The infinite and contradiction: A history of mathematical physics by dialectical approach

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January 18, 2021

Abstract

The following hypothesis is proposed: "In mathematics, the contradiction involved in the development of human knowledge is included in the form of the infinite." To prove this hypothesis, the author tries to find what sorts of the infinite in mathematics were used to represent the contradictions involved in some revolutions in mathematical physics, and concludes "the contradiction involved in mathematical description of motion was represented with the infinite within recursive (computable) set level by early Newtonian mechanics; and then the contradiction to describe discontinuous phenomena with continuous functions and contradictions about "ether" were represented with the infinite higher than the recursive set level, namely of arithmetical set level in second order arithmetic (ordinary mathematics), by mechanics of continuous bodies and field theory; and subsequently the contradiction appeared in macroscopic physics applied to microscopic phenomena were represented with the further higher infinite in third or higher order arithmetic (set-theoretic mathematics), by quantum mechanics".

1 Introduction

Contradictions found in set theory from the end of the 19th century to the beginning of the 20th, gave a shock called "a crisis of mathematics" to the world of mathematicians. One of the contradictions was reported by B. Russel: "Let w be the class [set][1] of all classes which are not members of themselves. Then whatever class x may be, 'x is a w' is equivalent to 'x is not an x'. Hence, giving to x the value w, 'w is a w' is equivalent to 'w is not a w'."[52]

Russel described the crisis in 1959:

> I was led to this contradiction by Cantor's proof that there is no greatest cardinal number. I thought, in my innocence, that the number of all things there are in the world must be the greatest possible number, and I applied his proof to this number to see what would happen. This process led me to the consideration of a very peculiar class. ... At first I thought there must be some trivial error in my reasoning. I inspected each step under a logical microscope, but I could not discover anything wrong. I wrote to Frege about it, who replied that arithmetic was tottering and that he saw that his Law V was false. Frege was so disturbed by this contradiction that he gave up the attempt to deduce arithmetic from logic, to which, until then, his life had been mainly devoted.... : Philosophers and mathematicians reacted in various different ways to this situation. Poincaré, who disliked mathematical logic and accused it of being sterile, exclaimed with glee, "it is no longer sterile, it begets contradiction." ... Some other mathematicians, who disapproved of Georg Cantor, adopted the March Hare's solution: "I'm tired of this. Let's change the subject." ... The contradiction about the greatest ordinal was discovered by Burali-Forti before I discovered my contradiction.... [53]

J. Diudonné, born in 1906, recollected the crisis in 1982:

[1] Words within the square parentheses are supplements by the present author. The same shall apply hereafter.

> There was a period – the famous crisis of foundations, which began in about 1895 and continued to 1930 – when many mathematicians were greatly troubled by the paradoxes and the difficulties of reasoning that seemed to emerge from everywhere. I believe that the mathematicians of that generation and of mine – which is later – experienced a personal crisis. During all an year, I spent my time to fabricate an logical system which satisfies me – I did not publish it, of course – , for I was so troubled that I feel the need to prove for myself that one can do mathematics by a totally coherent way.[15]

Speaking against the crisis, D. Hilbert said, "Mathematics in a certain sense develops into a tribunal of arbitration, a supreme court that will decide questions of principle – and on such a concrete basis that universal agreement must be attainable and all assertions can be verified." Concerning the use of the infinite which caused the crisis, he stated: "The right to operate with the infinite can be secured only by means of the finite." And to achieve the purpose, he proposed to express all the propositions and the inferences in mathematics with "an inventory of formulas that are formed from mathematical and logical signs and follow each other according to definite rules" and to prove consistency of the formal system by finite method. [33]

In 1931, however, K. Gödel showed that the formal system which contained mathematical induction process [with the lowest level of the infinite], could not prove its own consistency.[27] H. Weyl said in 1947: "Hilbert's hope to establish consistency of formally equivalent system to classic mathematics was crushed by discovery by K. Gödel in 1931. This discovery caused doubt about the whole project. Since then the general attitude was that of resignation. Ultimate foundation and ultimate meaning of mathematics are yet unsolved question." [60]

The above history of studies on foundations of mathematics shows that the thought since Plato, which seeks "a model of reliability and truth"[33] in mathematics, should be corrected. Then what is the role of mathematics in science or in the understanding of Nature by human beings? What is the objective meaning of the process of "proof", which has been assumed to guarantee the reliability of inference? These problems seem to be very important to those who are engaged in mathematical studies.

But these problems are now left to logicians or philosophers. Moreover, according to J. Diudonné, "95% of mathematicians have no interest in works of all the logicians and all the philosophers."[15]

Hilbert contrasted the crisis of mathematics with the confusion about infinitesimals in the 18th century. It seems that contemporary mathematicians, like those of the 18th century, are going forward putting aside problems about principles, also following the advice by d'Alembert, "Go forward! Then belief will come to you. (Allez en avant, et la foi vous viendra.)" [14]

As for the philosophy of mathematics which deals with problems of principles, A. Robinson reported in 1973:

> [In spite of technical progress in foundations of mathematics] the evolution of our understanding of the essential nature of mathematics has been hesitant and ambiguous, and in any case the conclusions that have been reached by one school of thought have been rejected by another.... There is no evidence that a more trenchant kind of progress in this area is at all likely."

And as one of possibilities of development in future, he said:

> For example, I can well imagine that a serious mathematical philosophy based on the dialectical approach will make its appearance. It seems to be this approach has already shown its great value in connection with our understanding of the evolution of scientific theories and their heuristic aspect. As far as detailed analysis of mathematics or of mathematical theory (e.g. the calculus) is concerned, my reading, beginning with Hegel's work in this area, had not led me to find anything that can stand up to serious criticism. It is quite possible that this situation will be remedied in the future."[54]

As described above, the classical viewpoint of mathematics was discredited by the problem of contradictions in set theory. On the other hand, dialectical philosophy positively accepts contradictions as the principle of the development of things. Hence, in particular, such problem should be resolvable by dialectical approach.

