On Abstraction in Mathematics and Indefiniteness in Quantum Mechanics (forthcoming in *Journal of Philosophical Logic*)

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Abstract

Abstraction turns equivalence into identity, but there are two ways to do it. Given the equivalence relation of parallelness on lines, the #1 way to turn equivalence into identity by abstraction is to consider equivalence classes of parallel lines. The #2 way is to consider the abstract notion of the *direction* of parallel lines. This paper developments simple mathematical models of both types of abstraction and shows, for instance, how finite probability theory can be interpreted using #2 abstracts as "superposition events" in addition to the ordinary events. The goal is to use the second notion of abstraction to shed some light on the notion of an indefinite superposition in quantum mechanics.

Contents

1 Introduction: Two Ways from Equivalence to Identity

Classical physics, if not our own intuitive concepts, consider reality to be objectively definite 'all the way down.' But quantum mechanics suggests that reality at the quantum level may be objectively or ontologically indefinite (not just subjectively or epistemologically indefinite). Since we seem to lack 'clear and distinct ideas' about objective indefiniteness, we need any help we can get, from any source, to build up those intuitions.

The purpose of this paper is seek some help by drawing out some intriguing and possibly illuminating analogies between abstraction in the philosophy of mathematics and the notion of superposition and objective indefiniteness in quantum mechanics (QM). Moreover, a mathematical model for abstractions (or paradigms) is proposed, and used to give a new type of "superposition event" in finite probability theory.

A well-known example of an abstraction principle is Frege's "direction principle" which Stewart Shapiro described as: for any lines l_1 and l_2 in some domain, the "direction of l_1 is identical to the direction of l_2 if and only if l_1 is parallel to l_2." [12, p. 107]

Abstraction turns equivalence into identity. But there are two different ways to turn this equivalence (i.e., parallelness) into identity. The version often used by the proverbial 'working mathematician' will be called the #1 *abstraction*, namely, just the equivalence class. If $[l]$ is the parallelism equivalence class of the line l, then the equivalence-to-identity principle is clearly satisfied: $[l_1] = [l_2]$ iff $l_1 \simeq l_2$ (where $\simeq$ is the equivalence relation of being parallel). But there is also what we may refer to as the #2 *type of abstraction* where the "direction of l" is an abstraction that captures what is common to parallel lines and *abstracts away from where they differ*.

The purpose of this paper is:

- to give a way to mathematically differentiate the #1 and #2 abstracts in a simple setting,
- to show that finite probability theory can be reformulated with the #2 abstracts as "superposition events" in addition to the #1 abstracts (i.e., the subsets as ordinary events), and then
- to show that the mathematical treatment of the #2 abstracts is essentially the same as is found in a rather different setting to describe superposition states in quantum mechanics–where the #2 abstracts-version of probability becomes quantum probability.

The goal is to use the new interpretation of probability theory using the mathematically modelled superposition events as #2 abstracts (rather than just subset-events) to build the bridge to QM and thus to better understand 'by analogy' the key superposition principle in QM.

2 Two Versions of Abstraction

One general form of an abstraction principle is given by Shapiro [12, p. 107] (taking @() as an abstraction operator):

$$(\forall a)\,(\forall b)\,(@\,(a) = @\,(b) \equiv E\,(a,b)).$$

1. the #1 version of the abstraction operator takes equivalent entities $E\,(a,b)$ to the equivalence class $@\,(a) = [a] = [b] = @\,(b)$, and;
2. the #2 version of the abstraction operator takes all the equivalent entities a, b such that $E\,(a,b)$ to the abstract entity $@\,(a) = @\,(b)$ that is definite on what is common in the equivalence class but is indefinite on how they differ (e.g., indefinite on all the other properties that distinguish them).

In Frege's well-known example from the *Grundlagen* [9, pp. 110-111], an equivalence class of parallel lines is a #1 type of abstraction out of some delimited class of lines, while the act of abstracting away from the differences between parallel lines (i.e., going from equivalence to identity) yields the #2 abstraction of direction.

W. T. Tutte provides a good example of the attitude of a working mathematician.

> Pure graph theory is concerned with those properties of graphs that are invariant under isomorphism, for example the number of vertices, the number of loops, the number of links, and the number of vertices of a given valency. It is therefore natural for a graph theorist to identify two graphs that are isomorphic. For example, all link-graphs are isomorphic, and therefore he speaks of the 'link-graph' as though there were only one. Similarly one hears of 'the null graph', 'the vertex graph', and 'the graph of the cube'. When this language is used, it is really an isomorphism class (also called an *abstract graph*) that is under discussion. ([15, p. 6 (original emphasis)]; quoted in: [10, p. 390])

For instance, a proof about a property of "the graph of the cube" is not a property of an isomorphism class of graphs but a property of the graphs in that class or of the "abstract graph" that abstracts away from the different instances in the isomorphism class. Often proofs that could be seen as, in effect, using the #2 abstract graph are formulated using systematic ambiguity, i.e., assuming an arbitrary graph in the isomorphism class and then using only the properties common to all members of the class (by showing that the proof was independent of the choice from the equivalence class)–which are precisely the properties in the #2 abstract graph.

Our purpose is to give clear and distinct models for these two types of abstracts, but first we might consider the two abstracts in a broader setting (without assuming an equivalence relation). This broader setting allows us to give a #2 abstract or superposition interpretation to "events" in finite probability theory–which, in turn, will facilitate the bridge over to QM.

Given *any* property $S(u)$ defined on the elements of U, two abstract objects can be defined as in Figure 1:

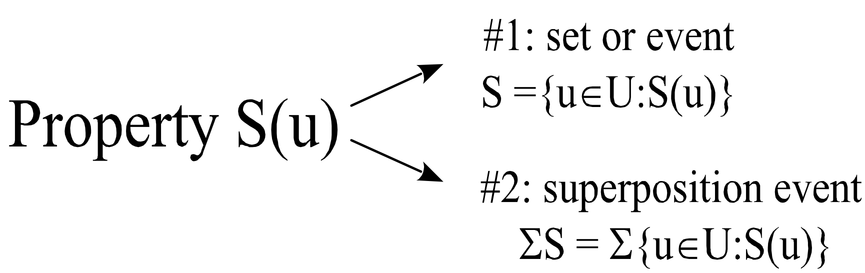

Figure 1: A property determines two types of abstract objects
(the 'blob-sum' or superposition-sum Σ is defined below).

The #1 abstract is just the set of elements S with that property while the #2 abstract object ΣS is '*the* S-entity' which is definite on the $S(u)$ property and indefinite on the differences between all the $u \in U$ such that $S(u)$.

We have a naming problem for these #2 abstracts like the problem of describing a glass as being half-full or half-empty. We could describe the #2 abstract ΣS according to the properties that are common to the entities in S and thus definite so it is a type of *paradigmatic* S-entity (the 'half-full' description). Or we could describe the #2 abstract ΣS as the *indefinite* S-entity that remains after all the properties that differentiate distinct S-entities are removed (the 'half-empty' description). For instance, in a logical context, the paradigm description might seem most appropriate while in the eventual application to quantum mechanics, it is the indefiniteness aspect of superposition states that is paramount.

3 Relations Between #1 and #2 Universals

In the version of finite probability theory developed below, the #2 abstracts or superposition events ΣS will supplement the #1 universals or ordinary events $S \subseteq U$.

For properties $S()$ defined on U, there is a 1-1 correspondence between the #1 and #2 universals:

$$\cup \{\{u_j\} | u_j \in U \& S(u_j)\} = S \longleftrightarrow \Sigma S = \Sigma \{u_j | u_j \in U \& S(u_j)\}.$$

If $T()$ another property defined on U implies $S()$ in the sense that $(\forall u \in U)[T(u) \Rightarrow S(u)]$, then in terms of #1 abstracts, this is the familiar $T \subseteq S$.

