Quantum Physics in Non-Separable Hilbert Spaces

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In *Mathematical Foundations of Quantum Mechanics* (1932) von Neumann made separability of Hilbert space an axiom. Subsequent work in mathematics (some of it by von Neumann himself) investigated non-separable Hilbert spaces, and mathematical physicists have sometimes made use of them. This note discusses some of the problems that arise in trying to treat quantum systems with non-separable spaces. Some of the problems are "merely technical" but others point to interesting foundations issues for quantum theory, both in its abstract mathematical form and its applications to physical systems. Nothing new or original is attempted here. Rather this note aims to bring into focus some issues that have for too long remained on the edge of consciousness for philosophers of physics.

1 Introduction

A Hilbert space $\mathcal{H}$ is separable if there is a countable dense set of vectors; equivalently, $\mathcal{H}$ admits a countable orthonormal basis. In *Mathematical Foundations of Quantum Mechanics* (1932, 1955) von Neumann made separability one of the axioms of his codification of the formalism of quantum mechanics. Working with a separable Hilbert space certainly simplifies matters and provides for understandable realizations of the Hilbert space axioms: all infinite dimensional separable Hilbert spaces are the "same": they are isomorphically isometric to $L^2_{\mathbb{C}}(\mathbb{R})$, the space of square integrable complex valued functions of $\mathbb{R}$ with inner product $\langle\psi,\phi\rangle := \int \overline{\psi}(x)\phi(x)dx$, $\psi, \phi \in L^2_{\mathbb{C}}(\mathbb{R})$. This Hilbert space in turn is isomorphically isometric to $\ell^2_{\mathbb{C}}(\mathbb{N})$, the vector space of square summable sequences $\mathcal{S}$ of complex numbers with inner product $\langle\psi,\phi\rangle := \sum_{n=1}^{\infty} x_n^* y_n$, $\psi = (x_1, x_2, ...), \phi = (y_1, y_2, ...) \in \mathcal{S}$. And all finite dimensional Hilbert spaces of dim N are the "same": they are isomorphically isometric to $\mathbb{C}^N$. But unless one believes that (quantum) Nature must be simple, a better reason for restricting to separable Hilbert spaces must be found. When *Mathematical Foundations* was published in 1932 it was certainly true that separable spaces sufficed for all of the extant applications of ordinary non-relativistic QM. But again this hardly justifies limiting future applications of the theory to those that can be treated using separable spaces. Did von Neumann suspect that something important would go awry in quantum theory if the axiom of separability were dropped, or was he simply being cautious and conservative? It will have to be left to the historians to supply an answer.[1]

It is useful at the beginning to have concrete examples where a non-separable Hilbert space comes into play in quantum theory. We will discover in the following section that unproblematic examples are hard to come by. Section 3 surveys properties of separable spaces that carry over directly to non-separable spaces as well as other properties that carry over with suitable modifications. Section 4 details cases of non-salvageable failures to carry over. The troubles that these failures make for applications of quantum theory in non-separable spaces are discussed in Section 5. Conclusions are contained in Section 6.

[1] Most likely the latter. As documented by Miklós Rédei, von Neumann thought of Hilbert space as a generalization of Euclidean space wherein the existence of a finite basis is replaced by a "minimum of topological assumptions (completeness + separability)"; see Rédei (2007, 115-116). It was not long after the publication of von Neumann's *Mathematical Foundations* that von Neumann's notion of Hilbert space was extended to the non-separable case; see Löwig (1934a, 1934b) and Rellich (1935).

2 Non-separability in applications of quantum theory[2]

For many physicists the appearance of non-separable Hilbert spaces in an application of quantum theory sets off alarm bells. One of our tasks here is to understand and assess this reaction. One idea motivating the alarm bells is that "A realistic space of distinguishable quantum states should be described by a separable Hilbert space" (Fairbairn and Rovelli 2004, p. 2808). The supporting reference that Fairbairn and Rovelli cite is Streater and Wightman's (1964) *PCT, Spin and Statistics, and All That*, a standard go-to reference used to brace those who might be tempted by the non-separable. That is a good place to begin our inquiry.

2.1 Streater, Wightman, and all that

Streater and Wightman opine that it is "wrong, or at best grossly misleading" (p. 86) to hold that quantum field theory (QFT) must employ a non-separable Hilbert space because it deals with a infinite number of degrees of freedom. This is certainly true for Wightman's original formulation of

[2]Terminology: In what follows a 'state' on a von Neumann algebra $\mathfrak{N}$ acting on a Hilbert space $\mathcal{H}$ is a normed positive linear functional ω on $\mathfrak{N}$. That ω is normal means that ω admits a density operator representation on $\mathcal{H}$. Equivalent characterizations of normality are that ω is completely additive on any family of mutually orthogonal projections in $\mathfrak{N}$, or ω is ultraweakly continuous. The state ω is a vector state on $\mathfrak{N}$ just in case there is a $\xi \in \mathcal{H}$ such that $\omega(A) = \langle \xi, A\xi \rangle$ for all $A \in \mathfrak{N}$. Vector states are normal. The state ω is mixed just in case there are states $\phi_1 \neq \phi_2$ and real numbers λ_1,λ_2 such that $0 < \lambda_1,\lambda_2 < 1$, $\lambda_1 + \lambda_2 = 1$, and $\omega = \lambda_1\phi_1 + \lambda_2\phi_2$. Otherwise ω is said to be pure. A state ω on $\mathfrak{N}$ is faithful in case $\omega(A) = 0$ implies $A = 0$ for any positive $A \in \mathfrak{N}$. The commutant $\{A\}'$ of a set of operators $\{A\}$ acting on $\mathcal{H}$ consists of all bounded operators that commute with the As. The center $\mathcal{Z}(\mathfrak{N})$ of a von Neumann algebra $\mathfrak{N}$ is $\mathfrak{N} \cap \mathfrak{N}'$. $\mathfrak{N}$ is a factor algebra if its center is trivial, i.e. $\mathcal{Z}(\mathfrak{N}) = \{cI\}$, $c \in \mathbb{C}$. The double commutant $\mathfrak{N}'' := (\mathfrak{N}')'$ of a von Neumann algebra is just $\mathfrak{N}$ itself. This section deals mainly with the simplest von Neumann algebra, the Type I factor $\mathfrak{B}(\mathcal{H})$, the algebra of all bounded operators acting on $\mathcal{H}$. Type I non-factors come into play when there are superselection rules; see below. A C^*-algebra $\mathcal{A}$ is an algebra with an involution ($*$-operation) and a norm. A representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a $*$-homomorphism $\pi : \mathcal{A} \to \mathfrak{B}(\mathcal{H})$. The weak closure of $\pi(\mathcal{A})$ or equalivalently (by von Neumann's double commutant theorem) the double commutant $\pi(\mathcal{A})''$ is a von Neumann algebra. The reader is referred to Bratelli and Robinson (1987) for a comprehensive survey of the application of von Neumann and C^*-algebras in quantum physics.

QFT, which not only does not require a non-separable Hilbert space but also entails separability. For it is assumed that the vacuum vector is cyclic[3] and that quantum fields are operator valued tempered distributions in $\mathcal{H}$. This implies that $\mathcal{H}$ is separable since the test function space is separable. Other test function spaces can be contemplated (see Wightman 1981). But the use of a non-separable test function space would potentially bring the Wightman theory into conflict with his assumption that the vacuum vector is separating as well as cyclic (see Section 4.1 below).[4]

Streater and Wightman also promote the widely shared notion that QFT deals with an indefinitely large number of particles (or excitations of the quantum field)—a *potential* rather than an *actual* infinity of particles. The Fock space used to treat this potential infinity uses a direct sum construction, which I will now briefly describe. The direct sum $\oplus_{\alpha\in\mathcal{I}}\mathcal{H}_\alpha$ space of the Hilbert spaces $\mathcal{H}_\alpha$ is defined for an index set $\mathcal{I}$ that may be finite, denumerable, or non-denumerable. It consists of vectors $\oplus_\vartheta := \oplus_{\alpha\in\mathcal{I}}\vartheta_\alpha$ defined by a family $\vartheta := \{\vartheta_\alpha\}$, $\alpha \in \mathcal{I}$ and $\vartheta_\alpha \in \mathcal{H}_\alpha$, provided that $\sum_{\alpha\in\mathcal{I}} ||\vartheta_\alpha||_{\mathcal{H}_\alpha} < \infty$.[5] The rules for scalar multiplication and vector addition can be summarized in one rule $c\oplus_\vartheta + d\oplus_\xi = c\oplus_{\alpha\in\mathcal{I}}\vartheta_\alpha + d\oplus_{\alpha\in\mathcal{I}}\xi_\alpha = \oplus_{\alpha\in\mathcal{I}}(c\vartheta_\alpha + d\xi_\alpha)$, $c, d \in \mathbb{C}$. This direct sum space is complete in the norm derived from the inner product $\langle\oplus_\vartheta, \oplus_\xi\rangle := \sum_{\alpha\in\mathcal{I}}\langle\vartheta_\alpha, \xi_\alpha\rangle_{\mathcal{H}_\alpha}$ and is, therefore, a Hilbert space. If $\dim(\mathcal{H}_\alpha) = D$ for all α then $\dim(\oplus_{\alpha\in\mathcal{I}}\mathcal{H}_\alpha) = D \cdot |\mathcal{I}|$. In particular, if the index set $\mathcal{I}$ is denumerable and the $\mathcal{H}_\alpha$ are all separable then so is their countable direct sum, so the direct sum construction can lead to non-separability only if there is non-separability to start with in the $\mathcal{H}_\alpha$s or else the sum over the $\mathcal{H}_\alpha$s is non-denumerable.

For sake of completeness I mention the direct integral of Hilbert spaces, which may be viewed as a generalization of the direct sum construction wherein the index set $\mathcal{I}$ of the direct sum is replaced a measure space (X, μ).[6] The component Hilbert spaces $\mathcal{H}_x$ of the direct integral Hilbert

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[3] For a von Neumann algebra $\mathfrak{N}$ acting on a Hilbert space $\mathcal{H}$ a vector $\xi \in \mathcal{H}$ is cyclic if $\{\mathfrak{N}\xi\}$ is dense in $\mathcal{H}$.

[4] For a von Neumann algebra $\mathfrak{N}$ acting on a Hilbert space $\mathcal{H}$ a vector $\xi \in \mathcal{H}$ is separating if $A\xi = 0 \Rightarrow A = 0$ for all positive $A \in \mathfrak{N}$.

