The physical impossibility of machine computations on sufficiently large integers inspires an open problem that concerns abstract computable sets $X \subseteq \mathbb{N}$ and cannot be formalized in the set theory ZFC as it refers to our current knowledge on $X$.

Sławomir Kurpaska, Apoloniusz Tyszka

Abstract. Edmund Landau’s conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite. Let $\beta = (((24!)!))!$, and let $\Phi$ denote the implication: $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, \beta]$. We heuristically justify the statement $\Phi$ without invoking Landau’s conjecture. The set $X = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$ satisfies conditions (1)–(4).

(1) There are a large number of elements of $X$ and it is conjectured that $X$ is infinite. (2) No known algorithm decides the finiteness/infiniteness of $X$. (3) There is a known algorithm that for every $n \in \mathbb{N}$ decides whether or not $n \in X$. (4) There is an explicitly known integer $n$ such that $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$. (5) There is an explicitly known integer $n$ such that $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$ and some known definition of $X$ is much simpler than every known definition of $X \setminus (-\infty, n]$. The following problem is open: Is there a set $X \subseteq \mathbb{N}$ that satisfies conditions (1)–(3) and (5)? The set $X = \mathcal{P}_{n^2+1}$ satisfies conditions (1)–(3). Let $\lfloor \cdot \rfloor$ denote the integer part function. For every explicitly given integer $m \geq 1$, the set $X = \left\{ k \in \mathbb{N} : \left\lfloor \frac{k}{m} \right\rfloor^2 + 1 \text{ is prime} \right\}$ contains $m$ consecutive integers and satisfies conditions (1)–(3). The statement $\Phi$ implies that both sets $X$ satisfy condition (5).

Key words and phrases: complexity of a mathematical definition, computable set $X \subseteq \mathbb{N}$, current knowledge on $X$, explicitly known integer $n$ bounds $X$ from above when $X$ is finite, infiniteness of $X$ remains conjectured, known algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in X$, large number of elements of $X$, mathematical statement that cannot be formalized in the set theory ZFC, no known algorithm decides the finiteness/infiniteness of $X$, physical impossibility of machine computations on sufficiently large integers.
1. Basic definitions and the goal of the article

Logicism is a programme in the philosophy of mathematics. It is mainly characterized by the contention that mathematics can be reduced to logic, provided that the latter includes set theory, see [3, p. 199].

**Definition 1.** Conditions (1)–(5) concern sets $X \subseteq \mathbb{N}$.

1. There are a large number of elements of $X$ and it is conjectured that $X$ is infinite.
2. No known algorithm decides the finiteness/infiniteness of $X$.
3. There is a known algorithm that for every $n \in \mathbb{N}$ decides whether or not $n \in X$.
4. There is an explicitly known integer $n$ such that $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$.
5. There is an explicitly known integer $n$ such that $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$ and some known definition of $X$ is much simpler than every known definition of $X \setminus (-\infty, n]$.

**Definition 2.** We say that an integer $n$ is a threshold number of a set $X \subseteq \mathbb{N}$, if $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$, cf. [6] and [7].

If a set $X \subseteq \mathbb{N}$ is empty or infinite, then any integer $n$ is a threshold number of $X$. If a set $X \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $X$ form the set $\{\max(X), \infty\} \cap \mathbb{N}$.

Edmund Landau’s conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite, see [4] and [5].

**Definition 3.** Let $\Phi$ denote the implication:

$$\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, (((24!)!)!)!)$$

Landau’s conjecture implies the statement $\Phi$. In Section 4 we heuristically justify the statement $\Phi$ without invoking Landau’s conjecture.

**Statement 1.** There is no explicitly known threshold number of $\mathcal{P}_{n^2+1}$. It means that there is no explicitly known integer $k$ such that $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, k]$.

Proving the statement $\Phi$ will falsify Statement 1. Statement 1 cannot be formalized in the set theory $\text{ZFC}$ because it refers to the current mathematical knowledge. The same is true for Statements 2–4 and Open Problem 1 in the next sections. It argues against logicism as Open Problem 1 concerns abstract computable sets $X \subseteq \mathbb{N}$. 