But what is the equivalent of $T \subseteq S$ for #2 universals? Intuitively ΣS is '*the* S-thing' that is definite on the S-property but is otherwise indefinite on the differences between the members of S. Those differences have been abstracted away from, blurred or 'blobbed' out, or rendered indefinite. If we make more properties definite, then in terms of subsets, that will in general cut down to a subset $T \subseteq S$, so ΣT would be a more definite version of ΣS. This "process" of changing from ΣS to a more definite ΣT, i.e., $\Sigma S \rightsquigarrow \Sigma T$ for $T \subseteq S$, might be called *projection* or *sharpening* (as in making a camera focus sharper or more definite) and symbolized:

$$\Sigma S \rhd \Sigma T \text{ (or } \Sigma T \lhd \Sigma S)$$

ΣS can be "sharpened" to ΣT by adding some definiteness.

These relations between #1 and #2 abstracts are summarized in Table 1.

$S()$ defined on U	#1 abstraction	#2 abstraction
Abstractions for $S()$	$S = \cup \{\{u_j\} \| u_j \in U \& S(u_j)\}$	$\Sigma S = \Sigma \{u_j \| u_j \in U \& S(u_j)\}$
$T()$ implies $S()$	$T \subseteq S$	$\Sigma S \rhd \Sigma T$

Table 1: Equivalents between #1 and #2 universals

In the language of Plato, the projection relation $\lhd$ is the relation of "participation" ($\mu\varepsilon\theta\varepsilon\xi\iota\varsigma$ or *methexis*) or entailment between universals. As Plato would say, '*the* T-thing' participates in or 'brings-on' ($\epsilon\pi\iota\phi\epsilon\pi\epsilon\iota$ or *epipherei* as in Vlastos [17, p. 102]) '*the* S-thing,' as in '*the* rocking chair' brings on '*the* chair,' i.e., $\Sigma T \lhd \Sigma S$, since '*the* chair' can be sharpened to '*the* rocking chair', i.e., the set T of rocking chairs is a subset of the set S of chairs.[1]

Like the #1 abstracts S, the #2 abstract entities ΣS, the *paradigm-universals*, are routinely used in mathematics.

4 Examples of Abstract Paradigms in Mathematics

There is an equivalence relation $A \simeq B$ between topological spaces which is realized by a continuous map $f : A \to B$ such that there is an inverse $g : B \to A$ so the $fg : B \to B$ is homotopic to 1_B (i.e., can be continuously deformed in 1_B) and gf is homotopic to 1_A. According to the 'classical' homotopy theorist, Hans-Joachim Baues, "Homotopy types are the equivalence classes of spaces" [2] under this equivalence relation. That is the #1 type of abstraction.

But the interpretation offered in homotopy type theory (HoTT) is expanding identity to "coincide with the (unchanged) notion of equivalence" in the words of the Univalent Foundations Program [16, p. 5] so it would refer to the #2 homotopy type, i.e., '*the* homotopy type' that captures the mathematical properties shared by all spaces in an equivalence class of homotopic spaces (abstracting away from the differences). Expanding identity to coincide with equivalence is another way to describe the #2 abstracting from the class S of equivalent entities to the abstract paradigm-universal entity ΣS which is not the same as the particular entities u in the equivalence class S.

[1] These non-mathematical everyday examples are used for the purpose of illustration and, perhaps, amusement.

For instance, '*the* homotopy type' is not one of the classical topological spaces (with points etc.) in the #1 equivalence class of homotopic spaces–just as Frege's #2 abstraction of direction is not among the lines in the equivalence class of parallel lines with the *same* direction.

> While classical homotopy theory is analytic (spaces and paths are made of points), homotopy type theory is synthetic: points, paths, and paths between paths are basic, indivisible, primitive notions. [16, p. 59]

Homotopy type theory systematically develops a theory of the #2 type of abstractions that grows out of homotopy theory and type theory into a new foundational theory.

> From the logical point of view, however, it is a radically new idea: it says that isomorphic things can be identified! Mathematicians are of course used to identifying isomorphic structures in practice, but they generally do so by "abuse of notation", or some other informal device, knowing that the objects involved are not "really" identical. But in this new foundational scheme, such structures can be formally identified, in the logical sense that every property or construction involving one also applies to the other. [16, p. 5]

In our terminology, "isomorphic things can be identified" means the 'blobbing together' of all the elements in an isomorphism class to create a single #2 abstract that is definite on what is common to all the isomorphs but is indefinite on where they differ.

Consider the homotopy example of '*the* path going once (clockwise) around the hole' in an annulus A (disk with one hole as in Figure 2), i.e., the abstract entity 1 in the fundamental group $\pi_0(A)$ of the annulus: $1 \in \pi_0(A) \cong \mathbb{Z}$:

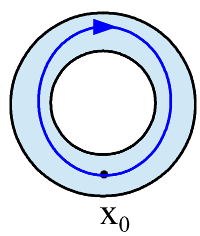

Figure 2: '*the* path going once (clockwise) around the hole'

Note that '*the* path going once (clockwise) around the hole' has the paradigmatic property of "going once (clockwise) around the hole" but is *not* one of the particular (coordinatized) paths that constitute the equivalence class of coordinatized once-around paths deformable into one another. It abstracts away from the coordinatizations that differentiate the paths in the homotopic equivalence class.

In a similar manner, we can view other common #2 abstractions such as: '*the* cardinal number 5' that captures what is common to the isomorphism class of all five-element sets; '*the* integer 1 mod (n)' that captures what is common within the equivalence class $\{..., -2n+1, -n+1, 1, n+1, 2n+1, ...\}$ of integers; '*the* circle' or '*the* equilateral triangle'–and so forth.

Category theory helped to motivate homotopy type theory for good reason. Category theory has no notion of identity between objects, only isomorphism as 'equivalence' between objects. Therefore category theory can be seen as a theory of *abstract* #2 objects, e.g., abstract sets, groups, spaces, etc., instead of the theory of the #1 abstracts, the isomorphism classes.

Our purpose is to model the theory of paradigm-universals ΣS and their projections or sharpenings ΣT–that is analogous to working with sets and subsets, e.g., in a Boolean algebra of subsets. That is all we will need to show that probability theory can be developed using paradigm entities or superposition events ΣS in addition to subset-events S, and then finally to cross the bridge to quantum mechanics.

5 The Connection to Interpreting Symmetry Operations

In the usual case of abstraction where S is an equivalence or isomorphism class, the #2 universal ΣS by definition abstracts away for the differences between the elements in the equivalence class. Hence if we consider any operation that takes one element u of an equivalence class $[u]$ to another element u' in the same class, then the induced operation on the #1 abstracts, $[u] \rightsquigarrow [u']$, is the identity, and the *same* holds for the #2 abstract ΣS since the two abstracts represent two different ways to get abstracts that in different ways disregard the differences between the elements in the equivalence class.

This can be visually illustrated in a simple example of the symmetry operation (defining an equivalence relation) of reflection on the aA-axis for an isosceles triangles as in Figure 3 that is taken as fully definite (all sides and angles labelled).

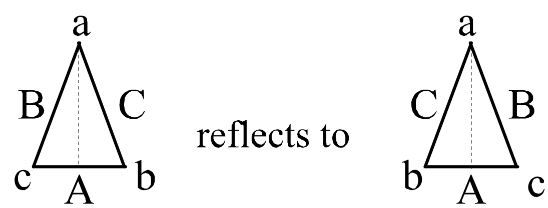

Figure 3: Reflection on vertical axis symmetry operation.

Thus the equivalence class of reflective-symmetric figures in the #1 or classical interpretation is the set in Figure 4.

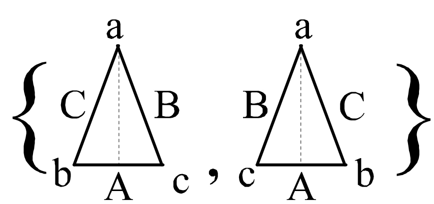

Figure 4: The #1 abstraction of equivalence class.

The set remains invariant under reflection applied to its elements, which is another way to say that the induced operation *on the equivalence classes* (or orbits) is the identity.