[5] This sum is understood as $\lim_F \sum_{\alpha\in F} ||\vartheta_\alpha||_{\mathcal{H}_\alpha}$ where the F are finite subsets of $\mathcal{I}$, and $\lim_F \sum_{\alpha\in F} ||\vartheta_\alpha||_{\mathcal{H}_\alpha} = L$ means that for any $\epsilon > 0$ there is a finite $F_0 \subset \mathcal{I}$ such that for any finite F with $\mathcal{I} \supset F \supset F_0$, $|\sum_{\alpha\in F} ||\vartheta_\alpha||_{\mathcal{H}_\alpha} - L| < \epsilon$.

[6] The direct integral construction was studied in detail by von Neumann (1949). von Neumann tells us that this paper was written in 1938: "Various other commitments prevented the author from effecting some changes, which he had intended to carry out before

space $\mathcal{H}^{\oplus} = \int_X^{\oplus} \mathcal{H}_x d\mu(x)$ are indexed by points $x \in X$. An element of $\mathcal{H}^{\oplus}$ is a function $f : X \to \cup_{x \in X} \mathcal{H}_x$ such that $f(x) \in \mathcal{H}_x$ for all $x \in X$ and $x \longmapsto \langle f(x), g(x) \rangle_{\mathcal{H}_x}$ is μ-integrable. The inner product on $\mathcal{H}^{\otimes}$ is given by $\langle f, g \rangle_{\mathcal{H}^{\oplus}} := \int_X \langle f(x), g(x) \rangle_{\mathcal{H}_x} d\mu(x)$. Two measures that are absolutely continuous with respect to one another give rise to isomorphically isometric direct integral spaces.[7] The main use for the direct integral construction is in proving results about von Neumann algebras, e.g. every von Neumann algebra acting on a separable Hilbert space is a direct integral of factor algebras. Note that if the $\mathcal{H}_x$ are separable and (X, μ) is a standard Borel space then $\mathcal{H}^{\oplus}$ is separable (Dixmier 1984 II.1.6 Corollary).[8]

Returning to the direct sum construction, the Fock space $\mathcal{F}(\mathcal{H})$ over the one-particle Hilbert space $\mathcal{H}$ is the direct sum space $\oplus_{n \in \mathbb{N}} \mathcal{H}_n$ where $\mathcal{H}_0 = \mathbb{C}$ (the no-particle or vacuum case), $\mathcal{H}_1 = \mathcal{H}$ (the one-particle case), and $\mathcal{H}_n = \mathcal{H} \otimes \mathcal{H} \otimes ... \otimes \mathcal{H}$, the n-fold tensor product of the one-particle space for $n \geq 2$ (the n-particle case). With an application to identical particles in mind, two subspaces of $\mathcal{F}(\mathcal{H})$ can be distinguished: the symmetric Fock space $\mathcal{F}_s(\mathcal{H}) = \oplus_{n \in \mathbb{N}} S_n \mathcal{H}_n$ (describing bosons) and the anti-symmetric Fock space $\mathcal{F}_a(\mathcal{H}) = \oplus_{n \in \mathbb{N}} A_n \mathcal{H}_n$ (describing fermions) where $S_n \mathcal{H}_n$ and $A_n \mathcal{H}_n$ stand respectively for symmetrized and anti-symmetrized tensor products for $n \geq 2$ while $S_n = A_n = \mathbb{I}_n$ for $n = 0, 1$. If, as is conventionally assumed, the one-particle Hilbert space $\mathcal{H}$ is separable, then $\mathcal{F}(\mathcal{H})$, $\mathcal{F}_s(\mathcal{H})$, and $\mathcal{F}_a(\mathcal{H})$ are separable. There is nothing sacred about this conventional choice and, as will be noted below, it can be illuminating to take the one-particle Hilbert space to be non-separable (see Halvorson 2007). But assuming that the conventional choice is made, the Fock space apparatus is in line with Streater and Wightman's position that relativistic QFT does not require non-separability.

Describing an *actual* as opposed to a *potential* infinity of particles—such as an infinite spin chain, where a countable infinity of spin sites extend literally to spatial infinity—calls for an infinite tensor product construction. Streater and Wightman do not explicitly discuss the infinite spin chain but they seem to have something like this example in mind when they say

> We shall not give the rather technical definition of infinite tensor

publishing the paper. This delayed the publication until the present." One can readily guess what these "other commitments" involved.

[7] See Takesaki (2001) and Dixmier (1984) for more details.

[8] Standard Borel means that there is a metric on X that makes it a complete separable metric space in such a way that the the μ-measurable sets are the Borel σ-algebra.

> product here but only remark that it is a natural generalization of the ordinary tensor product used to describe a composite system. Infinite tensor products of Hilbert spaces (of dimension greater than 1!) are always non-separable. (p. 87)

I will supply a brief outline of the missing technical definition.[9]

The infinite tensor product (ITP) construction was first described by von Neumann (1939).[10] A sequence $\xi := \{\xi_\alpha\}$, $\xi_\alpha \in \mathcal{H}_\alpha$ and $\alpha \in \mathcal{I}$, defines a C-vector $\otimes_\xi := \otimes_{\alpha\in\mathcal{I}}\xi_\alpha$ provided that $\Pi_{\alpha\in\mathcal{I}}||\xi_\alpha||_{\mathcal{H}_\alpha}$ converges.[11] The complete ITP Hilbert space $\otimes_{\alpha\in\mathcal{I}}\mathcal{H}_\alpha$ is constructed by forming finite linear combination of C-vectors and completing in the norm derived from the inner product $\langle\otimes_\xi,\otimes_\zeta\rangle := \Pi_\alpha\langle\xi_\alpha,\zeta_\alpha\rangle_{\mathcal{H}_\alpha}$ of C-vectors $\otimes_\xi$ and $\otimes_\zeta$. If $\dim(\mathcal{H}_\alpha) = D$ for all α then $\dim(\otimes_{\alpha\in\mathcal{I}}\mathcal{H}_\alpha) = D^{|\mathcal{I}|}$. In the simplest non-trivial case where $D = 2$ and $|\mathcal{I}| = \aleph_0$, as in the case of the infinite spin chain, $\dim(\otimes_{\alpha\in\mathcal{I}}\mathcal{H}_\alpha) = 2^{\aleph_0}$.

Make the default assumptions that the total algebra of observables for a system with state space the complete ITP space $\otimes_{\alpha\in\mathcal{I}}\mathcal{H}_\alpha$ is $\mathfrak{B}^\otimes := \mathfrak{B}(\otimes_{\alpha\in\mathcal{I}}\mathcal{H}_\alpha)$, the von Neumann algebra of all bounded operators acting on $\otimes_{\alpha\in\mathcal{I}}\mathcal{H}_\alpha$; and assume also that the algebra of observables associated with the α-component subsystem is $\mathfrak{B}(\mathcal{H}_\alpha)$. (As we will see below, further considerations may force a retreat from these default assumptions.) Each $\mathfrak{B}(\mathcal{H}_\alpha)$ has a natural extension $\mathfrak{B}_\alpha$ to the ITP space, and the smallest von Neumann algebra $\mathfrak{L}^\otimes$ that acts on $\otimes_{\alpha\in\mathcal{I}}\mathcal{H}_\alpha$ and that contains all the $\mathfrak{B}_\alpha$ is the weak closure of $\cup_{\alpha\in\mathcal{I}}\mathfrak{B}_\alpha$, which may be regarded as the algebra of local observables for the system described by the complete ITP space. Thus, the difference $\mathfrak{B}^\otimes - \mathfrak{L}^\otimes$ may be regarded as capturing the non-local observables. Of course, when the index set $\mathcal{I}$ is finite, $\mathfrak{B}^\otimes - \mathfrak{L}^\otimes$ is null since in this case $\mathfrak{L}^\otimes = \otimes_{\alpha\in\{1,\cdots,N\}}\mathfrak{B}(\mathcal{H}_\alpha) = \mathfrak{B}(\otimes_{\alpha\in\{1,2,\ldots N\}}\mathcal{H}_\alpha) = \mathfrak{B}^\otimes$ for $N < \infty$. Thus, something genuinely new emerges in the transition from finite to infinite tensor products, from the the separable to the non-separable, when $\mathfrak{B}^\otimes - \mathfrak{L}^\otimes$ can be non-empty.

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[9] There is no pretense of rigor in what follows. I aim only to sketch some of the main ideas. A readable summary of von Neumann's analysis can be found in Thiemann and Winkler (2001, Section 4), and physical applications are discussed in Thirring (1983).

[10] In 1939 von Neumann was still reserving the term 'Hilbert space' to denote a separable space. A non-separable Hilbert space was labeled a 'hyper-Hilbert space'. And von Neumann speaks of 'direct products' rather than tensor products.

[11] When $\mathcal{I}$ is uncountable $\Pi_{\alpha\in\mathcal{I}}||\xi_\alpha||_{\mathcal{H}_\alpha}$ is understood as $\lim_F \Pi_{\alpha\in F}||\xi_\alpha||_{\mathcal{H}_\alpha}$ where the F are finite subsets of $\mathcal{I}$, and $\lim_F \Pi_{\alpha\in F}||\xi_\alpha||_{\mathcal{H}_\alpha} = L$ means that for any $\epsilon > 0$ there is a finite $F_0 \subset \mathcal{I}$ such that for any finite F with $\mathcal{I} \supset F \supset F_0$, $|\Pi_{\alpha\in F}||\xi_\alpha||_{\mathcal{H}_\alpha} - L| < \epsilon$.