If mathematics is a system developed strictly with only formal logic, it should become a sterile system which only concludes what is already included in premises. But in fact, mathematics is creative, because it includes breaches of formal logic, that is, contradictions. In mathematics, the contradiction is included in the form of the infinite. And, definitions and proofs in mathematics, at least those which include infinite processes, are procedures which give the form of the infinite to the contradiction or inconsistency between established systems of concepts and new observations or concepts obtained through the progress of recognition.

H. Poincare also attributed creativity of mathematics to the infinite:

> The very possibility of mathematical science seems insoluble contradiction. ... If ... all the propositions which it enunciates may be derived in order by the rule of formal logic, how is it that mathematics is not reduced to a gigantic tautology? ...
>
> The process [to solve the contradiction] is proof by recurrence. We first show that a theorem is true for n=1; we then show that if it is true for n-1 it is true for n, and we conclude that it is true for all integers. [mathematical induction] ...
>
> When we take in hand the general theorem, it [mathematical induction] becomes indispensable. In this domain of Arithmetic we may think ourselves very far from the infinitesimal analysis, but the idea of mathematical infinity is already playing a preponderating part, and without it there would be no science at all, because there would be nothing general.

And concluded:

> We can only ascend by mathematical induction, for from it alone can we learn something new. ... Let me observe, this induction is only possible if the same operation can be repeated indefinitely [infinitely]. [50]

Human beings want to understand phenomena in the infinitely varying objective world using finite and fixed concepts and its quantitative extension, which correspond to measures of recognition already available to human beings. But when once they come across new observations which cannot be understood by means of the already existing system of concepts, the phenomena appear as contradictions in the system of concepts. Then we need to find new concepts to describe the new phenomena and the laws which rule the new concepts, and need to incorporate them into the existing knowledge system. To achieve these purposes, we need inference forms which extend existing concepts quantitatively as far as possible and, furthermore, allow qualitative change jumping over limits of application of the existing concepts. The inference forms using the infinite are the solutions to these problems.

For example, in the 5th century B. C., Pythagorean school found that if a side of a squire and its diagonal had any common scale, contradiction arose inevitably.[31] To represent the assumed ratio of the side to the diagonal (incommensurable quantity), a new concept "irrational number" should be introduced. And to prove that the same operations as rational numbers can be applied to irrational numbers, and to integrate them to real number system, the irrational number as a whole needed to be defined using infinite set of rational numbers.[16],[9] That is, the contradiction appeared in rational number system, or the qualitative jump from rational numbers to irrational numbers (or from discrete numbers to continuous quantity) was represented with the mathematical infinite through the definition of irrational numbers with infinite set of rational numbers.

Mathematics, as a science of quantity, has developed various forms of inference using the infinite and has offered them to other sciences as measures to connect and systematize qualitatively different concepts. On the other hand, as a result of pursuing universal validity of its inference forms, adopted only finite formal logic to be replaced by mechanical devices, with exception of the infinite which cannot be reduced to the formal logic. In mathematics, therefore, contradictions involved in the development of human knowledge or so called "mysteries of Nature" are required to be expressed as the infinite. Procedures that satisfy the requirement are proofs and definitions in mathematics.

Why the mathematical infinite can represent contradictions in systems of concepts? It is because the concept of the mathematical infinite itself contains a contradiction. The mathematical infinite is defined as something unreachable through iteration of some process. But when it is used in inference, it is treated as an object in the extension of the same process, that is, something already reached. It is a contradiction. The mathematical infinite is used at first unreachable "potential infinite" corresponding to a particular process in mathematics. But G. Cantor cut off the mathematical infinite from the particular process by creating set theory, and using the concept of set, presented abstract infinite itself, so called "actual infinite," as an object of mathematics. In other words, set theory can be said a field of mathematics to study "mysteries of Nature" abstracted and generalized in the form of the mathematical infinite.

As to the fact that by extending finite process to the infinite, qualitative change exceeding mere generalization can arise, G. Galilei already pointed out in 1638:

> These are some of the marvels...which should warn us against the serious error of those who attempt to discuss the infinite by assigning to it the same properties which we employ for the finite, the nature of the two having nothing in common,...
>
> I must tell you of a remarkable property ... which will explain the vast alteration and change of character which a finite quantity would undergo in passing to infinity. [28]

Galilei presented here the following example: the sequence of squired numbers: 1, 4, 9, 16, ...is a part of the sequence of natural numbers: 1, 2, 3, 4, ... In a finite range of the latter, the existing ratio of the former is less than 1, and as the range is prolonged it approaches to 0. But when the range is prolonged to the infinite, the ratio turns to 1. Because by corresponding every natural number to its squire, all the natural numbers correspond to all the squired numbers and the reverse is also true.

In the present paper, starting with Newton's case, the present author tried to work out what sort of the infinite were used to represent the contradictions involved in some revolutions in mathematical physics, in order to demonstrate historically the above hypothesis.

2 Newtonian mechanics

I. Newton summarized his early research in mathematics in his paper dated October 1666. [46]

His paper begins with the statement: "To resolve Problems by Motion these following Propositions are sufficient." Then Newton listed the propositions numbered from 1 to 8 like axioms. Among them, propositions No. 7 and No. 8 describe differential and integral calculus respectively as follows (Newton did not use the term "differential calculus" or "integral calculus"):

> 7. Having an Equation expressing y^e relation twixt two or more lines $x, y, z,$ &c: described in y^e same time by two or more moveing bodyes $A, B, C,$ &c: the relation of their velocitys $p, q, r,$ &c may bee thus found, viz:
>
> Set all y^e termes on one side of y^e Equation that they become equall to nothing. And first multiply each terme by so many times $\frac{p}{x}$ as x hath dimensions in y^t terme. Secondly multiply each terme by so many times $\frac{q}{y}$ as y hath dimensions in it. Thirdly (if there be 3 unknowne quantitys) multiply each terme by so many times $\frac{r}{z}$ as z hath dimensions in y^t terme, (& if there bee still more unknowne quantitys doe like to every unknowne quantity). The summe of all these products shall bee equall to nothing. w^ch Equation gives y^e relation of y^e velocitys $p, q, r,$ &c.
>
> ...
>
> 8. If two Bodys A & B, by their velocitys p & q describe y^e lines x & y. & an Equation bee given expressing y^e relation twixt one of y^e linens x, & y^e ratio $\frac{q}{p}$ of their motions q & p; To find y^e other line y...