2. The physical impossibility of machine computations on sufficiently large integers inspires Open Problem 1

**Definition 4.** Let $\beta = (((24!)!)!)!$.

**Lemma 1.** $\beta \approx 10^{10^{10^{25.16114896940657}}}$.

**Proof.** We ask Wolfram Alpha at [http://wolframalpha.com](http://wolframalpha.com). □

**Statement 2.** The set $X = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap P_{n^2+1} \neq \emptyset\}$ satisfies conditions (1)–(4).

**Proof.** Condition (1) holds as $X \supseteq \{0, \ldots, \beta\}$ and the set $P_{n^2+1}$ is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $P_{n^2+1}$ is greater than $\beta$, see [2]. Thus condition (2) holds. Condition (3) holds trivially. Since the set

\[ \{k \in \mathbb{N} : (\beta < k) \land (\beta, k) \cap P_{n^2+1} \neq \emptyset\} \]

is empty or infinite, the integer $\beta$ is a threshold number of $X$. Thus condition (4) holds. □

In Statement 2,

\[ \text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, \beta] \]

and the sets

$X = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap P_{n^2+1} \neq \emptyset\}$

and

$X \setminus (-\infty, \beta] = \{k \in \mathbb{N} : (\beta < k) \land (\beta, k) \cap P_{n^2+1} \neq \emptyset\}$

have definitions of similar complexity. The following problem arises:

**Open Problem 1.** Is there a set $X \subseteq \mathbb{N}$ that satisfies conditions (1)–(3) and (5)?

**Theorem 1.** Assume that for every positive integers $b$ and $s$, at some future day, machine computations will be possible on every integers from the interval $[-b, b]$ and this will be possible with the speed of $s$ FLOPS. These assumptions contradict the current paradigm of physics, although they alone have no consequences in mathematics formalized in ZFC. We claim that our assumptions alone imply that no set $X \subseteq \mathbb{N}$ will satisfy conditions (1)–(4) forever.

**Proof.** The proof goes by contradiction. Since conditions (2)–(4) will hold forever, the algorithm in Figure 1 never terminates and sequentially prints the following sentences:

\[ n + 1 \notin X, \ n + 2 \notin X, \ n + 3 \notin X, \ldots \] (T)
The sentences from the sequence (T) and our assumptions alone imply that for every explicitly given integer \( m > n \), at some future day, a computer will be able to confirm in 1 second or less that \( (n, m) \cap X = \emptyset \). Thus, at some future day, numerical evidence will support the conjecture that the set \( X \) is finite, contrary to the conjecture in condition (1). \( \square \)

3. Number-theoretic statements \( \Psi_n \)

Let \( f(1) = 2 \), \( f(2) = 4 \), and let \( f(n + 1) = f(n)! \) for every integer \( n \geq 2 \). Let \( \mathcal{U}_1 \) denote the system of equations which consists of the equation \( x_1! = x_1 \). For an integer \( n \geq 2 \), let \( \mathcal{U}_n \) denote the following system of equations:

\[
\begin{align*}
x_1! &= x_1 \\
x_1 \cdot x_1 &= x_2 \\
\forall i \in \{2, \ldots, n-1\}, \; x_i! &= x_{i+1}
\end{align*}
\]

The diagram in Figure 2 illustrates the construction of the system \( \mathcal{U}_n \).

**Fig. 2** Construction of the system \( \mathcal{U}_n \)

**Lemma 2.** For every positive integer \( n \), the system \( \mathcal{U}_n \) has exactly two solutions in positive integers, namely \((1, \ldots, 1)\) and \((f(1), \ldots, f(n))\).
Let
\[ B_n = \{ x_i! = x_k : i, k \in \{1, \ldots, n\} \} \cup \{ x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\} \} \]
For a positive integer \( n \), let \( \Psi_n \) denote the following statement: if a system of equations \( S \subseteq B_n \) has at most finitely many solutions in positive integers \( x_1, \ldots, x_n \), then each such solution \( (x_1, \ldots, x_n) \) satisfies \( x_1, \ldots, x_n \leq f(n) \). The statement \( \Psi_n \) says that for subsystems of \( B_n \) with a finite number of solutions, the largest known solution is indeed the largest possible. The statements \( \Psi_1 \) and \( \Psi_2 \) hold trivially. There is no reason to assume the validity of the statement \( \Psi_9 \), cf. Conjecture 1 in Section 4.

