On absolute units

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How may we characterise the intrinsic structure of physical quantities such as mass, length, or electric charge? This paper shows that group-theoretic methods—specifically, the notion of a free and transitive group action—provide an elegant way of characterising the structure of scalar quantities, and uses this to give an intrinsic treatment of vector quantities. It also gives a general account of how different scalar or vector quantities may be algebraically combined with one another. Finally, it uses this apparatus to give a simple intrinsic treatment of Newtonian gravitation.

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1. Introduction and motivation

In physics, much use is made of *quantities*—for example mass, charge, and length. Standardly, such quantities are represented by real numbers; however, it is generally acknowledged that doing so is merely a convenience, and that the real numbers have more structure than the quantities they represent. This raises a natural question: what might an "intrinsic" representation of such quantities look like, and how would laws formulated in terms of such intrinsic representations compare to more standard formulations? This paper offers an intrinsic account of the structure or scalar and vector quantities, and shows how this account may be used to offer a unit-independent formulation of Newtonian gravitational theory.

There are two motivations for this project. The first is the *intrinsicalist* motivation. If we use numerical representations, then we traffic in "mixed" mathematico-physical sentences, such as "the mass of this box is 5kg": that is, predicates which refer to both mathematical and physical entities. Even if we put aside questions about the *existence* of mathematical entities, there is something potentially unsatisfactory about describing physical entities in terms of entities external to them—especially when those descriptions are being used for the purposes of explanation and prediction. Presumably, this practice is licensed insofar as the physical and mathematical entities have some structure in common, and the explanations and predictions depend only on the structure of the mathematical entities that is shared by the physical entities. But showing that this is indeed the case requires giving an intrinsic description of the structure of the physical entities, and demonstrating that this intrinsic structure is sufficient for the purposes of physical theorising (albeit, presumably, at some cost in convenience).

Note that there are two ways in which this motivation is not the same as a full-blown nominalist motivation, of the kind driving Field (1980)'s program. First, as already mentioned, it applies even if one is a realist about mathematical entities. Second, the nominalist project is often taken to exclude not only reference to mathematical entities, but also reference to physical *properties*—at least, except insofar as such properties are actually manifested or instantiated. For example, Field writes that

> A possible approach to a coordinate-independent treatment of, say, temperature, would be to introduce a continuum of temperature properties, each one the property of having such and such specific temperature. One could then describe the structure of that system of properties not via numbers, but via certain intrinsic relations among them, say the relations of betweenness and congruence; and one could impose axioms on these notions to guarantee that there was a 1–1 function mapping the temperature properties into the reals, and that such a function was unique up to linear transformation. There is a certain conception of properties . . . on which this approach would be at least arguably a nominalistic one; but I prefer a different strategy, which doesn't invoke temperature properties but which makes do with space-time points (or more generally, space-time regions) as the only entities.[1]

I take the intrinsicalist motivation to be satisfied if we can, indeed, "describe the structure of [a] system of properties . . . via certain intrinsic relations among them" (and show that this structure suffices for stating the relevant laws), as that would suffice to delimit how much structure our theories are permitted to invoke. Whether this should be considered "nominalist" is not something that I will be concerned with.

[1](Field, 1980, p.p. 55–56)

The second motivation arises from the debate between *comparativism* and *absolutism* about quantity.[2] In rough terms, whereas the absolutist takes the fundamental facts about quantity to consist of facts attributing *absolute* values of quantities to objects, the comparativist takes the fundamental facts about quantity to consist of facts attributing *comparative* values of quantities to pairs of objects (typically, ratios). So, for example, the absolutist will consider the fact that this bag of flour has a mass of 500g to be fundamental, and in particular to be prior to the fact that it is half the mass of this bag of lentils. The comparativist, meanwhile, will take the fact of their masses standing in a 1:2 ratio as fundamental; according to taste, they will claim either that the facts about their individual masses are derivative on the ratio-facts, or that we should not think that there are any such facts at all.