Under the #2 indefiniteness-abstraction interpretation, the equivalence abstracts to the figure that is definite on what is the same, and indefinite on what is different between the definite figures in the equivalence class:

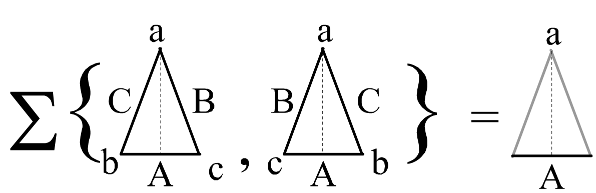

Figure 5: The #2 abstraction of an indefinite entity.

And the symmetry operation induced on the indefinite figure is also the *identity* as illustrated in Figure 5. As noted in the discussion of homotopy type theory, the movement from one space to a homotopic space leaves the "homotopy type" *the same* regardless of whether we think of the homotopy type as an equivalence class or as the #2 type of abstract considered in homotopy type theory.

A concrete example of the #1 and #2 ways to go from equivalence to identity is the derivation of the Maxwell-Boltzmann distribution and the Bose-Einstein distribution as in Feller [8, pp. 20-1] or Ellerman [5]. This treatment is illuminated by the classical and quantum version of a symmetry operation. Suppose we have two particles of the same type which are classically indistinguishable so, following Weyl, we artificially distinguish them using Mike and Ike labels. If each of the two particles could be in states A, B, or C, then the set of possible states is the set of nine ordered pairs $\{A,B,C\} \times \{A,B,C\}$. Applying the symmetry operation of permuting Mike and Ike, we have six equivalence classes (orbits) as in Table 2.

Equivalence classes under permutation	M-B
$\{(A,B),(B,A)\}$	2/9
$\{(A,C),(C,A)\}$	2/9
$\{(B,C),(C,B)\}$	2/9
$\{(A,A)\}$	1/9
$\{(B,B)\}$	1/9
$\{(C,C)\}$	1/9

Table 2: Maxwell-Boltzmann distribution.

The symmetry operation on the equivalence classes is the identity, but in (classical) Nature the primitive data are, as it were, the ordered pairs (the possible states), not the equivalence classes. When we assign the equal probabilities of $\frac{1}{9}$ to each ordered pair (i.e., to each distinct state), that results in the *Maxwell-Boltzmann distribution* on the equivalence classes. Nature counts states; we classically measure equivalence classes and find the M-B distribution.

But in the quantum case, the operation of going to the #2 abstract $\Sigma\{(A,B),(B,A)\}$ seems to be physically realized in an indefinite superposition state, i.e., the analogy: $\Sigma\{(A,B),(B,A)\} \approx |A\rangle \otimes |B\rangle + |B\rangle \otimes |A\rangle$, where the symmetry operation is the *identity.* Since there are *then* only six states, we assign the equal probabilities of $\frac{1}{6}$ to each state and obtain the *Bose-Einstein distribution* in Table 3. Nature again counts states, but the superposition states (seen as physically realizing a type of #2 abstract from the equivalence classes) reduces the number of states to six.

Six indefinite states	B-E
$\Sigma\{(A,B),(B,A)\} \approx |A\rangle \otimes |B\rangle + |B\rangle \otimes |A\rangle$	1/6
$\Sigma\{(A,C),(C,A)\} \approx |A\rangle \otimes |C\rangle + |C\rangle \otimes |A\rangle$	1/6
$\Sigma\{(C,B),(B,C)\} \approx |C\rangle \otimes |B\rangle + |B\rangle \otimes |C\rangle$	1/6
$\Sigma\{(A,A)\} \approx |A\rangle \otimes |A\rangle$	1/6
$\Sigma\{(B,B)\} \approx |B\rangle \otimes |B\rangle$	1/6
$\Sigma\{(C,C)\} \approx |C\rangle \otimes |C\rangle$	1/6

Table 3: Bose-Einstein distribution.

6 Modelling #1 and #2 Abstracts

But it will surely be asked:

> What is this crazy talk and loose analogy between forming an indefinite abstract in mathematics and a superposition state in QM?

It is a fine question, and surely one way to approach the question is to give 'clear and distinct' mathematical models of the two abstracts in a simple illustrative setting. We distinguish the #1 and #2 interpretations for a finite U as in Figure 6.

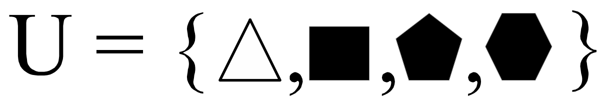

Figure 6: Universe U of figures

The polygons in Figure 6 can be characterized using two attributes, the number n of equal sides and being solid s or hollow h. Hence the universe U has the elements $U = \{u_1, u_2, u_3, u_4\} = \{3h, 4s, 5s, 6s\}$. The subset of *solid* figures $S = \{4s, 5s, 6s\} \subseteq \{3h, 4s, 5s, 6s\} = U$ might be represented by a one-dimensional column vector $|S\rangle = \begin{array}{cc} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} & \begin{array}{c} 3h \\ 4s \\ 5s \\ 6s \end{array} \end{array}$ (with the given ordering). But by moving up one dimension to a *two*-dimensional matrix, we can *represent* or *mathematically model* both the #1 and #2 versions of S as two types of incidence matrices. For $U = \{u_1, ..., u_n\}$, the *incidence matrix* $\mathrm{In}(R)$ of a binary relation $R \subseteq U \times U$ is the $n \times n$ matrix with $(\mathrm{In}(R))_{jk} = 1$ if $(u_j, u_k) \in R$ and 0 otherwise.

1. The #1 (classical) version of S (i.e., set of S-things or set of solid figures) is represented by the diagonal matrix $\mathrm{In}(\Delta S)$ that lays the column vector $|S\rangle$ along the diagonal: $\mathrm{In}(\Delta S) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ = representation of set S of distinct S-entities. $\mathrm{In}(\Delta S)$ is the incidence matrix of the diagonal relation $\Delta S \subseteq U \times U$ whose entries are the values of the characteristic function $\chi_{\Delta S}$ on $U \times U$.

2. The representation of the #2 (quantum-like) version of S (i.e., *the* S-thing ΣS) is the matrix $\mathrm{In}(S \times S)$ whose entries are the values of the characteristic function $\chi_{S \times S}$ on $U \times U$. Where $()^t$ signifies the transpose operation, this $n \times n$ incidence matrix can also be obtained as the product of the $n \times 1$ column vector $|S\rangle$ times the $1 \times n$ row vector $(|S\rangle)^t$: $\mathrm{In}(S \times S) = |S\rangle (|S\rangle)^t = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ = representation of one indistinct S-thing, '*the* solid figure' $\Sigma S = \Sigma \{4s, 5s, 6s\}$.

For (and only for) singletons $S = \{u_j\}$, the #2 'abstract' is just $\Sigma \{u_j\} = \{u_j\}$, and thus they have the same representation $\mathrm{In}(\Delta S) = \mathrm{In}(S \times S)$ as expected, but for $|S| > 1$, $\mathrm{In}(\Delta S) \neq \mathrm{In}(S \times S)$.

The two representations differ only in the off-diagonal entries. Think of the off-diagonal $\mathrm{In}(S \times S)_{j,k} = 1$'s as equating, cohering, blurring out, 'blobbing' out, or ignoring the differences (e.g., the number of sides) between u_j and u_k which have the common $S() =$ 'being a solid figure' property:

$$\mathrm{In}(S \times S) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \text{ says}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4s \overset{S}{\sim} 4s & 4s \overset{S}{\sim} 5s & 4s \overset{S}{\sim} 6s \\ 0 & 5s \overset{S}{\sim} 4s & 5s \overset{S}{\sim} 5s & 5s \overset{S}{\sim} 6s \\ 0 & 6s \overset{S}{\sim} 4s & 6s \overset{S}{\sim} 5s & 6s \overset{S}{\sim} 6s \end{bmatrix}.$$

Intuitively, the differences in the number of sides of the solid figures have been blurred out or rendered indefinite, so the only definite attribute of the paradigm entity is *the* solid-figure.