Theorem 2. For every statement \( \Psi_n \), the bound \( f(n) \) cannot be decreased.

Proof. It follows from Lemma 2 because \( \mathcal{U}_n \subseteq B_n \). \( \square \)

Theorem 3. For every integer \( n \geq 2 \), the statement \( \Psi_{n+1} \) implies the statement \( \Psi_n \).

Proof. If a system \( S \subseteq B_n \) has at most finitely many solutions in positive integers \( x_1, \ldots, x_n \), then for every integer \( i \in \{1, \ldots, n\} \) the system \( S \cup \{ x_i! = x_{n+1} \} \) has at most finitely many solutions in positive integers \( x_1, \ldots, x_{n+1} \). The statement \( \Psi_{n+1} \) implies that \( x_i! = x_{n+1} \leq f(n + 1) = f(n)! \). Hence, \( x_i \leq f(n) \). \( \square \)

Theorem 4. Every statement \( \Psi_n \) is true with an unknown integer bound that depends on \( n \).

Proof. For every positive integer \( n \), the system \( B_n \) has a finite number of subsystems. \( \square \)

4. A conjectural solution to Open Problem 1

Lemma 3. For every positive integers \( x \) and \( y \), \( x! \cdot y = y! \) if and only if \( (x + 1 = y) \lor (x = y = 1) \)

Lemma 4. (Wilson’s theorem, [1, p. 89]). For every integer \( x \geq 2 \), \( x \) is prime if and only if \( x \) divides \( (x - 1)! + 1 \).

Let \( \mathcal{A} \) denote the following system of equations:
\[
\begin{align*}
x_2! &= x_3 \\
x_3! &= x_4 \\
x_5! &= x_6 \\
x_8! &= x_9 \\
x_1 \cdot x_1 &= x_2 \\
x_3 \cdot x_5 &= x_6 \\
x_4 \cdot x_8 &= x_9 \\
x_5 \cdot x_7 &= x_8
\end{align*}
\]

Lemma 3 and the diagram in Figure 3 explain the construction of the system \( \mathcal{A} \).
Lemma 5. For every integer \( x_1 \geq 2 \), the system \( A \) is solvable in positive integers \( x_2, \ldots, x_9 \) if and only if \( x_1^2 + 1 \) is prime. In this case, the integers \( x_2, \ldots, x_9 \) are uniquely determined by the following equalities:

\[
\begin{align*}
x_2 &= x_1^2 \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_1^2)!)! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)! \\
x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 &= ((x_1^2)! + 1)! \\
x_9 &= (((x_1^2)! + 1))!
\end{align*}
\]

Proof. By Lemma 3, for every integer \( x_1 \geq 2 \), the system \( A \) is solvable in positive integers \( x_2, \ldots, x_9 \) if and only if \( x_1^2 + 1 \) divides \((x_1^2)! + 1\). Hence, the claim of Lemma 5 follows from Lemma 4.

Lemma 6. There are only finitely many tuples \((x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9\), which solve the system \( A \) and satisfy \( x_1 = 1 \). This is true as every such tuple \((x_1, \ldots, x_9)\) satisfies \( x_1, \ldots, x_9 \in \{1, 2\} \).

Proof. The equality \( x_1 = 1 \) implies that \( x_2 = x_2^2 = 1 \). Hence, for example, \( x_3 = x_2! = 1 \). Therefore, \( x_8 = x_3 + 1 = 2 \) or \( x_8 = 1 \). Consequently, \( x_9 = x_8! \leq 2 \).

Conjecture 1. The statement \( \Psi_9 \) is true when is restricted to the system \( A \).
Theorem 5. Conjecture \([1]\) proves the following implication: if there exists an integer \(x_1 \geq 2\) such that \(x_1^2 + 1\) is prime and greater than \(f(7)\), then the set \(P_{n^2+1}\) is infinite.