One of the main results of von Neumann's (1939) connects the ITP construction with the infinite direct sum construction. A Hilbert space $\mathcal{H}$ can be considered an internal direct sum if there is a family $\{\mathcal{H}_\beta\}$, $\beta \in \mathcal{J}$, of mutually orthogonal subspaces such that $\vee_{\beta\in\mathcal{J}}\mathcal{H}_\beta = \mathcal{H}$, for then $\mathcal{H}$ is isomorphic to $\oplus_{\beta\in\mathcal{J}}\mathcal{H}_\beta$ (see Kadison and Ringrose 1991, p. 124). von Neumann showed that an infinite tensor product space $\otimes_{\alpha\in\mathcal{I}}\mathcal{H}_\alpha$ has a canonical internal direct sum decomposition into what he called incomplete ITP's. A C_0-vector is a C-vector $\otimes_\xi$ such that $\sum_\alpha |\ ||\xi_\alpha||_{\mathcal{H}_\alpha} - 1|$ converges. Two C_0-vectors $\otimes_\xi$ and $\otimes_\zeta$ are said to be equivalent ($\xi \approx \zeta$) just in case $\sum_\alpha |\langle\xi_\alpha, \zeta_\alpha\rangle_{\mathcal{H}_\alpha} - 1|$ converges. It is shown that $\approx$ is in fact an equivalent relation; the equivalence class of ξ is denoted by $[\xi]$, and the set of equivalence classes is denoted by $\mathcal{S}$. For $[\xi] \in \mathcal{S}$ the Hilbert space $\mathcal{H}_{[\xi]}$ formed by taking the closure of finite linear combinations of $\otimes_{\xi'}$'s with $\xi' \in [\xi]$ is an incomplete ITP.[12] These incomplete ITPs all have the same dimension. Further, they are mutually orthogonal, for if $[\xi] \neq [\zeta]$ then $\langle\otimes_{\xi'}, \otimes_{\zeta'}\rangle = 0$ for all $\xi' \in [\xi]$ and $\zeta' \in [\zeta]$. von Neumann also showed that the closed set these incomplete ITP spaces determine is $\otimes_{\alpha\in\mathcal{I}}\mathcal{H}_\alpha$ and, hence, $\otimes_{\alpha\in\mathcal{I}}\mathcal{H}_\alpha = \oplus_{[\xi]\in\mathcal{S}}\mathcal{H}_{[\xi]}$. (von Neumann did not state the result in this format, presumably because he was not thinking in terms of infinite direct sums. He simply says that the complete ITP "splits up" into incomplete ITPs.)

Applying this to the infinite spin chain, the index set $\mathcal{I}$ is $\mathbb{N}^+$, and for each $n \in \mathbb{N}^+$ the component Hilbert space is the spin space $\mathfrak{h}_n = \mathbb{C}^2$.[13] The complete ITP space $\otimes_{n\in\mathbb{N}^+}\mathfrak{h}_n$ for the infinite spin chain has the direct sum decomposition $\oplus_{[\xi]\in\mathcal{S}}\mathfrak{h}_{[\xi]}$ where the $\mathfrak{h}_{[\xi]}$ are separable. In fact, for any $[\xi] \in \mathcal{S}$ there is a $\xi^0 \in [\xi]$ such that $||\xi^0|| = 1$, and $\mathfrak{h}_{[\xi]}$ is the closure of finite linear combinations of $\otimes_{\xi'}$'s such that $\xi' \in [\xi]$ and $\xi'_n = \xi^0_n$ for all but finitely many $n \in \mathbb{N}^+$.[14]

The direct sum decomposition (of which von Neumann's decomposition of ITP spaces is a special case) lends itself to a strategy for defanging non-separability. Suppose that a system of interest is described using a non-

[12] The infinite tensor product constructed in Bratelli and Robinson (1987, 144-145) corresponds to an incomplete ITP.

[13] I take the sum over $\mathbb{N}^+$ since I am imagining a spin chain that extends from the origin to spatial infinity along some direction. Nothing material is changed in what follows if one imagines that the spin chain extends to infinity in both directions.

[14] These are the spaces discussed in Sewell (2002, Sec. 2.3). He does not direct sum these spaces to get a space that is isomorphic to the complete ITP space for the spin chain, so all of the spaces he discusses are separable.

separable $\mathcal{H}$. And suppose that there is a direct sum decomposition $\oplus_{\beta\in\mathcal{J}}\mathcal{H}_\beta$ of $\mathcal{H}$ and that the relevant algebra of observables describing the system of interest is not $\mathfrak{B}(\oplus_{\beta\in\mathcal{J}}\mathcal{H}_\beta)$ but the smaller subalgebra $\oplus_{\beta\in\mathcal{J}}\mathfrak{B}(\mathcal{H}_\beta)$ consisting of all operators of the form $\oplus_{\beta\in\mathcal{J}}A_\beta$, where $A_\beta \in \mathfrak{B}(\mathcal{H}_\beta)$ and $\{||A_\beta||\}$ is bounded, acting on the direct sum space $\oplus_{\beta\in\mathcal{J}}\mathcal{H}_\beta$. Then the main ingredients for a superselection rule are in place. Coherent superpositions across the $\oplus$-sectors are impossible in the following sense: if $\oplus_\vartheta, \oplus_\xi \in \oplus_{\beta\in\mathcal{J}}\mathcal{H}_\beta$ are unit vectors whose only non-zero components belong to different sectors, then the transition probability $\langle\oplus_\vartheta, A\oplus_\xi\rangle$ is 0 for any $A \in \oplus_{\beta\in\mathcal{J}}\mathfrak{B}(\mathcal{H}_\beta)$. Further, the vector state $\omega_{c\oplus_\vartheta+d\oplus_\xi}$, $|c|^2+|d|^2=1$, corresponding to the superposition $c\oplus_\vartheta+d\oplus_\xi$ of these vector states is the mixed state $|c|^2\omega_{\oplus_\vartheta}+|d|^2\omega_{\oplus_\xi}$. The zero transition probabilities between sectors and the lack of interference terms between sectors are two of the key characteristic features of superselection rules.[15] It is usually also required for a superselection rule that the dynamics does not mix the selection sectors. If the dynamics is given by a continuous unitary group $V(t)$, $\in \mathbb{R}$, this means that for vectors $\oplus_\vartheta, \oplus_\xi$ belonging to different sectors $\langle\oplus_\vartheta, V(t)\oplus_\xi\rangle = 0$ for all t, which will be the case if the self-adjoint generator of the group $V(t)$ is an observable in the sense that its spectral projections belong to $\oplus_{\beta\in\mathcal{J}}\mathfrak{B}(\mathcal{H}_\beta)$.

When the conditions for superselection rules are in place and when the selection sectors are separable, the non-separability resulting from an uncountably infinite direct sum over the sectors is rendered innocuous, for quantum mechanicians can explore the physics of any given selection sector without fear that ignoring the other sectors will compromise the conclusions drawn for the target sector. This suggests a strategy for defanging non-separability that can in principle applied, from the inside out, to any non-separable Hilbert space since the space can always be written as a possibly non-countable internal direct sum of separable spaces. But there is no a priori reason for thinking that sound reasons for treating the $\oplus$-sectors of such a decomposition as su-

[15]'Superselection rule' is used in several different ways which are discussed in Earman (2008). It is sometimes said, incorrectly, that a superselection rule implies a limitation of the superposition principle. What is true is that a superselection rule disappoints our expectations about what a superposition represents. We normally think that pure states and vector states are coextensive and, thus, that a superposition of pure states is a pure state. It is this expectation that is disappointed when a superselection principle is at work. It should be emphasized that the pure vs. mixed state divide is decided not by the Hilbert space but the by algebra of observables. If the algebra of observables is $\mathfrak{B}(\mathcal{H})$, a Type I factor, then pure states and vector states coincide. For Type I non-factors—the case being considered here—some vector states are pure and some are not.

perselection sectors will be found except in special cases. Success or failure of defaging depends on what counts as the relevant algebra of observables. The point is illustrated by revisiting the infinite spin chain.

Introduce the Pauli spin algebra $\mathfrak{P}$ formed from finite sums and products of the Pauli spin operators $\sigma_n = (\sigma_{n,x}, \sigma_{n,y}, \sigma_{n,z})$ satisfying the following relations:

$$\begin{aligned} [\sigma_{n,x}, \sigma_{n,y}] &= 2i\sigma_{n,z} \ \ \text{etc} \\ [\sigma_{m,u}, \sigma_{n,v}] &= 0 \ \ \text{for } m \neq n, \ \ u, v = x, y, z \\ \sigma_n^2 &= 3\mathbb{I}_{\mathfrak{h}_n} \end{aligned} \tag{P}$$

where units of $\hbar/2$ have been used. Each of the incomplete ITP spaces $\mathfrak{h}_{[\xi]}$ in the direct sum decomposition $\oplus_{[\xi]\in\mathcal{S}}\mathfrak{h}_{[\xi]}$ of the complete ITP space of the infinite spin chain carries a representation of the algebra $\mathfrak{P}$. Choose a representation such that in the construction of a basis $\mathfrak{h}_{[\xi]}$ for the spin chain referred to above the $\xi^0 \in [\xi]$ is such that ξ_n^0 is the state of spin up along the z-axis for every site n (i.e. $\sigma_{n,z}\xi_n^0 = +\xi_n^0$ for all n) and the remaining basis vectors $\xi' \in \mathfrak{h}_{[\xi]}$ are states of spin up along the z-axis for all but a finite number of sites. The operator $M_N^z := \dfrac{1}{N}\sum\limits_{n=1}^{N} \sigma_{n,z}$ measures the net magnetization for the first N sites. And for the chosen representation we have, for any unit $\zeta \in \mathfrak{h}_{[\xi]}$, $Lim_{N\to\infty}\langle\zeta, M_N^z\zeta\rangle = +1$, so we can say that the net magnetization of the entire infinite spin chain is $+1$. More generally, for any value of $\lambda \in [-1, +1]$ there is a representation of $\mathfrak{P}$ and an incomplete ITP space $\mathfrak{h}_{[\vartheta_\lambda]}$ such that for each basis state, and thus for any unit vector $\zeta^\lambda \in \mathfrak{h}_{[\vartheta_\lambda]}$, $Lim_{N\to\infty}\langle\zeta^\lambda, M_N^{z,\lambda}\zeta^\lambda\rangle = \lambda$, where the λ-superscript has been added to distinguish representations.