Comparativists will often contend that absolutism is committed to empirically inaccessible structure. To support this claim, it has been argued that doubling (say) the mass of every object in the world would not induce any empirically accessible changes; hence, we should not take the masses themselves, but only their ratios (which are, of course, invariant under such a doubling) as reflecting genuine physical facts.[3] The overall argument here is an instance of a well-known argument from the philosophy of symmetry, which contends that models of a theory related by a symmetry transformation are empirically equivalent.[4] Unfortunately, however, the application of this argument to the case of a mass-doubling is misguided, since this operation is *not* a symmetry transformation—at least, not of the laws with which we are familiar. For example, a mass-doubling is not a symmetry of the laws of Newtonian gravitation; thus, an "overnight doubling" of all masses, in a world governed by such laws, would lead to empirically noticeable changes in the phenomena (the orbits of the planets would alter, pendulums would change their periods, etc.).[5]

That said, although mass-doubling is not a symmetry, certain other kinds of quantity-rescalings are: for example, it is a symmetry of Newtonian gravitation to double the masses of all bodies, *and* double the lengths of all distances, *and* double the durations of all processes. In fact, more generally, consider a transformation which rescales all masses by a factor μ, all lengths by a factor λ, and all durations by a factor τ, in such a way that $\lambda^3 = \mu\tau^2$. This is a symmetry of Newtonian gravitation, since—via Newton's Second Law—it induces a rescaling of all forces by a factor $\mu\lambda/\tau^2$, which means that

[2] Dasgupta (2013); (Wolff, 2020, chap. 8).
[3] Dasgupta (2013).
[4] Roberts (2008); Dasgupta (2016)
[5] In the context of the recent absolutism-comparativism debate, this observation was first made by Baker (nd); for an extended discussion, see Martens (2019).

Newton's law of gravitation is preserved: for,

$$\frac{GMm}{r^2} \mapsto \frac{\mu^2}{\lambda^2}\frac{GMm}{r^2} \tag{1}$$

and if $\lambda^3 = \mu\tau^2$, then $\mu^2/\lambda^2 = \mu\lambda/\tau^2$. Consequently, applying such a joint rescaling of mass, length and time does not induce any empirically accessible changes.[6] So the argument against absolutism is restored, albeit in a modified form: if an absolutist regards two models related by a joint rescaling (of this form) as representing different possible worlds, then she is committed to empirically inaccessible distinctions between possibilities, and hence (in some sense) to empirically inaccessible structure.

Unfortunately for the comparativist, however, it is not clear that this can be construed as a positive argument for comparativism: the comparativist quantity-ratios are invariant under *any* rescaling of mass, length, and time, not just those rescalings where $\lambda^3 = \mu\tau^2$.[7] This leaves the aspiring comparativist with two options. They can give up on trying to capture the full content of Newtonian gravitation, and seek to find some (hopefully empirically equivalent) alternative theory which admits arbitrary rescalings as symmetries, and hence is expressible purely in terms of these ratios.[8] Alternatively, they can seek some larger collection of "comparativist" quantities, such that these quantities are invariants *only* of the desired rescalings (or of some subclass thereof). For example, if the comparativist were to allow themselves not only the mass-, length- and time-ratios but also the mass-length, length-time and time-mass ratios, then they would have a collection of quantities invariant only under uniform rescalings—i.e., rescalings of the form $\lambda = \mu = \tau$. However, formulating a physics in terms of these quantities would be non-trivial; and even if it could be done, these "trans-quantity" comparativists would be committed to empirically inaccessible differences between possibilities, just as the absolutist is (since a non-uniform rescaling where $\lambda^3 = \mu\tau^2$ would be a symmetry of the dynamics, but would not leave the trans-quantity ratios invariant). Perhaps there is some yet more ingenious variant of comparativism that traffics in exactly those quantities invariant under the relevant class of rescalings, but it is not clear

[6] Empirically accessible, that is, by means of gravitational phenomena.