Proof. Suppose that the antecedent holds. By Lemma \([5]\) there exists a unique tuple \((x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8\) such that the tuple \((x_1, x_2, \ldots, x_9)\) solves the system \(\mathcal{A}\). Since \(x_1^2 + 1 > f(7)\), we obtain that \(x_1^2 > f(7)\). Hence, \((x_1^2)! > f(7)! = f(8)\). Consequently, \(x_9 = ((x_1^2)! + 1)! > (f(8) + 1)! > f(8)! = f(9)\). The inequality \(x_9 > f(9)\) implies that the system \(\mathcal{A}\) has infinitely many solutions \((x_1, \ldots, x_9)\) \(\in (\mathbb{N} \setminus \{0\})^9\). According to Lemmas \([5]\) and \([6]\) the set \(P_{n^2+1}\) is infinite. \(\square\)

Theorem 6. Conjecture \([7]\) implies the statement \(\Phi\).

Proof. It follows from Theorem \([5]\) and the equality \(f(7) = (((24!)!)!)!\). \(\square\)

Theorem 7. The statement \(\Phi\) implies Conjecture \([7]\).

Proof. By Lemmas \([5]\) and \([6]\) if positive integers \(x_1, \ldots, x_9\) solve the system \(\mathcal{A}\), then \((x_1 \geq 2) \land (x_5 = x_1^2 + 1) \land (x_5\text{ is prime})\)

or \(x_1, \ldots, x_9 \in \{1, 2\}\). In the first case, Lemma \([5]\) and the statement \(\Phi\) imply that the inequality \(x_5 \leq (((24!)!)!)! = f(7)\) holds when the system \(\mathcal{A}\) has at most \(n\) solutions in positive integers \(x_1, \ldots, x_9\). Hence, \(x_2 = x_5 - 1 < f(7)\) and \(x_3 = x_2! < f(7)! = f(8)\). Continuing this reasoning in the same manner, we can show that every \(x_i\) does not exceed \(f(9)\). \(\square\)

Statement 3. The set \(X = P_{n^2+1}\) satisfies conditions (1)–(3). The statement \(\Phi\) implies that the set \(X\) satisfies condition (5).

Proof. Since the set \(P_{n^2+1}\) is conjecturally infinite, condition (1) holds for \(X\). Condition (3) holds trivially. By Lemma \([1]\) due to known physics we are not able to confirm by a direct computation that some element of \(P_{n^2+1}\) is greater than \(f(7) = (((24!)!)!)! = \beta\), see \([2]\). Thus condition (2) holds for \(X\). Suppose that the statement \(\Phi\) holds. This implies that \(\beta\) is a threshold number of \(X = P_{n^2+1}\). Thus condition (4) holds for \(X\). The definition of \(P_{n^2+1}\) is much simpler than the definition of \(P_{n^2+1} \setminus (-\infty, \beta]\). The last two sentences imply that condition (5) holds for \(X\). \(\square\)

Let \([-\cdot]\) denote the integer part function.

Statement 4. For every explicitly given integer \(m \geq 1\), the set \(X = \{k \in \mathbb{N} : \left\lfloor \frac{k}{m} \right\rfloor^2 + 1\text{ is prime} \}\) contains \(m\) consecutive integers and satisfies conditions (1)–(3). The statement \(\Phi\) implies that the set \(X\) satisfies condition (5).

Proof. The set \(X\) contains \(m\) consecutive integers because the number 2 is prime and the equality \(\left\lfloor \frac{k}{m} \right\rfloor^2 + 1 = 2\) holds for every integer \(k \in \{m, \ldots, 2m - 1\}\). The rest of the proof goes as in the proof of Statement \([3]\) although the statement \(\Phi\) allows us to compute a threshold number of \(X\) that depends on \(m\). \(\square\)
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References


Sławomir Kurpaska
Technical Faculty
Hugo Kołłątaj University
Balicka 116B, 30-149 Kraków, Poland
E-mail: rtkurpas@cyf-kr.edu.pl

Apoloniusz Tyszka
Technical Faculty
Hugo Kołłątaj University
Balicka 116B, 30-149 Kraków, Poland
E-mail: rttyszka@cyf-kr.edu.pl