Why Newton's G is not a universal Constant of Nature

A reanalysis of Cavendish's experiment to determine the density of the Earth

by

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Abstract: A careful analysis of the methodology of Cavendish's classical experiment to determine the density of the Earth, [1], shows how G is constructed from realizations of Newton's definitions and axioms built into the methodology of the experiment. Therefore, as the physical constructs in these definitions and axioms are interrelated, Cavendish's methodology leads to three different but equivalent algebraic quantitative expressions, A, B, and C, each identified with G. This gives: $A^{-1}G=B^{-1}G=C^{-1}G=1$, and their inverses. This can be considered the historical starting point of constructing a quantitative philosophy of nature based on mathematical principles.

Each of the three equivalent expressions represents a realization of a combination of different physical constructs in the definitions and axioms used. The realizations of the relevant physical constructs and axioms built into Cavendish's methodology define the characteristics of what I call homogeneous gravitating spheres, HGS, for which the Earth is the quantitative reference standard. Realized in this reference standard are: Newton's operational definition 1 of mass in his Principia, his definition of centripetal acceleration, which implies the inverse square law for the centripetal surface acceleration of HGS, and his 2nd law of motion; the latter realized in a tangent space of the Earth. Taken together this defines a gravitational based standard of force for any arbitrary choice of a standard of mass. There is nothing universal in this. That this state of affairs has not been made clear since Cavendish read his report to the Royal Society of London on June 21 1798 is rather surprising, as the facts laid out here have far ranging consequences, particularly for the philosophy of science and nature w.r.t. constructing the Weltbild of our universe.

Keywords: Cavendish experiment, Newton's gravitational constant G, Planck's natural units.

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1. The motivation for analyzing Cavendish's experiment[2]

The motivation for my research into how and why physics works the way it does, began in my graduate student days, when, in a course given by Professor F. Kaempffer at UBC on Adv. Q.M., I chose as an assignment to derive and solve the Dirac equation for an electron by requiring the Lagrangian to be invariant under coordinate dependent transformations in a spherically symmetric space-time defined by the line element of the Schwarzschild solution to Einstein's field equations.

The theoretical framework was defined to follow the work of N.C. Yang and R.L. Mills[3], and in particular that of Ryoyu Utiyama[4], who extended it to the gravitational field. My solution for the strong field, in energy units, $E = m_e c^2 = 1$, was a perfect square, but centered at -1/2 in the Cartesian plane on the energy axis. This raised in me some suspicion, not regarding my solution, but of the role played by Pythagoras theorem in the geometric and algebraic representation of numbers and physical constructs in general.

I started to follow up on my thoughts after I retired and as time permitted; they have been very productive. Among them was the derivation of G from the condition of consistency between general relativity and quantum mechanics; achieved by transcending Newton's tautological relationship between mass and gravitation. This led eventually to my analysis of Cavendish's experiment and its role in the quantification of physical constructs; the determination of the density of the Earth and hence of its mass from its centripetal surface acceleration. Therefore, this experiment is a good place in the history of the physical sciences to begin the investigation of the development of a philosophy of nature based on mathematical principles, with the appropriate references to its base; geometry, mathematics, Newton's definitions and axioms.

The analysis of the Cavendish experiment and literature with G in the title indicates a lack of awareness of the basic aspects of Cavendish's methodology, and how it led to three relationships defined to be G. This seems to have dominated the conceptual structure surrounding G ever since.

* * *

Because G is only a universal constant by definition, the following <u>digression</u> is important and required section 11 of this paper. The reason is Newton's constant of gravitation plays a significant role in theories of gravitation as well as in other field theories, as indicated by the quotes below from [2], 1.4, p. 9, where M. Kaku gives his historical perspective on quantum field theory. I included his Eq. (1.1) from section 1.1 of his book, where M_p stands for the proton mass.

Kaku: section 1.4 "Ironically, although the gravitational interaction was the first of the four forces to be investigated classically, it was the most difficult one to be quantized. ….

The problem, however, was that quantum gravity, as seen from Eq. (1.1), had a dimensionful coupling constant and hence was nonrenormalizable. The coupling constant, in fact, was Newton's gravitational

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[2] The first two paragraphs, though important can be skipped by readers unfamiliar with field theory.

[3] N.C. Yang and R.L. Mills, "Conservation of Isotopic Spin and Isotopic Gauge Invariance", Phys. Rev. <u>96</u> 1, Oct. 1 1956

[4] Ryoyu Utiyama, "Invariant Theoretical Interpretation of Interaction", Phys. Rev.. 1<u>01</u>, 5, March 1. 1956

constant, the first important universal physical constant to be isolated in physics. Ironically, the very success of Newton's early theory of gravitation, based on the constancy of Newton's constant proved fatal for a quantum theory of gravity." (W. F.H. "investigated classically" could be debated.)

In his equation (1.1) from section 1.1, his one squiggle is replaced here by, $\approx$.

$$\begin{aligned} \alpha_{em} &\approx 1/137{,}0359895(61) \\ \alpha_{strong} &\approx 14 \\ \alpha_{weak} &\approx 1.02\mathrm{x}10^{-5} / M_P \\ G_{Newton} &\approx 5.9\mathrm{x}10^{-39} / M_P^2 \\ &\approx 6.67259(85)\mathrm{x}10^{-1}\, m^3 kg^{-1} s^{-2} \end{aligned} \qquad (1.1)$$

* * *

This paper is written so first year undergraduates in physics and mathematics can understand it. Hence, I point out that Kaku expression in line 5 of (1.1) is w.r.t. Planck's natural units; otherwise the factor 10^{-1} may look like a typographical error. In section 11, the reason for the "increase" in the value of G by a factor of 10^{10} in that line is explained.

The fine-structure constant α_{em} is constructed as, $\alpha_{em} = e^2/(4\pi\varepsilon_0 \times \hbar c)$, to be dimensionless. According to the conventions of physics, the function-value of α_{em} and the infinite other numbers that can be constructed from its powers, are therefore invariant under changes of the physical standards of mass, length and time. They are simply numbers. I prefer to view α_{em} in relation to line four of (1.1), not as shown there, but as explained below, otherwise row 4 and 5 in (1.1) may be confusing.

$$\alpha_{em} = 2\pi e^2/(4\pi\varepsilon_0 \times hc) \Rightarrow hc = e^2/(4\pi\varepsilon_0 \times \alpha_{em}/2\pi).$$

What the fourth row in (1.1) actually represents is shown below; $GM_P^2/\hbar c$ is dimensionless. Hence,

$$[G_{Newton} M_P^2/(e^2/4\pi\varepsilon_0)] \times \alpha_{em} = \frac{G_{Newton} M_P^2}{\hbar c}. \qquad (1.2)$$

This implies,

$$\frac{G_{Newton}}{\hbar c} \approx 5.9\mathrm{x}10^{-39} / M_P^2. \qquad (1.3)$$

The reader can easily verify (1.3) is simply the reciprocal of the square of Planck's natural unit of mass given in his 1900 paper [3], as stated in (1.4) below, after adjusting for the extra factor of 2π in $\hbar$. The values of these *physical constants have changed* somewhat since Planck's introduction.

$$\text{Planck in [3]; } \sqrt{\frac{hc}{G}} = 5.56\mathrm{x}10^{-5}\,g; \quad \text{Kaku's equivalent in[2]; } \sqrt{\frac{hc}{G}} = 5.456\mathrm{x}10^{-5}\,g. \qquad (1.4)$$

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2. A point of logic and a philosophy of nature based on mathematical principles

Physics, as a quantitative science, is the result of constructing a philosophy of nature based on mathematical principles. However, if physics is assumed to be more than just mathematics, its underlying mathematical logic structure should take account of the fact that Newton's definition of mass is tautological w.r.t. the phenomenology of gravitation. This is required if one wants to develop the tools to transcend this tautology w.r.t. Newton's Definition 1 of mass in his Principia, [4], p. 403, and w.r.t. the phenomenology of gravitation it was meant to explain. At one point in the book "Foundations of Physics" Margenau calls the view of mass in Definition 1, "ridiculous."

Definition 1, Newton's operational definition of mass, is fundamental to Cavendish's experiment and therefore quoted below, although my section 4 contains more relevant comments.

* * *

Newton: Definition 1: "Quantity of matter is a measure of matter that arises from its density and volume jointly." …'Furthermore, I mean this quantity whenever I use the term "body" or "mass" in the following pages. It can always be known from a body's weight, for – by making very accurate experiments with pendulums – I have found it proportional to the weight …"

* * *

The tautology referred to above is made clear by my point of logic stated below.

* * *

"*Using gravitational phenomenology to realize and quantify Newton's operational definition of mass, makes it ineligible as an explanation for the very phenomenology used to define and quantify the concept mass in the first place*."

* * *

The reason for accepting mass as explanation may be found in the basic doctrine of physics, explained by N.R. Campbell, quoted below from the end of chapter 1 of his book, [5], p. 36-37. It was first published in 1920 under the title; "Physics the Elements." Facing its Contents page is the motto,

"*It is not the facts, but the explanations of them, that matters*"

* * *

Campbell: "We have concluded that the subject matter of science consists of judgements concerning which universal agreement can be obtained, and that judgements of the external world received through the sensations are especially important for science because universal agreement can be obtained for them. What would happen if this universal agreement ceased, if we were unable to find any judgements which everyone would accept, or (more probably) if we found differences arising concerning space-, time- and number-judgements on which hitherto everyone has been agreed ? (……)

And even if the question were significant, any attempt to answer it would lead us to violate the fundamental principles of this volume. We are concerned to consider only what science is. If we considered what it might be, we should be forced to base the discussion on some doctrine more fundamental and more ultimate than any involved in actual science. For our present purpose it is necessary to deny that there is such a doctrine. *Science is what it is; it is a law to itself and admits no inquiry why it is what it is."*

* * *

In the quote below from Henry Margenau's 1941 article, [6], p. 181, he describes the mode of constructing the knowledge base of physics in more operational terms w.r.t. theoretical physics than Campbell. Margenau, with R. B. Lindsay, also wrote "Foundations of Physics" published in 1936. In Margenau's article, the reason for the doctrine stated by Campbell appears in a somewhat different light w.r.t. the quote below.

* * *

Margenau: "Sensed nature subsumes but a small fraction of the features which constitute physical experience. Of the remainder, we now single out all infra-susceptible elements of interest to the physicist. To name a few: *mass*, *force* (except when they are taken in an immediate particular sense; in physics, as everywhere else, a word often signifies a perception as well as the generalized, abstracted features of a class of similar perceptions), *energy, charge, wave-length of light, field strength, potential, probability, amplitude; - voltmeter, crystal* (last parenthesis repeated), *magnetic field, atom, photon, meson.* These typical examples have been listed in a special order on which we shall comment later. All such elements, which have the attributes of being infra-perceptible and occurring in some physical theory, are here called "*constructs of physical explanation*." This term was chosen to emphasize the following facts.

These elements are not and can never be part of sensed nature; they serve in fact the purpose of *explaining* nature. By physical explanation the physicist means nothing more than establishment of organized relations between constituents of nature and these elements. Every other interpretation, as a search for the Causes of Things, draws in more metaphysics than the minimum we are admitting. Moreover, these constructs do not partake of the spontaneity and immediacy that distinguishes the elements of nature; they can, in fact, be freely chosen, can be generated *ad libitum*. There is no criterion able to distinguish their suitability for physics *at birth*. It is to expose clearly this freedom in the manner in generating them that they are here called constructs. We wish to avoid ab initio the misconception that a good construct must be one that is visualisable or mechanical or any other such thing."