Of course, there are many representations for which the limit does not exist because the expectation value of M_N^z oscillates endlessly as N grows without bound; indeed, there is an uncountable number of them. But suppose for the moment that there is good physical motivation for the proposition that only states for the spin chain with a well-defined net magnetization are physically realizable. Then the relevant state space for the spin chain is the smaller but still non-separable $\widetilde{\mathcal{H}} = \oplus_{[\widetilde{\xi}]\in\mathcal{S}}\mathfrak{h}_{[\widetilde{\xi}]}$, where the sum is restricted to those $\mathfrak{h}_{[\widetilde{\xi}]}$ carrying a representation for which the net magnetization exists. This suggests that superselection is at work because the relevant algebra of observables is not $\mathfrak{B}(\widetilde{\mathcal{H}})$ but the smaller subalgebra $\oplus_{[\widetilde{\xi}]\in\mathcal{S}}\mathfrak{B}(\mathfrak{h}_{[\widetilde{\xi}]})$, or even some yet smaller subalgebra of this. The suggestion is supported by the following argument. If $\lambda_1 \neq \lambda_2$ then the representations with net magnetizations λ_1 and λ_2 are unitarily ineqivalent. For a reductio proof suppose that there is a unitary map $U : \mathfrak{h}_{[\vartheta_{\lambda_1}]} \to \mathfrak{h}_{[\vartheta_{\lambda_2}]}$ between the spaces $\mathfrak{h}_{[\vartheta_{\lambda_1}]}$ and $\mathfrak{h}_{[\vartheta_{\lambda_2}]}$ hosting the said representations such that $UM_N^{z,\lambda_1}U^{-1} = M_N^{z,\lambda_2}$. So if $\zeta^{\lambda_1} \in \mathfrak{h}_{[\vartheta_{\lambda_1}]}$ and $\zeta^{\lambda_2} \in \mathfrak{h}_{[\vartheta_{\lambda_1}]}$ are unit vectors such that $U\zeta^{\lambda_1} = \zeta^{\lambda_2}$ then $\lambda_1 = Lim_{N\to\infty}\langle\zeta^{\lambda_1}, M_N^{z,\lambda_1}\zeta^{\lambda_1}\rangle = Lim_{N\to\infty}\langle U\zeta^{\lambda_1}, (UM_N^{z,\lambda_1}U^{-1})U\zeta^{\lambda_1}\rangle = Lim_{N\to\infty}\langle\zeta^{\lambda_2}, M_N^{z,\lambda_2}\zeta^{\lambda_2}\rangle = \lambda_2$, a contradiction.[16] To continue the argument, note that for irreducible representations of a C^*-algebra $\mathcal{A}$ unitary inequivalence implies disjointness—no normal state of one representation is a normal state of the other (Kadison and Ringrose 1997, Prop. 10.3.13). So if $\{\pi_a, \mathcal{H}_a\}$ is a family irreducible representations of $\mathcal{A}$, any two of which are unitarily inequivalent, then the joint display of all the $\{\pi_a, \mathcal{H}_a\}$ on a common Hilbert space of which the $\mathcal{H}_a$ are subspaces must be of the form $\oplus_a \pi_a(\mathcal{A})$ acting on $\oplus_a \mathcal{H}_a$. Since the π_a are irreducible the von Neumann algebras $\pi_a(\mathcal{A})''$ determined by the π_a are $\mathfrak{B}(\mathcal{A})$. And since $(\oplus_a \pi_a(\mathcal{A}))'' = \oplus_\mathfrak{a} \pi_\mathfrak{a}(\mathcal{A})''$ for irreducible disjoint representations (Kadison and Ringrose 1997, Cor. 10.3.9) the upshot is that $\pi(\mathcal{A})'' = \oplus_a \mathfrak{B}(\mathcal{H}_a)$. Applying this to the case in point, if irreducible representations of the Pauli algebra are required then the relevant von Neumann algebra acting on the non-separable state space $\oplus_{[\widetilde{\xi}]\in\mathcal{S}} \mathfrak{h}_{[\widetilde{\xi}]}$ is indeed the superselection algebra $\oplus_{[\widetilde{\xi}]\in\mathcal{S}} \mathfrak{B}(\mathfrak{h}_{[\widetilde{\xi}]})$, and since the spaces $\mathfrak{h}_{[\widetilde{\xi}]}$ of the superselection sectors are separable, the non-separability of the state space $\widetilde{\mathcal{H}}$ has been defanged.

It is interesting that the idealization of an infinite spin chain leads to the emergence of an element of classicality in the form of the absence of interference effects between subsystems with different net magnetizations. But the story just sketched can hardly be counted as a successful illustration of the defanging strategy since it succeeds only by changing the target for defanging.

To return to Streater and Wightman, there is mention of another idealized case where non-separability rears its head, this one from quantum statistical mechanics (QSM); namely, the thermodynamic limit for a box of gas in which the number of particles in the box and the volume of the box both go to ∞ while the density ϱ of the gas remains constant. Streater and Wightman's analysis:

[16] This is a slight generalization of the argument given in Sewell (2002, pp. 15-18).

> Two states of the limiting system which have different densities actually differ by the presence of an infinite number of particles. One might expect them to be orthogonal and in fact that is the case in all examples worked out so far. Thus there is an orthonormal system labeled by a continuous parameter, the density, and the Hilbert space [say, $\mathcal{H}_{TL}$] is non-separable. (ibid., p. 87).

In contrast to the spin chain case no special pleading is needed to defang the non-separability in the present case. Because of the presence of an infinite number of particles in the thermodynamic limit, the von Neumann uniqueness theorem[17] for representations of the Weyl form of the canonical commutation relations (CCR) does not apply, and unitarily inequivalent representations of the CCR emerge, one for each value of the density ϱ. Thus, the density parameter labels superselection sectors, and since these sectors are separable, the non-separability of $\mathcal{H}_{TL}$ is defanged. I will return below in Section 4.3 to a more detailed discussion of the Weyl CCR and the von Neumann uniqueness theorem; but there the focus will not be on the failure of the theorem for an infinite number of degrees of freedom but rather on how non-separability can lead to non-uniqueness even for a finite number of degrees of freedom.

In the case of QFT there is a similar defanging of a threat of non-separability. The symmetric Fock space $\mathcal{F}_s(\mathcal{H})$ with the separable one-particle Hilbert space $\mathcal{H} \simeq {L^2}_{\mathbb{C}}(\mathbb{R})$ carries irreducible and strongly continuous unitary representations of the group of translations of $\mathbb{R}$ satisfying the the Weyl CCR. For a free relativistic field of mass $m > 0$ the representations corresponding to different values of m are unitarily inequivalent (see Blank et al. 1994, Section 12.3). If one wants to present these inequivalent representations of the free field all at once using an external direct sum to stitch them together then the direct sum Hilbert space is non-separable. But again the superselection story renders the non-separability innocuous.

Although it adds little new to our discussion, it is worth noting another swipe Streater and Wightman take at non-separability in QFT because some commentators have used it to motivate attempts to suppress non-separability at large.

> Since a (Bose) field can be thought of as an infinity of oscillators, one might think that such an infinite tensor product is the natural

[17] Sometimes referred to as the Stone-von Neumann uniqueness theorem. The proof was given by von Neumann (1931). A sketch of the proof was given by Stone (1930).

> state space. However, it is characteristic of [quantum] field theory that some of its observables involve the oscillators all at once and it turns out that such observables can be naturally defined only on vectors belonging to a tiny separable subset of the infinite tensor product. It is the space spanned by such a subset that is the natural state space rather than the infinite tensor product itself. (Streater and Wightman, ibid., p. 87)

Presumably the lesson we are supposed to learn is something like the following. In treating the free quantum field as an infinite assemblage of oscillators we might naively assume that the appropriate state space is the ITP space $\mathcal{P}(\mathcal{K}) := \otimes_{n\in\mathbb{N}}\mathcal{K}_n$ where $\mathcal{K}_0 = \mathbb{C}$ and $\mathcal{K}_n = \mathcal{K}$ all $n \in \mathbb{N}^+$, $\mathcal{K}$ being the one-oscillator space. The resulting ITP space is non-separable as long as $\dim(\mathcal{K}) \geq 2$. Then we realize that some of the observables that involve the oscillators "all at once" are naturally defined only on a separable subspace of $\mathcal{P}(\mathcal{K})$, an example being the particle number operator N where $N|_{\mathcal{K}_n} = n$ for all $n \in \mathbb{N}$.[18] Fairbairn and Rovelli (2004) supply the moral:

> Fock found a way to circumvent the problem by simply selecting the subspace $\mathcal{F}(\mathcal{K})$ of $\mathcal{P}(\mathcal{K})$ spanned by basis vectors where an arbitrary but finite number of n_i [of particles] differ from zero. It is $\mathcal{F}(\mathcal{K})$, called today Fock space, which is the appropriate state space for free QFT ... Thus, a straightforward and simple minded quantization strategy leading to a non-separable state space has been later corrected to get rid of the non-separability. (2004, p. 2803)[19]

The presumption of the moral seems to be when non-separability rears its head an effort should be make to get rid of it. But why? Separable Hilbert spaces have some advantages while non-separable spaces suffer corresponding disadvantages. In the present instance an advantage of using the separable $\mathcal{F}(\mathcal{K})$ is that unlike the non-separable ITP $\mathcal{P}(\mathcal{K})$ it "provides an *irreducible* representation of the field algebra of the creation and annihilation operators" (Fairbairn and Rovelli 2004, p. 2803). Later I will sharpen the point by

[18] Since N is unbounded it is not in $\mathfrak{B}(\mathcal{F}(\mathcal{K}))$. But as a self-adjoint operator all of its spectral projections are in $\mathfrak{B}(\mathcal{F}(\mathcal{K}))$. The domain of self-adjointness of N is $\{\psi_0 \oplus \psi_1 \oplus ... \oplus \psi_n \in \mathcal{F}(\mathcal{K}) : \sum_{n=1}^{\infty} n^2||\psi_n||^2 < \infty\}$ (Thirring 1983, p. 25).

[19] I have changed their notation to agree with that being used here.

indicating that non-separable Hilbert spaces cannot host representations of the Weyl CCR that are both irreducible *and* continuous (see Secs. 4.3 and 5.2). But I will argue that, whatever the advantages of separable spaces and the corresponding advantages of non-separable spaces, ultimately what matters is the algebra of observables needed to describe the system of interest and whether that algebra must act on a non-separable space.

To summarize the discussion thus far: Nothing was found to undermine Streater and Wightman's main thesis that "there is no evidence that separable Hilbert spaces are not the natural state spaces for quantum field theory" (ibid, p. 87). At the same time the discussion revealed that the tug-of-war between the separable and the non-separable is much more complicated and interesting than the unwary reader of Streater and Wightman might be led to believe. While some problematic features of non-separability have surfaced there is so far nothing to justify a righteous indignation against non-separability. But the absence of non-idealized examples of the need for non-separable Hilbert spaces in applications of quantum theory might encourage the attitude that the topic is one of merely idle speculation. This would, I think, be a mistake since to ban idealizations from physics would hamper if not cripple theorizing. Nevertheless, it would be reassuring to have less contrived examples of non-separability at work than the infinite spin chain. When some of these examples are brought up there are attempts to either defang or suppress the non-separability.