Since the #2 abstract paradigm entities are represented by a certain type of incidence matrix, we can mathematically represent the *blob-sum* or *superposition sum* #2 entity $\Sigma S = \Sigma \{u_i \in U | S(u_i)\}$ by the corresponding incidence matrix:

$$\mathrm{In}(S \times S).$$

For $S = \{u_2, u_4\}$, the blob-sum $\Sigma S = \Sigma \{u_2, u_4\}$ is represented by $\mathrm{In}(S \times S)$ where the blob-sum operation Σ means 'blobbing-out' the distinctions between entities in S (represented by the cross-terms in $\{u_2, u_4\} \times \{u_2, u_4\}$ which give the non-zero off-diagonal entries in the incidence matrix):

$$\begin{gathered} \mathrm{In}(S \times S) = \mathrm{In}(\{u_2, u_4\} \times \{u_2, u_4\}) \\ = \mathrm{In}(\{u_2\} \times \{u_2\}) \vee \mathrm{In}(\{u_4\} \times \{u_4\}) \vee \mathrm{In}(\{u_2\} \times \{u_4\}) \vee \mathrm{In}(\{u_4\} \times \{u_2\})^2 \\ = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \end{gathered}$$

To better understand abstraction in mathematics, the superposition events in probability theory (defined below), and superposition states in QM, we should become as comfortable with paradigms ΣS as with subsets S. The paradigms ΣS for $S \in \wp(U)$ form a Boolean algebra isomorphic to $\wp(U)$ under the mapping: for any Boolean binary operation $S\#T$ for $S, T \in \wp(U)$, $\Sigma S \# \Sigma T$ is the paradigm represented by $\mathrm{In}((S\#T) \times (S\#T))$.

- The *union or join of superposition events* is the blob-sum $\Sigma S \vee \Sigma T = \Sigma(S \cup T)$ which is the #2 abstract represented by:

$$\begin{gathered} \mathrm{In}((S \cup T) \times (S \cup T)) \\ = \mathrm{In}((S \times S) \cup (T \times T) \cup (S \times T) \cup (T \times S)) \\ = \mathrm{In}(S \times S) \vee \mathrm{In}(T \times T) \vee \mathrm{In}(S \times T) \vee \mathrm{In}(T \times S). \text{ (note as expected, for } T \subseteq S,\ \Sigma\{S \cup T\} = \Sigma S); \end{gathered}$$

- The *intersection or meet of superposition events* $\Sigma S \wedge \Sigma T = \Sigma(S \cap T)$ is represented by $\mathrm{In}((S \cap T) \times (S \cap T))$ where, as expected, for $T \subseteq S$, $\Sigma S \wedge \Sigma T = \Sigma T$;
- The *negation of a superposition event* $\neg \Sigma S = \Sigma(S^c)$ is represented by $\mathrm{In}(S^c \times S^c)$ (note as expected, $\neg \Sigma(S) \vee \Sigma S = \Sigma(S \cup S^c) = \Sigma U$).

The top ΣU and bottom $\Sigma \emptyset$ of the BA of superposition events are represented by the incidence matrices of all ones or all zeros respectively, and the partial order on the blobbed-out incidence matrices $\mathrm{In}(S \times S)$ is that induced by set inclusion [i.e., the entry-wise partial order $0 \leq 1$ on incidence matrices of the form $\mathrm{In}(S \times S)$]. If $T \subseteq S$, then $\Sigma T \lhd \Sigma S$, so moving down in the BA of superposition events represents 'sharpening' or rendering-more-definite just as a conditional probability $\Pr(T|S)$ is always for some event T (or $T \cap S$) below the conditioning event S in the partial order of ordinary events. The atomic elements $\Sigma\{u_j\}$ (corresponding to the singletons $\{u_i\}$) are the sharpest or most definite or determinate elements. When the events as subsets S of the

[2] The disjunction of incidence matrices is the usual entry-wise disjunction: $1 \vee 1 = 1 \vee 0 = 0 \vee 1 = 1$ and $0 \vee 0 = 0$, and similarly for conjunction.

sample space U, are replaced by the #2 abstracts ΣS, then this Boolean algebra structure on the set of superposition events ΣS in their $\mathrm{In}(S \times S)$ representation for $S \subseteq U$ is isomorphic to the usual BA of events S. Figure 7 illustrates the two BAs for $U = \{a, b, c\}$.

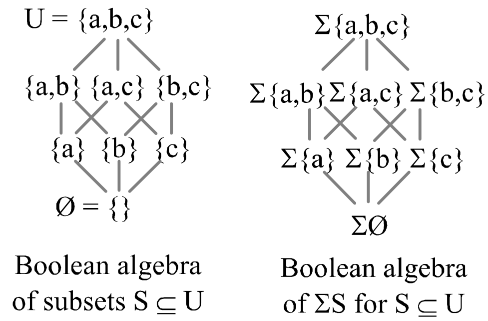

Figure 7: The Boolean algebras of ordinary events and superposition events for $U = \{a, b, c\}$.

7 The Projection Operation: Making an indefinite entity more definite

In the four figures example, suppose we classify or partition *all* the elements of U according to an attribute such as the parity of the number of sides, where a *partition* is a set of nonempty disjoint subsets (blocks) of U whose union is all of U. Let π be the partition of U with two *blocks* odd $O = \{3h, 5s\}$ and even $E = \{4s, 6s\}$ according to the parity of the number of sides.

The equivalence relation defined by π is referred to by Ellerman [4] as the set of *indistinctions*, $\mathrm{indit}(\pi) = (O \times O) \cup (E \times E)$, and the incidence matrix $\mathrm{In}(\mathrm{indit}(\pi))$ is formed by the usual disjunction of corresponding matrix entries:

$$\mathrm{In}(O \times O) \vee \mathrm{In}(E \times E) = \mathrm{In}(\mathrm{indit}(\pi))$$
$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The #1 (classical) operation of intersecting the set of odd-sided figures with the set of solid figures to give the set of odd-sided solid figures is represented as the conjunction:

$$\mathrm{In}(\Delta O) \wedge \mathrm{In}(\Delta S) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The #2 (quantum-like) operation of 'sharpening' or 'rendering more definite' '*the* solid figure' $\Sigma S = \Sigma\{u_2, u_3, u_4\}$ to '*the* odd-sided solid figure' $\Sigma\{u_3\} = \Sigma\{5s\} = \{5s\}$, so $\Sigma\{5s\} \triangleleft \Sigma S$ (suggested reading: $\Sigma\{5s\}$ is a projection or sharpening of ΣS) is represented as:

$$\mathrm{In}(O \times O) \wedge \mathrm{In}(S \times S) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \Sigma\{5s\}.$$

But there is a better way to represent 'sharpening' using matrix multiplication instead of just the logical operation $\wedge$ on matrices, and it foreshadows and illuminates the measurement operation in QM. For $E =$ even-sidedness, the matrix $\mathrm{In}(\Delta E) = P_E$ is a projection matrix, i.e., the diagonal matrix with diagonal entries $\chi_E(u_i)$ so $P_E|S\rangle = |E\cap S\rangle$. Then the result of the projection-sharpening can be represented as:

$$\begin{gathered}|E\cap S\rangle\left(|E\cap S\rangle\right)^t = P_E|S\rangle\left(P_E|S\rangle\right)^t = P_E|S\rangle\left(|S\rangle\right)^t P_E \\ = P_E \,\mathrm{In}(S\times S)\, P_E = \mathrm{In}(E\times E)\wedge \mathrm{In}(S\times S) = \mathrm{In}((E\cap S)\times(E\cap S)).\end{gathered}$$

Thus sharpening *the* solid-figure $\Sigma\{4s,5s,6s\}$ by the even number-of-sides attribute to obtain $\Sigma\{4s,6s\}$ is represented by pre- and post-multiplying the incidence matrix $\mathrm{In}(S\times S)$ by the projection P_E for evenness parity. Under the #2 interpretation, the parity-sharpening, parity-classifying, parity-differentiation, or *parity-measurement* of '*the* solid figure' by *both* the odd and even parities is represented as:

$$\begin{gathered}\mathrm{In}(\mathrm{indit}(\pi))\wedge \mathrm{In}(S\times S) \\ = \begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}\wedge\begin{bmatrix}0&0&0&0\\0&1&1&1\\0&1&1&1\\0&1&1&1\end{bmatrix} \\ = P_O\,\mathrm{In}(S\times S)\,P_O + P_E\,\mathrm{In}(S\times S)\,P_E \\ = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0&0\\0&1&0&1\\0&0&0&0\\0&1&0&1\end{bmatrix}=\begin{bmatrix}0&0&0&0\\0&1&0&1\\0&0&1&0\\0&1&0&1\end{bmatrix}.\end{gathered}$$ [3]