* * *

For *construct of physical explanation*, I use *physical construct*, including mass, length and time, when represented as mathematical variables. The conception as to what physics is, seems to have changed since Margenau's 1941 article, as made clear in Weinberg's quote below from 1996, [7].

* * *

Weinberg : "When you teach any branch of physics you must motivate the formalism - it isn't any good just to present the formalism and say that it agrees with experiment - you have to explain to the students why this the way the world is. After all, this is our aim in physics, not just to describe nature, but to explain nature. In the course of teaching quantum field theory, I developed a rationale for it, which very briefly is that it is the only way of satisfying the principles of Lorentz Invariance plus quantum mechanics plus one other principle."

* * *

With respect to my point of logic, I like to pass on a personal anecdote which indicates professional scientists also interpret the concept *explanation* in the sense as commonly understood. Many years ago, after completing my MA in applied mathematics, I worked during the summer for the astronomer who taught my Classical Mechanics undergraduate course in physics and mathematics. I was performing calculations relating to stellar atmospheres and the effect of ionization on energy transfer; in the good old days of FORTRAN and punched-cards. We kept periodically in touch. After he became Professor Emeritus, I also informed him of my research and explained my point of logic, that the concept mass as used w.r.t. gravitational phenomenology is tautological. It probably took about 3 minutes of discussion before it was clear to him I actually had a point. As a practicing scientist and teacher of physics, he found it difficult to acknowledge that the term mass, as operationally defined, cannot explain the phenomenology used to define it.

* * *

Cavendish's experiment to determine the density of the Earth,]1], identifies the point where Newton's operational definition of mass w.r.t. gravitational phenomenology in the tangent spaces of the Earth, established a quantitative realization of mass according to Newton's operational definition 1, and a gravitational standard of force for his inverse square law of gravitation.

3. Newton's Principia, I.B. Cohen, Definitions

To set the stage, I begin with some quotes from I.B. Cohen, from the marvelous new translation of Newton's Principia by I. B. Cohen, and Anne Whitman, assisted by Julia Budenz, [4], to remind readers of the context in which the laws of nature and of physics are defined.

* * *

Cohen: "Newton's Principia is a book of mathematical principles applied to nature insofar as nature is revealed by experiment and observation. As such, it is a treatise based on evidence. Never before had a treatise on natural philosophy so depended on an examination of numerical predictions and numerical evidence. The significance of Newton's numbers, in the context of the mission of *Principia*, was not only to provide convincing evidence of the correctness of the mathematical principles being applied to natural philosophy, but to explore the theoretical significance of possible conflicts between simple - perhaps overly simple – theory and the evidential universe."

* * *

Despite my point of logic, I agree with I. B. Cohen's view expressed in [4], that mass is defined only by implication in the Principia and is not tautological; but only in that context. My issue is not with the realization of mass in experimental physics, technology and their practices; but with the role mass is given in cosmology in connection with G; but also in astronomy, where apparent problems, related to mass and G, are attempted to be solved by inventing new physical constructs instead of examining whether the domain of applicability of the existing constructs has been exceeded. Cohen's reasons why mass in the Principia is not tautological are quoted below, [4] p. 93.

* * *

Cohen: "In the *Principia*, mass is related to force in two ways: (1) dynamically through the second law of motion: (2) gravitationally, through the law of gravity. From this point of view, the argument about circularity is wrongly conceived, being based on the supposition that in def. 1, mass is a secondary quantity defined in terms of the primary quantities, density and volume. But in the *Principia*, mass is a primary and not a secondary property, and is not explicitly or properly defined in terms of other quantities which are primary. Mass is defined only by implication. It has the property of inertia (def. 3). It appears (via momentum, or "quantity of motion") in the second law as the measure of a body's resistance to a change in motion (for impulsive forces) or as the measure of a body's resistance to acceleration (for continuous forces)."

* * *

With respect to the fact the fundamental reference concept for Newton's laws of motion is that of rest or uniform motion in a straight line, this expresses the condition of equilibrium, static or dynamical, of a body (mass) as stated in Newton's 1st law of motion, [4], p. 416. These reference states can always be related by a linear coordinate transformation in space and time in Euclidean geometry. His 2nd law relates the concept "change in motion" to the physical construct "force."

* * *

Newton: Law 1: "Every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by forces impressed."

* * *

Newton: Law 2 "A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed."

* * *

However, the physical construct of energy and momentum associated with the picture of a uniformly moving body and one at rest represent different states, considering the term body or mass to represent a system. This makes the assumption, that dynamic or static equilibrium w.r.t. transformations along a straight line, represents the natural state of things a very questionable one in Newton's axiomatic structure[5]. A likely consequence is the notion of primary and secondary physical constructs is only relevant within this system, whereas at a higher level they are all relative to one-another. The physical construct force, defined w.r.t. change in motion, in which the construct mass is only a coefficient in its mathematical realization, seems to be more primary than that of mass; i.e., $\text{force} = k \times \text{change in motion}$, where change in motion references uniform straight line motion as the natural state of the world w.r.t. Newton's axioms. Therefore, the two types of change in motion; along a straight line or perpendicular to a straight line are of particular interest, because of the roles the concept *perpendicular* is given in mathematics and physics and of course in Pythagoras theorem and in Minkowski's geometry of numbers.

* * *

4. Some implications of Newton's operational definition of mass

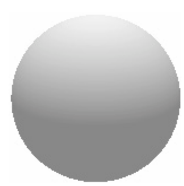
Newton's operational definition of body or mass likely arises from the fact objects of equal volumes can have different weights. This led to the concept of density w.r.t. Archimedes' principle. Taking water as an incompressible reference liquid, an object floating in equilibrium, if submerged in water on the surface of a gravitating sphere, has then the same density as water. The term density ρ is assumed here to refer to homogeneous bodies. The term body or mass, represented by Newton's definition 1, is viewed here as represented by HGS, as operationally as well as abstractly used in Cavendish's experiment to be analyzed, as indicated in figure 1a and 1b. The function value of the density ρ is relative to the density of water, taken as unity.

* * *

Newton: Definition 1: "Quantity of matter is a measure of matter that arises from its density and volume jointly." …'Furthermore, I mean this quantity whenever I use the term "body" or "mass" in the following pages. It can always be known from a body's weight, for – by making very accurate experiments with pendulums – I have found it proportional to the weight …"

* * *

$$M = \rho \frac{4\pi}{3} R^3 \text{; and}$$



(a) (b)

Figure 1: (a) A mathematical representation of Newton's operational Definition 1 of mass M as used by Cavendish in his experiment and as used in this analysis.
(b) The generic geometric representative element of mass in Newton's definition 1, as well as that of a HGS in the context of Cavendish's experiment. The concept, mass as such, is scale and orientation invariant and thus homogeneous as a generic physical construct.

* * *

[5] This is taken care of in modern physics by the Lorentz transformations which leave the speed of light invariant but lets the physical constructs of, mass, length, and time, vary. In this way conservation of energy and momentum is salvaged. A good read here is Einstein's 1934 Gibbs lecture.

In definition 1 of mass, any of the three physical constructs of M, ρ, and V, can be expressed in terms of the other two, i.e., $M = \rho V$, $\rho = M / V$, and, $V = M / \rho$. This fact implies homogeneity in the mathematical context by the scale invariance of definition 1, and means ρ is a constant w.r.t. M and V. This has mathematical consequences, i.e., (4.1) and (4.2), where m represents now the linear variable of mass, and R that of length, in physics. Then,

$$dm = \rho 4\pi R^2 dR \,. \tag{4.1}$$

Therefore, density can be eliminated from (4.1) in terms of the variables m and R, giving,

$$\frac{dm}{m} = 3\frac{dR}{R} = 3d\ln R \Rightarrow d\ln m = d\ln R^3 \,. \tag{4.2}$$

This indicates that in analysis, where mass and length are represented by linear variables, these variables are logarithmicly related w.r.t. Newton's definition 1, but this is hidden in physics by the use of dimensioned constants.

* * *

5. Newton's definition of centripetal force and centripetal acceleration

His definition of centripetal force, essential for understanding Cavendish's methodology, is introduced first from the perspective of experience by Newton in Definition 5, [4], 405. He defines the corresponding mathematical principle in proposition 4, theorem 4, [4], 449.

* * *

Newton: Definition 5 "*Centripetal force is the force by which bodies are drawn from all sides, are impelled, or in any way tend, toward some point as to a center.*

One force of this kind is gravity, by which bodies tend towards the center of the Earth, another is magnetic force, by which iron seeks a loadstone, and yet another is that force, whatever it may be, by which the planets are continually drawn back from rectilinear motion and compelled to revolve in curved lines.

A stone whirled in a sling endeavors to leave the hand that is whirling it, and by its endeavors it stretches the string, doing so the more strongly the more swiftly it revolves, and as soon as it is released, it flies away…."

* * *

Newton, Proposition 4: "The centripetal forces of bodies that describe different circles with uniform motion tend toward the centers of these circles and are to one another as the square of the arcs described in the same time divided by the radii of the circles."

* * *

Newton's definition of centripetal forces of bodies in uniform circular motion, UCM, of different radii is relative to the same bodies. Therefore the physical construct of mass cancels in its mathematical realization. This leads, using geometry and algebra, to the generic relationships in (5.1), between the period T of UCM, its radius R, and the speed v of a point moving in UCM. Here too, the definition of UCM, like that of mass or body, is defined by three terms of which any two of R, v, and T, can be used to express the third. Defining a function value to any one w.r.t. relevant physical standards defines a constraint between the other two, (5.1). This is the result of the definition of circular force and circular acceleration referencing the geometric structure of the circle; geodesics on HGS of radius R.

$$v = \frac{2\pi R}{T}; \; T = \frac{2\pi R}{v}; \text{ and; } R = \frac{Tv}{2\pi} \,. \tag{5.1}$$

Newton's definition of centripetal acceleration, g, leads, using (5.1), to (5.2). See Appendix I.

$$g = \frac{v^2}{R}\,;\; g = 4\pi^2\frac{R}{T^2}\,;\; g = 2\pi\frac{v}{T} \tag{5.2}$$

If in ((5.2), we substitute for v in its 3rd form from the first, we obtain the 1st Eq. in (5.3). The implication of its form, so obtained, is in my view underappreciated. Why, is made clear in section 6. The 3rd Eq. in (5.3) is defined to represent also the period of a simple pendulum for small oscillations, in which its r.h.s., according to the conventions of physics, is invariant under changes of the standards of length, can be used as a standard of time $\{T\}$, where $\tilde{t}_p$ is the function value of T.

$$g = 2\pi\frac{\sqrt{gR}}{T} \Rightarrow \sqrt{\frac{g}{R}} = 2\pi\times\frac{1}{T} \Rightarrow T = \tilde{t}_p\{T\} = 2\pi\sqrt{\frac{R}{g}} \tag{5.3}$$

* * *

6. The simple pendulum and the period of surface grazing orbits on HGS

To make this point and its consequences clear, it is useful to introduce the symbols, $\{M\}$, $\{L\}$, and $\{T\}$, for the three fundamental physical standards of, mass, length and time, I define their physical dimension to be implicit to these symbols. In this analysis the physical standards are respectively identified with,

$$\{M\} \leftrightarrow 1\text{kg} \rightarrow [M],\;\; \{L\} \leftrightarrow 1\text{m} \rightarrow [L],\;\; \{T\} \leftrightarrow 1\text{s} \rightarrow [T]. \tag{6.1}$$

As they are formally identified with the number 1, they are assumed to be real and positive. I represent the dimensioned variables of mass m for example, when appropriate as, $m = \tilde{m}\{M\}$. A tilde over the symbol m, e.g., $\tilde{m}$, represents a positive number, which for a particular object or measurement depends of course on the particular physical standard, $\{M\}$. I call $\tilde{m}$ in this context its function value. This notation is also used here for higher order physical constructs. This makes it easier to establish the boundary between mathematics and physics also in other areas.