First get the valor of $\frac{q}{p}$. Which if it bee rationall & its Denominator consist of but one term: Multiply y^{t} valor by x & divide each terme of it by y^{e} logaritheme of x in y^{t} terme y^{e} quote shall bee y^{e} valor of y. As if $ax^{\frac{m}{n}} = \frac{q}{p}$. Then is $\frac{na}{m+n} x^{\frac{m+n}{n}} = y$ ...

But this eighth Proposition may bee ever thus resolved mechanically. viz:

Seeke y^{e} Valor of $\frac{q}{p}$ as if you were resolving y^{e} equation in Decimall numbers either by Division or extraction of rootes or Vieta's Analyticall resolution of powers; This operation bee continued at pleasure, y^{e} farther the better. & from each terme ariseing from this operation may bee deduced a parte of y^{e} valor of y, (by pte y^{e} 1st of this prop).

Example 1. If $\frac{a}{b+cx} = \frac{q}{p}$. Then by division is

$$\frac{q}{p} = \frac{a}{b} - \frac{acx}{bb} \frac{+accxx}{b^3} - \frac{ac^3x^3}{b^4} + \frac{ac^4x^4}{b^5} - \frac{ac^5x^5}{b^6} + \frac{ac^6x^6}{b^7} \text{ \&c.}$$

And consequentry

$$y = \frac{ax}{b} - \frac{acxx}{2bb} + \frac{accx^3}{3b^3} - \frac{ac^3x^4}{4b^4} + \frac{ac^4x^5}{5b^5} \text{ \&c.}$$

Newton expanded functions in question into power series, and applied termwise differential or integral calculus.[47] Therefore it is sufficient only to apply such simple methods of calculus as the above propositions 7 and 8.

V. I. Arnol'd wrote: "Newton understood by analysis the investigation of equations by means of infinite series. In other words, Newton's basic discovery was that everything had to be expanded in infinite series." [1]

The velocities p, q, r... occurring in those equations are quantities varying with time, and in the case of accelerated motion, distances moved actually with these velocities at each moment are zero. Consequently, these values cannot be calculated by the conventional definition of velocity: the ratio of distance and time of motion. Newton calculated the velocities $p, q, r, \ldots$ from the coordinates $x, y, z, \ldots$ using the above methods of calculus.

The referred paper shows that Newton's differential and integral calculus are parts of dynamics, and entirely developed to represent motions. Therefore, in his paper, independent variables are always time "t," and the quantity $\frac{dy}{dx}$ in modern style is written as $\frac{q}{p}$. Afterwards Newton named the coordinates $x, y, z, \ldots$ of moving bodies "fluence," and the velocities $p, q, r, \ldots$ varying with time (velocities at each moment) "fluxion," and represented them with symbols $\dot{x}, \dot{y}, \dot{z}, \ldots$.

Fluxion or velocity in a moment was essential to represent accelerated motion mathematically, and to represent the relation between force and motion: "force = mass × acceleration," and hence to develop mathematical dynamics. As Zeno of Elea stated: "The moving body moves neither in the place where it exists, nor where it does not exit"[17], motion cannot be described by directly observable concepts such as existence, non-existence, position or time.

In the same paper dated October 1666, Newton justified the fluxion method (differential and integral calculus) using the infinitesimal o, the sign of the infinitely small quantity, as follows.

To demonstrate the proposition 7, he presented an equation relating the fluences x and y (coordinates of moving bodies A and B respectively): $x^3 - abx + a^3 - dyy = 0$ and define p and q as fluxions of x and y respectively. Then he obtained the relation between p and q:

I may substitute $x + po$, & $y + qo$ into y^{e} place of x & y; because (by y^{e} lemma) they as well as x & y, doe signify y^{e} lines described by y^{e} bodys A & B. By doeing so there results

$$x^3 + 3poxx + 3ppoox + p^3o^3 - dyy - 2dqoy - dqqoo = 0$$

$$-abx \quad - abpo$$

$$+a^3$$

But $x^3 - abx + a^3 - dyy = 0$ (by supp). Therefore there remains onely

$$3poxx + 3ppoox + p^3o^3 - 2dqoy - dqqoo = 0$$

$$-abpo$$

Or dividing it by o tis

$$3px^2 + 3ppox + p^3oo - 2dqy - dqqo = 0.$$

$$-abp$$

Also those termes are infinitely little in w$^{\text{ch}}$ o is. Therefore omitting them there rests

$$3pxx - abp - 2dqy = 0$$

The like may bee done in all other equations.

Hence I observe. First y$^{\text{t}}$ those termes ever vanish w$^{\text{ch}}$ are not multiplyed by o, they being y$^{\text{e}}$ propounded equation. Secondly those termes also vanish in w$^{\text{ch}}$ o is of more y$^{\text{n}}$ one dimension, because they are infinitely lesse y$^{\text{n}}$ those in w$^{\text{ch}}$ o is but of one dimension. Thirdly y$^{\text{e}}$ still remaining termes, being divided by o will have y$^{\text{t}}$ form w$^{\text{ch}}$, by y$^{\text{e}}$1$^{\text{st}}$ rule in Prop 7$^{\text{th}}$, ...

....