2.2 Other cases of non-separability

For examples that don't carry the taint of idealization of the infinite spin chain case one can turn to current research programs in quantum gravity, loop quantum gravity (LQG) and loop quantum cosmology (LQC) in particular. The "background independent" kinematical Hilbert space $\mathcal{H}_{kin(rs)}$ for LQG constructed by Rovelli and Smolin (1995) is non-separable.[20] This could be brushed aside if the physical Hilbert space that results from reducing the kinematical space by the Hamiltonian constraint is separable (see Rovelli 1998). But even at the kinematical level Fairbairn and Rovelli (2004) propose to restore separability by replacing $\mathcal{H}_{kin(rs)}$ with a separable $\mathcal{H}_{kin}$ that removes a perceived redundancy in $\mathcal{H}_{kin(rs)}$. A basis for $\mathcal{H}_{kin(rs)}$ is

[20] $\mathcal{H}_{kin(rs)}$ is typically labeled $\mathcal{H}_{diff}$ since it is supposed to consist of diffeomorphically invariant quantum states.

provided by s-knot states which are labeled by continuous moduli parameters, resulting in a non-countable basis. It is contended, however, that s-knot states that differ only by their moduli values are not distinguishable by physical measurements (and so, in the terminology used above, some of the self-adjoint elements of $\mathfrak{B}(\mathcal{H}_{kin(rs)})$ do not correspond to genuine observables). Fairbairn and Rovelli propose to enlarge the diffeomorphism group *diff*, the gauge invariance group of the theory, in a way that washes away the moduli in the basis states, restoring separability. Some skepticism has been expressed about how this proposal meshes with the rest of program in LQG (see Asktekar and Lewandowski 2004 and Barbero et al. 2014). In particular, it is worrisome that the proposed enlarged gauge group for LQG has no apparent classical counterpart. And one would like to see how the proposal to enlarge *diff* affects one of the notable results in LQG, viz. the proof of the existence of a unique *diff*-invariant state on the holonomy-flux algebras of LQG (Lewandowski et al. 2006), the GNS representation of which yields a non-separable Hilbert space.[21]

Another avenue to non-separability in LQG lies in the construction of semi-classical LQG states that approximate QFT in classical general relativistic spacetime backgrounds, a necessary step in showing the LQG has classical general relativity as its classical limit. Thiemann and Winkler (2001) and Sahlmann et al. (2001) argue that these semi-classical states require the construction of the infinite tensor product of the Hilbert spaces associated with the edges of a graph that fills a time slice Σ of the spacetime. If Σ is non-compact there will be a countably infinite number of edges so that the resulting tensor product space will be non-separable as long as the component spaces have dim ≥ 2. Unlike the infinite spin chain, the need for a infinite tensor product here does not result from an idealization.

Finally, the strategy of defanging non-separability for non-ITP spaces by partitioning the Hilbert space into separable superselection sectors that are not mixed by the dynamics has met with mixed success in LQG and LQC (compare Ashtekar et al. 2007 and Kreienbuehl and Pawlowski 2013).

It is hard to draw any firm conclusions about the status of separability

[21] The algebra referred to here is an abstract C^*-algebra which gets represented via the GNS construction as a concrete von Neumann algebra acting on a Hilbert space. The GNS representation theorem shows that if ω is a state on a C^*-algebra $\mathcal{A}$ then there is a unique (up to unitary equivalence) representation π_ω of $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}_\omega$ and a cyclic vector $\xi_\omega \in \mathcal{H}_\omega$ such that $\omega(A) = \langle \xi_\omega, \pi_\omega(A)\xi_\omega \rangle$ for all $A \in \mathcal{A}$. GNS representations will make other appearances below in Sections 4 and 5.4.

vs. non-separability in a field that is in flux, but the fact that non-separable Hilbert spaces enter into actively discussed issues in contemporary quantum gravity research is enough to motivate inquiring into the challenges that would arise if quantum physics is to be done with non-separable Hilbert spaces. The discussion thus far indicates that many physicists are leery of non-separability and seek to avoid it or to defang it, but little has been learned about why they do, or should, adopt a wary stance.

The first order of business in such an inquiry is to get a better feel for what features of separable Hilbert spaces do, and what features do not, carry over to non-separable spaces.

3 Successes and salvageable failures

3.1 Successes: cases where properties of separable spaces transfer to non-separable spaces.

Ex. 1. Even if $\mathcal{H}$ is non-separable it has an orthonormal basis. But the proof is non-constructive, requiring the axiom of choice, Zorn's lemma, or something equivalent. The dimension of a Hilbert space is the cardinality of an ON basis. This is a well-defined notion since all ON bases of a Hilbert space, separable or non-separable, have the same cardinality (see Dunford and Schwartz 1988, Theorem IV.4.14).

Ex. 2. The spectral theorem carries over to non-separable spaces (see Rellich 1935). For self-adjoint A acting on a separable or non-separable $\mathcal{H}$ there is a unique projection valued measure $E_A(x)$ such that $A = \int_{\mathbb{R}} x dE_A(x)$.

Ex. 3. If a von Neumann algebra admits a cyclic and separating vector—and thus necessarily acts on a separable Hilbert space—it is $*$-isomorphic to an algebra in standard form.[22] Eventually it was shown that every von Neumann algebra—whatever the dimensionality of the Hilbert space on which it acts—is $*$-isomorphic to a von Neumann algebra in standard form (Haagerup 1975).

3.2 Salvageable failures: cases where suitable modifications/generalizations allow the transfer.

[22] The standard form of von Neumann algebras will be discussed below in Section 5.4.1.

Ex. 1. Gleason's theorem was originally formulated for ordinary QM and a separable $\mathcal{H}$. In the language of algebraic quantum theory, if the algebra of observables is $\mathfrak{B}(\mathcal{H})$ the statement of Gleason's theorem becomes: For separable $\mathcal{H}$ with $\dim(\mathcal{H}) > 2$ any countably additive quantum probability measure on the projection lattice $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ of $\mathfrak{B}(\mathcal{H})$ extends uniquely to a normal state on $\mathfrak{B}(\mathcal{H})$.

This version of Gleason's theorem fails for a non-separable $\mathcal{H}$. But the natural generalization of Gleason's theorem for non-separable spaces does hold. When $\mathcal{H}$ is non-separable a normal state on $\mathfrak{B}(\mathcal{H})$ induces a not merely countably additive but a *completely additive* probability measure on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$. So the natural question is: For general $\mathcal{H}$ with $\dim(\mathcal{H}) > 2$ does every completely additive quantum probability measure on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ extend uniquely to a normal state on $\mathfrak{B}(\mathcal{H})$? Yes! And this generalization can be further extended to cover any non Neumann algebra not containing a Type I_2 summand. It is worth noting that a countably additive quantum probability measure on the projection lattice $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ is completely additive unless $\dim(\mathcal{H})$ is as large as the least measurable cardinal, so the original Gleason theorem holds for a non-separable $\mathcal{H}$ unless $\dim(\mathcal{H})$ is bigger than any of the familiar Cantorian infinities (see Eilers and Horst 1975). This is not true for more general von Neumann algebras whose projections are not in one-one correspondence with the closed subspaces of the Hilbert space on which the algebra acts.[23]

Ex. 2. If $\mathcal{H}$ is separable then for any unbounded self-adjoint A there is a unitary $U : \mathcal{H} \to \mathcal{H}$ such that $D(UAU^*) \cap D(A) = \emptyset$ (von Neumann 1929). Elst and Sauer (2015) provide a counterexample for a non-separable space. But but they also provide a suitable reformulation that holds in a general Hilbert space.

Ex. 3+. Other examples of properties of separable Hilbert spaces that, upon appropriate reformulation/generalization, extend to non-separable spaces: the characterization of two-sided ideals (Luft 1968), the characterization of the distance of an operator to a set of unitary operators (Elst 1990), and the block diagonalization of operators (Mikkola 2009).

From this cursory survey of successes and salvageable failures it might seem that with the help of proper navigation there will be smooth sailing from the separable to the non-separable. But there are non-salvageable failures

[23] E.g. Type III von Neumann algebras which contain only infinite dimensional projections.

that make for stormy weather for quantum physics.

4 Non-Salvageable Failures

4.1 Faithless states

A non Neumann algebra is σ-finite iff every family of mutually orthogonal projections is countable.[24] The Type I factor $\mathfrak{B}(\mathcal{H})$ used in ordinary QM is σ-finite iff $\mathcal{H}$ is separable. For a general von Neumann algebra $\mathfrak{N}$ acting on $\mathcal{H}$, the separability of $\mathcal{H}$ entails that $\mathcal{H}$ is σ-finite. But the converse is not true. Let $\mathfrak{N}_1$ be a σ-finite algebra acting on a separable $\mathcal{H}_1$. Tensor on to $\mathcal{H}_1$ a non-separable $\mathcal{H}_2$ and let $\mathfrak{N} = \mathfrak{N}_1 \otimes \mathfrak{N}_2$ act on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ where $\mathfrak{N}_2$ is the von Neumann algebra generated by the identity operator on $\mathcal{H}_2$. $\mathfrak{N}$ is a σ-finite von Neumann algebra acting on a non-separable $\mathcal{H}$. What makes this example uninteresting is that the same algebra, i.e. a $*$-isomorphic algebra, can act on a separable Hilbert space; for example, $\mathfrak{N}_1 \otimes \mathfrak{N}_2$ is $*$-isomorphic to $\mathfrak{N}_1 \otimes \mathfrak{N}_3$ acting on $\mathcal{H}_1 \otimes \mathcal{H}_3$ where $\mathcal{H}_3$ is separable and $\mathfrak{N}_3$ is the von Neumann algebra generated by the identity operator on $\mathcal{H}_3$. However, there are more interesting examples of the failure of the converse; specifically, there are σ-finite algebras that are not $*$-isomorphic to any von Neumann algebra acting on a separable Hilbert space (see Blackadar 2006, pp. 226 and 277). But if $\mathfrak{N}$ is separable (i.e. is generated by a countable family in $\mathfrak{N}$ that is dense in the strong operator topology) then it is $*$-isomorphic to a von Neumann algebra acting on a separable Hilbert space. To see this note first that if $\mathfrak{N}$ is separable then it is σ-finite, and the latter implies that $\mathfrak{N}$ admits a faithful normal state ω. Construct the GNS representation $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ of $\mathfrak{N}$. Since ω is faithful the GNS representation $\pi_\omega(\mathfrak{N})$ of $\mathfrak{N}$ is $*$-isomorphic to $\mathfrak{N}$. The representation acting on the GNS vector $\pi_\omega(\mathfrak{N})\xi_\omega$ is dense in $\mathcal{H}_\omega$, and since $\mathfrak{N}$ is separable there is a countable generating set $G \subset \mathfrak{N}$ dense in $\mathfrak{N}$ and $\pi_\omega(G)\xi_\omega$ is a countable dense subset of $\mathcal{H}_\omega$. Hence, $\pi_\omega(\mathfrak{N})$ is a von Neumann algebra $*$-isomorphic to $\mathfrak{N}$, and it acts on a separable $\mathcal{H}_\omega$.[25]

For many, if not most, of the von Neumann algebras used in physical applications, it is true that if $\mathfrak{N}$ acts on a non-separable Hilbert space and is not $*$-isomorphic to a von Neumann algebra acting on a separable Hilbert space then $\mathfrak{N}$ is not σ-finite. Hence, the need to use a non-separable Hilbert space

[24] Countably decomposable is another name for σ-finiteness.

