Ad hoc identity, Goyal complementarity, and counting quantum phenomena

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Abstract

I introduce a thin concept of ad hoc identity – distinct from metaphysical accounts of either relative identity or absolute identity – and an equally thin account of concepts and their content. According to the latter minimalist view of concepts, the content of a concept has behavioral consequences (though it may not be identical to those consequences), and so content can be bounded if not determined by appeal to linguistic and psychological evidence. In the case of counting practices, this evidence suggests that the number concept depends on a notion of identity at least as strong as ad hoc identity. In the context of nonrelativistic QM in particular, all of the counting procedures appealed to in the existing literature on nonindividuality are shown to involve ad hoc identity. I then show that Goyal Complementarity (Goyal 2019) and the associated derivation of a strong symmetrization principle in QM can be understood in terms ad hoc identity. Specifically, persistence and non-persistence models of quantum processes are seen to be complementary in the sense that they involve two relations of ad hoc identity that occasionally overlap in empirically meaningful ways. Finally, I attempt to draw out consequences for theories of nonindividuality from the above conceptual analysis. The upshot is that counting and definite cardinality are incompatible with nonindividuality, and that none of the counting procedures cited for quantum phenomena offer positive support for an interpretation in terms of nonindividuals.

1 Introduction

There is less a coherent debate about identity in quantum mechanics (QM) than there are three overlapping conversations. The Discernibilists are concerned with the extent to which Leibniz's Principle of the Identity of Indiscernibles (PII) is threatened by the quantum theory of particles of the same type (e.g., electrons).[1] PII asserts that indiscernible things are identical, and since quantum particles of the same type seem to be distinct yet utterly indiscernible, they look like counterexamples. Those engaged in the Discernibilist debate are attempting to settle whether some or all quantum particles of the same type might yet be "weakly discerned" by relations alone.[2] The interlocutors in this discussion presume that it makes sense to speak of the identity of the entities concerned, after all PII cannot be expressed without identity. In fact, the Disceriniblists take the basic units of consideration to be formal objects, "values of variables bound by quantification and subject to predicative identity-criteria, that can in principle be described in elementary predicative formal languages, incorporating elementary predicate logic" (Muller and Saunders 2008, p13). What's at issue is whether – given a framework of formal objects – Leibniz's PII can be maintained in some nontrivial form.

The Nonindividualists, on the other hand, are concerned with a distinct ontological puzzle. Their attention is fixed on an ostensibly underdetermined

[1] In the physics literature, there is an unfortunate tendency to call particles of the same type – and thus possessed of the same intrinsic properties – "identical". See, e.g., (Messiah and Greenberg 1964).

[2] For the core of the debate, see (Muller and Saunders 2008; Muller and Seevinck 2009; Caulton and Butterfield 2012; Muller 2015)

choice of metaphysical packages: the Received View and Primitive Identity.[3] Despite its conservative sounding name, the Received View is quite radical. It asserts that quantum entities of the same type are *nonindividuals* in the sense that they do not stand in relations of identity or nonidentity with one another. They cannot be labeled, cannot be tracked through time, and cannot be the value of a logical variable subject to predicative identity-criteria — in other words, they are *not* formal objects. Alternatively, one might insist that quantum entities – though they are utterly indiscernible – nonetheless possess primitive identities. The point of contention among the disputants in this conversation is how best to flesh out the Received View as a coherent view, and whether or not the underdetermination between the available metaphysical packages can be broken.[4]

Finally, there are the Conceptualists. These folks have been arguing that identity is conceptually fundamental in one way or another. The primary issue is not whether one or another metaphysical package should be preferred but rather whether one can coherently describe such a package without identity. We reason with concepts, and we express this reasoning in language. But not all language successfully expresses a concept. Furthermore, there may be concepts beyond our grasp. The conceptualists worry that any theory which attempts to describe quantum objects with no identity must deploy a concept that simply isn't accessible and that the language of nonindividuality fails to express a concept. A primary motivation for this concern is counting. The proponents of nonindividuality claim that there can be definite (finite) numbers of entities with no identity. Conceptualists are concerned with whether such a description expresses a possible concept. In particular, they worry about the separability

[3]Ontic Structural Realism (OSR) is now typically included as a third option, but won't directly concern us here.

[4]See, for instance, (French and Krause 2006; Da Costa and Krause 2007; Krause and Arenhart 2012; Dorato and Morganti 2013; Arenhart and Krause 2014; Krause and Arenhart 2016).

of the concepts of number (or count) and identity.[5]

I'm one of the Conceptualists. In previous work, I tried to show the ways in which identity and cardinality are intertwined semantically, and argued that their separation is ill-motivated by the empirical facts (Jantzen 2011, 2019). My aim here is both more narrow and more broad. It's narrower in the sense that I wish to explicitly restrict attention to finite collections for which the notion of cardinality and count coincide. It is more broad in that I intend to expand the range of evidence against the separability of the concepts of counting and identity. Conceptual separability is an empirical question. Whether or not two concepts—in this case, identity and counting—can be divorced and left more or less intact is a question about what concepts human beings are capable of possessing, and the latter may only be inferred through similarly contingent facts about the ways in which humans think, speak, and behave. The same applies to the subsidiary question as to whether there can exist a concept similar to counting but without identity.

I argue below that the answer to both questions is robustly negative. The phenomena of psychology and linguistics both imply that some notion of identity is part of the core of the concept of counting, and so one cannot frame a concept of counting without identity. I isolate the requisite notion of identity, and show that it is routinely and systematically at play in the empirical study of quantum systems. In a nutshell, even in QM we count with identity. Finally, I argue that this thin notion of identity is implicit in Philip Goyal's proposed complementarity between models of QM with and without persistent particles. This proposed complementarity has as a consequence exactly the constraints on systems of particles of the same type that are actually observed—all other putative derivations of the salient principle have to additionally rule out unobserved possibilities (so-called paraparticles) by fiat. This is good reason to take Goyal's

[5]See (Bueno 2014; Berto 2017; Krause and Arenhart 2018; Arenhart and Krause 2018).

proposal seriously. But doing so further reinforces the conceptual connection between identity and counting and diminishes the motivation for interpreting many-particle states in terms of nonindividuals.