Prop 8$^{\text{th}}$ is y$^{\text{e}}$ Converse of this 7$^{\text{th}}$ Prop. & may be therefore Analytically demonstrated by it. [46]

To represent motion mathematically, Newton introduced the concept of fluxion, and thus established the foundation of the fluxion method using infinitesimals. That is, he represented the contradiction of motion with the infinite in the form of the infinitesimal "o".

In 1734, G. Berkeley criticized the fluxion method :

> Hitherto I have supposed that x flows, that x hath a real increment, that o is something. And I have proceeded all along on that supposition, without which I should not have been able to have made so much as one single step. From that supposition it is that I get at the increment of x^n, that I am able to compare it with the increment of x, and that I find the proportion between the two increments. I now beg leave to make a new supposition contrary to the first, i.e. I will suppose that there is no increment of x, or that o is nothing; which second supposition destroys my first, and is inconsistent with it, and therefore with every thing that supposeth it. I do nevertheless beg leave to retain nx^{n-1}, which is an expression obtained in virtue of my first supposition, which necessarily presupposeth such supposition, and which could not be obtained without it: All which seems a most inconsistent way of arguing,....
>
> ..., as it is impossible to conceive velocity without time or space, without finite length or duration, it must seem above the powers of men to comprehend even the first fluxions. [2]

The criticism by Berkeley shows that fluxion (velocity in a moment) or fluxion method (differential calculus) which was created by Newton to describe motion mathematically, resolving the contradiction of motion pointed out by Zeno of Elea, contains contradictions for conventional systems of concepts which apply formal logic to measurable quantities such as position, time, or distance.

The confusion around the infinitesimals was ended by the studies of A. L. Cauchy published in 1820s.

Cauchy wrote in the preface of his text book of infinitesimal calculus : "Those who read my book will be convinced, I hope, that the principles of differential calculus and its most important applications can be easily explained without the help of series."[11]

To avoid the use of infinitesimals which were the target of controversy in those days, J. L. Lagrange and S. F. Lacroix algebraically defined differentiation or the derivative as the coefficient of power series expansion of functions. Cauchy's phrase "the help of series" indicate this situation.

In his textbook of analysis published in 1821, Cauchy introduced the following intermediate-value theorem.

> If the function $f(x)$ is continuous with respect to the variable x between the limits $x = x_0$ and $x = X$, and b denotes a quantity between $f(x_0)$ and $f(X)$, we may always satisfy the equation
>
> $$f(x) = b$$
>
> by one or more real values of x contained between x_0 and X. [12]

Then in the textbook of infinitesimal calculus published in 1823, whose preface was quoted above, Cauchy defined derivative function $f'(x)$ as the limit of the ratio of infinitely small increases

$$\frac{f(x+i) - f(x)}{i}$$

and proved the following mean-value theorem using the intermediate-value theorem.

> If the function $f'(x)$ is continuous between the limits $x = x_o$ and $x = X$, we designate by A the minimum, and by B the maximum of values which the derivative function $f'(x)$ takes in that interval, then the ratio of finite differences:
>
> $$\frac{f(X) - f(x_0)}{X - x_0}$$
>
> should be comprised between A and B. [11]

From this theorem he derived the following equation:

$$\frac{f(x+h) - f(x)}{h} = f'(x + \theta h)$$

where h means small increase of x, and θ designates a positive number smaller than 1.

Also by writing Δx instead of h in this equation he gave the equation:

$$f(x + \Delta x) - f(x) = f'(x + \theta \Delta x)\Delta x$$

Cauchy defined the definite integral also using the intermediate-value theorem. [11]

Later F. Klein commented, "To be sure, the differential quotient was defined as a limit [by other mathematicians], but there was lacking a method for estimating, from it, the increment of the function in a finite interval. This was supplied by the mean-value theorem; and it was Cauchy's great service to have recognized its fundamental importance and to have made it starting point accordingly of differential calculus. And it is not saying too much if, because of this, we adjudge Cauchy as the founder of exact infinitesimal calculus in the modern sense." [35]

Note that, classical analysis created by Newton and founded by Cauchy, dealt with only continuous and differentiable functions.

Hence, it can be concluded that Cauchy established the foundation of classical analysis created by Newton, by means of the intermediate-value theorem.

Then with what sort of the infinite did Newtonian mechanics represent the contradiction of motion mathematically?

Since the 1970s H. Friedman[26], S. G. Simpson and others have been developing "Reverse mathematics," whose main question is "Which set existence axioms are needed to prove the theorems of ordinary mathematics?"[56]

Reverse mathematics shows that the intermediate-value theorem is proved by RCA_0: the system of Recursive Comprehension Axioms in second order arithmetic (ordinary mathematics). [56]

RCA_0 consists of basic axioms of logic and arithmetic, plus the following two axioms:

1. Δ^0_1 comprehension scheme:

$$\forall n(\phi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n(n \in X \leftrightarrow \phi(n))$$

where set variable X is not free in $\phi(n)$.

2. Σ^0_1 induction:

$$\phi(0) \wedge \forall n(\phi(n) \rightarrow \phi(n+1)) \rightarrow \forall n \phi(n)$$

In the above two axioms,

$\phi(n)$ is the formula which does not contain the universal quantifier $\forall$, and contains an existential quantifier $\exists$ for number variables, that is to say, $\phi(n)$ is Σ^0_1 formula.

$\psi(n)$ is the formula which does not contain $\exists$, and contains an $\forall$, that is to say, $\psi(n)$ is Π^0_1 formula.

$\phi(n)$ and $\psi(n)$ do not contain $\forall$ and $\exists$ for set variables. [56]

Informally;

The axiom 1 means "for all n, if a Σ^0_1 formula $\phi(n)$ is equivalent to a Π^0_1 formula $\psi(n)$, then there exists a set X of n which satisfies $\phi(n)$.