[7] The same observation applies to Roberts' proposal that "The reasonable comparativist will not say that just any old rescaling of dimensional quantities will leave everything unchanged; she says only that *uniform* rescalings do" (Roberts, 2016, p. 11), where a uniform rescaling is (roughly) one which rescales the relevant constants of nature along with the quantities, in such a way as to preserve the laws. Certainly, holding that only uniform rescalings leave things the same is a much more compelling position; but the problem is whether the comparativist can indeed be reasonable in this fashion, given that two worlds related by a *non*-uniform rescaling will agree on all quantity-ratios.

[8] See Martens (nd) for an exposition and critique of such a theory.

to me what that might look like.

There is, however, an alternative.[9] I noted above that we can revive the argument against absolutism, *if* the absolutist is committed to regarding models related by joint rescalings as representing distinct possibilities. Is the absolutist permitted to reject that commitment? To answer this question, consider the distinction that Dewar (2019) draws between the "sophisticated" and "reduced" reformulations of theories with symmetries. To reduce a theory is to reformulate it purely in terms of symmetry-invariant quantities, with the result that applying the symmetry transformation to a model yields a numerically identical model (indeed, in reduced theories, there is a sense in which the symmetry transformation in question cannot even be expressed). To sophisticate a theory, on the other hand, is to find a formulation of the theory such that symmetry-related models are isomorphic to one another (rather than, as in the reduced theory, identical). Thus, for example, suppose that we start with electromagnetism expressed in terms of the electromagnetic potential; to reduce this theory by its gauge symmetry would be to reformulate it in terms of the electromagnetic field, whereas to sophisticate it by that symmetry would be to reformulate it in terms of $U(1)$ fibre bundles.

Dewar (2019) argues—following similar arguments for "sophisticated substantivalism"[10]—that a sophisticated formulation is sufficient for escaping the argument from empirical inaccessibility, as we may legitimately regard isomorphic models as representing the same possibility.[11] If we accept that claim, then we have a motivation to seek a formulation of Newtonian gravitation which uses absolute quantities, but is formulated in such a way that models related by a joint rescaling (but not by an arbitrary rescaling) are isomorphic to one another. Doing this will, in the first instance, require doing away with the structure of the quantities that goes beyond what is invariant under a rescaling. But this structure presumably coincides with the "intrinsic" structure of those quantities. So the two motivations push us in the same direction: giving a formulation of physical theory in terms intrinsic to the quantities in question.

The paper proceeds as follows. In the next section, I introduce "numerical quantities", and justify their use for representing "pure" ratio-quantities (e.g. the mass-ratio between two objects). In section 3, I use numerical quantities to define the structure of scalar quantities (without any notion of a choice of unit); section 4 shows how such quantities may be algebraically combined, to form product- and ratio-quantities. Sections 5 and 6 extend these ideas to include vector quantities: how such quantities are

[9] Wolff (2020) also defends this "third way" in the absolutism-comparativism debate, and gives a much more detailed discussion of the metaphysical (and meta-metaphysical) issues that are at stake here.
[10] Pooley (2006)
[11] For a critical discussion of this proposal, see Martens and Read (2020).

to be defined, and how vectors may be multiplied or divided by scalars.[12] Section 7 shows how the quantities associated with space, time, and motion can then be defined; whilst section 8 presents a formulation of Newtonian gravitational theory (for point masses) in terms of these intrinsic quantities. Section 9 concludes, by evaluating this theory against the two motivations discussed here.

2. Numerical quantities

The first kind of quantity we will consider is that of *numerical* quantities. Such quantities are represented by positive real numbers—without any redundancy or surplus of representation. More exactly, we may say that such quantities take their values in $\mathbb{R}^+$, where $\mathbb{R}^+$ is (by definition) the set of real numbers strictly greater than 0:

$$\mathbb{R}^+ = \{x : x \in \mathbb{R}, x > 0\} \tag{2}$$

One might be concerned that this is not a sufficiently intrinsic characterisation of such quantities, since it proceeds via the real numbers.[13] However, it is easy to remedy this defect: we define a numerical quantity as a quantity whose values take the form of a complete ordered positive semifield. In Appendix A, it is shown that $\mathbb{R}^+$ (equipped with the usual order, addition, and multiplication structures) is a complete ordered positive semifield, and that any complete ordered positive semifield is uniquely isomorphic to $\mathbb{R}^+$. Thus, we may identify the value-range of any numerical quantity with $\mathbb{R}^+$, whilst still abiding by the intrinsicalist scruples of §1: all we are noting is that in light of the unique isomorphism, we may regard $\mathbb{R}^+$ as providing uniquely assigned labels for the possible values of any numerical quantity. (That is, the requirement that the labelling scheme be an isomorphism suffices to determine which label belongs to which numerical quantity-value.)