[25] It follows that $\mathfrak{N}$ is σ-finite does not entail that $\mathfrak{N}$ is separable.

typically signals non-σ-finiteness of the algebra of observables. There are at least two problems for physical applications that result from non-σ-finiteness: (a) Every σ-finite algebra admits faithful normal states. But a non-σ-finite $\mathfrak{N}$ admits no faithful states, normal or not. For such a state would have to assign a value in $(0,1]$ to each member of any family of mutually orthogonal projections, but this cannot happen in the non-σ-finite case because in order to normalize the state can assign a non-zero value to at most a countable number of the uncountable family members. (b) For a non-σ-finite $\mathfrak{N}$ acting on $\mathcal{H}$ there is no separating vector $\xi \in \mathcal{H}$ since such a ξ would determine a state $\omega_\xi(\bullet) := \langle \xi, \bullet\xi \rangle$ that is faithful and normal; and, thus, there is no vector in $\mathcal{H}$ that is cyclic for the commutant $\mathfrak{N}'$ since such a vector would be separating for $\mathfrak{N}$. Furthermore, a non-σ-finite $\mathfrak{N}$ is not $*$-isomorphic to an algebra that admits a cyclic and separating vector (Bratelli and Robinson 1987, Prop. 2.5.6). These consequences of non-σ-finiteness make trouble for QFT by undermining the existence of vacuum states possessing features that are generally taken for granted. They also make for bigger trouble for QSM and modular theory (see Section 5.4 below).

4.2 The split property

Algebraic QFT (AQFT) assumes that there is an assignment of $\mathcal{O} \mapsto \mathfrak{N}(\mathcal{O})$ of local von Neumann algebras $\mathfrak{N}(\mathcal{O})$ to open bounded regions $\mathcal{O}$ of Minkowski spacetime with $\longmapsto$ having the net property that $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathfrak{N}(\mathcal{O}_1) \subset \mathfrak{N}(\mathcal{O}_2)$. Generically the local algebras are of AQFT Type III. The quasi-local global algebra is the smallest von Neumann algebra generated by the $\mathfrak{N}(\mathcal{O})$ as $\mathcal{O}$ ranges over the open bounded regions of Minkowski spacetime. A pair $(\mathfrak{N}_1, \mathfrak{N}_2)$ of von Neumann algebras is a split inclusion iff there is a Type I factor $\mathfrak{F}$ such $\mathfrak{N}_1 \subseteq \mathfrak{F} \subseteq \mathfrak{N}_2$. The net $\mathcal{O} \mapsto \mathfrak{N}(\mathcal{O})$ has the split inclusion property iff for any $\mathcal{O}$ and $\widehat{\mathcal{O}}$ with $\widehat{\mathcal{O}} \supset \mathcal{O}$, the pair $(\mathfrak{N}(\mathcal{O}), \mathfrak{N}(\widehat{\mathcal{O}}))$ is a split inclusion. If the net of local algebras acts on a non-separable $\mathcal{H}$ then it cannot be the case that both (i) there is a vector that is cyclic and separating for all the local algebras, and (ii) the split property holds (Halvorson 2007).

4.3 The CCR and the von Neumann uniqueness theorem

For one degree of freedom the von Neumann uniqueness theorem (von Neumann 1931) for the Weyl CCR takes the form[26]:

> *Theorem* (von Neumann). Let $U(a), V(b)$, $a, b \in \mathbb{R}$, be strongly continuous[27] groups of unitary operators acting on a separable[28] Hilbert space $\mathcal{H}$ and satisfying the Weyl relations
>
> $$U(a)U(b) = U(a+b), \quad V(a)V(b) = V(a+b)$$
> $$U(a)V(b) = e^{iab}V(b)U(a).$$
>
> Then there are closed subspaces $\mathcal{H}_j \subseteq \mathcal{H}$, $j \in \mathcal{I}$ (with $\mathcal{I}$ countable) such that
>
> (i) $\mathcal{H} = \oplus_{j\in\mathcal{I}}\mathcal{H}_j$
>
> (ii) $U(a) : \mathcal{H}_j \to \mathcal{H}_j$ and $V(b) : \mathcal{H}_j \to \mathcal{H}_j$ for all $a, b \in \mathbb{R}$
>
> (iii) for each j there is a unitary $T_j : \mathcal{H}_j \to L^2_{\mathbb{C}}(\mathbb{R})$ such
>
> that $(T_jU(a)T_j^{-1}\psi)(x) = e^{iax}\psi(x)$ and
> $(T_jV(b)T_j^{-1}\psi)(x) = \psi(x-b), \psi(x) \in L^2_{\mathbb{C}}(\mathbb{R})$.

By Stone's theorem[29] the strongly continuous $U(a), V(b)$ have self-adjoint generators Q, P, and there is a common and invariant dense domain $D \subset \mathcal{H}$ on which these generators satisfy the Heisenberg form of the CCR: $PQ -$

[26] The version quoted here is from Reed and Simon (1980, Theorem VIII,14) and Blank et al. (1994, 8.2.4 Theorem). For an excellent survey of the history and implications of von Neumann' uniqueness theorem see Summers (2001).

[27] Here weak continuity suffices since for a unitary group weakly continuous implies strongly continuous.

[28] Here separable means that $\dim(\mathcal{H}) = \aleph_0$ since a strongly continuous representation of the Weyl CCR is not possible if $\dim(\mathcal{H}) < \infty$. The generators of a strongly continuous representation of the Weyl CCR satisfy the Heisenberg CCR (see below), which cannot be satisfied by bounded operators, whereas all linear operators on finite dimensional $\mathcal{H}$ are bounded.

[29] Stones's theorem shows that if $T(t)$, $t \in \mathbb{R}$, is a strongly continuous unitary group acting on a Hilbert space $\mathcal{H}$ then there is a unique self-adjoint A such that $T(t) = e^{itA}$. A theorem of von Neumann shows that if $\mathcal{H}$ is separable and $\langle T(t)\varphi, \psi\rangle$ is measurable for all $\varphi, \psi \in \mathcal{H}$ then $T(t)$ is strongly continuous and, thus, is of the form e^{itA} for a unique self-adjoint A.

$QP = -iI$. In the $L^2_{\mathbb{C}}(\mathbb{R})$ representation referred to in (iii) of the theorem, Q, P take their familiar form Schrödinger form, viz. $(Q\psi)(x) = x\psi(x)$ and $(P\psi)(x) = -i\hbar\dfrac{d}{dx}\psi(x)$. Extra conditions are needed to guarantee that exponentiating the Q, P satisfying the Heisenberg CCR leads to a satisfaction of the Weyl CCR (see Reed and Simon 1980, p. 275).

There is a generalization of the von Neumann theorem to cover any number of finite of degrees of freedom, but the theorem breaks down when there is an infinite number of degrees of freedom. This situation is commonly parsed by saying that for a system with a finite number of degrees of freedom irreducible and strongly continuous representations of the Weyl CCR are all unitarily equivalent and, in particular, equivalent to the Schrödinger representation, whereas unitarily inequivalent representations arise for systems with an infinite number of degrees of freedom. This gloss is perfectly true, but it leaves out an some important addenda.

The first is a corollary of the von Neumann uniqueness theorem, showing that the hypothesis of a separable $\mathcal{H}$ in the above formulation is unnecessary:

> *Corollary*: If a Hilbert space $\mathcal{H}$ carries an irreducible and strongly continuous representation of the Weyl CCR then $\mathcal{H}$ is separable.

To see this suppose that $U(a), V(b)$, $a, b \in \mathbb{R}$, are strongly continuous groups of unitary operators that act on a Hilbert space $\mathcal{H}$ and satisfy the Weyl relations. Consider a (non-trivial) separable closed subspace $\mathcal{H}^s \subseteq \mathcal{H}$. The $U(a), V(b)$ are strongly continuous on $\mathcal{H}_s$, so by the Theorem $\mathcal{H}^s = \oplus_{j\in\mathcal{I}}\mathcal{H}^s_j$ where the $\mathcal{H}^s_j$ are necessarily separable. Since by conclusion (ii) of the theorem $U(a) : \mathcal{H}^s_j \to \mathcal{H}^s_j$ and $V(b) : \mathcal{H}^s_j \to \mathcal{H}^s_j$ for all $a, b \in \mathbb{R}$, the groups $U(a), V(b)$ would act reducibly on $\mathcal{H}$ if any of the $\mathcal{H}^s_j$ were a proper subspace of $\mathcal{H}$.

When von Neumann published his uniqueness theorem in 1931 this Corollary was not something that would have occurred to him because he was assuming that Hilbert spaces for use in QM are separable. But if he had allowed for the possibility of non-separable spaces then the Corollary might have reinforced his decision to make separability a postulate of his *Mathematical Foundations of Quantum Mechanics*. Doing a *modus tollens* on the Corollary reveals a price tag for working with a non-separable $\mathcal{H}$: if the $U(a)$ and $V(b)$ satisfy the Weyl CCR and together act irreducibly on $\mathcal{H}$ then $U(a)$ and $V(b)$ cannot both be continuous. The mischief this makes for trying to do quantum dynamics on a non-separable Hilbert space will be discussed below.

In addition, it is also notable that even when a finite number of degrees of freedom are involved, non-continuous representations of the Weyl CCR on a non-separable $\mathcal{H}$ can be unitarily inequivalent (see Emch 1981). An example is given in Section 5.2 below.

4.4 Unitary representations of groups

The $U(a)$ and $V(b)$, $a, b \in \mathbb{R}$, of the von Neumann uniqueness theorem are unitary representations of the translation group of $\mathbb{R}$,[30] a special case of a locally compact topological group, and there is now a large mathematical literature on the unitary representations of such groups. But what is relevant for our purposes is that the translation group of $\mathbb{R}$ is separable, and for *any* such group we have

> *Lemma* (Bekka and Harpe 2019): If a Hilbert space $\mathcal{H}$ carries an irreducible and strongly continuous unitary representation of a separable topological group then $\mathcal{H}$ is separable.