The axiom 2 means "if $\phi(0)$ is true, and for all n if $\phi(n+1)$ is derived from $\phi(n)$, then for all n $\phi(n)$ is true."

The set existence axioms of RCA_0 are strong enough to prove the existence of a recursive (computable) set of natural numbers.

RCA_0 is viewed as a formal version of computable or constructive mathematics.[56] To prove a theorem in RCA_0, the algorithm (computing procedure) to computing it should be composed. For example, in the case of the intermediate-value theorem, because the intermediate-value can be computed by the algorithm to divide ranges into 2 parts successively, this theorem is valid in RCA_0.[57] Though the theory of continuous function can be developed within RCA_0, further general analysis cannot be developed within it.

In conclusion, Newtonian mechanics in the 17th century, which described motion mathematically for the first time, with functions to be expanded into power series, represented the contradiction of motion with the infinite of not exceeding recursive set level.

3 Mechanics of continuous bodies; theory of fields

Mathematicians and physicists in the 18th century who succeeded Newton and Leibniz, faced new contradictions along with the further evolution of mathematical physics in the fields of mechanics of continuous bodies, for example in the theory of vibrating strings.

B. Riemann described the situation:

> The shape of a string under tension that is vibrating in a plane is determined by the partial differential equation
>
> $$\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$$
>
> where x is the distance of an arbitrary one of its points from the origin and y is the distance from the rest position at time t. Furthermore α is independent of t, and also of x for a uniform thickness.
>
> D'Alembert was the first to give a general solution to this differential equation.

He showed that ....

$$y = f(\alpha t + x) - f(\alpha t - x)$$

[to be the general solution of the problem.]

Euler made a basic advance, giving a new presentation of d'Alembert's work ....

Daniel Bernoulli presented a third treatment of this topic, which was quite different from the previous two....

The observation that a string could simultaneously sound different notes now led Bernoulli to remark that the string (by the theory) could also vibrate in accordance with the equation

$$y = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi\alpha}{l}(t - \beta_n)$$

Further, since all observed modifications of the phenomenon could be explained by this equation, he considered it the most general solution....

It seemed impossible to represent an algebraic curve, or in general an nonperiodic analytically given curve, by the above expression. Hence Euler thought that the question must be decided against Bernoulli.... [2]

This induced the young, and then little known, mathematician Lagrange to seek the solution of the problem in a completely new way, by which he reached Euler's result.... Concerning Bernoulli's result, all three agreed not to consider it as general. ...

Almost fifty years had passed without a basic advance having been made in the question of analytic representation of arbitrary function. Then a remark by Fourier threw a new light on the topic. A new epoch in the development of this part of mathematics began, which soon made itself known in a wonderful expansion of mathematical physics. ...

Fourier, in one of his first papers on heat, which was submitted to the French academy (December 21, 1807) first announced the theorem, that an arbitrary (graphically given) function can be expressed as a trigonometric series. This claim was so unexpected to the aged Lagrange that he opposed it vigorously. [51]

Fourier stated in his "Analytical Theory of Heat" (1822):

The series formed of sines or cosines of multiple arcs are therefore adapted to represent, between definite limits, all possible functions, and ordinates of lines or surfaces whose form is discontinuous. Not only has the possibility of these developments been demonstrated, but it is easy to calculate the terms of the series; the value of any coefficient whatever in the equation:

$$\phi(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + ... + a_i \sin ix + \text{etc.}$$

is that of a definite integral:

$$\frac{2}{\pi} \int \phi(x) \sin ix dx$$

Whatever be the function $\phi(x)$, or the form of the curve which it represent, the integral has a definite value which may be introduced into the formula. The values of these definite integrals are analogous to that of the whole area $\int \phi(x) dx$ included between the curves and the axis in a given interval, or to the values of mechanical quantities, such as the ordinates

[2] "The question then at once arose whether d'Alembert's arbitrary function was capable of expansion into such a sine series. To Euler this seemed unthinkable. It was, so to speak, against the laws of the game, it was contrary, to the rules of analysis that arbitrary, non-periodic functions could be represented in terms of periodic functions." E. B. Van Vleck [58]

> of the centre of gravity of this area or of any solid whatever. It is evident that all these quantities have assignable values, whether the figure of the bodies be regular, or whether we give to them an entirely arbitrary form.
>
> If we apply these principles to the problem of the motion of vibrating strings, we can solve difficulties which first appeared in the researches of Daniel Bernoulli. The solution given by this geometrician assumes that any function whatever may always be developed in a series of sines or cosines of multiple arcs. Now which consists in actually resolving a given function into such a series with determined coefficients. [25]

Y. Kondou remarked:

> The reason why such a great mathematician as Euler could not discern the remarkably excellent ability of the representation of trigonometric series presented by Bernoulli, may be that the series were such common and familiar ones as trigonometric series. Now when these common beings make such a composition of infinitely many terms as infinite series, they show the great power with a further leap in quality. This situation could not be discerned from the view point of mathematicians of the 17 and 18th centuries who could see the infinite only as a simple extension of the finite. [36]

Thus in the 18th century, along with the progress in knowledge of Nature, newly appeared contradictions in conventional continuous expression applied to discontinuous phenomena or in conventional periodic expression applied to non-periodic phenomena. And these contradictions were again represented with the infinite through the introduction of infinite trigonometric series (Fourier series).

Then with what sort of the infinite did Fourier series represent the above contradictions?