The result is the mixture or sum of incidence matrices for '*the* even-sided solid figure' $\Sigma\{4s,6s\}$ *and* '*the* odd-sided solid figure' $\Sigma\{5s\}$. The important thing to notice is the action on the off-diagonal elements where the action $1 \rightsquigarrow 0$ in the j,k-entry means that a *distinction* between u_j and u_k has been created; u_j and u_k have been deblobbed, decohered, distinguished, or differentiated–in this case by parity in the number of sides:

$$\begin{gathered}\mathrm{In}(S\times S)\rightsquigarrow \mathrm{In}(\mathrm{indit}(\pi))\wedge \mathrm{In}(S\times S) \\ = P_O\,\mathrm{In}(S\times S)\,P_O + P_E\,\mathrm{In}(S\times S)\,P_E \\ = \begin{bmatrix}0&0&0&0\\0&1&1\overset{decohered}{\rightsquigarrow}0&1\\0&1\overset{decohered}{\rightsquigarrow}0&1&1\overset{decohered}{\rightsquigarrow}0\\0&1&1\overset{decohered}{\rightsquigarrow}0&1\end{bmatrix}\end{gathered}$$

Differentiating solid figures by parity.

We could also classify the figures as to having 4 or fewer sides (few sides) or more (many sides) so that partition is $\sigma = \{\{u_1,u_2\},\{u_3,u_4\}\} = \{\{3h,4s\},\{5s,6s\}\}$ which is represented by:

$$\mathrm{In}(\mathrm{indit}(\sigma)) = \begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{bmatrix} \text{ and}$$

$$\begin{gathered}\mathrm{In}(\mathrm{indit}(\sigma))\wedge(\mathrm{In}(\mathrm{indit}(\pi))\wedge \mathrm{In}(S\times S)) \\ = P_{few}\left(\mathrm{In}(\mathrm{indit}(\pi))\wedge \mathrm{In}(S\times S)\right)P_{few} + P_{many}\left(\mathrm{In}(\mathrm{indit}(\pi))\wedge \mathrm{In}(S\times S)\right)P_{many}\end{gathered}$$

[3] This classifying or measuring operation using the pre- and post-multiplication by projection matrices foreshadows the Lüders mixture representation of projective measurement in QM (see below).

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathrm{In}\,(\Delta S).$$

Thus the parity and the few-or-many-sides partitions suffice to classify the solid figures uniquely and thus to yield the representation $\mathrm{In}\,(\Delta S)$ of the distinct elements of $S = \{u_2, u_3, u_4\} = \{4s, 5s, 6s\}$. Thus making all the distinctions (i.e., decohering the entities that cohered together in ΣS) takes $\mathrm{In}\,(S \times S) \rightsquigarrow \mathrm{In}\,(\Delta S)$.

In QM jargon, the parity and few-or-many-sides attributes constitute a "complete set of commuting operators" (CSCO) so that measurement of the 'pure,' blobbed-out, superposition figure, '*the* solid figure,' by those observables will sharpen '*the* solid figure,' to the 'mixture' of the three separate solid 'eigen-figures':

- '*the* few- and even-sided solid figure' (the square $u_2 = 4s$),
- '*the* many- and odd-sided solid figure' (the pentagon $u_3 = 5s$), and
- '*the* many- and even-sided solid figure' (the hexagon $u_4 = 6s$).

8 From Incidence to Density Matrices

To move from Boolean logic to probability theory for ordinary and superposition events, we move from incidence matrices to density matrices. The incidence matrices $\mathrm{In}\,(\Delta S)$ representing the subset S and $\mathrm{In}\,(S \times S)$ representing the paradigm or superposition state ΣS can be turned into *density matrices* by dividing through by their trace (sum of diagonal elements):

$$\rho\,(\Delta S) = \frac{1}{\mathrm{tr}[\mathrm{In}(\Delta S)]}\,\mathrm{In}\,(\Delta S) \text{ and } \rho\,(S) = \frac{1}{\mathrm{tr}[\mathrm{In}(S \times S)]}\,\mathrm{In}\,(S \times S).$$

In terms of probabilities, this means treating the outcomes in S as being equiprobable with probability $\frac{1}{|S|}$. But now we have the #1 and #2 interpretations of the sample space for finite discrete probability theory.

1. The #1 interpretation, represented by $\rho\,(\Delta U)$, is the classical version with U as the sample space of six equiprobable outcomes. For instance, the 6×6 diagonal matrix with diagonal entries $\frac{1}{6}$ is "the statistical mixture describing the state of a classical dice [die] before the outcome of the throw" [1, p. 176];
2. The #2 interpretation replaces the "sample space" with the one indefinite '*the* sample outcome' ΣU represented by $\rho\,(U)$ (a 6×6 matrix with the $\frac{1}{6}$ diagonal entries 'blobbed out' to fill the whole matrix with $\frac{1}{6}$ entries) and, in a trial that distinguishes the six outcomes, the indefinite outcome ΣU 'sharpens to' or becomes a definite outcome$\{u_i\} \subseteq U$ with probability $\frac{1}{|U|}$.

Let $f : U \to \mathbb{R}$ be a real-valued random variable with distinct values ϕ_i for $i = 1, ..., m$ and let $\pi = \{B_i\}_{i=1,...,m}$ where $B_i = f^{-1}\,(\phi_i)$, be the partition of U according to the f-values as in [5]. As before with incidence matrices, we want the classification or differentiation of $\rho\,(S)$ according to the different f-values. It could be obtained as $\mathrm{In}\,(\mathrm{indit}\,(\pi)) \wedge \rho\,(\psi)$ where the meet takes the minimum of the corresponding entries of the matrices. But if P_{B_i} is the diagonal (projection) matrix with diagonal elements $(P_{B_i})_{jj} = \chi_{B_i}\,(u_j)$, then the classified, differentiated, or measured density matrix is also obtained by the *Lüders mixture operation* of pre- and post-multiplying $\rho\,(S)$ by the projection matrices P_{B_i} [1, p. 279] to get the mixed-state density matrix:

$$\hat{\rho}\,(S) = \sum_{i=1}^{m} P_{B_i} \rho\,(S)\, P_{B_i},$$

and the probability of a trial returning ϕ_i is:

$$\Pr(\phi_i|S) = \mathrm{tr}\left[P_{B_i}\rho(S)\right].$$

There are two interpretations of that probability corresponding to the #1 or #2 abstracts:

1. It is the probability that given the #1 abstract, i.e., the event S, a trial leads to the #1 abstract, the event $B_i \cap S$, occurring, or
2. It is the probability that given the #2 abstract, i.e., the entity ΣS, a trial or π-measurement leads to (or sharpens to) the #2 superposition event, the entity $\Sigma(B_i \cap S)$ that is definite on the 'eigen' f-value of ϕ_i.

For instance, in the previous example, where $f : U \to \mathbb{R}$ gives the parity partition π with the two values ϕ_{odd} and ϕ_{even}, then:

$$P_{even}\rho(S) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

so $\mathrm{tr}\left[P_{even}\rho(S)\right] = \frac{2}{3}$ which, under the #2 (quantum-like) interpretation, is the conditional probability that a trial or 'parity-of-sides-measurement' sharpens '*the* solid figure' to '*the* even-sided solid figure'. And under the #1 (standard) interpretation, $\Pr(\phi_{even}|S) = \mathrm{tr}\left[P_{even}\rho(\Delta S)\right] = \frac{2}{3}$ is the probability of a trial yielding *an* even-sided solid figure starting with the *subset* of equiprobable solid figures represented by $\rho(\Delta S)$. Thus we have two different interpretations of events in finite probability theory, the conventional one using the #1 events S and the new superposition events interpretation using #2 abstracts ΣS.