The paradigm under discussion in (5.3) is one of the generic representations between T, R and g of centripetal acceleration. On the other hand, it is defined to give the relationship between the length and period of a simple pendulum in the tangent space of any gravitating sphere, where g is defined to be its surface gravitational acceleration. I demonstrate later how this is done. Naturally, the Earth is used as a particular example. The reason I will discuss a simple pendulum whose period T is defined to be $T = 2\{T\}$, is because Cavendish, defined his reference pendulum, to calibrate his torsion pendulum in his experiment, to swing from one extreme to the other in 1s.

The physical standard of, mass, length and time, are identified with the first natural number, e.g., $\{T\} = 1$, however chosen, which I take to be 1, despite what Russell says in his "Introduction to Mathematical Philosophy." Cavendish defined the function value of the period of his reference pendulum to be, $\tilde{t}_p$=2. As g is defined to be a constant gravitational acceleration on the Earth surface as a HGS, this constant, I denote by g_E. Having defined the function values for T and defining g_E to be an unknown constant w.r.t. $\{T\}$; remember the r.h.s. (5.3) is independent of the choice of any standard $\{L\}$, one has to let gravitational phenomenology define the length of the pendulum. Let this be $R = \tilde{R}_p\{L\}$. For the pendulum to have a period of 2s its length R_p had to be 39.14 inches, or $0.994156\{L\}$ at Cavendish's geographical location. This allows us to calculate, g_E.

$$1\{T\} = \pi\sqrt{\frac{39.14\text{ inches}}{g_E}} = \pi\sqrt{\frac{0.994156\{L\}}{g_E}} \tag{6.2}$$

After squaring (6.2), and rearranging, g_E, for the physical standards used is,

$$1^2 g_E = \frac{\pi^2 \times 3.261666667\text{ft}}{\{T\}^2} = 32.19135969\text{ft}/\{T\}^2 \ , \tag{6.3a}$$

$$1^2 g_E = \frac{\pi^2 0.994156\{L\}}{\{T\}^2} = 9.811926433\{L\}/\{T\}^2 \tag{6.3b}$$

It is unusual to express physical data and the result of calculations as if the data is exact. The reason this is necessary here emerges shortly. We can now use Newton's definition of centripetal acceleration to calculate the period T_E of the Earth's surface grazing orbit using Cavendish's value for the Earth's radius, R_E, expressed w.r.t. $\{L\}$, viewing the Earth as a HGS without an atmosphere to retard motion. The period T_E is then given by (5.3), as about, 84.4 min without having used the physical construct mass or any standard $\{M\}$, however chosen, so far.

$$T_E = 2\pi\sqrt{\frac{R_E}{g_E}} = 2\pi\sqrt{\frac{6370320\ \{L\}\{T\}^2}{9.811926433\ \{L\}}} = 5062.713498\{T\} \tag{6.4}$$

* * *

It is now appropriate to quote Cavendish, defining the relationship between his reference simple pendulum used above and his methodology w.r.t. his torsion balance [1], p. 509.

* * *

Cavendish: "The first thing is, to find the force required to draw the arm aside, which, as was before said, to be determined by the time of one vibration. (underlining by W.F.H.)

The distance of the centers of the two balls from each other is 73,3 inches, and therefore the distance of each from the center of motion is 36.65, and the length of a pendulum vibrating seconds, in this climate, is 39,14; therefore, if the stiffness of the wire by which the arm is suspended is such, that the force which must be applied to each ball is such, that in order to draw the arm aside by the angle A, is to the weight of that ball as the arch of A to the radius, the arm will vibrate at the same time as a pendulum whose length is 36.65 inches, that is in $\sqrt{36.65/39.14}$ seconds; and therefore, if the stiffness of the wire is such as to make it vibrate in N seconds, the force that must be applied to each ball, in order to draw it aside by the angle A, is to the weight of the ball as the arch of $A \times \frac{1}{N^2} \times \frac{36.65}{39.14}$ to the radius."

* * *

At the end of section 5, I stated that the r.h.s. of $T = 2\pi\sqrt{\frac{R}{g}}$, according to the conventions of physics, is invariant under changes of the physical standards of length. Having calculated the function values of its r.h.s. for two different standards of length in (6.3a) and (6.3b), it is obvious what this actually means in Cavendish's experiment; namely the constraint shown below,

$$\frac{g_E}{L_P} = \frac{\pi^2}{\{T\}^2} = \frac{32.19135969\text{ ft}}{3.261666667\text{ ft}}\{T\}^{-2} \ , \tag{6.3a}$$

$$\frac{g_E}{L_P} = \frac{\pi^2}{\{T\}^2} = \frac{9.811926433}{0.994156}\frac{\{L\}}{\{L\}\{T\}^2} \ . \tag{6.3b}$$

This means that in Cavendish's experiment, $\tilde{g}_E / \tilde{L}_P = \pi^2$, is the constraint imposed by the standard of time as used here w.r.t. the paradigm of the simple pendulum between the function values of $\tilde{g}_E$ and $\tilde{L}_P$ by the gravitational phenomenology at the Earth's surface; it must be invariant under changes of the standard of length as pointed out earlier.

* * *

As the period of a simple pendulum on a gravitating sphere does not depend on the weight of the pendulum bob, i.e., simple pendulums of the same length but with bobs of different weight have the same period under the assumed ideal conditions, it is not quite clear what the experiment will actually achieve w.r.t. the quantification, i.e., realization, of Newton's axioms and definitions. The remainder of this paper will make that clear; it defines a standard of force w.r.t. any choice of, $\{M\}$, $\{L\}$ and $\{T\}$.

* * *

7. Another aspect of Cavendish's methodology

To make the conceptual context of Cavendish's historic experiment clear, I think it best to let Cavendish describe first the purpose of this experiment himself, using the quote below, taken from his introductory remarks in his paper, [1], read June 21, 1798 to The Royal Society of London.

* * *

Cavendish: "Many years ago, the late Rev. John Michell, of this Society, contrived a method of determining the density of the Earth, by rendering sensible the attraction of small quantities of matter; but, as he was engaged in other pursuits, he did not complete the apparatus till a short time before his death, and did not live to make any experiments with it. After his death, the apparatus came to the Rev. Francis John Hyde Wollaston, Jacksonian Professor at Cambridge, who, not having conveniences for making experiments with it, in the manner he could wish, was so good to give it to me."

* * *

The significance of his experiment is that he determined the mean density of the Earth relative to that of water. It took a few calculations to interpret correctly his description of a particular part of his methodology. I was not clear on his use of "10.64 spherical feet of water" in the quote below, [1], p. 510. A numerical correction factors arising from a particular geometric constraint of his equipment, 0.44948, I replaced by, Ξ, It includes the ratio of the squares of the perpendicular distances of the center of an abstract spherical foot of water and of the deflecting lead weight from the equilibrium position of the arm of the torsion balance, i.e., $(6/8.85)^2 = 0.45964$. (Figure 2 is mine.)

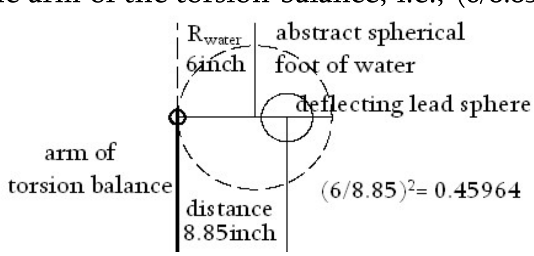


Figure 2: This clarifies Cavendish's underlying conceptual structure. He used the spherical foot of water as an abstract HGS as standard of <u>weight</u> w.r.t. the <u>weights</u> of deflecting lead spheres. Use of this abstract standard and this correction factor in Ξ shows Cavendish's experiment was designed for a quantitative realization of Newton's inverse square law of gravitation w.r.t. Newton's operational definition of mass. The dimension and density of the deflecting lead weights was

irrelevant, viewed as HGS, analogous to a spherical foot of water as an abstract standard of a HGS. Only their relative weights were significant, and that the density of water was taken to be unity.

* * *

Cavendish: "Each of the weights weighs 2439000 grains, and therefore is equal in weight to *10,64 spherical feet of water*; and therefore its attraction on a particle placed at the center of the ball, is to the attraction of a spherical foot of water on an equal particle placed on its surface, as $10.64 \times \Xi$ * to 1. The mean diameter of the Earth is 41800000 feet; and therefore, if the mean density of the Earth is to that to water as D to one, the attraction of the leaden weight on the ball will be to that of the Earth thereon, as $10.64 \times \Xi$ to $41800000\,D$ as 1 to $8739000\,D$." (The foot note * on Ξ reads)

* In strictness, we ought, instead of the mean diameter of the earth, to take the diameter of that sphere whose attraction is equal to the force of gravity in this climate; but the difference is not worth regarding.)

* * *

The reader should be clear on Cavendish's methodology. 10.64 *spherical feet of water* refers to the number of individual spheres of water, 1 foot in diameter, each taken as an abstract *physical reference standard of a HGS of water*, not to a sphere of water with a diameter of 10.64 ft. No physical sphere of water is used in the experiment. The context of relative squares of the distances contained in Ξ as a correction factor, I indicated in fig. 2. The perpendicular distance of 8.85 inches was the closest the deflecting lead sphere could come to the rest position of the arm of the torsion pendulum in its swing. The inverse square law was thus built into Cavendish's methodology; by treating every spherical weight as a HGS, defined as a realization of Newton's inverse square law of gravitation in this experiment, as well as of his definition 1 of mass or object.

* * *

To simplify the quantitative analysis for the reader, the relative weight of the large lead weights w.r.t. a spherical foot of water is converted here into their relative masses in kg; the constants of proportionality cancel. This is of course the reason G has more than one definition.

$$1\text{grain} = 0.0648\text{gram}; \qquad m_{Pbsph} = 158.047\{M\}; \qquad \rho_{H_2O} = 1\text{g}/\text{cm}^3 = 1000\text{kg}/\text{m}^3$$

The *spherical foot of water* serves *as an abstract model of a HGS*. The density of water I take as: $1\text{g}/\text{cm}^3$. By Definition 1, its volume and density define its mass, given below in kg.

$$1\text{ft} = 30.48\text{cm}, \qquad V_{sp}(1\text{ft}) = 14827.6662\text{cm}^3, \qquad M_{sp}(H_2O) = 14.8277\text{kg}.$$

With respect to this abstract standard, each deflecting lead sphere has a weight equivalent to that of 10.6589 spheres of water of 1ft diameter. From the above data, Cavendish effectively used a density of water which was almost 2 parts/thousand lower than, $1\text{g}/\text{cm}^3$. With respect to, $M = \rho V$, the Earth, the deflecting lead spheres, and the abstract spherical foot of water, are considered HGS in the context of an inverse square law, made clear by his methodology and the correction factor in Ξ and the fact the diameter of the Earth, is taken as inversely proportional to its density. This has also consequences for the representation of G as we will see shortly.