This sort of the infinite can not be of recursive set level, which contains at most continuous functions. Reverse mathematics shows that general analysis requires the infinite higher than that of recursive set level. That is to say, general analysis requires the following arithmetical comprehension axiom:

$$\exists X \forall n (n \in X \leftrightarrow \phi(n))$$

Where $\phi(n)$ represents an arbitrary arithmetical formula, which does not contain $\forall$ nor $\exists$ for set variables, and the set variable X does not occur as a free variable in $\phi(n)$. [56]

Informally, the above formula means "there exists the set X of n which satisfies an arbitrary arithmetical formula $\phi(n)$." Reverse mathematics show that ACA_0, the system of Arithmetical Comprehension Axioms, which consists of the system of axioms RCA_0 plus the above arithmetical comprehension axiom, is equivalent to the Bolzano/Weierstrass theorem: "Every bounded sequence of real numbers contains a convergent subsequence," and also to fundamental theorems of analysis such as "Every Cauchy sequence of real numbers is convergent," and " Every bounded sequence of real numbers has a least upper bound." [56] (The infinite sequence $x_1, ...x_n...$ of numbers is named Cauchy when $|x_m - x_n| < \epsilon$ is valid for arbitrary positive ϵ for sufficiently large m, n). These theorems are essential to develop the theory of Fourier series. For example, U. Bottazzini pointed out that G. Cantor's work on the uniqueness of the convergence of Fourier series contains as its "indispensable premise" his theories of real numbers and of derived set [5], which are based on Cauchy sequence ("Fundamentalreihen") and the Bolzano/Weierstrass theorem ("Häufungspunkt") respectively.[7] Therefore Fourier series is considered to represent contradictions between discontinuous phenomena and continuous expression, or between non-periodic phenomena and periodic expression, at least with the infinite of arithmetical set level in second order arithmetic (ordinary mathematics).

The system of axioms ACA_0 including the arithmetical set can prove the consistency of the system of axioms RCA_0 including the recursive set.[56] As an consequence of Gödel's incompleteness theorem [27], consistency of a system of axioms including the infinite can be proved only by another system of axioms with larger consistency strength. And the consistency strength increases with the height of the infinite (cardinals or ordinals) contained in the axiom system. G. Gentzen proved the consistency of elementary number theory using transfinite induction process, with transfinite ordinals (the infinite) up to the ordinal ϵ_0 [29], and predicted that the consistency of analysis and of set theory can be proved using corresponding higher ordinals respectively. [30]

In Gentzen-style proof theory, consistency strength of the system of axioms RCA_0 including the recursive set corresponds to the transfinite ordinal (provable ordinal) ω^ω and that of ACA_0 including the arithmetical set corresponds to the transfinite ordinal ϵ_0. Since $\epsilon_0 = sup(\omega, \omega^\omega, \omega^{\omega^\omega}, ...)$ namely $\omega < \omega^\omega < \omega^{\omega^\omega} ... < \epsilon_0$, the consistency strength of ACA_0 is higher than that of RCA_0. And in Gentzen-style proof theory also, ACA_0 can prove the consistency of RCA_0.[56] Thus the infinite of arithmetical set level is higher than that of recursive set level. Both of ω^ω and ϵ_0 are ordinals of second number-class with the cardinal $\aleph_o$. [8]

Hence it is concluded that to represent contradictions between discontinuous phenomena and continuous expression required the infinite higher than that used to represent the contradiction of motion by Newton. K. Tanaka said that reverse mathematics is the study which draws contour lines in the world of mathematical propositions by means of set existence axioms.[57] It can be considered also as the reflection of the hierarchy of the knowledge of physics or of Nature.

In 1847, H. Helmholtz expressed the mechanical views of Nature (based on Newtonian mechanics), which prevailed in those days: "The subject of physical natural science is to reduce natural phenomena to invariable attraction and repulsion with strength depending only on distance."[32] But these views led to a contradiction in explanation of electromagnetic phenomena which attracted the attention of physicists in 18-19th centuries. In particular, "ether," needed to explain the propagation of electromagnetic wave, could not be the material ruled by Newtonian mechanics. [20]

While, J. C. Maxwell constructed the theory of electromagnetic field with the action through medium, on the base of experimental studies by M. Faraday, [23] instead of the action at a distance of Newtonian mechanics. In 1873, Maxwell wrote "Faraday, in his mind's eye, saw lines of force traversing all space where the mathematicians saw centres of force attracting at a distance: Faraday saw a medium where they saw nothing but distance." in the preface of his "A treatise on electricity and magnetism." Maxwell also wrote, " Faraday's method of conceiving the phenomena was also a mathematical one, ... I also found that these methods were capable of being expressed in the ordinary mathematical forms."[41]

In special relativity theory introduced in 1905, A. Einstein extended the 3 dimensional Euclidean space for Newtonian mechanics to Minkowski space including time dimension, and resolved contradictions appeared in Newtonian mechanics applied to electromagnetic phenomena. Then instead of "ether," he considered the field with the action through medium to be Physical Reality. Attributing this development of the conception of Physical Reality to Maxwell's work, Einstein commented:

> Before Maxwell, Physical Reality was thought of as consisting in material particles, whose variation consist only in movements governed by ordinary differential equations. Since Maxwell's time, Physical Reality has been thought of as represented by continuous fields, governed by partial differential equations, and not capable of any mechanical interpretation. This change in the conception of Physical Reality is the most profound and the most fruitful that physics has experienced since the time of Newton. [21]

Mechanics of continuous bodies is the forerunner of theory of fields, in the sense of physical quantities distributed in space. While, the existence axiom of the arithmetical set is equivalent to the definition of real number through fundamental sequence ("Fundamentalreihen") by Cantor, as mentioned above. Therefore, as the mechanics of continuous bodies requires functions of real variables based on the infinite of arithmetical set level, theory of fields constructed with the action through medium, requires also functions of real variables or the real number system based on the infinite of arithmetical set level, since both theories requires Fourier series to describe propagation of wave.