The remainder of the essay is structured as follows. In sections 2 and 3, I lay out a weak theory of concepts and their observable consequences as well as a novel but thin notion of identity. Section 4 traces the contours of the concept of counting as demarcated by pscyhological and linguistic phenomena. The psycho-linguistic case against the possibility of separating this concept from one of identity is then made in 5. In section 6, I show that counting practices in QM offer no support for separability either. Finally, the role of identity in Goyal complementarity and the consequences of the preceding considerations for theories of nonindividuality are then taken up in the final two sections.

2 Concepts

My thesis is about the relation between two concepts – counting and identity – and so my argument must turn in part on some assumptions about concepts. It's a fool's errand to try and sketch a full theory of concepts that all parties will find compelling. Instead, I'll adopt a few particulars designed to be compatible with as many views on the nature of concepts as possible. Chief among these is dependence: concepts have, if not a compositional structure akin to the grammar of natural languages (as, e.g., Fodor (1975) would have it), then at least a structure of dependence. Some concepts depend upon others in the thin sense that one cannot possess or "grasp" a dependent concept without first possessing the concepts on which it depends. [6] This may be because the dependent concept has other concepts for constituents, or it may be that one ability requires

[6]The survey article by Margolis and Laurence (2019) provides a useful map of the space of views in which to situate the set of assumptions I make about concepts in this section.

another as a precondition. I won't defend any particular interpretation of this dependence, only its existence.

Examples are plentiful at the extremes: the concept RAVEN and the concept WRITING DESK seem to be entirely independent in that one can possess one with all its nuance and rich relations to other concepts without possessing the other. At the extreme of dependence we find the stipulated concepts of mathematics. One cannot have the concept of TRIANGLE without LINE SEGMENT and ANGLE.

If one concept X depends upon another Y, say that Y is a subconcept of X. We want to distinguish subconcepts from what we might call modifiers – concepts that are typically associated with or which enrich a given concept. The intended distinction is easiest to illuminate by illustration. Consider the concept of RAVEN again. Ravens are Maniraptoran dinosaurs. Yet it seems reasonable (since most competent speakers of English are unaware of this fact) to think that BIRD does not depend upon MANIRAPTORAN DINOSAUR; the latter is not a subconcept of the former. Or consider the case of CRYSTAL. Crystals are typically described in museum displays and textbooks in terms of the orderly arrangement of their atoms. But the concept of CRYSTAL predates the Daltonian concept of ATOM,[7] and so it seems reasonable to suggest that the former does not depend upon the latter.

Perhaps the better way to look at these examples is in terms of the evolution of a concept possessed of a hard core and capable of accreting or losing components without significant alteration of this core. So BIRD is really a family of concepts, some including the relation of the modern clade to their extinct dinosaurian cousins, and others not, but all sharing a more or less hard core of content. In these terms, a central question of this paper is whether IDENTITY

[7]Nicolaus Steno, for instance, was using very much the modern concept of CRYSTAL in the mid-seventeenth century while eschewing commitment to any sort of atomism (Steno and Winter 1916).

can be separated from the hard core of the concept of COUNTING. [8]

Another feature of concepts I'll take for granted is that they have behavioral correlates. This is *not* to insist that concepts amount to abilities (pace (Kenny 2010)), or that concepts are identical to a constellation of behavioral dispositions. Rather, it is the far weaker and less controversial claim that possession of a concept tends to manifest in behavioral dispositions, propositional attitudes, and linguistic practices. For instance, if one possesses the concept BIRD, then unlike someone without the concept, one will likely be able to recognize a bird in one's immediate environment, one will be inclined to act in particular ways toward objects that fit the concept (such as not exhibiting surprise when one takes flight), and one is likely to deploy linguistic terms that others recognize as naming or being about birds (i.e., if one has the concept of BIRD, one can talk intelligibly about birds). Again, this is not to equate the concept with the behavior (though one could choose to do so). Nor does it mean that distinct concepts are guaranteed to produce empirically distinct outcomes in possessors of those concepts. But it does mean that different behavioral suites suggest distinct concepts.

Finally, and most controversially, I assert that concepts can be pre-linguistic in either an ontogenetic or phylogenetic sense. That is, I take it that the possession of concepts is at least compatible with any agent capable of intelligent interaction with its environment. This includes human infants who have yet to develop linguistic abilities as well as an enormous range of species that possess no language, or at least no unbounded generative grammar like our own. While

[8]It may be objected that I have implicitly assumed that conceptual content and the relations between concepts are invariant across situations, unaffected by cognitive aims, goals, abilities, and so forth (my thanks to Ted Parent for pointing this out). On the one hand, I do think concepts possess an invariant core, though I'm amenable to a pragmatist reading of what that content is. To make this case, however, would require a discussion of identity conditions for concepts that is beyond the scope of this essay. On the other hand, even if concepts do vary, one can expect empirically accessible manifestations of these variants. In which case, the central question of this paper can simply be amended to ask whether there is any evidence that IDENTITY can be separated from COUNTING in any situation.

this claim conflicts with say, the a priori (and somewhat opaque) concepts of Davidson (1975), there is a surging literature of concept acquisition and content in animals including everything from non-human primates, to birds, molluscs, and insects. [9] So while this claim may be controversial, it is at least not unconventional or fringe.

A consequence of the above assumptions is that we can operationalize the notion of concept dependence. Specifically, we can ask whether the semantics of human languages tend to suggest a well-posed cluster that can be associated with the concept of counting, whether practices within a language permit talk of counting without implications for identity, whether the set of languages suggest that the concepts can be separated, whether behavioral dispositions reflected in psychological experiments support separability or dependence, and whether what's known of the development of the concept of counting ontogenetically comports with its separability from the concept of identity.

3 Ad hoc identity

Concepts of identity fall along at least one axis of disputation at one end of which is absolute (or standard) identity, and at the other is pure relative identity. *Absolute identity* is often glossed as the relation that any given entity stands in with itself and no other.[10] Taken at face value, this gloss is circular since "no other" means "no other non-identical entity". But some essential characteristics of the notion can be stated in a consistent fashion. In a modest break with tradition, I want to frame these characteristics in terms of what I'll call contexts. In the sense in which I'll use it, a *context* is a particular collection of entities

[9] For number concepts alone, see (Dehaene 2011; Koehler 1950; Pepperberg 1987; Skorupski et al. 2018).