* * *

The role of the simple pendulum is actually two-fold; (a): as a physical reference device for Cavendish to calibrate the torsion pendulum of his experiment, and (b): to provide a paradigm to relate his reference pendulum to the gravitational phenomenology at the Earth surface, to the data obtained by "rendering sensible the attraction of small quantities of matter" and the density of the Earth so determined, i.e., w.r.t. the forces sensible on its surface.

To summarize, Cavendish decided on a simple pendulum oscillating with a half-period of 1 second to calibrate his torsion pendulum. Its length is then determined by the phenomenological manifestation of gravitation g_E at his location in a tangent space of the Earth; the acceleration due to gravity. This was identified earlier with the paradigm of centripetal acceleration g. At his geographical location, he determined the length L of his reference simple pendulum must be 39.14 inches in order to oscillate with a half period of 1s. Later, I use Cavendish's data, including the value of 41,800,000. ft he used for the diameter of the Earth needed to determine its average density w.r.t. water; 5.48, and the conversion, 1ft = 12inches, 1inch= $0.0254\{L\}$. This gives the Earth's radius as: $6{,}370.320\text{km}$. I take this value for the radius R_E of the Earth as an idealized HGS.

An important fact is the period T of a simple pendulum is a function of its length L and the centripetal acceleration g_E at the Earth's surface, but is independent of the mass of its bob. The function value of its length and period depends of course on the choice of the physical standards of length and time in general, but subject to the constraint (6.3), for this particular choice of $\{T\}$.

* * *

In quantifying g_E by means of a simple pendulum, only its period T or its length L_P can be arbitrarily defined. The other quantity is then defined by phenomenology and the paradigm for the simple pendulum. In the tangent space of the Earth as a HGS, g_E is considered a constant vector, which for any m defines a gravitational standard of force in its tangent spaces for the variable m, and leads from Cavendish's experiment to Newton's inverse square law of gravitation. However, the inverse square law, $g_E m / m = M_E G / R_E^2$, applies only to, $g_E = M_E G / R_E^2$; as $m = \tilde{m}\{M\}$, is a free parameter in, $g_E(m/m) = M_E G / R_E^2$, as both sides are independent of any standard of mass $\{M\}$.

* * *

In the following, sections, I make the same idealizing assumption as Cavendish regarding the Earth; assuming it to be a HGS, for which thanks to his experiment, I take its average density as 5.48 times that of his equivalent spherical feet of water. In this context, the loci of UCM, with g_E their centripetal acceleration, are great circles on the Earth's surface I call surface grazing orbits. Section 8 is a necessary digression to show how Newton's 2nd law of motion is used to define the paradigm required to relate Cavendish's simple pendulum to the minute forces in his experiment and to those at the surface of the Earth as a HGS.

* * * * *

8. Tangent spaces on a gravitating sphere and Newton's 2nd Law

In the tangent spaces of a HGS, it is legitimate to consider g_E a <u>constant vector</u>, normal to a tangent plane at the point of tangency. The Earth, and gravitating bodies in general, do not correspond to the abstract model of HGS Therefore, the surface centripetal acceleration g_E depends on the location of the tangent space of the Earth and includes effect of rotation and inhomogeneities that don't enter Newton's definitions or mathematical principles.

Matching a mathematical paradigm to the phenomenology exhibited by physical simple pendulums include such effects in g_E. In that context, the mathematical structures developed, using the data from Cavendish's experiments, are treated as belonging to a stationary HGS. There are no indications in his report that the Earth rotation was considered a factor in his experiments, except perhaps addressed in the footnote quoted w.r.t. Ξ.

Therefore, as is usual in deriving the equation describing the simple pendulum for small oscillations, only the 2-D tangent subspace defined by the plane in which the pendulum oscillates is considered. The vector decomposition diagram of g_E, and a corresponding force decomposition diagram are shown in fig. 3, viewing the Earth as a stationary gravitating sphere.

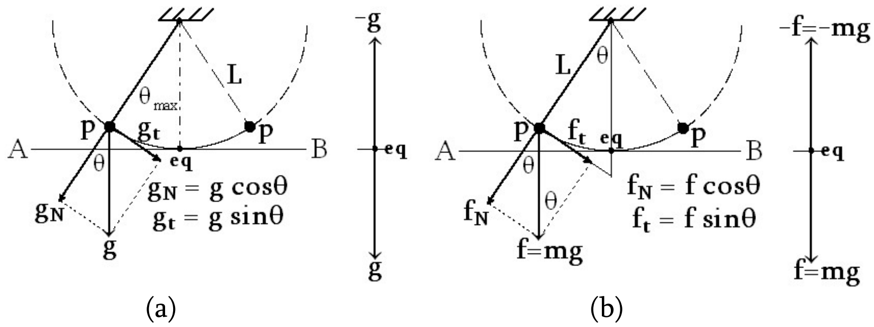


Figure 3: Decomposition diagrams in the tangent space of a HGS in which the constant centripetal acceleration vector is g w.r.t. the constrained motion of the pendulum bob. In the corresponding force diagram; m is an independent variable w.r.t. g. However, Cavendish's experiment is designed to link that variable m to centripetal acceleration, (5.3), as "cause" of centripetal force making it agree with Newton's 2nd law of motion in the sphere's tangent plane. The line AB represents a tangent to a surface grazing orbit, Eq. (6.4), at the point labeled, eq ; the pendulum's equilibrium or rest position where $g_t = 0$.

* * *

For small amplitudes of oscillations, Newton's 2nd law is applied to the periodic motion of the pendulum bob, *<u>but perpendicular</u> to* g_E, i.e., along the tangent to a surface grazing orbit. For very small oscillations, it is legitimate to approximate $\sin\theta = g_t / L$ by, $\theta = x / L$, about the equilibrium position eq of the pendulum bob, in terms of radian measures, *where each of x and L represents a length*.

For oscillations of very small amplitude, the motion of the pendulum bob is viewed as analogous to that of an object of mass m attached to a linear spring on a frictionless surface oscillating back and forth; effectively in the tangent plane to the Earth. No mathematical magic changes the fact this analogue is perpendicular to the vectors g_E and mg_E, respectively representing the gravitational acceleration and force acting on the pendulum bob. This leads to the form of G, $G = 3\pi / \rho_E T_E^2$; T_E is the period of a frictionless surface grazing Earth orbit; on a HGS of density ρ_E.

In the analogue of a linear spring oscillating with an attached object m on a frictionless surface, the restoring force on m is always proportional and opposite to its displacement x from the equilibrium position of the spring, identified with the point eq in fig. 3, expressed as, $f = -kx$. Then, Newton's 2nd law takes the form (8.1), where x / L is the dimensionless ratio of two perpendicular lengths; i.e., sides of right-angled triangles.

$$m \frac{d^2x}{dt^2} + mg \frac{x}{L} = 0 . \tag{8.1}$$

From the viewpoint of mathematics, the symbol m in (8.1) is irrelevant. For that reason, m is left out of equation (8.1), turning it into (8.2). The imposition or realization of the conceptual structure consistent with Newton's 2nd law of motion is thus only formally bypassed. The context here corresponds to defining any $\{M\} = 1$, as a physical standard of mass in the tangent space of the Earth by means of its gravitational phenomenology; the formal identification of $\{M\}$ with 1 is independent of the particular physical standard chosen: each term can always be thought of as multiplied or divided by integer powers of 1, representing the identity of any physical construct whatsoever. This is one fact that makes mathematics to effective in physics; as illustrated below.

$$m\frac{d^2x}{dt^2} + mg\frac{x}{L} = 0\,; \;=\; \frac{d^2x}{dt^2} + g\frac{x}{L} = 0\,; \;=\; \frac{m}{m}\frac{d^2x}{dt^2} + g\frac{x}{L} = 0\,; \;=\; \frac{d^2x}{dt^2} + \frac{m}{m}g\frac{x}{L} = 0 \qquad (8.2)$$

Superposing the physical concepts of length and time on the variables x and t, then the 1st order ordinary differential, $dx/dt = v$, represents the generic concept of instantaneous speed v in the direction x, in a tangent plane perpendicular to g_E. Similarly, $d^2x/dt^2 = \frac{d}{dt}v$, represents the instantaneous change in v along the line x, i.e., an instantaneous acceleration along x as formulated in $f = -kx$; which is zero at eq. This formulation of a mathematical paradigm for oscillation of small amplitudes of the simple pendulum is defined valid in the tangent spaces of HGS. That the phenomenological acceleration, g_E, and mg_E, are perpendicular to the projected motion about eq in the tangent planes defines g_E as a realization of Newton's definition of centripetal acceleration.

* * *

In constructing a natural philosophy based on mathematical principles, physical dimensions are assigned to mathematical variables, constants and differential operators etc., which then represent physical constructs. Their physical dimensions are intrinsically associated with the physical standards of mass, length and time however chosen. In the context discussed here, the differential operator, d^2x/dt^2, is thus given the physical dimension of an acceleration intrinsic to the symbolic combination of physical standards $\{L\}/\{T\}\{T\}$.

* * *

In the term $g\frac{x}{L}$ of (8.2), g is defined to be a constant vector in all tangent spaces of HGS; it represents the centripetal acceleration of all objects in its surface grazing orbits in the absence of frictional forces. The coefficient of g, x/L, represents the ratio of the opposite side to a very small angle θ of a right angled triangle whose hypotenuse is L, defining a dimensionless ratio of lengths. The dimension of the vector, g, is intrinsic to $\{L\}/\{T\}\{T\}$. One convention of physics decrees all terms in an equation involving physical constructs have to have the same physical dimension.

Magic occurs when the divisor of x namely L in the approximation of $\sin\theta$ by, $\theta = x/L$, about the equilibrium point eq for small oscillations, is moved in $g\frac{x}{L}$ to become the divisor of g, i.e., the term becomes $\frac{g}{L}x$. The ratio of the constant vector g_E of the gravitational manifestation in the tangent spaces of gravitating spheres, divided now by the length L of a simple pendulum becomes, one may say magically, a physical construct invariant w.r.t. the choice of the standard of length, as demonstrated earlier, qv Eq. (6.3a). Here, it becomes the reciprocal of the square of time, whereas x maintains its role as a physical construct of length. If this is not magic, I don't know what is.

Whether magic or not, equation (8.2) in the form (8.3), makes it clear the coefficient to the variable x is a constant, representing the square of an inverse time, i.e., $t^2 = L/g$, i.e., $t = \sqrt{L/g}$. This ratio is of course familiar from sections 5 and 6.

$$\frac{d^2x}{dt^2} + \frac{g}{L}x = 0. \tag{8.3}$$

There is nothing wrong with magic as long as one knows what one is doing and why. This may be easier in the context of experimental than in the context of theoretical physics. To relate this magic to Cavendish's experiment, we need a solution for (8.3). Although trivial, the method of finding its solution is important enough to be demonstrated in the next section.