The paradigmatic examples of numerical quantities are *ratio* quantities, such as the mass-ratio between two massive objects. In stating that the bag of lentils is twice as massive as the bag of flour, we do not make some hidden appeal to a choice of unit, or standard of measurement: it is simply true that the number 2 is uniquely apt to represent the mass-ratio in which these two objects stand. Indeed, one can provide a

[12]There is a notable element missing here: the issue of how one is to multiply vectors by one another, i.e. how to define scalar or tensor products. I leave this issue for future work.

[13]This is an interestingly different kind of non-intrinsicality to the kind discussed in §1: the issue is not that $\mathbb{R}$ has surplus *structure* relative to a given numerical quantity, but rather that it has surplus *members*. In model-theoretic terminology, the issue is that we want a numerical quantity to have the form of a substructure of $\mathbb{R}$, not that we want it to have the form of a reduct of $\mathbb{R}$.

straightforward argument for the use of the positive real numbers (equivalently, the use of the elements of a complete ordered positive semifield). I will give the argument for length-ratios, but it may easily be applied to other extensive ratios such as mass or charge.[14]

Suppose that we have two objects A and B, and we wish to determine the ratio between their lengths. We take it as given that we can determine when two objects are the same length. Idealising, we assume that given some object, we can produce arbitrarily many objects of the same length. From this, it follows that we can determine when an object is half the length of another (by observing that if the former object is placed end-to-end with an equally long object, the two objects together are the same length as the latter object).

Without loss of generality, suppose that A is shorter than B. First, we see how many times A fits in B. Call this number n_0. We then determine how many times a rod half the length of A fits in the remainder of B. Call this n_1. We then determine how many times a rod one-quarter of the length of A fits in the remainder of the remainder, and call that number n_2; and so on, an infinite number of times. It follows that, where l_A is the length of A and l_B the length of B,

$$\begin{aligned} l_A &= n_0 l_B + \frac{1}{2} n_1 l_B + \frac{1}{4} n_2 l_B + \cdots \\ &= n_0 l_B + \sum_{k=1}^{\infty} \frac{n_k}{2^k} l_B \end{aligned}$$

For any $k > 0$, n_k is bounded by $n_k < 2$. Therefore, the sum converges. Hence, the ratio is given by

$$\frac{l_A}{l_B} = n_0 + \sum_{k=1}^{\infty} \frac{n_k}{2^k} \tag{3}$$

It follows that the length-ratios are given, in general, by positive real numbers (i.e. that length-ratios are a numerical quantity): a convergent sum of rational numbers will yield some positive real number, since the positive real numbers are the completion of the positive rational numbers.

[14] That said, there is one important limitation: this argument is limited to ratios of *extensive* quantities. How to justify the structure of *intensive* quantities, such as temperature, is a further question beyond the scope of this paper. For a more detailed discussion of the concept of extensive quantity (and an argument that the above is too glib in treating mass and length analogously), see Perry (2015).

3. Scalar quantities

Next, we consider *scalar quantities*. These quantities may also be represented by $\mathbb{R}^+$, but only once a choice of unit has been made: that is to say, once some particular scalar value has been chosen to be represented by the number 1. This immediately suggests one way to characterise a scalar structure: regarding $\mathbb{R}^+$ as a multiplicative group, a scalar structure is a *principal homogeneous space* (also known as a *torsor*) for this group. To explain this, we need to recall a little bit of group theory.