[10] What I'm calling "absolute identity" is how so-called "standard identity" (Hodges 1983; Krause and Arenhart 2018) is often interpreted. Deutsch and Garbacz (2018) use the term "absolute identity", paralleling Geach's phrase "strict, absolute, unqualified identity" (Geach 1972, p240).

with a definite collection of relations in which they can stand. A context is analogous to a structure in classical model theory. But they differ in two ways. First, structures involve not just the specification of classes and relations, but the interpretation of the nonlogical symbols of a formal language. A context does not; we can consider contexts regardless of whether any linguistic expressions are in question. Second, for the first- and second- order languages typically discussed in debates over identity, structures are given in terms of sets. I prefer to remain as neutral as possible about the nature of the collections that comprise contexts.

In terms of contexts then, absolute identity involves the following features:

1. In any context, the extension of the identity relation is the diagonal of the collection. That is, for every entity x, $\langle x, x \rangle$ is in the extension of the identity relation, and nothing else is.

2. Leibniz's Law (LL) of the indiscernibility of identicals is presumed to hold, such that, for any entities x and y and any property P,

$$x = y \rightarrow P(x) \leftrightarrow P(y) \tag{LL}$$

3. The relation of identity is *not* first-order definable.

4. The relation holds across contexts in the sense that there is a fact of the matter whether an element of one context is identical with an element of a distinct context. Put differently, for any other collection of entities, there is a fact of the matter whether the intersection between it and the given context is non-empty. If so – if there are elements shared between contexts – then if for one such element, x, $< x, x >$ is in the extension of identity in the first context, then $< x, x >$ is in the extension of identity in the second, and LL holds for the joint context that constitutes the union

of the two contexts. Consequently, the identity of entities is independent of context in that, if two entities are non-identical in one context, they are non-identical in all contexts. So the identity of contexts may depend upon the identity of their constituent entities, but not vice versa.

In contrast, the most radical version of *relative identity* denies the existence of any such relation as absolute identity requires. More specifically, relative identity denies the existence of an identity relation simpliciter. Instead, there is only identity relative to a substantive predicate. One cannot say "x is identical to y" but only "x is the same F as y", since it may be the case that "x is not the same G as y." For my purpose here, I again want to frame this view in terms of contexts:

1. If an identity relation obtains for a context, there must be at least one substantive predicate that also applies; a context with no properties or relations other than identity is similarly devoid of identity.

2. An identity relation (there may be many) in a context is the diagonal of the subcontext consisting of all the entities to which a particular substantive predicate applies.

3. Identity relations obey a limited version of LL for each substantive predicate F such that, for any two entities that are F's, there are *some* additional properties P_i such that

$$x = y \to P_i(x) \leftrightarrow P_i(y) \qquad (\mathrm{RLL}(P_i))$$

4. Each relation of relative identity holds across contexts in the sense that there is a fact of the matter whether an element that is F in one context is the same F as an element of a distinct context. The restricted version of

LL appropriate for F (i.e., RLL(F)) holds for the union of the F's across contexts.

For both absolute and relative identity, there is a presumption of at least partial indentifiability of elements across contexts. What I will call *ad hoc identity* makes no such assumption. Ad hoc identity says that:

1. In any context, the extension of the identity relation is the diagonal of the collection of entities. That is, for every entity x, $\langle x, x \rangle$ is in the extension of the identity relation, and nothing else is.

2. The relation of identity obtains amongst elements of the context regardless of which, if any, substantive predicates they belong to.

3. The identity relation obeys LL *for the entities and relations of the given context.* That is, when restricted to the universe of a context, the identity relation that obtains there respects LL.

4. There need not be any fact of the matter whether any entity in a context is identical with an entity outside of that context. The identity of contexts is prior to the identity of entities within them.

Note that ad hoc identity is compatible with both absolute and relative identity. If it is supplemented with the additional claim that there is always a fact of the matter whether entities in different contexts are identical, then it is equivalent to absolute identity. If supplemented with the claims that (i) at least one substantive predicate applies in every context; and that (ii) for a given substantive that applies in more than one context, there is a fact of the matter whether entities in different contexts to which that substative predicate applies are identical, then is it equivalent to relative identity. So in this sense, ad hoc identity is a weak or thin concept of identity.

Ad hoc identity is, however, strong enough to be inconsistent with the rhetoric of theories of nonindividuality. Within any given context, entities that stand in relations of ad hoc identity can be labeled, can be the values of a logical variable, and *can be counted*. For a collection of nonindividuals, none of these is supposed to apply.

4 The concept of counting in psychology and linguistics

Given that the content of concepts can be probed empirically and given a relatively precise but weak notion of identity, it remains to examine whether ad hoc identity is part of the hard core of the concept of counting. The concept of counting is perhaps best viewed as a pair of related concepts, one narrow and one broad. The narrow concept of counting involves putting entities of a collection in a one-to-one correspondence with the natural numbers – typically via a shift of focus from item to item accompanied by either an internal or external recitation of the natural numbers – so as to assign a natural number as the cardinality or size of the collection using the last numeral recited. The broad notion of cardinality involves only the assignment of numerosity – regardless of how it's accomplished – with the implication that, in principle, one could explicitly enumerate or otherwise tally the entities in the collection (e.g., by making a tally mark for each). In both cases, we're talking about finite cardinalities – the mathematician's generalized concept which can encompass infinite cardinalities is beyond the scope of the ordinary concept, though intimately related to it. The broad sense of counting, along with the ability to systematically compare numerosities, makes up the core of the number concept deployed by psychologists and linguists.

For assessing the relation between identity and counting it is essential to distinguish counting from measuring. Rothstein (2017, p3) summarizes the distinction this way: " Counting is putting individual entities in one-to-one correspondence with the natural numbers and this involves individuating the entities which are to be counted, while measuring involves assigning to a body (plurality or substance) an overall value on a dimensional scale which is calibrated in certain units." In psychology, this distinction is operationalized by controlling for the spatial or temporal size, the visual or auditory intensity, and the specific geometry or frequency structure of the items to be counted. In other words, if subjects' responses depend upon how much area is covered by the items to be counted, or what shape they are, or how long and loudly a tone is played, then the subjects are not counting – they're responding to something other than the *number* of objects. Such a response constitutes instead a sort of magnitude estimate or measuring operation.