9. Newton's 2nd law and the paradigm of the simple pendulum

Being aware that equations (8.1) to (8.3) represent the mathematical formulation of a periodic motion, with or without an object attached to a linear spring on a frictionless plane; one chooses a periodic function as a solution. Figure 3 indicates the use of the sine function as a solution, by making its argument a function of time as in (9.1). This function is then matched to equation (8.3) by differentiating it twice w.r.t. the variable t, and then substituting for x from (9.1), and for its 2nd derivative from (9.2) in (8.3). This gives (9.2). For (9.1) to be a solution of (8.3), equation (9.3) has to agree with (8.3). This imposes the condition (9.4) between the constants, ω, g, and L.

$$x = A\sin(\omega t + \delta) \tag{9.1}$$

$$\frac{d^2x}{dt^2} = \frac{d^2A\sin(\omega t + \delta)}{dt^2} = -\omega^2 A\sin(\omega t + \delta) \tag{9.2}$$

$$-\omega^2 A\sin(\omega t + \delta) + \frac{g}{L}A\sin(\omega t + \delta) = 0 \tag{9.3}$$

$$\omega^2 = \frac{g}{L}; \; = \; ;\omega^2 = \frac{g}{L}\frac{A\sin(\omega t + \delta)}{A\sin(\omega t + \delta)}; \; = \; \omega^2\frac{A\sin(\omega t + \delta)}{A\sin(\omega t + \delta)} = \frac{g}{L} \tag{9.4}$$

* * *

A circular function like $A\sin(\omega t + \delta)$ is defined to have a period of 2π radians. The period of the variable t is then defined by the constants in (9.4). This leads from (9.4). to (9.5), and its common equivalent forms in (9.6) which define the same relationships as in (5.1) to (5.3).

$$(\omega t)^2 = 4\pi^2 \Rightarrow t^2 \to T^2 = \frac{4\pi^2}{\omega^2} = 4\pi^2\frac{L}{g} \tag{9.5}$$

$$T = 2\pi\sqrt{\frac{L}{g}}\text{; or, } T^2 = 4\pi^2\frac{L}{g}\text{; or, } g = 4\pi^2\frac{L}{T^2}\text{; or, } L = \frac{T^2 g}{4\pi^2}, \quad v = \frac{2\pi R}{T} = \frac{gT}{2\pi}, \text{ and to, } g = \frac{2\pi v}{T} \tag{9.6}$$

These are the same generic relationships that arise from Newton's definition of centripetal acceleration, with $L = R$. However, here they arose from a generic realization of his 2nd law of motion in the tangent space of a gravitating sphere; by identifying the infinitesimal motion about the equilibrium position of a simple pendulum with the motion of an object attached to a linear spring on a frictionless plane perpendicular to g. Now, physical meaning still has to be injected into these relationships w.r.t. the phenomenology we believe makes the pendulum oscillate.

* * *

A brief synopsis: In the preceding I showed Newton's definition of centripetal acceleration of UCM, is easily translated into the context of mathematics; section 5. In section 8 and this one, it was shown that with particular assumptions, how the state of a simple pendulum at, $\theta = 0$, figure 3, can, be turned into the same paradigm describing UCM, exact for zero amplitude, using the projection of the oscillations about $\theta = 0$, onto the tangent plane to a gravitating sphere at this point, and realizing Newton's 2nd law of motion in this space. The assumption g is a constant vector in the tangent space into which the oscillations about $\theta = 0$ were projected, define the two paradigms as identical at the points of tangency. Then, identifying this paradigm with any standard of time will define a function value of g for any gravitating sphere, as demonstrated in section 6. For $\theta \neq 0$ the period of the simple pendulum must therefore be given by a series of continuous corrections based on θ_{max} as indicated in (9.7).

$$T = 2\pi\sqrt{\frac{L}{g}}(1 + \frac{1}{2^2}\sin^2\frac{\theta_{max}}{2} + \frac{1}{2^2}\frac{3^2}{4^2}\sin^4\frac{\theta_{max}}{2} + \cdots) \qquad (9.7)$$

The periods given by (9.7) are still independent of the mass of the pendulum bob. The period of surface grazing orbits on HGS are of course also independent of the objects assumed to be in these orbits. We are dealing with realization of mathematical definitions and assumptions called axioms.

Despite having kept the level of abstraction to that of senior high school algebra, physics and calculus, or perhaps to those courses in the first year of college or university, I still feel obliged to dissect the mathematical structure arising from Cavendish's experiment w.r.t. Newton's law of gravitation, to show the consequence of defining Newton's G to be a universal constant of nature.

* * *

10. Why G is not a universal constant of nature

If Newton's law of universal gravitation is a law of nature, gravitational spheres exist. One such gravitational sphere, realized by Newton's relevant definitions and axioms in Cavendish's experiment is the Earth. This gravitational sphere, idealized just as the simple pendulum and its paradigm, is then characterized by five constants, each representing particular physical constructs, whose function values are defined by the Earth viewed as a HGS; its centripetal acceleration, its radius, the period of its surface grazing orbit, its density w.r.t. that of water taken as unity, and its mass. I denote these respectively by: g_E, R_E, T_E, ρ_E, and M_E. The triple, g_E, R_E, T_E, is related by Newton's definition of centripetal acceleration. The triple, R_E, ρ_E, and M_E, is related by Newton's definition 1 of mass.

How the function values of g_E and T_E in the first triple was obtained w.r.t. $\{L\}$ and $\{T\}$ was explained in the earlier sections using the data defined by Cavendish's reference pendulum and the radius he used for the Earth, i.e., (10.1) represents the paradigm of a linear spring perpendicular to the constant force/acceleration vector in a tangent plane to the Earth as a sphere; a realization of Newton's 2nd law of motion. Equation (10.1) shows the relationship of the paradigm used by Cavendish to use the pendulum as a clock w.r.t. to $\{T\}$ and its use in the quantification of centripetal acceleration.

$$g_E = \frac{\pi^2 0.994156\{L\}}{1^2\{T\}^2} = 9.811926433\{L\}/\{T\}^2 . \qquad (10.1)$$

Equation (10.2) is a consequence of the period of a simple pendulum being independent of the mass of its bob, or, restating a feature of (10.1), the centripetal acceleration in independent of mass. These axioms and definitions applied to a great circle on the Earth as a HGS defines the relationship (10.2), which is hence also independent of the construct mass.

$$T_E = 2\pi\sqrt{\frac{R_E}{g_E}} = 2\pi\sqrt{\frac{6370320\ \{L\}\{T\}^2}{9.811926433\ \{L\}}} = 5062.713498\{T\} \qquad (10.2)$$

Newton's definition of mass, using Cavendish's value obtained for, ρ_E, gives M_E.

$$M_E = \frac{4\pi}{3} R_E^3 \rho_E = \frac{4\pi}{3}(6370320\{L\})^3 \times 5.48 \times 1000 \frac{\{M\}}{\{L\}^3} . \qquad (10.3)$$
$$= 5.934073403 \text{x} 10^{24} \{M\}$$

* * *

Let us put this together, to determine the relationship of Newton's constant of gravitation G to these definitions and axioms. Because the physical constructs used in these definitions and axioms are interrelated, as well as in their realization w.r.t. the Earth viewed as a HGS, we obtain three equivalent quantitative algebraic structures of equivalent physical dimension and by definition, the three corresponding dimensionless ones shown on the r.h.s..

$$G = g_E \frac{R_E^2}{M_E} = A = 6.71002080\,6 \text{x} 10^{-11} \{L\}^3 \{M\}^{-1} \{T\}^{-2}; \quad 1 = \frac{GM_E}{g_E R_E^2} \qquad (10.4)$$

$$G = A = \frac{4\pi^2 R_E}{T_E^2} \times \frac{R_E^2}{M_E} = B = 6.710020804 \text{ x} 10^{-11} \{L\}^3 \{M\}^{-1} \{T\}^{-2}; \quad 1 = \frac{GM_E}{R_E^2} \times \frac{T_E^2}{4\pi^2 R_E} \qquad (10.5)$$

$$G = A = B = \frac{3\pi}{\rho_E T_E^2} = C = 6.710020805 \text{x} 10^{-11} \{L\}^3 \{M\}^{-1} \{T\}^{-2}; \; 1 = \frac{G\rho_E T_E^2}{3\pi} \qquad (10.6)$$

An interesting feature of algebraic structures like (10.4) – (10.6), is that 1 is the first natural number, and the basic construct on which much of mathematics is founded. Hence, the identity on the l.h.s. of these equations represents the r.h.s. Defining G to be an absolute quantitative invariant w.r.t. a particular set of, $\{M\}, \{L\}$ and $\{T\}$, gives these relationships in principle several degrees of freedom, without violating the underlying definitions and axioms w.r.t. its reference object, the Earth, and w.r.t. the physical standards referenced. This makes it possible for other objects, viewed as HGS, to obtain function values for their five characteristics.

Defining the interrelationships of these definitions and axioms to represent laws of Nature and universal quantitative invariants may be useful, as illustrated in the appendixes, but defining each to be a universal constant of nature may create illusions that will eventually collide with the reality of the physical world, unlikely to be tautological like those arising from our definitions and axioms if extended beyond their domain of applicability, namely our solar system. I believe we have already seen signs of this for a while, but this is a topic for a separate communication.

However, the magical transformation constants G with their degrees of freedom can be used to enforce quantitative invariance w.r.t. to the conventions of physics and mathematics in transferring relevant physical constructs to other gravitational structures such as planets and stars. This yields first approximation of HGS, with the Earth and Cavendish's spherical foot of water as prototypes. The rest becomes mathematical physics. The efficacy of such first approximation is illustrated in the appendixes. In connection with other sciences and experimental physics, this provides a powerful tool to develop technologies and useful knowledge bases, consistent with the underlying axioms, definitions and the conventions of physics within a particular domain of applicability which is currently being explored.

The reader may be able to view Newton's constant of Gravitation now as the reciprocal of, $g_E R_E^2 / M_E$ and its other equivalent forms. Defining these as quantitative invariants w.r.t. to a particular set of, $\{M\}$, $\{L\}$ and $\{T\}$, constrains their product to be 1, which has of course consequences for the relationships of the physical constructs characterizing HGS. The above indicates how Cavendish's experiment enabled theoretical physicists to begin the quantitative construction of the solar system and then the Universe. This may also start to explain what Wigner called "The unreasonable effectiveness of mathematics in the natural sciences" in his 1960 article, [8]. From my reading, there was really no clarification of the issue.

* * *

11. Comments on Natural Units

A careful reading of Cavendish's experiment to determine the mean density of the Earth shows his methodology was designed to realize Newton's inverse square law of gravitation, his operational definition of mass and his definition of relative centripetal force. These definitions are interrelated by the characteristics of an abstract HGS and leads to three equivalent quantitative relationships between different physical constructs, being characteristics of the Earth as a HGS, each identified with G. They lead to the mathematical invariants, $GA^{-1} = GB^{-1} = GC^{-1} \equiv 1$, i.e., to expressions lacking physical dimensions.

None of them are universal in any way but are realizations of definitions and axioms. The mathematical efficacy of defining these relationships invariant arises from the resulting degrees of freedom in defining constraints between these characteristics when applied to other objects viewed as HGS. This is illustrated in the appendixes. Physicists who have read Planck section 26 in his 1900 paper, in which he introduced the concept of natural units, may now realize the relationship of the function value of G w.r.t. the individual physical standards of mass, length and time, differ for each of its version defined respectively by A, B, and C. However, if only the physical dimension of these algebraic structures are considered as the reciprocal of G as a physical constant, this is lost.

