\
x_8 &= (x_1^2)! + 1 \\
x_9 &= ((x_1^2)! + 1)!
\end{align*}
\]

**Proof.** By Lemma 4 for every integer $x_1 \geq 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1!)! + 1$. Hence, the claim of Lemma 5 follows from Lemma 4. \qed

**Lemma 6.** There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$. This is true as every such tuple $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \in \{1, 2\}$.

**Proof.** The equality $x_1 = 1$ implies that $x_2 = x_1 \cdot x_1 = 1$. Hence, $x_3 = x_2! = 1$. Therefore, $x_4 = x_3! = 1$. The equalities $x_5! = x_6$ and $x_5 = 1 \cdot x_5 = x_3 \cdot x_5 = x_6$ imply that $x_5, x_6 \in \{1, 2\}$. The equalities $x_8! = x_9$ and $x_8 = 1 \cdot x_8 = x_4 \cdot x_8 = x_9$ imply that $x_8, x_9 \in \{1, 2\}$. The equality $x_3 \cdot x_7 = x_8$ implies that $x_7 = \frac{x_8}{x_5} \in \{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{2}{2}\} \cap \mathbb{N} = \{1, 2\}$. \qed

**Conjecture 1.** The statement $\Psi_9$ is true when is restricted to the system $\mathcal{A}$.

**Theorem 5.** Conjecture 7 proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set $\mathcal{P}_{n^2+1}$ is infinite.

**Proof.** Suppose that the antecedent holds. By Lemma 5 there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{A}$. Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geq f(7)$. Hence, $(x_1^2)! \geq f(7)! = f(8)$. Consequently,

\[
x_9 = ((x_1^2)! + 1)! \geq (f(8) + 1)! > f(8)! = f(9)
\]

Conjecture 7 and the inequality $x_9 > f(9)$ imply that the system $\mathcal{A}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 5 and 6 the set $\mathcal{P}_{n^2+1}$ is infinite. \qed

**Theorem 6.** Conjecture 7 implies the statement $\Phi$.

**Proof.** It follows from Theorem 5 and the equality $f(7) = (((24!)!)!)!$. \qed
Theorem 7. The statement $\Phi$ implies Conjecture 1.

Proof. By Lemmas 5 and 6 if positive integers $x_1, \ldots, x_9$ solve the system $\mathcal{A}$, then

$$(x_1 \geq 2) \land (x_5 = x_1^2 + 1) \land (x_5 \text{ is prime})$$

or $x_1, \ldots, x_9 \in \{1, 2\}$. In the first case, Lemma 5 and the statement $\Phi$ imply that the inequality $x_5 \leq (((24!)!)!)! = f(7)$ holds when the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_9$. Hence, $x_2 = x_5 - 1 < f(7)$ and $x_3 = x_2! < f(7)! = f(8)$. Continuing this reasoning in the same manner, we can show that every $x_i$ does not exceed $f(9)$.

Statement 3. The statement $\Phi$ implies that the set $X = \{1\} \cup \mathcal{P}_{n^2+1}$ satisfies conditions (1)-(5).

Proof. The set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. There are 219984223892 primes of the form $n^2 + 1$ in the interval $[2, 10^{28}]$, see [5]. These two facts imply condition (1). By Lemma 1 due to known physics we are not able to confirm by a direct computation that some element of $\{1\} \cup \mathcal{P}_{n^2+1}$ is greater than $f(7) = (((24!)!)!)! = \beta$, see [2]. Thus condition (2) holds. Condition (3) holds trivially. The statement $\Phi$ implies that $\beta$ is a threshold number of $X = \{1\} \cup \mathcal{P}_{n^2+1}$. Thus condition (4) holds. The following condition:

$$k - 1 \text{ is a square and } k \text{ has no divisors greater than } 1 \text{ and smaller than } k$$

defines in $\mathbb{N}$ the set $\{1\} \cup \mathcal{P}_{n^2+1}$. This proves condition (5).

Proving Landau’s conjecture will disprove Statement 3.

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References


Sławomir Kurpaska
Technical Faculty
Hugo Kołłątaj University
Balicka 116B, 30-149 Kraków, Poland
E-mail: rtkurpas@cyf-kr.edu.pl

Apoloniusz Tyszka
Technical Faculty
Hugo Kołłątaj University
Balicka 116B, 30-149 Kraków, Poland
E-mail: rttyszka@cyf-kr.edu.pl