A typical experiment to get at the capacity for counting in a verbal subject would involve either showing a subject two collections of shapes simultaneously and asking the subject to choose which contains more or, asking of a single image, "How many?" For pre- or non-verbal subjects, proxies such as time spent gazing at one or another image are used to assess surprise or interest, and are used to assess whether the subject has noticed a difference in numerosity. This procedure of showing images and eliciting a response is repeated many times with a range of images that either keep the overall area covered by the shapes and their integrated intensity constant while varying the number of spatially distinct shapes, or else fixing the number while varying the other, non-cardinal features.[11] Note that distinguishing numerosities in this way does not require the subject to be able to determine whether, say, a particular dot is the same

[11] See, for instance, (Starkey and Cooper 1980; Strauss and Curtis 1981). More often than not, the same pair of contrasting numerosities is presented in many exposures across which all other features are varied.

or different from a dot in a previous experimental exposure, nor does it presume that subjects have such a concept. In the terminology of section 3, each experimental visual scene is a context (provided, presumably, by the nature of our visual systems) in which there *is* presumed to be a relation equivalent to the diagonal of the domain of visually distinct objects, but that relation of identity need not extend beyond that particular image presentation. Thus, the psychological concept of counting involves an identity at least as strong as ad hoc identity. Different practitioners may take stronger views on the nature of the identity relation involved, but one needn't do so to make sense of the experiments or the theories invoked to explain them.

In linguistics, it is widely recognized that the distinction between counting and measuring[12] is gramatically encoded in a broad and typologically diverse set of languages. In some languages like English, the distinction is marked grammatically by a lexical distinction in noun constructions – this is the distinction between count and mass terms. Count terms in English are nouns, noun phrases, or determined noun phrases (noun phrases to which an article or cardinal quantifier has been appended) that can be pluralized and can take an indefinite article. For example, "philosopher" is a count term. It can be pluralized ("philosophers") and can appear with an indefinite article ("a philosopher"). Mass terms, on the other hand, can occur with indefinite quantifiers like "much" or "little" but cannot be pluralized or occur with cardinal or indefinite determiners. Thus, "ice" is a mass term. You can say "a lot of ice" but not "an ice". You can assert there is "much ice" in a glacier but would garner some puzzled looks if you tried to refer to "three ice". Semantically, expressions with number terms (what Rothstein calls "numericals") that are interpreted as measure expressions have the following characteristics (Pelletier 2009) they divide their reference in that they apply to the parts if they apply to the sum of the

[12] I'm using the terminology of Rothstein (2017) in denoting the two semantic categories.

parts, they are cumulative in their reference in that they apply to the sum if they apply to the parts, they refer to stuff that cannot be counted, and they refer to stuff that can be measured in one or more units.

As I said, what sets counting uses of numericals apart from measure uses is that they denote properties of collections of individuated entities – they say "how many." Importantly, this denotation is context dependent; the grammatical context determines the individuation of entities to be assigned a collective cardinality. As Rothstein (2010, pp353-354) sees it, "the crucial difference between count nouns and mass nouns is that count nouns make a set of atoms grammatically accessible, while mass nouns do not. Count nouns do this since they 'presuppose' a set of atoms, and this presupposition makes the set of atoms salient in the discourse and available for the semantics to make use of." By experimentally manipulating grammatical context, one can elicit different interpretations and thus truth assignments for assertions involving count nouns. [13] But this means that to interpret a count expression requires a context in exactly the sense of ad hoc identity; in such a context, we then count by identity.[14]

The semantic distinction between count and measure expressions is robust, though the particulars of grammatical encoding are not. Many languages – like Mandarin Chinese – lack a mass/count distinction at the level of noun phrases,

[13] The relationship between grammatical count and mass terms and the nature of their denotations is rather complex. It is not as simple as mass terms referring to stuff that is naturally unindivuated. "Furniture" for example is a mass term, but the stuff it refers to is clearly naturally atomic. One way to look at it is in terms of whether or not a natural division is gramatically accessible – we can talk about divisible or individuated collections as though they were a stuff by making the salient units inaccessible in grammar, as with furniture (Rothstein 2017). Similarly, substances like water can be addressed with count terms that refer to a standard unit or quantity of the stuff as objects in a collection (e.g., glasses of water). What is constant are these two very different domains of denotation.

[14] Some recent philosophical work suggests there are clear counterexamples to counting with identity. Liebesman (2015), for instance, argues that these include cases such as those emphasized by Salmon (1997) in which we assign fractions as the answer to "how many" as in, e.g., "There are two and half bagels on the table". Liebesman argues from the purported truth conditions of such phrases – if we were counting by identity we have to say either that there are two bagels or three upon the table. But Liebesman's truth conditions are simply not empirically plausible, nor theoretically compelling. See, for instance, Snyder and Barlew (2019), for a comprehensive treatment of Salmon's puzzle that does not require making fractional cardinalities coherent or rejecting the association between counting and identity.

and instead encode the difference in other ways. Mandarin, for instance, encodes the distinction through the use of either classifier phrases headed by a sortal classifier (equivalent to English count nouns) or bare nouns (equivalent to English mass nouns) (Rothstein 2017, section 6.2). And there do exist languages that do not mark the semantic distinction at all in their grammar. In a recent study[15] by Rothstein (2017), the only language encountered without such a distinction was Yudja, a Tupi language spoken by the indigenous Yudja people of the southern Amazonian basin. Importantly for our inquiry, that language has only a counting denotation. In other words, although Yudja has no grammatical mass/count distinction, all nouns are treated as count nouns; even nouns referring to substances like water can be modified by cardinals. If, say, the equivalent of "two" is attached to the Yudja word for "water" one gets an expression that is best interpreted as something like "two (portions of) water". The denotation is a set of individuated portions of water where the nature of those portions is dictated by context and where interestingly, the portions need not be of even approximately the same size. Thus, the only exception to the rule is a language which only has denotations dependent on identity.

Finally and perhaps unsurprisingly, there is research that directly bridges the cognitive psychology of counting and number representation with the linguistic dichotomy of count and measure expressions. In one particularly influential set of experiments, Soja et al (1991) demonstrate that, contra Quine, the prior conceptual categories of object and substance in young children who have yet to master the grammatical mass/count distinction of English strongly guides or determines their induction of word meaning. That is, there is a compelling body of experimental evidence that suggests that the robust cross-linguistic distinction between counting and measuring is underwritten by a pre-linguistic cognitive

[15] This was not a comprehensive cross-linguistic survey, but a project which did aim to canvas a typologically diverse sample.

distinction between number and magnitude. In sum then, the psychological and linguistic literature jointly suggest a concept of counting that involves individuating entities, and this individuation requires a notion of identity at least as strong as ad hoc identity.