For example, the factors defining the function value of G in A are, $g_E = 9.811926433\{L\}/\{T\}^2$, $(6370320\{L\})^2$, and $(5.934073403\text{x}10^{24}\{M\})^{-1}$. The only arbitrarily chosen physical standard was $\{T\} = 1$ in Cavendish's experiment, to match the definition of centripetal acceleration at the Earth surface applied to the paradigm for the simple pendulum; the rest followed from definitions and axioms defining the methodological context.

$$G = g_E \frac{R_E^2}{M_E} = A = 6.71002080\,6\,\text{x}10^{-11} \frac{\{L\}^3}{\{M\}\{T\}^2}$$

The factors defining the value of G in B are, $4\pi^2 R_E, T_E^{-2}$, R_E^2, and M_E^{-1}, in which the first two terms are related to Newton's operational definition of centripetal acceleration and to Cavendish's simple reference pendulum paradigm, scaled to UCM on a great circle or geodesic on the Earth viewed as a HGS.

$$G = \frac{4\pi^2 R_E}{T_E^2} \times \frac{R_E^2}{M_E} = B = 6.710020804\ \text{x}10^{-11} \frac{\{L\}^3}{\{M\}\{T\}^2}$$

The factors defining the value of G in C are, 3π, ρ_E^{-1}, and T_E^{-2}.

$$G = \frac{3\pi}{\rho_E T_E^2} = C = 6.710020805\ \text{x}10^{-11}\ \{L\}^3\{M\}^{-1}\{T\}^{-2}$$

The physical dimensions in these expressions, all necessarily equal, may be used in a formal separation of the physical standards. However, mass, length and time have different quantitative relationship to G in each of its representations; related by the phenomenology and paradigm of the simple pendulum used in the construction of the model of the Earth as a HGS as illustrated.

I have the greatest respect for Max Planck; as person and theoretical physicist. Having read his 54 page summery of his researches “Regarding Irreversible Radiation-Processes”, Ann. d. Phys, of 1900, [9], several times, I think it is one of the best reasoned theoretical physics papers I read. One is naturally intrigued, if not seduced, by the concept of “natural units” he introduced at the end of his paper in § 26, titled “Natural Units of Measurement.” The problem with this concept is, it depends (a): on one’s belief in the existence of universal dimensioned constants of nature, and (b); that their algebraic combinations, so that they have respectively the dimensions of, length, mass, time, and temperature, and normalizing their function values are physically meaningful. My translation of his thoughts regarding the significance of his natural units are presented after the quote from Hilbert’s introduction to his 1913 lectures in Göttingen on electromagnetic oscillations,[10]. (The underlining in Hilbert’s quote is his.)

With respect to the constants a and b in Planck’s paper [9], it may be useful to point out that Planck gave the name Boltzmann constant to a, and b is now called Planck’s constant h. They are related in [9] by the *mechanical equivalent of heat*, for which the reference material is water, quantitatively expressed w.r.t. its unit volume. I think the reader should be aware of this somewhat analogous situation w.r.t. to the use of the abstract sphere of water, as the model of a HGS in [1], to define the mass-density of the earth and its mass w.r.t. that of an equal volume of water, whereas h is related to a unit volume of water as reference material to relate the constructs of energy and temperature. However, this is not the place to show how the function value of Planck’s constant h is related to the earth and photons.

* * *

Hilbert: “Geometry, since ancient times is a field of the discipline of mathematics; the experimental foundations it must use, are so obvious and generally accepted, that right-away and directly it assumed the role of a theoretical science. However, I believe it is the highest renown of every science to be assimilated by mathematics, and that now, <u>theoretical physics</u> is about to achieve this renown. This applies foremost to <u>relativistic mechanics</u> or four-dimensional electrodynamics, which I have been convinced for a long time as belonging to mathematics. Meanwhile, it appears as if theoretical physics is finally being totally and completely absorbed by electrodynamics, insofar as each and every question however particular must, in the last instance, appeal to electrodynamics.”

* * *

I inserted Planck’s equation (41) into his quote so the reader can see what it stands for. S in (41) is the entropy of an electromagnetic resonator with frequency, ν, energy U, and damped only by the emission of electromagnetic radiation; e is the base of the natural logarithm Planck added for later convenience; a and b are in Planck’s view universal constants. The labeling of the natural units i.e., 1, by, $\{\mathfrak{L}\}$, $\{\mathfrak{M}\}$, $\{\mathfrak{T}\}$, and clarifying by (..) to which constant the functions values belongs, are my additions; I prefer to call $\{\mathfrak{L}\}$, $\{\mathfrak{M}\}$, and $\{\mathfrak{T}\}$ theoretical physical standards of length, mass, and time. I ignore the fourth, temperature, as not relevant to the discussion of G.

* * *

Planck: "By comparison it may not be without interest to remark, that with the help of the two constants a and b appearing in equation (41) for the radiation entropy, the possibility is given to define units for length, mass, time and temperature, which independent of specific objects or substances, retain necessarily their significance for all times and for all, also for extraterrestrial and non-human cultures, and which can therefore be considered as "Natural Units of Measurement."

$$S = \frac{U}{a\nu} \log \frac{U}{eb\nu} \tag{41}$$

The means for the determination of the four units is given by the two mentioned constants, a and b, and further by the magnitude of the speed of propagation of light c in vacuum and by the gravitational constant f. With respect to the centimeter, gram, second, and Celsius degrees, the function values of these constants are the following:"

$$a = 0.4818\text{x}10^{-10}[\text{sec} \times \text{Celsius degree}]$$

$$b = 6.885\text{x}10^{-27}[\text{cm}^2\text{g sec}^{-1}];\ (\text{Planck's constant h})$$

$$c = 3.00\text{x}10^{10}[\text{cm sec}^{-2}];\ (\text{speed of light})$$

$$f = 6.685\text{x}10^{-8}[\text{cm}^3\ \text{g}^{-1}\ \text{sec}^{-2}];\ (\text{Newton's gravitational constant G})$$

If now one chooses the "natural units" so that in the new system of measurement each of the above four constants assume the value 1, then one obtains as unit of length, the magnitude:

$$\sqrt{\frac{bf}{c^3}} = 4.13\text{x}10^{-33}\,\text{cm}, = \{\mathfrak{L}\}$$

as unit of Mass:

$$\sqrt{\frac{bc}{f}} = 5.56\text{x}10^{-5}\,\text{g}, \quad = \{\mathfrak{M}\}$$

as unit of time:

$$\sqrt{\frac{bf}{c^5}} = 1.38\text{x}10^{-48}\,\text{sec}, \quad = \{\mathfrak{T}\}$$

as unit of temperature:

$$a\sqrt{\frac{c^5}{bf}} = 3.50\text{x}10^{32}\ \text{C}^0.$$

* * *

One problem with these units is the unit of speed in these units is one-hundred-thousand times the speed of light, i.e., $\mathfrak{L}/\mathfrak{T} = 2.992\text{x}10^{15}\,\text{cm}/\text{s} = 10^5\text{c}$, apart from the fact that substituting the various meanings of G into the above is in my view no longer appropriate.

However, I will use these units to make a different point. The concept of "abstract space of physical dimension," denote by $\mathfrak{D}$ in my general researches, appears to be an underappreciated construct in physics. I define its elements as the normalized projection of physical constructs to $\mathfrak{D}$; indicated below for Planck's natural units referencing the physical standards $\{M\}, \{L\}$ and $\{T\}$. The reason I take Planck's values for b, c and f, as exact and express Planck's natural units to 5 decimal places w.r.t., $\{M\}, \{L\}$ and $\{T\}$, will become clear below.

$$\frac{1}{4.12877\text{x}10^{-35}}\sqrt{\frac{bf}{c^3}} \rightarrow \mathfrak{D} \equiv \{L\}$$

$$\frac{1}{5.55856\text{x}10^{-8}}\sqrt{\frac{bc}{f}} \rightarrow \mathfrak{D} \equiv \{M\}$$

$$\frac{1}{1.37626\text{x}10^{-48}}\sqrt{\frac{bf}{c^5}} \rightarrow \mathfrak{D} \equiv \{T\}$$

I explore here briefly the magic of these natural units, taking G as an example. In the usual units, $G = \tilde{f}\{L\}^3\{M\}^{-1}\{T\}^{-2}$. The corresponding constant expressed in natural units, is shown in (11.1a). The l.h.s. corresponds to $\mathfrak{G} = 1$, as imposed by Planck.

$$\mathfrak{G} = \frac{\{\mathfrak{L}\}^3}{\{\mathfrak{M}\}\{\mathfrak{T}\}^2} = \frac{(4.12877\text{x}10^{-35})^3\{L\}^3}{(5.55856\text{x}10^{-8})\{M\}(1.37252\text{x}10^{-48})^2\{T\}^2} = \tilde{\mathfrak{G}}\frac{\{L\}^3}{\{M\}\{T\}^2} \qquad (11.1\text{a})$$

The function value of $\tilde{\mathfrak{G}}$ in (11.1a) is, $\tilde{\mathfrak{G}} = 6.68506\text{x}10^{-1}$. Then, the ratio $\tilde{\mathfrak{G}}/\tilde{f}$ is given by (11.1b) as,

$$\frac{\tilde{\mathfrak{G}}}{\tilde{f}} = \frac{6.68506\text{x}10^{-1}}{6.685\text{x}10^{-11}} = 1.00000.\text{x}10^{10}. \qquad (11.1\text{b})$$

The calculations for the fundamental constants expressed in the cgs system must give the same results according to the conventions of physics, i.e., the rules defining the function value of physical constructs when referencing a different physical standard of the same kind. This applies of course to the other two examples below.

* * *

Let us do a similar calculation for the speed of light, taking Planck's data again as exact. Then, $c = \tilde{c}\{L\}\{T\}^{-1}$, and for, $\mathfrak{c} = \{\mathfrak{L}\}\{\mathfrak{T}^{-1}\}$.

$$\mathfrak{c} = \{\mathfrak{L}\}\{\mathfrak{T}^{-1}\} = \frac{4.21877\text{x}10^{-35}\{L\}}{1.37626\text{x}10^{-48}\{T\}} = \tilde{\mathfrak{c}}\frac{\{L\}}{\{T\}};\ \tilde{\mathfrak{c}} = 2.99999\text{x}10^{13} \qquad (11.2\text{a})$$

This leads to the natural unit of speed below, I refer to as warp-speed 5.

$$\frac{\tilde{\mathfrak{c}}}{\tilde{c}} = \frac{2.99999\text{x}10^{13}}{3.00\text{x}10^{8}} = 0.99999\text{x}10^{5} \qquad (11.2\text{b})$$

* * *

Let us also do the similar calculation for Planck's constant h, and for, $\mathfrak{h} = \{\mathfrak{M}\}\{\mathfrak{L}\}^2\{\mathfrak{T}\}^{-1} = 1$. Then, $\tilde{\mathfrak{h}} = 6.88498\text{x}10^{-29}$, w.r.t., $\{M\}$, $\{L\}$ and $\{T\}$. This leads to (11.3).

$$\frac{\tilde{\mathfrak{h}}}{\tilde{h}} = \frac{6.88498\text{x}10^{-29}}{6.885\text{x}10^{-34}} = 0.999998\text{x}10^{5} \qquad (11.3)$$

The last two calculations may imply it is not h or c individually that are significant w.r.t. G in the physicists construction of the universe based on mathematical principles, but that it may be their product hc.

$$\frac{\tilde{\mathfrak{h}}\tilde{\mathfrak{c}}}{\tilde{h}\tilde{c}} = 10^{10} \Rightarrow \log_{10}(\tilde{\mathfrak{h}}\tilde{\mathfrak{c}} - \tilde{h}\tilde{c}) = 10;\ \text{and,}\ \log_{10}[\log_{10}(\tilde{\mathfrak{h}}\tilde{\mathfrak{c}} - \tilde{h}\tilde{c})] = 1. \qquad (11.4)$$

This may be an indication of the relationship between the role of dimensioned transformation constants of physics, their relationship to the natural numbers identified with the linear Euclidean geometric continuum of the straight line, and the identification of the physical standards, $\{M\}$, $\{L\}$, and $\{T\}$, with the first natural number 1.