First, given a group G and a set Ω, an *action* of G on Ω is an association of each $g \in G$ with a bijection $a \mapsto g * a$ of Ω to itself, such that $gh * a = g * (h * a)$. This action is said to be *free* if for every $g \neq h$ and every $a \in \Omega$, $g * a \neq h * a$; and it is said to be *transitive* if for every $a, b \in \Omega$, there is a $g \in G$ such that $g * a = b$. If G's action on Ω is both free and transitive, then for every $a, b \in \Omega$, there is exactly one $g \in G$ such that $g * a = b$; in such a case, the action is said to be *regular*. We will refer to this unique group element g as the *ratio* of b to a, and denote it by $\frac{b}{a}$. Ω is a principal homogeneous space for G exactly if the action of G on Ω is regular.

One useful way to think of such a principal homogeneous space Ω is that it looks just like G, except that we "forget where the origin is": if we pick (arbitrarily) some point $a \in \Omega$ as the origin, then we can regard Ω as a group with group multiplication defined by

$$bc := \left(\frac{b}{a}\frac{c}{a}\right) * a \tag{4}$$

and show that this group is isomorphic to G, via the isomorphism $b \mapsto \frac{b}{a}$.

Moreover, suppose that G carries some further structure which is compatible with its group structure, in the sense of being invariant under group multiplication: if this non-group structure took the form of an n-ary relation R, for instance, then compatibility would require that for any $g_1, \ldots, g_n, h \in G$, $R(g_1, \ldots, g_n)$ iff $R(hg_1, \ldots, hg_n)$. Then we may regard Ω as also carrying this structure, by transferring said structure via one of the above-mentioned isomorphisms; the compatibility requirement will entail that the transferred structure is independent of which isomorphism we choose. The relation R, for instance, would be transferred over by stipulating that for any $a_1, \ldots, a_n \in \Omega$, $R(a_1, \ldots, a_n)$ iff $R(\frac{a_1}{b}, \ldots, \frac{a_n}{b})$, for any arbitrarily chosen $b \in \Omega$. It is straightforward to check that this definition is well-formed, since for any $c \in \Omega$, $R(\frac{a_1}{b}, \ldots, \frac{a_n}{b})$ iff $R(\frac{a_1}{c}, \ldots, \frac{a_n}{c})$.

Thus, when we say that a scalar quantity $\mathcal{S}$ has the structure of a principal homogeneous space for the multiplicative group $\langle \mathbb{R}^+, \cdot \rangle$, we mean that we take as given some action $a \mapsto x * a$ of $\mathbb{R}^+$ on $\mathcal{S}$ which is such that for any $a, b \in \mathcal{S}$, there is a unique $x \in \mathbb{R}^+$ for which $x * a = b$. As in the general case, we shall refer to this x as the ratio of b to a, and denote it by $\frac{b}{a}$. And, indeed, the ratio of b to a is just the sort of thing that we would think of as the ratio of one scalar quantity to another: for instance, if $\mathcal{S}$ were a mass-scale, then these ratios are mass-ratios (in the ordinary sense of the term); and as we have seen, mass-ratios inhabit $\mathbb{R}^+$. Moreover, subject to a choice of unit,[15] $\mathcal{S}$ may be identified with $\mathbb{R}^+$. This, of course, is just the observation we started this section with.

Being a principal homogeneous space over $\mathbb{R}^+$, $\mathcal{S}$ inherits any of the structure of $\mathbb{R}^+$ which is invariant under the multiplicative action of $\mathbb{R}^+$ on itself. This includes, in particular, the *additive* and *order* structure of $\mathbb{R}^+$, as these are both invariant under multiplication: for any $x, y, z \in \mathbb{R}^+$,

$$x \cdot (y + z) = x \cdot y + x \cdot z \tag{5}$$

$$x < y \text{ iff } z \cdot x < z \cdot y \tag{6}$$

More specifically, the addition of scalars is defined, in terms of the addition of ratios of scalars (elements of $\mathbb{R}^+$) by

$$a + b = \left(\frac{a}{c} + \frac{b}{c}\right) * c \tag{7}$$

where c is an arbitrarily chosen scalar; and the order-relation on scalars is defined, in terms of the order-relation on ratios of scalars, by

$$a < b \text{ iff } \frac{a}{c} < \frac{b}{c} \tag{8}$$

where, again, c is an arbitrarily chosen scalar. These definitions are easily shown to be well-formed, i.e., independent of the choice of c.