5 Evidence for separability?

Though the concepts are clearly associated, does our understanding of the psychology of number or the linguistics of counting offer reason to believe that identity is not part of the hard core of the concept of counting and can therefore be separated, at least in principle? Do we have reason to believe that the concepts *can* come apart, even if they happen not to do so in the cognitive lives of ordinary people? Let's start with a closer look at the current picture of number in psychology. Adult humans, pre-verbal human babies, and an ever-broadening range of nonhuman animals – including primates, birds, and bees – have a capacity for "subitizing" (Jensen, Reese, and Reese 1950; Beran, Perdue, and Evans 2015; Agrillo 2015; Skorupski et al. 2018). This is the "sudden" apprehension of cardinalities between one and three. That is, it's a capacity for immediately assessing – without the use of short-term memory or recitation of names – how many objects there are in a scene when the scene involves one, two, or three such objects. Generally, the ability to judge the cardinality of collections of objects increases only very slowly as the number of objects increases from one to three; in effect, subitizing requires only a constant amount of time – which would not be the case if one were implicitly enumerating each item in an internal recitation of the cardinals – and yet it is highly accurate and precise.

Beyond three things, members of the same menagerie of animals that includes human adults possess a fuzzy sense of pre-linguistic numerosity. That is, non-verbal animals like rats as well as pre-verbal infant humans and adult humans that are given only short exposures to a scene, can nonetheless perceive differences in cardinality or estimate cardinality within some margin of error. For comparative tasks, i.e., tasks involving saying which of two images contains more objects, that error grows as the cardinalities of the collections involved grows and as the difference between collections diminishes. We can easily tell in an eye-blink the difference between 10 things and 20 things, but accuracy plummets to random when asked to discern a difference between 100 and 101 objects. So how do we apprehend exact cardinalities greater than three? Only humans do this, and only humans with mastery of numeral words at that. We do it by counting in the sense you learn in elementary school – by literally pointing or by mentally directing our attention to each object in a collection, reciting a cardinal name, and continuing until every object has been named once and only once. The final name recited is the size of the collection.

Given this fuller picture of the psychology of counting, can the concepts of counting and identity come apart? The process for exact counting clearly requires a relation of identity – we have to know that each object has been labeled, and labeled only once. But the process of subitizing seems on the face of it to support the possibility that identity and counting can come apart. There is agreement that subitizing answers the question, "How many?" The answers are sharp and exact and comprise cardinalities. And yet, there is no possibility of having systematically enumerated each object by attending to them sequentially – there seems to be no need for a relation of identity.[16] But this isn't the case. As Trick and Pylyshyn (1993, 1994) demonstrated, subitizing fails when there is not a clear spatial separation between the objects to be counted. In other words, subitization as a visual process works only when the objects to be counted are

[16] Some have proposed that subitizing is really rapid, subconscious exact counting, involving the serial direction of one's attention to each object counted. But there are some folks for whom it is impossible to apply serial attention – making exact counting of sets containing more than three things impossible – and yet they can subitize with normal acuity (Dehaene 2011, pp 58-59).

individuated by a spatial separation that is significant to our visual systems – our eyes dictate the context in which a relation of ad hoc identity applies. Take away that separation, and we can't subitize. In terms of contexts, our visual system acts through pre-attentive processes to present as distinct entities those spatial regions that do not overlap in contour. It then counts with respect to the ad hoc identity relation of this context.

To make the point even finer, development of an exact number sense requires children to have experience with a counting practice. Kids learn to sequentially point and speak the names of cardinals before they develop any facility in comparing those cardinals (greater than three) (Sarnecka, Goldman, and Slusser 2015). Thus, it seems that for a full-blown count concept to develop at all, the ability to identify objects – deploying a relation of at least ad hoc identity – is essential.

If not psychology, then does linguistics offer any reason to suppose that identity can be excised from the concept of counting? What would affirmative evidence look like? The most obvious would be a culture that speaks a language containing numerical terms expressing finite cardinalities (and a competence in using them to describe arithmetic relations amongst those cardinalities) but which lacks a counting practice, i.e., a practice of matching items one-to-one with number terms, body parts, tally marks, or some other such placeholder. But no such culture is to be found. On the other hand, there do exist cultures with counting practices for limited yet exact numerical assessments, despite not having any counting words. Take, for instance the Mairassis, an indigenous people of the interior of New Guinea. According to a mid-nineteenth century report, they apparently lack any sort of numerals, but can determine modest cardinalities exactly via finger tallies, uttering the single word, "awari", whenever a cardinality is displayed by show of fingers (Conant 1931, p 10). Natives of the islands of the Torres straight similarly lack count words, but tally exactly by pointing at parts of the body according to one of a few systems, depending on the particular island. On Muralug Island, they begin with the little finger of the left hand, point to each digit on that hand to represent up to five, then proceed to point to and name the left wrist, elbow, shoulder, and breast, then the sternum (which corresponds to 10, and then likewise down the right side in reverse order to reach a representation of 19 at the little finger of the right hand (Conant 1931; Everett 2017). Australian aboriginal peoples that speak Warlpiri or Anindilyakwa do not possess counting words. Nonetheless, Warlpiri- and Anindilyakwa-speaking children were found to perform equivalently to English speaking children in number development tasks that require replicating exact numerosities (Butterworth and Reeve 2008). That is, children were shown an array of objects which was then obscured, and children were tasked at placing as many objects as they had seen onto a mat. Unlike English speaking children, speakers of Warlpiri or Anindilyakwa typically replicated the exact spatial arrangement of the set they had seen. The implication, as Butterworth and Reeve (2008) see it is that these children use spatial memory rather than count words to tag items. That is, rather than associate an item with a count word and use this association to reproduce cardinality, the Warlpiri and Anindilyakwa children tag objects with recollected spatial locations.

Where does this leave us? If the concepts of counting (and finite cardinality) and identity are independent, then we should find either a linguistic or psychological context in which the former but not the latter concept is present. In other words, we should be able to find a case in which count numerals and comparative expressions involving numerals exists in the absence of a counting practice, or we should find tasks for which humans can assign or compare numerosity (count, in some sense) without the expectation of the ability to individuate. But we don't. Instead we find universally that development of counting practices occurs without numeral systems but never vice versa, and on an individual basis, we find that a full sense of number magnitude follows (and perhaps depends upon) learning a counting procedure in child development. The psychological and linguistic evidence thus suggests that a notion of identity at least as strong as ad hoc identity is part of the hard core of the concept of counting.