Having read Heilbron's "Max Planck", [11], it is clear Max Planck truly believed he was doing more than constructing the universe based on mathematical principles. This is indicated in the Planck quote below, from a 1908 lecture presented in Leiden titled "The unity of the physical Weltbild[6]" contained in [12]; my translation and my italics.

* * *

Planck: "For the laws of heat radiation in free-space, it is especially noteworthy, that the constants appearing in it have a universal character, *just like the gravitational constant*, insofar as

[6] A lecture given, 9. 12. 1908, at the faculty of natural sciences at Leiden University.

they are independent with respect to any particular substance or any particular body. Therefore, with their help, the possibility is given to define units of length, time, mass, and temperature, whose significance must necessarily hold for all times and for all, also for extraterrestrial and non-human cultures."

* * *

My view of G differs from Planck's. I wonder if his would have changed, if had been clear on Cavendish's methodology and the fact G is defined w.r.t. the spherical of water as a HGS scaled up to the Earth as a HGS with a different density; (11.5).

$$G = \frac{3\pi}{\rho_E T_E^{\,2}} = \frac{g_E R_E^2}{M_E} = \frac{4\pi^2 R_E}{T_E^2} \times \frac{R_E^2}{M_E} \tag{11.5}$$

An interesting aspect of G is that its expressions not only contain linear variables and their powers whose multiplicative identities are by convention representative perpendicular elements of a Minkowski lattice, but that these expressions contain also geometric constants without physical dimensions. In general these cannot be eliminated by canceling physical dimensions to obtain Planck's natural units; they also differ between the different equivalent expressions of G.

* * *

I hope first year students in physics and mathematics, by this reanalysis of [1], may obtain a better understanding of theoretical physics. Section 11. may have shed a different light on natural units. Natural units indicate we use a decimal system and that Pythagoras theorem plays a role in representing numbers in terms of elements of Euclidean geometry. The dimensioned physical constant G acts in two ways as a magical transformation operator upon unfolding from its dimensionless algebraic structure. On its own, upon multiplying a physical construct, it transforms that construct into a different one. The other is to constrain the function value of the physical construct created, referencing now a different combination of physical standards of the same set. Only its form A in (10.4) seems to be known. The other forms are also used in the appendixes.

Appendix I

The mathematical formulation of Newton's definition of centripetal force

Newton's defines centripetal force as relative w.r.t. UCM of the same objects. This requires two mathematical models of UCM of different R and v and T, of which one has to be taken as arbitrary standard. For these we use (A1.1).

$$T_0 = \frac{2\pi R_0}{v_0}\text{, and. } \frac{2\pi R_0}{T_0} = v_0\text{; } T_1 = \frac{2\pi R_1}{v_1}\text{, and, } \frac{2\pi R_1}{T_1} = v_1 \tag{A1.1}$$

The concept "arcs described in the same time in different circles", requires a reference circle for UCM, as a metaphorical reference standard of time, for different UCM, i.e., with different periods and radii. Let this reference UCM be characterized by, R_0, T_0, and v_0, where, $T_0 = \tilde{T}_0\{T\}$.

Let t_0 be a very small time interval w.r.t. T_0, i.e., $t_0 = \frac{T_0}{n}; n >> 1$. The arc described by the motion of a point in the time-interval t_0 in the circle of radius R_0 for n is then given by,

$$\frac{2\pi R_0 t_0}{T_0} \equiv v_0 t_0 = \frac{2\pi R_0}{n}.$$

Following Newton's definition, the square of such arc divided by its radius is given by (A1.2).

$$\frac{4\pi^2 R_0 t_0^2}{T_0^2} = \frac{v_0^2 t_0^2}{R_0} = \frac{4\pi^2 R_0}{n^2} \qquad \text{(A1.2)}$$

For the UCM in a circle of a radius, $R_1 \neq R_0$, of period T_1, the arc described during the same time-interval t_0, with the implicit assumption time is a linear variable, is given by,

$$v_1 t_0 = \frac{2\pi R_1}{T_1} t_0 = \frac{2\pi R_1}{T_1} \frac{T_0}{n}.$$

The squares of the arcs of the respective UCM describe in the same time interval t_0, divided by their respective radii are given in (A1.3). These are abstract mathematical operations to create a mathematical realization of Newton's definition. No real time intervals are as yet involved.

$$\frac{4\pi^2 R_0 t_0^2}{T_0^2} = \frac{v_0^2 t_0^2}{R_0}, \text{ and, } \frac{4\pi^2 R_1}{T_1^2} t_0^2 = \frac{v_1^2 t_0^2}{R_1}. \qquad \text{(A1.3)}$$

In their corresponding ratios, the terms t_0^2 cancel. In the ratios of their respective centripetal force, the physical construct mass also cancels. We are left with the generic ratio, v^2 / R, for the centripetal acceleration. This is as a generic invariant, i.e., a constant, of each UCM, defined either by, R and T, by R and v, or, v and T. The centripetal acceleration of the UCM on great circle on the surface of HGS of different radii are related by,

$$\frac{4\pi^2 R_0}{T_0^2} / \frac{4\pi^2 R_1}{T_1^2} = g_0 / g_1, \text{ and, } \frac{v_0^2}{R_0} / \frac{v_1^2}{R_1} = g_0 / g_1. \qquad \text{(A1.4)}$$

* * *

Appendix II

An example: showing a HGS of the density of the Earth has the same T_E.

Although obvious directly from one of the forms of G, it may still be surprising. The density of water is, $1\text{kg} / 1000\,\text{cm}^3$. From Newton's Definition 1, $M = \rho \times V$, a kg of material having a density 5.48 times that of water has a volume of, $1000\text{cm}^3 / 5.48$, i.e., a volume of, $V = 182.48175518\text{cm}^3$. A spherical volume of matter of this density and mass $\{M\}$, has a radius of,

$$R_{sphKg} = \sqrt[3]{\frac{3 \times V}{4\pi}} = 0.035186570 84\, \{L\}$$

The centripetal acceleration in its tangent spaces *is given by dividing any of the three forms of* G by, R_{sphKg}^2. This follows *from separating* $\{M\}$ *from* G w.r.t. the convention of physics. Therefore, *neither* G, *nor mass*, has explanatory value in gravitational theories in general. Hence, mass and force, can be assumed as intrinsic to density in Cavendish's experiment.

$$g_{sphkg} = \frac{1\text{Kg} \times G}{R_{sphKg}^2} = 5.419634249 \;\; \text{x}\, 10^{-8} \{L\}\{T\}^{-2}$$

The period of a surface grazing orbit of this spherical kilogram of density, $\rho = 5.48\text{kg} / 1000\text{cm}^3$, w.r.t. these mathematical paradigms, must be equal to T_E; no surprise here!

$$T_{sphkg} = 2\pi \sqrt{\frac{R_{sphkg}}{g_{sphkg}}} = 2\pi \sqrt{\frac{0.035186570 84\, \{L\}\{T\}^2}{5.419634249 \;\text{x}\, 10^{-8}\, \{L\}}} = 5062.713498\, \{T\}$$

This is as simple as I can make the clarification of what Newton's constant of gravitation represents and why. Newton's achievements are rightly considered those of a genius; in my view, especially for taking *"rest and/or uniform motion along a straight line"* as the basic reference concept for his other axiom and definitions. However, it may become apparent it may also be UCM, geodesic on HGS, and their tangent spaces, that define operational reference standards of force for his axiomatic system.

* * * * *

Appendix III

Three ways of calculating the radius of the geostationary orbit of Cavendish's Earth

Data from Cavendish's experiment:

$$\left.\begin{array}{l} \tilde{R}_E\{L\} = 6.370320\text{x}10^6\ \{L\} \\ \tilde{g}_E\{L\}\{T\}^{-2} = 9.811926433\{L\}\{T\}^{-2} \\ \tilde{T}_E\{T\} = 2\pi\sqrt{\dfrac{R_E}{g_E}} = 5{,}062.713498\{T\} \end{array}\right\}$$ These are related by the definition of centripetal acceleration.

$\tilde{T}_{ES}\{T\} = 86{,}164.1\{T\}$: The sidereal period of rotation of the Earth about its axis is an observed quantity. The radius of its geostationary orbit, R_{ES}, is sought.

$\tilde{g}_E\{L\}\{T\}^{-2} = 9.811926433\{L\}\{T\}^{-2}$ is defined to vary inversely as the distance from R_E, for $R \geq R_E$, where, R_E, is defined to be the multiplicative identity of length.

From (10.7), $\sqrt{\dfrac{3\pi}{\rho_E G}} = T_E$. where, $\rho_E = 5.48\text{x}10^3\{M\}\{L\}^{-3}$, $G = 6.710020805\text{x}10^{-11}\{L\}^3\{M\}^{-1}\{T\}^{-2}$.

$$T_E = 2\pi\sqrt{\frac{R_E}{g_E}} = \sqrt{\frac{3\pi}{\rho_E G}}\ .$$

* * *

Let the first method to determine R_{ES} be labeled (A), then,

(A): From, $g_E = \dfrac{M_E G}{R_E^2}$, the acceleration at a distance R_{ES} from the center of this HGS is given by (A.1), w.r.t. Newton's inverse square law of gravitation. However, the function value of $M_E G$ plays no role except as a constant of proportionality, k.

$$g_E = \frac{M_E G}{R_E^2} \Rightarrow g_E\frac{R_E^2}{R_{ES}^2} = \frac{M_E G}{R_{ES}^2} = g_{ES}\text{, i.e., } g_E\frac{R_E^2}{R_{ES}^2} = \frac{k}{R_{ES}^2} = g_{ES} \tag{A.1}$$

Then, $g_E R_E^2 = k = g_{ES} R_{ES}^2$, and the period for a circular orbit of radius $R_{ES} \geq R_E$ is,

$$T_{ES} = 2\pi\sqrt{\frac{R_{ES}}{g_{ES}}} = 2\pi\sqrt{\frac{R_{ES}^3}{k}} = 2\pi\sqrt{\frac{R_{ES}^3}{R_E^2 g_E}} \tag{A.2}$$

For a geostationary orbit T_{ES} is taken as a constant given by the sidereal rotation of the Earth about its axis. Thus,

$$T_{ES}^2 = 4\pi^2 \frac{R_{ES}^3}{R_E^2 g_E},$$

and hence,

$$R_{ES} = \sqrt[3]{\frac{T_{ES}^2 R_E^2 g_E}{4\pi^2}} = \sqrt[3]{\frac{(8.61641x10^4)^2 x(6.37032x10^6)^2 x9.811926433}{4\pi^2}}\{L\} \qquad (A.4)$$

$$R_{ES} = 4.214925655x10^7\{L\}. \qquad (A.4)$$

$$g_{ES} = 4\pi^2 \frac{R_{ES}}{T_{ES}^2} = 0.224128427\{L\}\{T\}^{-2} \qquad (A.5)$$

It should be clear that R_{ES} was calculated from two dimensioned constants, R_E and g_E. The function value of T_{ES} and g_E are determined by $\{T\}$, the length of Cavendish's reference pendulum expressed w.r.t. w.r.t. some {L} and a paradigm defining its period in the infinitesimal context of a tangent space w.r.t. Newton's 2nd law of motion and to centripetal acceleration of UCM in the large.