To see this start from the fact that a continuous map between topological spaces carries a dense set onto a dense set, so that the continuous image of a separable space is separable. Now let G be a separable group and let $U : G \ni g \mapsto U(g)$ be a unitary representation of G on the Hilbert space $\mathcal{H}$. If U is strongly continuous then $U(G)$ is separable and, thus, for a non-zero $\xi \in \mathcal{H}$ the subspace $\mathcal{K} = \overline{U(G)\xi}$ generated by $U(G)$ is separable. Since $\mathcal{K}$ is invariant under $U(G)$, irreducibility implies that $\mathcal{K} = \mathcal{H}$.

The Corollary of the preceding section can be derived from the Lemma by applying it to the Weyl group formed by combining the $U(a)$ and $V(b)$ of the von Neumann uniqueness theorem into $W(a, b) := \exp(-\frac{1}{2}ab)U(a)V(b)$ with the multiplication rule $W(a, b)W(c, d) = \exp(\frac{1}{2}(ad - bc))W(a+c, c+d)$.

[30] The only continuous unitary representation of the translation group of $\mathbb{R}$ that acts irreducibly on a Hilbert space is one-dimensional. By Schur's lemma a unitary $U(a)$, $a \in \mathbb{R}$, acts irreducibly iff $\{U(a) : a \in \mathbb{R}\}' = \mathbb{C}I$. If $U(a)$ is continuous then $U(a) = \exp(iaA)$ for some self-adjoint A. Since $\{U(a) : a \in \mathbb{R}\}' = \{A\}'$ we get $\{A\}' = \mathbb{C}I$ and, thus, $A = cI$ for some $c \in \mathbb{C}$. This does not rule out that a pair of continuous unitary representations of the translation group of $\mathbb{R}$ act irreducibly on a Hilbert space and produce non-one dimensional representations—if the space is separable.

5 Problematics of quantum physics in non-separable spaces

5.1 Split

The split property expresses a strong form of the independence of algebras associated with relatively spacelike regions; e.g. it implies that if $\mathcal{O}_1$ and $\mathcal{O}_2$ are relatively spacelike regions then the von Neumann algebra generated by the local algebras $\mathfrak{N}(\mathcal{O}_1)$ and $\mathfrak{N}(\mathcal{O}_2)$ associated respectively with $\mathcal{O}_1$ and $\mathcal{O}_2$ is $*$-isomorphic to $\mathfrak{N}(\mathcal{O}_1)\overline{\otimes}\mathfrak{N}(\mathcal{O}_2)$, the von Neumann algebra generated by the tensor product of the local algebras. The split property itself is implied by the nuclearity property, which is thought to be a feature of well behaved models of QFT.[31] So abandoning the split property has some unpalatable consequences; but giving up on a cyclic and separating vector for local algebras is also unpalatable since these are features that the vacuum state is often postulated to possess. As noted in Section 4.2, in a non-separable Hilbert space one of these unpalatable consequences must be swallowed.

5.2 P's, Q's and quantum dynamics

To repeat once more, if $\mathcal{H}$ is non-separable then an irreducible representation of the Weyl CCR on $\mathcal{H}$ cannot be strongly continuous in both $U(a)$ and $V(b)$. But the representation can be continuous in one of them: if it is strongly continuous in $U(a)$ (respectively $V(b)$) then by Stone's theorem the generator Q of $U(a)$ (respectively the generator P of $V(b)$) is a self-adjoint operator. These two representations—the position representation and the momentum representation—are unitarily inequivalent. And there are other unitarily inequivalent representations, in fact an infinite array of them. An explicit construction in which Q but not P, or vice versa, has a complete set of orthonormal eigenfunctions in a non-separable $\mathcal{H}$ is given in Halvorson (2001). The Hilbert space used is $\ell^2_{\mathbb{C}}(\mathbb{R})$, the vector space of square summable sequences of functions $\psi : \mathbb{R} \to \mathbb{C}$ with inner product $\langle\psi_1, \psi_2\rangle := \sum_{x\in\mathbb{R}} \overline{\psi}_1(x)\psi_2(x)$. Note that for each $\psi \in \ell^2_{\mathbb{C}}(\mathbb{R})$ the set of points $x \in \mathbb{R}$ at that $\psi(x) \neq 0$ is at most countably infinite.

[31] The nuclearity condition expresses a precise form of the idea that as the energy of a system increases the energy level density should not increase too rapidly (Buchholz and Wichmann 1986).

So if $\mathcal{H}$ is non-separable there is the problem of choosing among the inequivalent representations of the Weyl CCR. But more fundamentally there is a problem that in no representation are Schrödinger-type Hamiltonians, requiring both self-adjoint operators Q and P, well-defined. The familiar procedure for obtaining a unitary dynamics by exponentiating the Hamiltonian operator is foiled.

Reaction 1. To obtain a dynamics do an end run. Assume that the system of interest is characterized by a von Neumann algebra $\mathfrak{N}$ of observables acting on an $\mathcal{H}$, separable or non-separable. Time translation invariance is then expressed by a one-parameter group α_t, $t \in \mathbb{R}$, of automorphisms of $\mathfrak{N}$ (algebraic form of Heisenberg dynamics in which the observables evolve). For some algebras the $*$-automorphism group α_t is implemented by a unitary group $W(t)$ acting on $\mathcal{H}$, i.e. $\alpha_t(A) = W(t)AW^*(t)$ for all $A \in \mathfrak{N}$. In fact, any von Neumann algebra $\mathfrak{N}$ is $*$-isomorphic to an algebra $\widetilde{\mathfrak{N}}$ in standard form, and for $\widetilde{\mathfrak{N}}$ a $*$- automorphism group is always unitarily implementable (Haagerup 1975). If $W(t)$ is strongly continuous then its generator H is a self-adjoint operator that serves as the system's Hamiltonian, albeit of non-Schrödinger-type if $\mathcal{H}$ is non-separable. If $W(t)$ is not strongly continuous then there is no self-adjoint generator H, and if α_t is not unitarily implementable there is no unitary dynamics. But in any case there is still Heisenberg dynamics as given by α_t.

Problems: That an automorphism group α_t is implemented by a unitary group $W(t)$ does not mean that the automorphism group is inner, i.e. that $W(t) \in \mathfrak{N}$. If the group is not inner then its generator, should it exist, does not count as an observable by the lights of $\mathfrak{N}$ which is supposed to serve as the observable algebra for the system. More fundamentally, without knowing from the beginning what the system's Hamiltonian is, how does one know *which* automorphism group of $\mathfrak{N}$ expresses time translation invariance? Put another way, how does one know that the parameter of the autormophism group α_t, suggestively labeled 't', represents time. (A proposal by Connes and Rovelli (1994) for answering such questions for σ-finite algebras will be considered below in Sec. 5.3.) For that matter how does one know that the system is time translation invariant? Does such an invariance necessarily hold (i.e. is the existence of a Heisenberg dynamics a priori)?

Reaction 2. Attack the problem by turning it into a problem that can be solved. An illustration of this reaction is found in LQG. The Hilbert space $\mathcal{H}_{poly}$ ('poly' for polymorphic) which appears in LQG research is isomorphic to the non-separable $\ell^2_{\mathbb{C}}(\mathbb{R})$ space mentioned above. The Hamiltonian constraint cannot be directly implemented since self-adjoint P and Q satisfying the Heisenberg CCR are not both available. The non-existence of a P operator of Schrödinger form $-i\hbar d\ /dx$ is not alarming from the perspective of LQG since one should not expect such an operator to exist if, as LQG implies, space is not continuous (see Ashtekar et al. 2003, p. 1038). It is thus natural to work in the position representation in which a self-adjoint Q is available. Then one way forward is to "regularize momentum," which means approximating the missing P by a self-adjoint difference operator coupling two points separated by $2q_0$, where q_0 is a regularization constant. It is argued that the freedom in choosing the regulator can be associated with a length scale that results from the discreteness of space (see Ashtekar 2009 and Ashtekar et al. 2003). In the toy example of a harmonic oscillator it is claimed that by taking the length scale small enough the standard Schrödinger quantum mechanical treatment can the approximated to arbitrary precision. A somewhat different point of view of what is involved in the approximation takes the standard Schrödinger quantum mechanics to be the continuum limit of effective theories at different scales (see Croichi et al. 2007a, 2007b).

5.3 Unitary representations of symmetry groups

Doing a *modus tollens* on the Lemma of Section 4.4 reveals that a unitary representation of any separable topological group acting on a non-separable $\mathcal{H}$ must either be reducible or non-continuous. As a consequence non-separable Hilbert spaces cause a headache for the way physicists usually treat symmetry groups in quantum physics. Physicists are usually interested irreducible unitary representations, which for a separable group must be non-continuous on a non-separable $\mathcal{H}$. This clashes with the standard procedure of promoting elements of the Lie algebra of the group to self-adjoint operators that are to serve as the generators of a unitary representation of the group, which is perforce a continuous representation.

5.4 Modular theory and quantum statistical mechanics

5.4.1 Modular theory and the standard form of a von Neumann algebra

A von Neumann algebra $\mathfrak{N}$ acting on $\mathcal{H}$ is said to be in standard form if it has the following property:

> (SF) There is a conjugation $J : \mathcal{H} \to \mathcal{H}$ such that $J\mathfrak{N}J = \mathfrak{N}'$ and $JZJ = Z^*$ for $Z \in \mathcal{Z}(Z)$ (center of $\mathfrak{N}$).[32]

Any von Neumann algebras $\mathfrak{N}_i \subseteq \mathfrak{B}(\mathcal{H}_i)$, $i = 1, 2$, satisfying (SF) have the nice feature that a $*$-isomorphism $\alpha : \mathfrak{N}_1 \to \mathfrak{N}_2$ is implemented by a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$, i.e. $\alpha(A) = UAU^*$ for $A \in \mathfrak{N}_1$, while a $*$-anti-isomorphism γ is implemented by and anti-unitary $V : \mathcal{H}_1 \to \mathcal{H}_2$, i.e. $\gamma(A) = VA^*V^*$.[33]

Modular theory shows that any von Neumann algebra is $*$-isomorphic to an algebra satisfying (SF).[34] In brief outline, the proof goes as follows for σ-finite algebras. Not every σ-finite von Neumann algebra admits a cyclic and separating vector, but every σ-finite von Neumann algebra is $*$-isomorphic to an algebra that admits a cyclic and separating vector. Choose a faithful normal state φ on a σ-finite $\mathfrak{N}$. The GNS representation $\pi_\varphi : \mathfrak{N} \to \mathfrak{B}(\mathcal{H}_\varphi)$ yields a $*$-isomorphism of $\mathfrak{N}$ onto $\pi_\varphi(\mathfrak{N})$. The algebra $\pi_\varphi(\mathfrak{N})$ admits a cyclic and separating vector $\xi \in \mathcal{H}_\varphi$. Define the operator S acting on $\mathcal{H}_\varphi$ by $SA\xi = A^*\xi$. Modular theory proves that S admits a polar decomposition of the form $S = J\Delta^{1/2}$ where J is a conjugation of $\mathcal{H}_\varphi$ and Δ is a densely defined positive self-adjoint operator. Furthermore, J satisfies (SF) for $\pi_\varphi(\mathfrak{N})$ acting on $\mathcal{H}_\varphi$.