6 Counting quantum entities

So why the effort to describe counting without identity? To great extent, the efforts of the Discernibilists and Nonindividualists have been motivated by ostensible oddities involved in counting quantum entities. At the very least, Nonindividualists have claimed that quantum phenomena *do* offer the kind of affirmative empirical evidence for the separability of counting and identity we were seeking in the last section. I therefore wish here to review in brief the empirical foundation for these assessments. How exactly do we count in quantum experiments?

Broadly speaking, there are two experimental approaches to assigning a finite cardinality to a system of quantum entities. The first, which I'll call the *direct method*, involves counting something in the ordinary sense and then assigning the result to a collection of quantum entities. The *indirect method* involves measuring something and then assigning an approximate numerosity though an algebraic inference. As one example of the direct method, consider the experiments of Frasinski et al. (2013) who blasted a jet of helium atoms with short pulses from a high-energy x-ray laser to create "hollow" atoms with their inner electron orbitals emptied out. Electrons freed from their respective atoms passed through a "magnetic bottle" that converted their energy differences into time-of-flight differences as they ultimately impacted a microchannel plate (MCP) detector. The latter is a disk perforated with many tiny pores whose long axis is at an angle to the surface of the plate. When the top and bottom surfaces of the semiconductor plate are held at high voltage, electrons tend to strike the sides of the pores and cause a cascade of new, lower energy electrons, like an array of tiny charge-multiplier tubes (Wiza 1979). The result is a macroscopically detectable surge in charge (or change in voltage) that can be registered with electronic amplifier circuits. As long as the electrons are of high enough energy (so that their times of flight through the magnetic bottle are sufficiently different) they can be counted by counting pulse trains from the MCP. [17] To be clear, what are being counted directly are spatiotemporally separated events – temporally separated voltage spikes indicative of temporally (and usually spatially) separated charge cascades precipitated by quantum events.

I should note that such a procedure of stripping off electrons that are then irretrievably absorbed (and so cannot be counted twice) was described in abstract terms by Krause and Arenhart (2019) as an empirical schema that suggests the possibility of describing counting without identity. However, it is unclear why it should be taken as as such. What's being counted are voltage events, and these are spatiotemporally individuated, i.e., in the context of the experiment, they clearly stand in relations of at least ad hoc identity or nonindentity. If one wants to infer from these voltage spikes that one is counting electrons (as Frasinski et al. (2013) obviously do), then one must suppose that each event corresponds to a distinct electron detection. The relations of identity that apply to the voltage events provide an ad hoc relation of identity for the electron detection events. Thus, one counts the electrons by appeal to a relation of at least ad hoc identity.

[17] I'm deliberately simplifying the use of MCPs. When the electron energies are lower (near the main photoelectron line), multiple impacts close together produce a single charge (or voltage) spike. Since there cannot be an exact charge associated with each impact event, teasing these overlapping pulses apart is subtle. In the simplest approach, it works like the indirect method I describe later in the text. With more complex readout hardware, simultaneously arrivals can be separated spatially.

This does not entail that the electrons are identifiable in some other context; it does not entail, for instance, that each electron involved in a detection event can be retrieved from the sea of electrons in the lab or even identified in the distinct context of all matter in the laboratory. But insofar as one is counting – assigning a finite cardinality to a collection of electron-detection events – one does so by way of a suitable ad hoc identity relation.[18]

Perhaps a more visceral example of the direct approach – and one not usually raised in the philosophical literature – is the observation of individual atoms through scanning-tunneling microscopy (STM). Atoms of course, are composite quantum entities, but quantum entities nonetheless (witness the Bose-Einstein condensate). In STM, a sharp metal tip (typically one or a few atoms across at it's narrowest), is held a few Angstroms away at high electrical potential relative to some conductive sample of interest, such as the surface of a mass of gold. Once close enough, electrons will pass from tip to sample through the quantum process of tunneling. The rate of tunneling (and thus the measurable current) will depend on the spatial separation between tip and sample. Thus the height of the sample surface can be measured through the tunneling current, and the surface topography can be mapped with a resolution on the order of Angstroms by sweeping the tip across the sample. This allows individual atoms in a lattice to be resolved and, consequently, counted. Note that here, too, counting proceeds in very much the sense sketched above by enumerating spatially distinct regions corresponding to atoms. That is, the ad hoc relation of identity in this case is not provided by time but by space. All in all, the direct approach thus offers no empirical evidence that counting and identity can or do separate.

[18]Though I didn't use the same terminology, I argued elsewhere (Jantzen 2011) that the procedure of Domenech and Holik (2007) for ostensibly counting without individuality does in fact introduce a relation of ad hoc identity – namely membership in a quasi-singleton. That is, y is ad hoc identical to x just in case y belongs to the quasi-singleton of x. The counting procedure they advocate is then just ordinary counting with respect to this ad hoc identity relation.

What of the indirect approach? This is well illustrated by the procedure imagined in broad strokes by Krause and Arenhart (2019, p 69): "...given that we know *the kind of particles* we have in a state, and given that we know the mass of each such element ... we can determine how many objects there are." It's true that such a procedure does not require counting. But to believe that the result provides a *count* still requires an implicit relation of identity. To see why, consider what is actually done in the laboratory in such a context. In fact, the famous Milikan oil drop experiment follows pretty much this recipe (though Millikan was inferring the charge per unit and the count simultaneously) (Bishop, Xian, and Feller 2019). As any erstwhile physics undergraduate can attest, this and similar experiments require the painstaking determination of a real-valued magnitude – in this case the charge of an individual oil droplet in Coulombs. So the actual experimental act is one of measuring. This measure can only be converted to a count if one presupposes a collection of distinct entities with identical masses (and, of course, that these masses compose linearly). Consider this inference in ordinary contexts. If I weigh a container of water, I can divide its mass by that of a standard "cup" of water and arrive at a number indicative of the number of cups of water in the container. But the proceeding locution is still understood in the measure sense – I've merely stipulated a particular unit of measure in terms of some canonical quantity. I have not counted anything. Now suppose that we wish to count jellybeans by weighing a jar full of them, as many an elementary school student has done. If – for sake of argument – all jellybeans have the same known mass, we can get at the number of the jellybeans by dividing the weight of the aggregate by the weight of single jellybean. Typically, one would interpret the result as a count, not a measure. But that's because we have already presupposed that jellybeans can be individuated and counted – that they are naturally divided. To sum up, experimental acts of weighing (or assessing aggregate charge, etc.) are acts of measurement. From such acts, we infer counts, but only by implicitly assuming that the measures pertain to individuated collections. Thus, at the very least, such a procedure does *not* provide a counterexample – it is not an empirical manipulation that must be understood as separating counting from identity, nor is there any guidance in such an experiment as to how one could achieve such a separation.