Einstein pointed out: "It is well known that Maxwell's electrodynamics, when applied to moving bodies, leads to asymmetries that do not seem to attach to the phenomena." at the beginning of his paper, which introduced special relativity.[22] In this theory, he revived mechanics and described both mechanics and electromagnetic phenomena consistently, using the coordinate transformation (Lorentz transformation) which kept the velocity of light in vacuum "c" (a constant in Maxwell's equation) invariable. Lorentz transformation was originally introduced to keep Maxwell's equations invariable for coordinate systems moving each other with constant velocities. [59]

Special relativity theory is, therefore, natural extension of Maxwell's electromagnetic theory, a theory of fields. Thus contradictions appeared in Newtonian mechanics applied to electromagnetic phenomena were resolved by special relativity theory, a theory of fields which is based on the infinite of arithmetical set level or higher level. While they could not be resolved by Newtonian mechanics, based on the infinite within recursive set level, as mentioned previously. These facts show that the contradictions appeared in Newtonian mechanics applied to electromagnetic phenomena, were represented with the infinite of arithmetical set level or higher level in second order arithmetic (ordinary mathematics). General theory of relativity, which claims that the laws of physics are invariable between the systems moving each other with acceleration, is the theory of field of gravity.

4 Quantum mechanics

At the beginning of the 20th century, physicists faced many contradictions appeared with the progress in science and technology. They were, for example, the contradiction of black-body radiation[49], the contradiction of the stability of electron orbits in atoms[4], and contradictions between the particle picture and the wave picture for light and for the electron.[6] These contradictions showed that classical macroscopic physics could not be applied to newly observed microscopic phenomena.

N. Bohr pointed out: "a uniform formulation of quantum theory in classical terms is impossible. .... nevertheless, all experience must ultimately be expressed in terms of classical concepts neglecting the quantum of action."[3] Thus although based on concepts of classical physics which are essential for observation, physicists developed quantum mechanics consisting of new concepts and laws, using mathematics with the infinite. Then with what sort of the infinite did quantum mechanics represent the contradictions which appeared in classical physics?

Simpson limited the object of his reverse mathematics to "ordinary" or "non-set-theoretic" mathematics such as geometry, number theory, calculus, differential equations, real and complex analysis, and countable algebra. He excluded "set-theoretic" mathematics including abstract set theory and abstract functional analysis, from his reverse mathematics.[56] P. A. M. Dirac, one of the pioneers of quantum mechanics, introduced "δ function" as an essential tool for quantum mechanics which he systematized. His δ function cannot be included in "ordinary mathematics" as Dirac himself said, because its value is always 0 except at one particular point, while the value of its integration over all of the real number region is 1. [18]

Instead of δ function, which could not be dealt with in ordinary mathematics, J. v. Neumann sought the mathematical foundation of quantum mechanics in the abstract Hilbert space (functional space) with infinite dimensions.[44] That is, he represents states of physical systems, which are object of quantum mechanics, with vectors in Hilbert space (square integrable functions) and corresponds physical quantities to operators of functions and observable values of physical quantities to eigenvalues of the operators.

H. Yukawa remarked: "Even to draw an intuitive picture for the quantum mechanical system, Hilbert space has become essential background. It can be compared to three-dimensional Euclid space as the background of Newtonian mechanics and four-dimensional space as that of Einstein's relativity theory." [63]

The "ordinary mathematics" included in Simpson's reverse mathematics belongs to second order arithmetic, whose objects are real numbers namely sets of natural numbers. (First order arithmetic is elementary number theory, whose objects are natural numbers). But abstract functional analysis, which deals with abstract Hilbert space, belongs to third or higher order arithmetic, whose objects are functions namely sets of sets of natural numbers. That is to say, quantum mechanics, founded on the abstract Hilbert space, represented the contradictions appeared in classical macroscopic physics faced newly observed microscopic phenomena, with the infinite exceeding the level of second order arithmetic.

As Dirac said "The bra and ket vectors that we now use form a more general space than a Hilbert space." [18], quantum mechanics by his method is thought to be based on the infinite of the same or higher level than that of Hilbert space. Dirac's δ function was mathematically justified as one of the "generalized function," and represented by an infinite set of functions (for example the sequence of functions: $\{(n/\pi)^{1/2}exp(-nx^2)\}$.[39] Therefore δ function also belongs to third or higher order arithmetic, whose objects are functions.

In quantum mechanics, path integral method developed by R. P. Feynman is an popular alternative of the operator method in Hilbert space. For path integrals, the probability $P(b, a)$ for a particle which starts from a point x_a at time t_a to arrives x_b at t_b is represented by the square of the absolute value of the amplitude $K(b, a)$. The amplitude $K(b, a)$ is calculated by the integral summing the contribution from all (infinite) paths from a to b:

$$K(b, a) = \int_a^b \mathcal{D}x(t)e^{(i/\hbar)S[b,a]}$$

The definition of this integral is based on the infinite set of functions, and it means that path integral method is also belongs to third or higher order arithmetic. Furthermore, because of the "uncertainty principle," when the position of the particle is fixed, its momentum cannot be fixed and the curve of the path becomes unable to differentiate everywhere.[24] While in classical physics, the path from a to b is limited to the smooth curve with the least action S.

Y. Nambu commented "My first question concerns the two formulations of quantum mechanics: the canonical (Heisenberg- Schrödinger) formulation [operator method] and functional integration (Feynman) formulation [path integral method] .... they are generally regarded as equivalent, at least in a naive sense. However, one may assert that the functional formalism [path integral method] is, in many ways, more general than canonical one [operator method]."[43] This comment indicates that path integral method requires the same or higher level of infinite as the operator method.