The mixed state density matrix $\hat{\rho}(S)$ resulting from 'measuring' or classifying the solid figures according to the parity of their sides is:

$$\begin{aligned} \hat{\rho}(S) &= P_{even}\rho(S)P_{even} + P_{odd}\rho(S)P_{odd} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &+ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}. \end{aligned}$$

9 Logical Entropy

In the density matrix formulation of classical or quantum logical information theory ([6]; [7]), the *logical entropy* of a density matrix ρ is: $h(\rho) = 1 - \mathrm{tr}\left[\rho^2\right]$. Intuitively, logical information is information-as-distinctions. Since the non-zero off-diagonal amplitudes in a density matrix represent indistinctions–whose squares are indistinction probabilities–the gain in logical entropy due to the measurement or classification process is the sum of the squares of the non-zero off-diagonal terms that are zeroed, i.e., turned into distinctions, in the change $\rho(S) \rightsquigarrow \hat{\rho}(S)$.

In the example, there were four off-diagonal terms that were zeroed in the parity classification each with an amplitude of $\frac{1}{3}$, so the change in logical entropy is $4 \times \left(\frac{1}{3}\right)^2 = \frac{4}{9}$. This can be checked by directly computing the two logical entropies. All density matrices have trace 1 and a *pure state* density matrix is one where $\rho^2 = \rho$; otherwise it is a *mixed state* density matrix. The initial state $\rho(S)$ is a pure state since $\rho(S)^2 = \rho(S)$ so $\operatorname{tr}\left[\rho(S)^2\right] = 1$ and $h(\rho(S)) = 1 - 1 = 0$. For the post-classification density matrix $\hat{\rho}(S)$, we have:

$$\hat{\rho}(S)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{9} & 0 & \frac{2}{9} \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & \frac{2}{9} & 0 & \frac{2}{9} \end{bmatrix}$$

so $\operatorname{tr}\left[\hat{\rho}(S)^2\right] = \frac{5}{9}$ and $h(\hat{\rho}(S)) = 1 - \frac{5}{9} = \frac{4}{9}$. Logical entropies always have the interpretation of getting a distinction in two independent trials, so in this case, the probability of *the* solid figure sharpening to solid figures of distinct parities in two independent trials is $\frac{4}{9}$. This can be intuitively checked since the probability of getting two odd-parity solid figures is $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$ and of getting two even-parity solid figures is $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$ so the probability of getting different parities is: $1 - \left(\frac{4}{9} + \frac{1}{9}\right) = \frac{4}{9}$.

These two interpretations of finite discrete probability theory extend easily to the case of point probabilities[4] p_j for $u_j \in U$ (instead of equiprobable points), where $\Pr(S) = \sum_{u_j \in S} p_j$:

1. $(\rho(\Delta S))_{jj} = \chi_S(u_j)\, p_j / \Pr(S)$, so $\operatorname{tr}\left[P_{even}\rho(\Delta S)\right]$ = probability of getting *an* even-sided solid figure starting with the set of solid figures, and

2. $(\rho(S))_{j,k} = \chi_S(u_j)\,\chi_S(u_k)\sqrt{p_j p_k} / \Pr(S)$, so $\operatorname{tr}\left[P_{even}\rho(S)\right]$ = probability of getting '*the* even-sided solid figure' starting with '*the* solid figure.'

The whole of finite discrete probability theory can be developed in this manner, *mutatis mutandis*, for the #2 abstract superposition events in addition to the usual #1 events.

10 Density matrices in Quantum Mechanics

This indefiniteness interpretation of finite probability with superposition events leads directly to the use of probability in finite-dimensional quantum mechanics. The jump to quantum mechanics (QM) is to replace the reals $\sqrt{p_j p_k}$ in the density matrices by complex amplitudes. Instead of the set S represented by a column $|S\rangle$ of real 'amplitudes' $\sqrt{p_j}$, we have a normalized column $|\psi\rangle$ of complex numbers α_j whose absolute squares are probabilities: $|\alpha_j|^2 = p_j$, e.g.,

$$|S\rangle = \begin{bmatrix} 0 \\ \sqrt{p_2} \\ \sqrt{p_3} \\ \sqrt{p_4} \end{bmatrix} \rightsquigarrow |\psi\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

where $\alpha_1 = 0$ and $|\alpha_j|^2 = p_j$ for $j = 2, 3, 4$.

1. The *density matrix* $\rho(\Delta\psi)$ has the absolute squares $|\alpha_j|^2 = p_j$ laid out along the diagonal.

2. The *density matrix* $\rho(\psi) = |\psi\rangle\langle\psi|$ (where $\langle\psi|$ is the conjugate-tranpose of $|\psi\rangle$) has the j,k-entry as the product of α_j and α_k^* (complex conjugate of α_k), so the diagonal entries are $p_j = \alpha_j^*\alpha_j = |\alpha_j|^2$.

Thus:

[4] Point probabilities are given by a probability density function $p : U \to [0,1]$ where $p(u_j) = p_j$ and $\sum p_j = 1$.

$$\rho(\Delta\psi) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix} \text{ and } \rho(\psi) = |\psi\rangle\langle\psi| = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & p_2 & \alpha_2\alpha_3^* & \alpha_2\alpha_4^* \\ 0 & \alpha_3\alpha_2^* & p_3 & \alpha_3\alpha_4^* \\ 0 & \alpha_4\alpha_2^* & \alpha_4\alpha_3^* & p_4 \end{bmatrix}.$$

Some modern quantum mechanics texts, such as [3, Vol. 1, p. 302] or [1], call attention to the special significance of the "coherences" represented by the non-zero off-diagonal terms.

> [The] off-diagonal terms of a density matrix...are often called *quantum coherences* because they are responsible for the interference effects typical of quantum mechanics that are absent in classical dynamics. [1, p. 177]

In the analogy between paradigm-universals in mathematics and superposition states in QM, *the point* is that an indefinite superposition QM state is a single entity that 'blobs out,' 'blurs out,' renders indefinite, or coheres together the differences between the definite eigenstates in the superposition. We previously noted that there is both the 'half-full'/paradigm description or the 'half-empty'/indefinite description of the same entity ΣS. It is the indefiniteness description that best applies to the quantum case, not the classical-Platonic notion of a 'paradigm.' The notion of #2 abstraction could be applied to any collections of distinct entities. In the quantum case, it is also not a zero-one affair whether two elements are equated as in the incidence matrices $\mathrm{In}(S \times S)$ of the 'blobbed-out' sets; the off-diagonal elements in the density matrix give the 'amplitude' of the equating or cohering together of the eigenstates in the superposition state.

The classifying or measuring operation $\mathrm{In}(\mathrm{indit}(\pi)) \wedge \rho(\psi)$ could still be defined taking the minimum of corresponding entries in absolute value, but in QM it is obtained by what Auletta et al. [1, p. 279] call the *Lüders mixture operation.* If $\pi = \{B_1, ..., B_m\}$ is a partition according to the eigenvalues $\phi_1, ..., \phi_m$ on $U = \{u_1, ..., u_n\}$ (where U is an orthonormal basis set for the observable being measured), let P_{B_i} be the diagonal (projection) matrix with diagonal entries $(P_{B_i})_{jj} = \chi_{B_i}(u_j)$. Then $\mathrm{In}(\mathrm{indit}(\pi)) \wedge \rho(\psi)$ is obtained as:

$$\hat{\rho}(\psi) = \sum_{B_i \in \pi} P_{B_i} \rho(\psi) P_{B_i}$$

The Lüders mixture.

The probability of getting the result ϕ_i is:

$$\Pr(\phi_i | \psi) = \mathrm{tr}[P_{B_i} \rho(\psi)].$$

These results are summarized in Table 4 (where $P_{|u_i\rangle}$ is the projection to the subspace generated by $|u_i\rangle$, and $P_{\{u_i\}}$ is the corresponding projection to the subset $\{u_i\}$).

Table 4: Parallel operations in probability theory with superposition events and in quantum mechanics

11 Intuitive Example: Distinguishing States

The two versions of $S = U$ give us two versions of the starting point in this expanded finite discrete probability theory. The #1 version of U is the classical sample space of possible outcomes, and the #2 version of U is ΣU which represents *the* indefinite sample outcome.