7 Counting and Goyal complementarity

Though there is a dearth of evidence suggesting that identity can be separated conceptually from counting, there is affirmative reason not to do so. Specifically, some recent work in the foundations of QM suggests that counting with identity in the right way may resolve the mystery of the Symmetrization Postulate (SP). SP asserts that there are only two allowable symmetry types for quantum states (in the labeled tensor-product formulation) – exactly symmetric and exactly antisymmetric. Why should this be? In the physics literature, the typical approach is to assume that swapping particle labels can have no observable consequences and that states are distinct if and only if they differ with respect to some observable (Messiah and Greenberg 1964; Hartle and Taylor 1969). Philosophically, this indifference to particle names in quantum state representations is a prime motivator of the Received View of nonindividuality. However, these postulates are collectively insufficient to entail SP. Instead, the weaker theory they represent allows for not only many-particle quantum states with the kinds of symmetries we observe, but also a slew of alternative state symmetries corresponding to so-called 'paraparticles'. These in turn need to be ruled out by an additional postulate with no obvious independent motivation. The upshot is that we're left with as much of a mystery as we started with, even if the other postulates are granted as somehow intuitively obvious.

In a pair of papers that introduce what I'll call the principle of *Goyal complementarity*, Philip Goyal implicitly appeals to ad hoc identity to derive the SP in the Feynman formulation (2015, 2019). [19] In broad strokes, Goyal asks us to consider two scenarios described by what he calls the *persistence model* and the *nonpersistence model*. In both models, a pair of point detections (of some cluster of properties like mass and charge) are made at locations l_1 and l_2 at time t_1, and then again at locations m_1 and m_2 at time $t_2 > t_1$. According to the persistence model, there are entities—call them particles—that persist through time and that link the measurements in one of two ways: (i) the particle observed at l_1 at t_1 is the same particle that winds up at m_1 at t_2 and mutatis mutandis for the other particle, or (ii) the particle detected initially at l_1 winds up ultimately at m_2 and the other particle goes from l_2 to m_1. QM in the Feynman formulation attributes an amplitude to (i) and (ii). Call these amplitudes α_{12} and α_{21}, respectively.

In the nonpersistence model, we observe the same sequence of events – namely a pair of detections at t_1 and a pair at t_2 – but there is no presumption of the existence of distinct, persistent particles with their own intrinsic properties. Instead, we treat the spatial region of detection along with the observed events as manifestations of a single abstract system. In other words, we posit only the existence of a quantum system evolving in time, the properties of which include particle-like clusters of properties that are spatially localized when measured. The transition amplitude for this system to go from detections at l_1 and l_2 at t_1 to detections at m_1 and m_2 at t_2 is denoted α.

One of Goyal's principal assumptions – which he calls the operational indistinguishability postulate (OIP) – asserts that $\alpha = H(\alpha_{12}, \alpha_{21})$, where H is a

[19] To be more specific, the conceptual framework on which this essay draws appears in (Goyal 2019), while the mathematical results on which it depends are worked out in (Goyal 2015).

continuous complex-valued function. Qualitatively, the OIP "establishes a relation between the theoretical description of two different experiments, positing that the amplitude of a process involving several indistinguishable subsystems (hereafter referred to as particles) is determined by the amplitudes of all possible transitions of these particles when treated as distinguishable" (Goyal, 2015). In other words, the relevant amplitude in the nonpersistence model is some function of the amplitudes describing the persistence model. Or put yet another way, we must ultimately describe the same empirical outcome whichever perspective we adopt.[20]

The second key assumption – the *isolation condition* – is that there are circumstances in which, for the persistence model, it is known (or can be safely assumed) that one only transition amplitude is significantly different from 0. I take this to be an empirically motivated premise. As long as the probabilities of event detection are spatially localized – i.e., within some local region of interest, there is a vanishing probability of detecting more than one particle-like event – then behavior must act as though one of the amplitudes, α_{12} or α_{21} is 0. But that means that, in such circumstances,

$$|H(\alpha_{12}, \alpha_{21})| = |\alpha_{12}|^2. \tag{1}$$

From these assumptions, Goyal is able to exploit the fact that the amplitude from the nonpersistence perspective can sometimes be computed in two ways for multistage experiments. That is, one can consider experiments in which measurements are made at *three* successive times, $t_1, t_2, \text{and}, t_3$. For two detections at each time, the persistence model requires four distinct amplitudes, one for each of the two ways a particle could move from t_1 to t_2 (as above), and one for each of the two ways each could then move from

[20] Note that this generalizes to the case of arbitrarily many particles.

t_2 to t_3. If the latter transition amplitudes are labeled β_{12} and β_{21}, then the four amplitudes of the persistence model for transitions from t_1 to t_3 are $\alpha_{12}\beta_{12}, \alpha_{12}\beta_{21}, \alpha_{21}\beta_{12},$ and $\alpha_{21}\beta_{21}$. OIP entails the existence of a function G such that the amplitude of the transition from t_1 to t_3 in the nonpersistence model is given by $G(\alpha_{12}\beta_{12}, \alpha_{12}\beta_{21}, \alpha_{21}\beta_{12}, \alpha_{21}\beta_{21})$. But since we can also split that transition into time steps, the Feynman formulation tells us that the amplitude in the nonpersistence model can also be represented by a product of amplitudes involving the function H defined above. We thus get a functional equation:

$$G(\alpha_{12}\beta_{12}, \alpha_{12}\beta_{21}, \alpha_{21}\beta_{12}, \alpha_{21}\beta_{21}) = H(\alpha_{12}, \alpha_{21})H(\beta_{12}, \beta_{21})$$

From the set of functional equations of this form, and the assumptions above, Goyal is able to derive Feynman's symmetrization rule for two particles:

$$\alpha = \alpha_{12} \pm \alpha_{21} \tag{2}$$

Importantly, this derivation generalizes to arbitrary finite numbers of particles and admits *only* totally symmetric or totally antisymmetric states. In other words, OIP and the localization principle are sufficient to secure the equivalent of SP; there is no need to rule out paraparticles by fiat.