Since non-σ-finite algebras do not admit faithful states this construction does not suffice to show that such algebras can be put into standard form. But a work-around can be found since any algebra does admit faithful normal semi-finite weights[35] and since the GNS construction can be applied to

[32] A conjugation $J : \mathcal{H} \to \mathcal{H}$ is a conjugate linear isometry such that $J^2 = I$.

[33] A $*$-anti-isomorphism reverses the order of products. An anti-unitary map is a conjugate linear isometry.

[34] For the relevant mathematical background see Stratila and Zsido (2019)

[35] A weight on $\mathfrak{N}$ is a map from the positive elements $\mathfrak{N}_+$ of $\mathfrak{N}$ to $[0, \infty]$ such that for $A, B \in \mathfrak{N}_+$, $\phi(A + B) = \phi(A) + \phi(B)$, $\phi(\lambda A) = \lambda\phi(A)$ for $\lambda \in [0, \infty]$ with $(0\infty := 0)$, and $\phi(0) = 0$. Per usual faithful means that $\phi(A) = 0$ implies $A = 0$. Normal means that ϕ is

such weights. The above construction can be carried out in the GNS representation $\pi_\phi(\mathfrak{N})$ determined by a faithful normal semi-finite weight ϕ. The upshot is that every von Neumann algebra, σ-finite or not, can be put into standard form where $*$-isomorphisms (respectively, $*$-anti-isomorphisms) are unitarily (respectively, anti-unitarily) implementable.

5.4.2 Modular theory and QSM

In what Leibniz would have counted as an example of preestablished harmony and Wigner would have counted as an illustration of the unreasonable effectiveness of mathematics, the piece of pure mathematics sketched above turns out to be relevant to QSM. For QFT and other systems with an infinite number of degrees of freedom physicists want an analog of the Gibbs equilibrium states used in classical statistical mechanics. They are convinced that they have found it in the notion of KMS (for Kubo-Martin-Schwinger) states (see Haag et al. 1967). Let α_s, $s \in \mathbb{R}$, be a weakly continuous group of automorphisms of $\mathfrak{N}$. A state φ on $\mathfrak{N}$ is said to be a KMS state at inverse temperature $0 < \beta < \infty$ with respect to α_s if for any $A, B \in \mathfrak{N}$ there is a function $f_{A,B}(s)$ analytic on the strip $\{z \in \mathbb{C} : 0 < \operatorname{Im} z < \beta\}$ such that $f_{A,B}(s) = \varphi(\alpha_s(A)B)$ and $f_{A,B}(s + i\beta) = \varphi(B\alpha_s(A))$ for all $s \in \mathbb{R}$.[36] For σ-finite algebras the construction described above supplies KMS states galore. Ignore for ease of presentation the difference between an algebra $\mathfrak{N}$ and its $*$-isomorphic GNS representation $\pi_\varphi(\mathfrak{N})$ induced by a faithful normal state φ. Then we can say that the positive self-adjoint operator Δ defined above gives rise to the automorphism group $\alpha_s^\varphi(A) := \Delta^{is} A \Delta^{-is}$, $A \in \mathfrak{N}$ and $s \in \mathbb{R}$, of $\mathfrak{N}$. And remarkably it turns out that φ is a KMS state at some inverse temperature β with respect to α_s^φ. Note that β can be eliminated by rescaling the group parameter: φ is a (α_s, β)-KMS state if and only if it is a $(\alpha_u, -1)$-KMS state, where $u = -\beta s$. Without any loss of generality mathematicians set $\beta = -1$ and call this form of the resulting KMS condition the modular condition. The situation is summarized in the Tomita-Takesaki theorem:

Theorem (Tomita-Takesaki). Let $\mathfrak{N}$ be a von Neumann algebra

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lower semi-continuous with respect to the ultraweak topology on $\mathfrak{N}_+$. Semi-finite means that if $\phi(A) = \infty$ and $\lambda < \infty$ there is a $B \leq A$ such that $\lambda \leq \phi(B) < \infty$.

[36] To say that β is inverse temperature means that $\beta = 1/kT$ where k is Planck's constant.

acting on a Hilbert space $\mathcal{H}$ and let φ be a faithful normal state on $\mathfrak{N}$. Then there exists a unique one-parameter group of inner automorphisms α_s^φ , $s \in \mathbb{R}$, of $\mathfrak{N}$ such that φ is a $(\alpha_s, -1)$-KMS state.

To be applicable to a physical system it has to be argued that the parameter "s" in α_s^φ denotes time in the sense that α_s^φ does indeed provide the (Heisenberg) dynamics of the system. One famous application provides the mathematical basis of the Unruh effect: the restriction of the Minkowski vacuum state to the algebra of observables associated with a Rindler wedge region of Minkowski spacetime is a KMS state with respect to the automorphism group generated by the Lorentz boosts, leading to the claim that an observer in hyperbolic acceleration through the Minkowski vacuum will find himself in a thermal bath.[37]

The Connes-Rovelli thermal time hypothesis, crudely put, is that it is not necessary that the temporal dynamics be supplied by independent means because time—not just the direction of time but time itself—arises from statistical considerations and that in appropriate circumstances the state φ fixes the time in that parameter "s" of α_s^φ is to be identified with the physical time that governs macroscopic thermodynamical processes when the system is in state φ (see Connes and Rovelli 1994).[38]

There are many fascinating developments that flow from these considerations, but in the present context the important point is that the story of KMS states applies only to σ-finite algebras since the story told above requires that the algebra admits faithful states. The work-around used to show that even non-σ-finite algebras can be put in standard form is of no avail here since σ-finite weights are not capable of representing physical states of anything. Is one to conclude that an alternative approach is needed to describe equilibrium states for systems whose algebras of observables are non-σ-finite? Or is the darker conclusion that systems whose algebras of observables are non-σ-finite simply have too many degrees of freedom to admit equilibrium states?

[37] The temperature associated with an acceleration that a human observer could tolerate is tiny. For an overview of the Unruh effect see Fulling and Matas (2014).

[38] This can be numbered among the "end run" maneuvers mentioned in Section 5.2 for obtaining a dynamics.

6 Conclusion

Various problems with doing quantum physics in non-separable Hilbert spaces have been discussed. These problems show that non-separable spaces disappoint expectations formed from operating in the separable arena and force deviations from the usual ways of doing business in this arena, but none of them presents a crippling roadblock to quantum physics. In any case, it seems shortsighted to lay the blame for perceived problems at the feet of non-separable Hilbert spaces. In parceling out praise or blame the focus should initially be on the algebra rather than the Hilbert space on which it acts. If an adequate description of the observables of a system requires the use of a von Neumann algebra that can only act on a non-separable space (i.e. the algebra at issue is not $*$-isomorphic to an algebra acting on a separable space) then that's the way it is, and it is the algebra that bears responsibility for the resulting difficulties in operating in the non-separable space. Of course, if $\mathfrak{N}$ acts on a non-separable $\mathcal{H}$ but there is an algebra $\mathfrak{M}$ that is $*$-isomorphic to $\mathfrak{N}$ and that acts on a separable $\mathcal{H}'$ then the non-separable $\mathcal{H}$ commits the sin of containing an uncountable number of superfluous dimensions. Otherwise the non-separable $\mathcal{H}$ is guilty of nothing but doing its job of providing a concrete realization of the algebra.

It is not obvious how to restate the intuition that "A realistic space of distinguishable quantum states should be described by a separable Hilbert space" (Fairbairn and Rovelli 2004, p. 2808) as an intuition about observables in such a way as to redound against non-separable Hilbert spaces. Even using the debatable premise that a realistic algebra of distinguishable observables should be σ-finite doesn't suffice since some σ-finite algebras can act only on non-separable Hilbert spaces. In any case, rather than imposing a priori restrictions on "realistic" algebras of observables, and thereby on the spaces on which they act, it seems preferable to allow the full range of mathematical expression in formulating theories and then to judge the theories by the usual criteria of how well they save the phenomena and how well they explain and unify. If theories using algebras that implicate non-separable Hilbert spaces prove wanting in this regard then there is a solid reason to rethink what we count as a genuine observables so as to restore separability. Otherwise there is no good motivation for seeing non-separability as an ill that has to be cured, and the inconveniences entailed by having to work with a non-separable Hilbert space have to be tolerated. If superselection rules are in play then non-separability may be rendered innocuous if the selection sectors are separable. But there is no reason to think that superselection will always or often come to the rescue of separability.

Turning from alleged demerits of non-separable Hilbert spaces to positive virtues, they deserve praise for illustrating the range of conceptual possibilities in quantum theory and for helping to clarify foundations issues. We have already noted that the non-separable $\ell^2_{\mathbb{C}}(\mathbb{R})$ shows how a self-adjoint position (or a momentum operator but not both) can have eigenvectors with point eigenvalues and how unitarily inequivalent representations of the CCR can arise even for systems with a finite number of degrees of freedom (Halvorson 2001, 2004). And despite various no-go results that militate against quantum fields defined at spacetime points, the standard second quantization procedure starting from $\ell^2_{\mathbb{C}}(\mathbb{R})$ yields a Fock space in which field operators at a spacetime point can be given a mathematically rigorous definition (Halvorson 2007). There are no doubt many other examples illustrating the utility of non-separable Hilbert spaces.

von Neumann's wariness regarding non-separable Hilbert spaces was justified in the following sense: if you venture beyond the separable expect surprises, some of them disconcerting and unpleasant. But to step beyond the separable is not to step into a mathematically ill-defined realm but into a territory, some of which is occupied by idealized applications of quantum theory and, possibly, even some non-idealized applications as well. And even that part of the territory that is currently occupied by purely conceptual possibilities is worth exploring for the light it can shed on the foundations of our most successful but most elusive physical theory. That having been said, one is left to wonder whether the lack of successful applications of non-separable spaces to non-idealized systems is nature's way is trying to tell us something about the granularity of quantum systems.

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