1. The #1 classical version of flipping a fair coin where $U = \{H, T\}$and getting head or tails with equal probability (Figure 8)–like the mixed state:

$$\frac{1}{2}[|H\rangle\langle H| + |T\rangle\langle T|] = \rho(\Delta U) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Table 4	Probability theory	Quantum mechanics
Sample space	$U = \{u_1,...,u_n\}$ set of outcomes	$U = \{u_1,...,u_n\}$ orthonormal basis
#1 Abstract	Event = set of outcomes $S \subseteq U$	Equal mixture of eigenstates S
#2 Abstract	$\Sigma S = \Sigma\{u_i \vert u_i \in S\}$ blob-sum of S = superposition event	Equal superposition $\vert\psi\rangle = (1/\sqrt{\vert S\vert})\sum\{\vert u_i\rangle : u_i \in S\}$
#1 density matrix	Diagonal $\rho(\Delta S)_{ii} = (1/\vert S\vert)\chi_S(u_i)$	$\rho(\Delta\psi) = (1/\vert S\vert)\sum\{\vert u_i\rangle\langle u_i\vert : u_i \in S\}$ equal mixture of eigenstates of S
#2 density matrix	$\rho(S)_{ij} = (1/\vert S\vert)\chi_S(u_i)\chi_S(u_j)$	$\rho(\psi) = \vert\psi\rangle\langle\psi\vert$ pure state of equal superposition of eigenstates of S
Non-degenerate measurement	$\sum_i P_{\{u_i\}}\rho(S)P_{\{u_i\}} = \rho(\Delta S)$ Lüders mixture	$\sum_i P_{\vert u_i\rangle}\rho(\psi)P_{\vert u_i\rangle} = \rho(\Delta\psi)$ Lüders mixture
Probability of u_i	$\mathrm{tr}[P_{\{u_i\}}\rho(S)]$	$\mathrm{tr}[P_{\vert u_i\rangle}\rho(\psi)]$

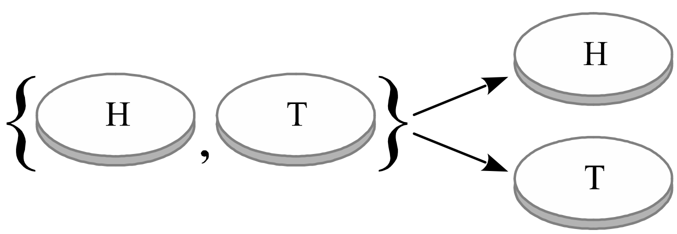

Figure 8: Outcome set for classical coin-flipping trial.

2. The #2 superposition version starts with the indefinite entity ΣU, '*the* indefinite outcome', and a trial renders it into one of the definite outcomes $\{u_i\}$ with some probability p_i so that ΣU could be represented by the density matrix $\rho(U)$ where $(\rho(U))_{jk} = \sqrt{p_j p_k}$. In the case at hand, this is like a coin ΣU with the difference between heads or tails rendered indefinite, blurred out, or superposed (which in QM is the pure state with the blobbed-out cross-terms $|H\rangle\langle T|$ and $|T\rangle\langle H|$ in the density matrix), and the trial results in it sharpening or 'decohering' to definitely heads or definitely tails with equal probability (Figure 9):

$$\frac{1}{\sqrt{2}}\left[|H\rangle + |T\rangle\right]\left[\langle H| + \langle T|\right]\frac{1}{\sqrt{2}} = \rho(U \times U) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

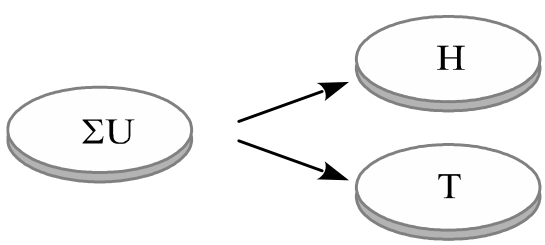

Figure 9: '*the* outcome state' for the coin-flipping trial.

By a heads-or-tail trial or measurement, one *cannot* distinguish $\rho(\Delta U)$ from $\rho(U \times U)$. The probability of getting heads in each case is:

$$\Pr(H|\rho(\Delta U)) = \operatorname{tr}\left[P_{\{H\}}\rho(\Delta U)\right] = \operatorname{tr}\left[\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0\\0 & \frac{1}{2}\end{bmatrix}\right] = \operatorname{tr}\begin{bmatrix}\frac{1}{2} & 0\\0 & 0\end{bmatrix} = \frac{1}{2}$$
$$\Pr(H|\rho(\Sigma U)) = \operatorname{tr}\left[P_{\{H\}}\rho(\Sigma U)\right] = \operatorname{tr}\left[\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & \frac{1}{2}\end{bmatrix}\right] = \operatorname{tr}\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\0 & 0\end{bmatrix} = \frac{1}{2}$$

and similarly for tails. They both give heads and tails with probability 1/2. This is not a bug but a feature since the same thing happens in QM. To distinguish such states in QM, we need to measure in a different basis. But for finite probability theory with both ordinary and superposition events, there is no 'different basis.'

However, that can be changed by moving to the pedagogical model of *quantum mechanics over sets* or *QM/Sets* [5] using vector spaces $\mathbb{Z}_2^n$ over the base field of $\mathbb{Z}_2 = \{0,1\}$. In the two dimensional version, $\mathbb{Z}_2^2$, we can take the computational U-basis as $\{H\}$ and $\{T\}$. But there is a different basis of $U' = \{H', T'\}$ where $\{H'\} = \{H, T\}$ and $\{T'\} = \{T\}$ since $\{H'\} + \{T'\} = \{H, T\} + \{T\} = \{H\}$ (mod 2) so all the non-zero states can also be expressed in the U'-basis. The vector $\{H\}$ is expressed in the U-basis by the column vector $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_U$ (the subscript indicating the basis) and in the U'-basis by the column vector $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]_{U'}$. The basis conversion matrix is

$$C_{U\to U'} = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} \text{ so } \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}_U = \begin{bmatrix}1\\1\end{bmatrix}_{U'}.$$

Hence converting the superposition $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]_U$ or $\{H, T\}$ to the U'-basis gives:

$C_{U\to U'}\begin{bmatrix}1\\1\end{bmatrix}_U = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}_U = \begin{bmatrix}1\\0\end{bmatrix}_{U'}$ or $\{H'\}$ so its density matrix (computing in the reals) is $\begin{bmatrix}1\\0\end{bmatrix}_{U'}\begin{bmatrix}1 & 0\end{bmatrix}_{U'} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}_{U'}$. The classical mixed event U is the half-half mixture of $\{H\}$ and $\{T\}$. The basis conversion for $\{H\}$ gives $C_{U\to U'}\begin{bmatrix}1\\0\end{bmatrix}_U = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}_U = \begin{bmatrix}1\\1\end{bmatrix}_{U'}$ so the associated real density matrix is:

$$\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}_{U'}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}_{U'} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & \frac{1}{2}\end{bmatrix}_{U'}.$$

For $\{T\}$, the basis conversion gives $C_{U\to U'}\begin{bmatrix}0\\1\end{bmatrix}_U = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}_U = \begin{bmatrix}0\\1\end{bmatrix}_{U'}$ so its real density matrix is:

$$\begin{bmatrix}0\\1\end{bmatrix}_{U'}\begin{bmatrix}0 & 1\end{bmatrix}_{U'} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}_{U'}.$$

Their half-half mixture has the density matrix in the U'-basis:

$$\frac{1}{2}\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & \frac{1}{2}\end{bmatrix}_{U'} + \frac{1}{2}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}_{U'} = \begin{bmatrix}\frac{1}{4} & \frac{1}{4}\\\frac{1}{4} & \frac{3}{4}\end{bmatrix}_{U'}.$$