* * *

(B): Let us now calculate R_{ES} using the same input data. We start with the two constant periods T_E and T_{ES} of the corresponding UCM which are known. They are given by,

$$T_E = 2\pi\sqrt{\frac{R_E}{g_E}}, \text{ and, } T_{ES} = 2\pi\sqrt{\frac{R_{ES}}{g_{ES}}}.$$

By definition, the gravitational acceleration g varies inversely proportional to the square of the distance for $R > R_E$, where R_E is the unit of length w.r.t. the linear parameter λ; R_E is its multiplicative identity. The period of the UCM centered on this HGS becomes a function of λ w.r.t. R_E as the unit of length, i.e.,

$$T_{ES}(\lambda) = 2\pi\sqrt{\frac{\lambda R_E}{g_E/\lambda^2}} = 2\pi\sqrt{\frac{\lambda^3 R_E}{g_E}}. \qquad (B.1)$$

Thus, $R_{ES} = \lambda R_E$, for the orbital period to be T_{ES}, and similarly, $g_{ES} = g_E/\lambda^2$, for this orbit. As T_E and T_{ES} are known quantities, their ratio defines the function value of, λ. Though formally dimensionless, it is just a number, its reference standard of length is the radius of the HGS defined as the universal reference standard, the Earth, i.e., R_E.

$$\lambda = \left(\frac{T_{ES}}{T_E}\right)^{2/3} = \left(\frac{86{,}164.1}{5062.713498}\right)^{2/3} = 6.61650538 \qquad (B2)$$

(B2) makes clear that the variable of length and time as used in the realization of Newton's relevant axioms in the execution and interpretation of the Cavendish experiment are not linear with respect to each other. Here, time is non-linear w.r.t. variables in Minkowski's geometry of numbers and Pythagoras theorem.

From (B2) we obtain for, R_{ES}, as before,

$$R_{ES} = 6.61650538 \times R_E = 4.214925656x10^7\{L\}. \qquad (B3)$$

From, $g_{ES} = g_E/\lambda^2$, we obtain also as before,

$$g_{ES} = g_E/(6.61650538)^2 = 0.224128426\{L\}\{T\}^{-2}. \qquad (B4)$$

* * *

(C): If the mass of the Earth were uniformly redistributed within the sphere of radius R_{ES} of the geostationary orbit, the ratio of the respective mass-densities of these spheres is given by (B.1) as,

$$\frac{\rho_{ES}}{\rho_E} = (M_E/V_{ES})/(M_E / V_E) = \frac{R_E^3}{R_{ES}^3} = \frac{1}{\lambda^3} = \frac{1}{(6.616250538)^3}. \tag{C.1}$$

Then, R_{ES}, as before,

$$R_{ES} = \lambda R_E = R_E \times 6.616250538 = 42{,}147.63386\,\text{km}. \tag{C2}$$

$$\rho_{ES} = \frac{\rho_E}{(6.616250538)^3} = \frac{5.48\text{x}10^3}{(6.616250538)^3}\{M\}\{L\}^{-3} = 18.92102814\{M\}\{L\}^{-3}.$$

$$g_{ES} = g_E/(6.61650538)^2 = 0.224128426\{L\}\{T\}^{-2} \tag{B4}$$

* * *

This illustrates the Earth is the reference standard of a universal HGS, as a realization of some of Newton's axioms. The phenomenology of its surface acceleration defines a gravitationally standard of force w.r.t. any set of, $\{M\}$, $\{L\}$, and $\{T\}$, as made clear in this analysis of Cavendish's experiment. The rest is mathematical physics.

* * *

I indulge in a brief digression; defining G to be a universal constant implies each of the three equivalent forms is a universal constant. This also implies that time, as quantified in physics by its physical standard as used w.r.t. Cavendish's reference pendulum for example, is the only linear variable in gravitational theories. Below, I show the splitting of elements belonging to the square root geometry of Minkowski's geometry of numbers, w.r.t. to G. In the expressions below, physical constructs are represented by positive function values w.r.t. the reference standards, e.g., $\rho_E = \tilde{\rho}_E\{M\}\{L\}^{-3}$; The reader may take another look at section 10. As we treat time as a linear progression, i.e. $T_E = 5062.713498\,\{T\}$ is based on a periodic phenomenon. As relationships between the construct of area and time exist in Kepler's laws, Pythagoras Theorem may play a role here!

$$T_E^2 = \frac{3\pi}{\rho_E G} \Rightarrow \tilde{T}_E\{T\} \times \left[\sqrt{\frac{\tilde{\rho}_E}{3\pi}}\,\frac{\{M\}^{1/2}}{\{L\}^{3/2}}\right] = \left[\sqrt{\frac{1}{\tilde{G}}}\,\frac{\{M\}^{1/2}}{\{L\}^{3/2}}\right] \times \{T\};\;\; T_E = 2\pi\sqrt{\frac{R_E}{g_E}}$$

The only experimentally verified inverse square law of force is in my view that of electrostatics. Based on an analysis of Cavendish's experiment, that of gravitation appears to be a defined one for bulk matter, but whose origin is likely that of electric charge and electromagnetic phenomenology.

* * *

Appendix IV

The density, mass, and surface acceleration of the Sun; from the Earth as a HGS standard

The analysis of Cavendish's experiment made it clear why defining the mass of the sun requires the function value of the semi-major axis $\mathfrak{A}_E$ of the earth's orbit and its sidereal orbital period, T_{EO}, and the sun's radius, R_{Sun}. These are obtained from astronomical observations, taken from [12]. For the gravitating reference sphere, I still take Cavendish's data for the Earth, which defines it as such in these examples. It should be mentioned that the data for the Earth in [12] is not quite consistent with the Earth being a HGS. This is not surprising as the Earth is neither homogeneous nor a perfect sphere. The important point is the realization of definitions and axioms, provided they can be related in some manner to a phenomenological aspect quantified w.r.t. to the definitions.

Considerations regarding rotation of HGS were deliberately omitted to present a clear picture of the meaning and role of Cavendish's experiment in realizing some of Newton's definition and axioms relevant to its methodology. Their iterations so to speak, to celestial objects themselves, become then the domain of mathematical physics w.r.t. solid and fluid bodies, using additional physical constructs.

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Data: $\mathfrak{A}_E = 1.49598\text{x}10^{11}\,\text{m}$; $T_{EO} = 365.25\text{days} = 3.15576\text{x}10^{7}\,\text{s}$; $R_{Sun} = 6.96\text{x}10^{8}\,\text{m}$.

From the methodology of Cavendish's experiment and its analysis, the physical construct mass is realized w.r.t. UCM, and is associated with a constant force, defined by the choice of any $\{M\}$ in a tangent space of the Earth, and on the other hand, by the choice of $\{T\}$ for the simple pendulum in a tangent space of the Earth viewed as a HGS. In this context, the length of the pendulum L_P is defined by gravitational phenomenology while the relationship between L_P and g_E is constrained as shown in (6.3) and is invariant w.r.t. the choice of $\{L\}$.

For these reasons, and by defining the Earth as the physical standard of a HGS, the concept of uniforms circular motion and its paradigm can be used to define the mean centripetal acceleration of the earth's orbit, and to scale the characteristic properties of the Earth as a HGS to that of the Sun and other celestial objects in the solar system as illustrated for the sun, using the above data.

$$g_{EO} = 4\pi^2 \frac{\mathfrak{A}_E}{T_{EO}^2} = 5.9303\text{x}10^{-3}\,\text{ms}^{-2}. \qquad \text{(IV.1)}$$

To find the centripetal acceleration at the sun's surface, we need $\mathfrak{A}_E$ in terms of the radius of the Sun, R_{Sun}, as the unit of length, because we know how the definition of mass is related to the centripetal acceleration on the surface of the standard HGS. This gives, $\mathfrak{A}_E = 214.94 R_S$.

$$\frac{\mathfrak{A}_E}{R_{Sun}} = \frac{1.49598\ \text{x}10^{11}\,\text{m}}{6.96\text{x}10^{8}\,\text{m}} = 214.94 \qquad \text{(IV.2)}$$

By definition, the sun's surface centripetal acceleration decreases as the square of the distance w.r.t. its radius. Therefore, the sun's surface acceleration is defined in constructing this Weltbild as,

$$g_{EO}\left(\frac{\mathfrak{A}_E}{R_{Sun}}\right)^2 = 5.9303\text{x}10^{-3}\,\text{ms}^{-2} \times (214.94)^2 = 273.97\,\text{ms}^{-2}. \qquad \text{(IV.3)}$$

From its surface acceleration the period of its surface grazing orbits is,

$$T_{Sun} = 4\pi^2 \frac{R_{Sun}}{g_{Sun}} = 2\pi\sqrt{\frac{6.96\text{x}10^{8}\,\text{m}}{273.96\text{ms}^{-2}}} = 1.00145\ \text{x}10^{4}\,\text{s} = 2.782\ \text{hr}. \qquad \text{(IV.4)}$$

The Earth is the reference HGS w.r.t. defining the three forms of G to be a universal constant, then (IV.5) defines the Sun's mean density, and hence its mass.

$$\frac{3\pi}{\rho_E T_E^2} = \frac{3\pi}{\rho_{Sun} T_{Sun}^2}.\ \text{i.e.,}\ \rho_E T_E^2 = \rho_{Sun} T_{Sun}^2 \qquad \text{(IV.5)}$$

Therefore, the sun's mean density is given by (IV.6) as having about 1.4 times the density of water.

$$\rho_S = \rho_E \frac{T_E^2}{T_S^2} = 5.48\text{x}10^{3}\,\frac{\text{kg}}{\text{m}^3} \times \frac{2.5631068\ \ \text{x}10^{7}}{1.002900\ \ \text{x}10^{8}} = 1.40052\ \text{x}10^{3}\,\frac{\text{kg}}{\text{m}^3} \qquad \text{(IV.6)}$$

The mass of the sun obtained from Cavendish's experiment and the above data is thus,

$$M_S = 1.9779x10^{30}\,\mathrm{kg}\,. \tag{IV.7}$$

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The mean density of the earth in [12] is about 6 /1000 greater than Cavendish's, and lists, M_S as,

$$M_S = 1.9891\,x10^{30}\,\mathrm{kg}\,.$$

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Using the equivalent to the above paradigm gives of course the same answer for the mean acceleration of the earth orbit as above. Scale independent mathematical definition, once realized, can of course be scaled w.r.t. the conventions of physics as a quantitative science in constructing the pictures of our Universe, our physical Weltbild.

$$g_{EO} = \frac{M_{Sun}G}{\mathfrak{A}_E^2} = \frac{1.9779\,x10^{30} \times 6.7100298\,x10^{-11}}{(1.49598x10^{11})^2}\,\mathrm{ms}^{-2} = 5.9303x10^{-3}\,\mathrm{ms}^{-2}$$

Having a gravitational standard of force and mass and quantitative realization of the underlying axioms and definitions, made it possible to build mathematical models of the relationship between the celestial objects of the solar system. This made the exploration of the solar system by spacecrafts possible without having to know anything about the origin of the phenomenology we call gravity. A good explanation of Weltbild is given in [11], 508, "Der Kausalbegriff in der Pysik."

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