Goyal describes this derivation as involving a complementarity in Bohr's sense of the synthesis of two mutually exclusive models, in this case persistence and nonpersistence (hence the term "Goyal complimentarity").[21] I suggest that this synthesis can be understood as a consequence of related identity claims. Let's start with the notion of a trajectory implicit in the persistence model. A trajectory in the weak sense in which I intend it is an ordered set of spatial

[21] It is worthwhile to compare with (Dieks 2020).

locations indexed by time. In other words, it is a set of locations X and a mapping, $x : T \to V$, from times $t \in T$ to X. This is more compactly denoted in functional form: $x(t)$. [22] A collection of trajectories (like the paths of multiple particles moving through space) is then a collection of such ordered sets of spatial locations, that is, a set of locations and a set of n maps, $x_i(t)$.

The sets of trajectories described by a persistence model constitute a context in the technical sense introduced in section 3. And in such contexts, there is a relation of ad hoc identity. That is, each trajectory is ad hoc identical with itself and no other in the context, though there need not be any fact of the matter how these trajectories might relate to other things in a different or broadened context. In fact, so far as the Feynman formulation goes, each trajectory is individuated by its sequence of positions (the function $x_i(t)$). Trajectories can cross and share points in common, but no two trajectories are qualitatively indistinguishable. That means that trajectories can be counted in the usual sense. Insofar as such trajectories are idealizations of empirically realizable states of affairs – such as tracks in bubble chambers – this would involve counting of ordinary, spatiotemporally distinguished entities.

Given a context containing trajectories, each time t picks out a subcontext of spatially located events that inherit the ad hoc identity of their parent trajectories. For example, in a context with two trajectories, $x_1(t) \not\equiv x_2(t)$ (where $\not\equiv$ means "not ad hoc identical"), the collection of spatial locations $x_1(t_0), x_2(t_0)$ for a particular time t_0 is also a context. Even if the values of the spatial locations are identical (i.e., if $x_1(t_0) = x_2(t_0)$), it is still the case that $x_1(t_0) \not\equiv x_2(t_0)$ given the relation "is a member of the same trajectory." These spatially located events can thus also be counted in the usual sense, and since there is a one-to-one mapping from entities related by the ad hoc identity relation of the supercontext

[22] The set of times is only required to be non-empty, and may in fact be only finite. In a classical setting by contrast, each object is presumed to have a corresponding trajectory characterized by a continuous mapping from $\mathbb{R}$ to V.

to entities related by the corresponding ad hoc identity relation in the subcontext, the counts must give the same answer. In other words, two spatial events are ad hoc identical if and only if the trajectories of which they are constituents are ad hoc identical. Thus, counting spatial events must yield the same answer as counting trajectories.

In these terms then, we can see that, when the isolation condition is satisfied, the number of trajectories in a persistence model must be the same as the number of spatial detection events in the corresponding nonpersistence model. This isn't quite enough to imply the OIP. But it does mean that the degrees of freedom in one model are fixed by those of the other in certain circumstances. And it does demonstrate that Goyal complimentarity presumes at least an ad hoc identity relation among trajectories and spatial events. It's manifest in equation 1 where the dimensionality of the function H is determined by the number of trajectories summarized by α_{12}. If one took away this identity, the OIP would be implausible.

Goyal complementarity is explanatorily powerful (it gets Bose-Einstein and Fermi-Dirac statistics while ruling out parastatistics) in a way that neither metaphysical packages is (i.e., Primitive Individuality or Nonindividuality). And it depends on conceiving of entities as related by ad hoc identity. This is reason alone to take ad hoc identity seriously as conceptually sufficient. Depending on one's views of theory choice, it's also reason to take seriously the possibility that this concept accurately describes the world.

8 Conclusion

I've introduced thin characterizations of the concepts of counting and identity, and argued that the psychological and linguistic evidence supports the view that ad hoc identity is part of the hard core of the counting concept. If this is right, then these concepts cannot be separated, even in principle. Without the notion of identity, counting has no content. Put differently, there is no concept nearby or similar to the concept of counting that does not invoke at least a weak notion of identity.

What should we take away from this result? I'm inclined to think it deflates the dispute between the ostensibly competing metaphysical packages of Nonindividuality and Primitive Identity. Nonindividuality presumes that the concepts of identity can be pared off the concept of counting. Primitive Identity insists that all identity is absolute. Thus, the former package attempts to deploy a concept of counting that is simply unavailable, and the latter adopts a much stronger view of identity than is necessary.

Furthermore, though proponents of nonindividuality have carefully insisted that either metaphysical package can be made consistent with quantum theory, the space of options is supposed to be motivated by quantum phenomena. But neither view offers satisfying answers to basic questions about the physics of quantum particles. In particular, neither explains why SP obtains. But there do exist approaches that do not require an unavailable counting concept. Goyal complimentarity – which offers a path between the formalism of labeled tensor-product states and a switch to a Fock space formalism devoid of labels – is one such. In the Feynman formalism of QM we can represent the evolution of quantum systems from one of two perspectives. Goyal's OIP insists that these perspectives yield the same answers in certain circumstances (they are re-descriptions of the same thing). But as I suggested above, the OIP can also be seen as a consequence of the inheritance of an ad hoc identity relation from a context of trajectories to a context of instantaneous spatial detections. In other words, embracing the relation between counting and ad hoc identity is not merely consistent with the quantum facts, but serves as a key premise in deriving otherwise mysterious quantum phenomenon. Given the relative sparsity of assumptions involved, this seems to be a superior starting point for building interpretive metaphysical theories.

But of course, there is room yet for debate. It's true that participants in the three conversations I sketched at the outset tend to talk past one another. For instance, the discussion of weak discerinibility makes no sense from the perspective of nonindividuality and vice versa. And it's also true that there is ineliminable variation in the degree to which philosophers will privilege semantics over metaphysics or vice versa. But perhaps derivations like Goyal's and an increased attention to the actual content of entrenched concepts can help us all agree on a bare set of facts from which to build (or refrain from building) metaphysical theories. One such bare fact appears to be the dependence relation between the concepts of counting and identity.

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