By way of contrast, the multiplicative structure of $\mathbb{R}^+$ is not transferred to $\mathcal{S}$, since that structure is not invariant under multiplication: in general, for arbitrary $x, y, z \in \mathbb{R}^+$,

$$x \cdot (y \cdot z) \neq (x \cdot y) \cdot (x \cdot z) \tag{9}$$

Again, this makes intuitive sense: we expect statements about adding masses together, or about which mass is greater than another, to be well-formed; but not statements about what mass one obtains by multiplying two masses together.[16]

[15]There is something of a pun here: we may read this as the choice of a "unit of quantity" (e.g. the choice of some particular quantity of mass to serve as our unit of mass), or as the choice of an element of $\mathcal{S}$ to serve as multiplicative unit (i.e. group identity).

[16]This isn't to say that no sense can be made of multiplying two masses together: just that the answer,

However, although characterising scalar structures as principal homogeneous spaces of a certain kind is a convenient (and potentially illuminating) way of proceeding, one might feel uncomfortable with it as a method: it might be felt that this treatment is insufficiently intrinsic, and that we should instead be characterising these structures through a set of appropriate axioms. I am not wholly convinced by this worry. Effectively, this approach to scalar structures amounts to claiming that any scalar structure is characterised by a certain kind of binary relation that holds between pairs of scalar magnitudes—it is just that the binary relation's possible values have the structure of $\mathbb{R}^+$, rather than that of the set $\{\text{True}, \text{False}\}$.[17] It is, therefore, a relation with a determinate-determinable structure, just as mass itself is a property with a determinate-determinable structure.

Nevertheless, for those who do share this worry, an axiomatic treatment of the desired kind is available. In Appendix A, it is shown that scalar structures can alternatively be characterised as complete dense ordered positive semigroups, using a set of axioms that (in essence) are the same as those given by Hölder (1901). That is, we can show that any principal homogeneous space over $\mathbb{R}^+$ obeys the relevant axioms, and that any model of the axioms admits a regular action of $\mathbb{R}^+$. I will take this to legitimate the analysis of scalar structures in terms of regular actions of $\mathbb{R}^+$.

4. Scalar algebra

Thus, a scalar quantity $\mathcal{S}$ is characterised by the fact that it comes equipped with a regular action of $\mathbb{R}^+$. We now use this action to define products and ratios of scalar quantities: this will enable us to define quantities such as total momentum (the product of mass with speed) or density (the ratio of mass to volume). We already have—indeed, we began with—ratios of a given scalar quantity to itself; now, though, we will see how to define products of scalar quantities, and ratios between distinct scalar quantities. So suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are two scalar quantities, each equipped with the canonical action. We will denote the product quantity by $\mathcal{S}_1 \cdot \mathcal{S}_2$. To define it, let $(*, *^{-1})$ denote the following action of $\mathbb{R}^+$ on $\mathcal{S}_1 \times \mathcal{S}_2$: for all $x \in \mathbb{R}^+$, $a_1 \in \mathcal{S}_1$, $a_2 \in \mathcal{S}_2$, acting on (a_1, a_2)

whatever it is, will not itself be a mass. Indeed, in section 4 we shall see how to construct the quantity of "mass-squared", in which the products of masses live.