We then measure by the partition $\sigma = \{\{H'\}, \{T'\}\}$ with half-half probabilities so the probability of H' for the superposition event $\{H, T\}$ or $\{H'\}$ in the U'-basis is:

$$\mathrm{tr}\left[P_{\{H'\}}\begin{bmatrix}1&0\\0&0\end{bmatrix}_{U'}\right]=\mathrm{tr}\left[\begin{bmatrix}1&0\\0&0\end{bmatrix}_{U'}\begin{bmatrix}1&0\\0&0\end{bmatrix}_{U'}\right]=\mathrm{tr}\begin{bmatrix}1&0\\0&0\end{bmatrix}_{U'}=1$$

and for the classical mixture of half $\{H\}$ and half $\{T\}$which in the U'-basis is the mixture of half $\{H',T'\}$and half $\{T'\}$, is:

$$\mathrm{tr}\left[P_{\{H'\}}\begin{bmatrix}\frac{1}{4}&\frac{1}{4}\\\frac{1}{4}&\frac{3}{4}\end{bmatrix}_{U'}\right]=\mathrm{tr}\left[\begin{bmatrix}1&0\\0&0\end{bmatrix}_{U'}\begin{bmatrix}\frac{1}{4}&\frac{1}{4}\\\frac{1}{4}&\frac{3}{4}\end{bmatrix}_{U'}\right]=\mathrm{tr}\begin{bmatrix}\frac{1}{4}&\frac{1}{4}\\0&0\end{bmatrix}_{U'}=\frac{1}{4}.$$

The first calculation makes intuitive sense since the superposition $\{H,T\}$ in the U-basis is the singleton event $\{H'\}$ in the U'-basis, so measuring in the U'-basis for the event $\{H'\}$ will give $\{H'\}$ with probability 1. The second calculation makes intuitive sense since it is half-half in the mixture whether we get the $\{T'\}$ event or the $\{H',T'\}$ event and then the probability of getting H' is zero for the $\{T'\}$ event and $\frac{1}{2}$ for the $\{H',T'\}$ event so the overall probability of $\{H'\}$ is $\left(\frac{1}{2}\times 0\right)+\left(\frac{1}{2}\times\frac{1}{2}\right)=\frac{1}{4}$. Thus the two events, the classical mixture of half $\{H\}$ and half $\{T\}$, and the superposition $\{H,T\}$, which cannot be distinguished by measurements in the U-basis, can be distinguished by measurement in the U'-basis.

12 Simplest Quantum Example

Consider a system with two spin-observable σ eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ (like electron spin up or down along the z-axis) where the given normalized superposition state is $|\psi\rangle=\frac{1}{\sqrt{2}}|\uparrow\rangle+\frac{1}{\sqrt{2}}|\downarrow\rangle=\begin{bmatrix}\alpha_\uparrow\\\alpha_\downarrow\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$ so the density matrix is $\rho(\psi)=\begin{bmatrix}p_\uparrow&\alpha_\uparrow\alpha_\downarrow^*\\\alpha_\downarrow\alpha_\uparrow^*&p_\downarrow\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{bmatrix}$ where $p_\uparrow=\alpha_\uparrow\alpha_\uparrow^*$ and $p_\downarrow=\alpha_\downarrow\alpha_\downarrow^*$. Using the Lüders mixture operation, the measurement of that spin-observable σ goes from the pure state $\rho(\psi)$ to

$$P_\uparrow\rho(\psi)P_\uparrow+P_\downarrow\rho(\psi)P_\downarrow$$
$$=\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}p_\uparrow&\alpha_\uparrow\alpha_\downarrow^*\\\alpha_\downarrow\alpha_\uparrow^*&p_\downarrow\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}+\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}p_\uparrow&\alpha_\uparrow\alpha_\downarrow^*\\\alpha_\downarrow\alpha_\uparrow^*&p_\downarrow\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}$$
$$=\begin{bmatrix}p_\uparrow&0\\0&p_\downarrow\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&0\\0&\frac{1}{2}\end{bmatrix}=\rho(\Delta\psi).$$

The gain in quantum logical entropy $h(\rho(\Delta\psi))-h(\rho(\psi))$ due to the spin measurement is the sum of the (absolute) squares of the off-diagonal terms that were zeroed in the change: $\rho(\psi)\rightsquigarrow\rho(\Delta\psi)$. In this case, that is $2\times\left(\frac{1}{2}\right)^2=\frac{1}{2}$. That is also $h(\rho(\Delta\psi))$ since $\rho(\psi)$ is a pure state so $h(\rho(\psi))=0$.

Experimentally, it is not possible to distinguish between the #1 and #2 versions *by σ-measurements*– since, in either case, the result will be spin up or spin down (heads or tails) with equal probability. But in QM the two states $\rho(\Delta\psi)$ and $\rho(\psi)$ *can* be distinguished by measuring other observables like spin along a different axis as emphasized by Auletta et al. [1, p. 176] and as we illustrated above using QM/Sets.

13 Conclusions

We have approached the paradigm/indefinite interpretation of probability theory with superposition events by starting with the logical situation of a universe U of distinct entities. Given a property $S(x)$ on U, we can associate with it:

1. the #1 abstract object $S=\{u_i\in U|S(u_i)\}$, the set of $S(x)$-entities, or

2. the #2 abstract object $\Sigma S = \Sigma\{u_i \in U | S(u_i)\}$ which is the abstract paradigm-entity expressing the properties common to the $S(x)$-entities but "abstracting away from," "rendering indefinite," "cohering together," or "blobbing or blurring out" the differences between those entities.

We argued that the mathematical machinery that could *distinctly* treat *both* abstractions was incidence matrices in logic and density matrices in probability theory:

1. #1 representation as $\mathrm{In}(\Delta S)$ in logic or $\rho(\Delta S)$ in probability theory; and
2. #2 representation as $\mathrm{In}(S \times S)$ in logic or $\rho(S)$ in probability theory.

Quantum mechanics can be equivalently formulated using wave-function state vectors or using density matrices [11, p. 102]. Our development above, using the analogy with #2 abstractions, dovetailed precisely into density-matrix mathematical treatment in QM where the state vector $|\psi\rangle$ is rendered as $\rho(\psi)$ which can be interpreted as an *objectively* indefinite state (according to the off-diagonal elements). This exemplifies the objectively-indefinite or literal interpretation of QM proposed by Abner Shimony.

> From these two basic ideas alone – indefiniteness and the superposition principle – it should be clear already that quantum mechanics conflicts sharply with common sense. If the quantum state of a system is a complete description of the system, then a quantity that has an indefinite value in that quantum state is objectively indefinite; its value is not merely unknown by the scientist who seeks to describe the system. [13, p. 47]
> But the mathematical formalism ... suggests a philosophical interpretation of quantum mechanics which I shall call "the Literal Interpretation." ...This is the interpretation resulting from taking the formalism of quantum mechanics literally, as giving a representation of physical properties themselves, rather than of human knowledge of them, and by taking this representation to be complete.[14, pp. 6-7]

This objective-indefiniteness or literal interpretation of QM could also be described as *density-matrix realism* since, as we have tried to show, density matrices can be interpreted as representing an objectively indefinite reality (the attempt to interpret the wave function as representing some sort of physical wave has been abandoned for almost a century now). Unfortunately, this natural (but hard to intuitively imagine) interpretation of QM is ignored in the literature of the philosophy of quantum mechanics in favor of fantasies about 'many worlds' or last-gasp attempts to retain the image of reality as definite 'all the way down' in Bohmian mechanics.

Since the ancient Greeks, we have had the #2 Platonic notion of the abstract paradigm-universal '*the* S-entity', paradigmatically definite on what is common to the entities with the property $S()$, and indefinite on where they differ, i.e., abstracting away from how they differ. By using incidence and density matrices to differentiate the #1 abstraction (e.g., the equivalence class of distinct but parallel lines) and the #2 abstraction (e.g., *the* direction of the lines), we can cross the conceptual bridge to better understand indefiniteness in quantum mechanics by seeing the analogy:

$$\text{The paradigm } \Sigma S, \text{ `}\textit{the } S\text{-entity' represented by } \mathrm{In}(S \times S)$$
$$\approx \text{the superposition state } \psi \text{ represented by the density matrix } \rho(\psi).$$

This may recall Whitehead's quip that Western philosophy is "a series of footnotes to Plato." [18, p. 39]

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