In 1998, E. Witten summarized the history of quantum theory:

> The quantum theory of particles – which is more commonly called non relativistic quantum mechanics – was put in its modern form by 1925 and has greatly influenced the development of functional analysis, and other areas.
>
> But the deeper part of quantum theory is the quantum theory of fields, which arises when one tries to combine quantum mechanics with special relativity ... This much more difficult theory, developed from the late 1920s to present, encompasses most of what we know of the laws of physics, except gravity. In its seventy years there have been many milestones, ranging from the theory of "antimatter," which emerged around 1930, to more precise description of atoms, which quantum field theory provided by 1950, to the "standard model of particle physics" (governing the strong, weak, and electromagnetic interactions), which emerged by the early 1970s, to new predictions in our own time that one hopes to test in present and future accelerators. [62]

As mechanics of continuous bodies and classical theory of fields required higher infinite than Newtonian mechanics within second order arithmetic, quantum theory of fields with continuously infinite degrees of freedom requires higher infinite than quantum theory of particles which does not deal with emerging and diminishing of particles. This more higher infinite in third or higher order arithmetic represents the contradictions resolved by quantum theory of fields for the first time, for example, contradictions between the particle picture and the wave picture for light. [55]

Concerning the role of the infinite in quantum theory of field, Y. Ohnuki wrote:

> Another remarkable characteristic of quantum theory of fields is that it is the quantum theory of a system with infinite degrees of freedom. Conventional quantum mechanics is,

> so to speak, quantum mechanics of mass point systems, and its degrees of freedom are finite from the beginning. But since the field can be regard as a kind of continuous body, degrees of freedom in such a system, of course, should be infinite, that is to say, its quantum theoretical description also should be the quantum theory with infinite degrees of freedom. This infinite degrees of freedom ... are not only the limit of finite degrees of freedom fully increased in number. This character, in fact, relates to representation of operators for fields, and it brings new quality that has never been seen, into the dynamics of operators for fields. It can be said that this situation remarkably enriches the contents of quantum theory of fields, and gives the theory great power and various possibility. Whole aspects of the theory yet can not be said to be exhausted, and new developments in many phases are expected in the future. [48]

The infinite degrees of freedom is critical for "spontaneous symmetry breaking" in elementary particle physics [42], [40], which resolved the contradiction that the fundamental theory (the Hamiltonian of a system) was symmetry whereas its realization (its ground state) should be asymmetry, and opened the way to the "standard model of particle physics." It is another case supporting the hypothesis that the the mathematical infinite represents the contradiction involved in the development in physics.

The study of relation between the mathematical infinite and quantum mechanics or quantum theory of fields, requires reverse mathematics in third or higher order arithmetic. But higher order reverse mathematics is not studied in detail as that of the second order arithmetic. The present author knows only the paper titled "higher order reverse mathematics," which relates mathematical theorems not with set existence axioms but with a kind of functional [37], and it is difficult to study the relation between the infinite and quantum theories through reverse mathematics for the present author.

In this paper, it is possible only to point out that quantum mechanics represented contradictions involved in the development of knowledge with higher infinite than classical physics, namely with the infinite in third or higher order arithmetic, and to suggest that quantum theory of fields represented the contradictions with further higher infinite than quantum mechanics of particles in third or higher order arithmetic, just as classical theory of fields requires higher infinite than mechanics of mass points in the second order arithmetic.

5 Summary

To sum up, the contradictions involved in the development of the knowledge of Nature were represented with the infinite of recursive set level, by early Newtonian mechanics, then with the infinite higher than recursive set level, namely that of arithmetical set level or the higher infinite in second order arithmetic, by mechanics of continuous bodies and classical theory of fields. Then the contradictions were represented with the further higher infinite in third or higher order arithmetic, by quantum mechanics. The above summary shows that the contradictions between the newly observed phenomena and the existing system of concepts become more serious along with the evolution of physics, and we require the higher infinite in mathematics to represent the contradictions or mysteries of Nature.

P. Cohen warned, "as one postulates the existence of larger sets one also comes closer to paradoxes arising from the set of all sets, etc." [13] In fact, in 1971 K. Kunen proved that a contradiction arose when the elementary embedding, "One of the standard ways of postulating large cardinal axioms" beyond the standard axiomatic set theory (Zermelo-Fraenkel set theory with axiom of choice), were applied maximally.[38] A. Kanamori and F. R. Drake illustrated in their charts that ascending stages of the higher infinite ultimately reached this contradiction. [34], [19]

As an consequence of Gödel's incompleteness theorems, consistency of a system of axioms including the infinite can be proved only by another system of axioms with the higher infinite. To complete the prove of consistency of any axiom system, therefore, we need to ascend stages of the infinites, and ultimately come to the contradiction. This situation suggests that as the level of the required infinite becomes higher, contradictions inherent in the infinite become more explicit, and in this context, it is no wonder that contradictions can appear in the lower level of the infinite than that postulated in Kunen's proof. T. Nishimura and K. Namba referred to this possibility:

> Whether "borderline" of the contradiction of the axiom of infinity reaches huge cardinal, compact cardinal or measurable cardinal [large cardinals listed in descending order] should be decided in the future development. But if the borderline reaches lower cardinals, set theory will come across to the period of a new crisis or to the starting point of a new language. [45]

According to J. A. Wheeller's recollection of Bohr and his institute, "One of Bohr's favorites was his definition of a 'great truth': a truth whose opposite is also a great truth. ... The central idea of the institute was clear: 'No progress without a paradox'."[61] This philosophy of a pioneer of quantum physics suggests that the essential progress in physics can be achieved only through dialectical thinking including contradictions, and naturally it should be reflected by mathematical theories of physics.

Cantor said that he undertook his studies on point-set not only from speculative interest but also considering application in mathematical physics and other sciences. But he was misled to infer that the "body material" had "the first cardinal [Mächtigkeit]" of his set theory and "ether material" had "the second cardinal," through his study in physics in those days.[10] Hilbert tried to find in the physical world the things that realized the infinite, but he could not find it. [33]

The present author also tried to relate the mathematical infinite to the real physical world, or to find the physical foundation of set theory, and proposed the hypothesis that the mathematical infinites represent contradictions involved in development of mathematical physics. It was demonstrated historically in this paper.

The present author would like to acknowledge Mr. T. Irie for his discussion and his suggestion about path integral method and the quantum theory of fields, and Prof. K. Ogawa, former president of Yokohama City University, for his encouragement to submit this paper to JHM.

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