[17] Indeed, it has been suggested that the real numbers might be *defined* as magnitude-ratios, i.e. as the values of this relation (Forrest and Armstrong, 1987; Michell, 1994). For further discussion of the idea that scalar quantities may be understood as magnitudes characterised by their standing in ratio-relations, see Michell (2004).

with x yields

$$(x * a_1, x^{-1} * a_2) \tag{10}$$

It is straightforward to verify that this is indeed a group action. We then define $\mathcal{S}_1 \cdot \mathcal{S}_2$ as the quotient of $\mathcal{S}_1 \times \mathcal{S}_2$ by this action, that is

$$\mathcal{S}_1 \cdot \mathcal{S}_2 := {}^{\mathcal{S}_1 \times \mathcal{S}_2}\!\big/_{(\mathbb{R}^+/\mathbb{R}^+)} \tag{11}$$

In other words, we define the product quantity $\mathcal{S}_1 \cdot \mathcal{S}_2$ as a set of pairs—with the proviso that the product defined by the pair (a_1, a_2) is the same as the product defined by the pair $(xa_1, x^{-1}a_2)$. We will denote elements of $\mathcal{S}_1 \cdot \mathcal{S}_2$ by expressions of the form $a_1 \cdot a_2$, thereby denoting the equivalence class to which the pair (a_1, a_2) belongs: thus, $a_1 \cdot a_2 = xa_1 \cdot x^{-1}a_2$.

So far, however, $\mathcal{S}_1 \cdot \mathcal{S}_2$ is a mere set. In order to give it some structure, we now equip it with an action of $\langle \mathbb{R}^+, \cdot \rangle$: namely, for any $x \in \mathbb{R}^+, a_1 \in \mathcal{S}_1, a_2 \in \mathcal{S}_2$,

$$x * (a_1 \cdot a_2) = (x * a_1) \cdot a_2 \tag{12}$$

First, this action is well-defined (i.e., is independent of the choice of a_1 and a_2): a few lines of algebra shows that for any $x, y \in \mathbb{R}^+$,

$$x * (y * a_1 \cdot y^{-1} * a_2) = x * (a_1 \cdot a_2) \tag{13}$$

Second, the action is regular:

Proposition 1. Consider any two products $a_1 \cdot a_2$ and $a'_1 \cdot a'_2$ (with $a_1, a'_1 \in \mathcal{S}_1$ and $a_2, a'_2 \in \mathcal{S}_2$). There is a unique $z \in \mathbb{R}^+$ such that $z * (a_1 \cdot a_2) = a'_1 \cdot a'_2$.

Proof. Since the action of $\mathbb{R}^+$ on $\mathcal{S}_1$ and $\mathcal{S}_2$ is regular, there exists a unique x such that $x * a_1 = a'_1$, and a unique y such that $y * a_2 = a'_2$. So immediately,

$$\begin{aligned}(xy) * (a_1 \cdot a_2) &= (xy * a_1) \cdot a_2 \\ &= (x * a_1) \cdot (y * a_2) \\ &= a'_1 \cdot a'_2\end{aligned}$$

Thus, we have existence (i.e., the action is transitive). For uniqueness, suppose that $z * (a_1 \cdot a_2) = a'_1 \cdot a'_2$; i.e., $(z * a_1) \cdot a_2 = a'_1 \cdot a'_2$. This means that there is some $r \in \mathbb{R}^+$

such that

$$rz * a_1 = a_1'$$
$$r^{-1} * a_2 = a_2'$$

Since x and y are unique, we immediately have that $rz = x$ and $r^{-1} = y$; thus, $z = r^{-1}x = xy$. So the action is also free. □

Hence, this action makes the product quantity $\mathcal{S}_1 \cdot \mathcal{S}_2$ into a principal homoegenous space for $\mathbb{R}^+$: or, in other words, the product of two scalar quantities is itself a scalar quantity (as one would expect). As a result, $\mathcal{S}_1 \cdot \mathcal{S}_2$ is equipped with both additive and order structure, just as any scalar quantity is.

The *ratio* quantity $\frac{Q_1}{Q_2}$ is defined by a similar method to that used to define the product. This time, let $(*, *)$ denote the following action of $\mathbb{R}^+$ on $\mathcal{S}_1 \times \mathcal{S}_2$: for all $x \in \mathbb{R}^+$, $a_1 \in \mathcal{S}_1$, $a_2 \in \mathcal{S}_2$,

$$x(*, *)(a_1, a_2) = (x * a_1, x * a_2) \tag{14}$$

We then define

$$\frac{\mathcal{S}_1}{\mathcal{S}_2} := \mathcal{S}_1 \times \mathcal{S}_2 \big/ (*, *) \tag{15}$$

Thus, the idea is that the ratio defined by the pair (a_1, a_2) is the same as the ratio defined by the pair $(x * a_1, x * a_2)$; note the analogy to defining the rational numbers in terms of the natural numbers. Members of $\frac{Q_1}{Q_2}$ will be denoted by expressions of the form $\frac{a_1}{a_2}$, this denoting the equivalence class of (a_1, a_2).

Again, we define an action of $\mathbb{R}^+$ on $\frac{Q_1}{Q_2}$, according to:

$$r * \frac{a_1}{a_2} = \frac{r * a_1}{a_2} \tag{16}$$

By an analogous proof to that used for the product quantity, we can show that this action is regular. So $\frac{Q_1}{Q_2}$ is also a principal homogeneous space for $\mathbb{R}^+$; thus ratios of scalars are themselves scalars, and any ratio quantity possesses both additive and order structure.

There is a special case, however, which is worth remarking upon. If $\mathcal{S}_1 = \mathcal{S}_2$, then the condition $a_1 = a_2$ is well-posed. Moreover, this condition is preserved under the action $(*, *)$; note, by contrast, that the action $(*, *^{-1})$ does not preserve this condition. This means that in the ratio quantity $\frac{\mathcal{S}}{\mathcal{S}}$, the element $\frac{a}{a}$ is naturally privileged. It is therefore a natural choice of unit, which can be used to upgrade $\frac{\mathcal{S}}{\mathcal{S}}$ from a principal homogeneous space for $\mathbb{R}^+$ to (a canonical copy of) $\mathbb{R}^+$ itself, by equipping it with multiplicative structure. But this makes sense, since quantities of the form $\frac{a_1}{a_2}$ where $a_1, a_2 \in \mathcal{S}$, are the "pure" ratios that we met earlier: that is, the ones which are not just a scalar quantity, but a numerical quantity.

5. Vector quantities

Now, we turn to *vector* quantities.[18] We will define a vector quantity to be a quantity whose values constitute a vector space *equipped with a scalar-valued Euclidean norm*. For example, the quantity of spatial displacement takes values in a length-valued three-dimensional vector space; the quantity of velocity takes values in a three-dimensional speed-valued vector space; and the quantity of temporal displacement takes values in a one-dimensional duration-valued vector space. In general, I will use double-struck typeface to indicate vector quantities, e.g. $\mathbb{V}$, and boldface to indicate values of those quantities, e.g. $\mathbf{v}$. (Similarly, the zero vector in $\mathbb{V}$ will be denoted $\mathbf{0}$.) I will use the same letter, but in script, to denote the scalar quantity in which the vector quantity's norm takes values: thus, the vector quantity $\mathbb{V}$ will have a $\mathcal{V}$-valued norm.

To say that $\mathbb{V}$ has a $\mathcal{V}$-valued Euclidean norm means the following. First, we extend the structure $\mathcal{V}$ to its *null completion* $\mathcal{V}_0$: this is an expansion of $\mathcal{V}$ to $\mathcal{V} \cup \{0\}$, with $<$ and $+$ extended to 0 as follows:

- For all $a \in \mathcal{V}$, $0 < a$
- For all $a \in \mathcal{V}$, $0 + a = a + 0 = a$

This is necessary to allow for the possibility that a vector has null magnitude. We extend the action of $\mathbb{R}^+$ to $\mathcal{V}_0$ by stipulating that

$$\lambda \cdot 0 = 0 \tag{17}$$

and use this definition to extend the scalar algebra of §4 to null completions; we find that $\mathcal{S} \cdot \mathcal{V}_0 = \mathcal{S}_0 \cdot \mathcal{V}_0 = (\mathcal{S} \cdot \mathcal{V})_0$, with $s \cdot 0 = 0$ and $0 \cdot 0 = 0$; and $\mathcal{V}_0/\mathcal{S} = (\mathcal{V}/\mathcal{S})_0$, with $0/s = 0$.

Second, a $\mathcal{V}$-valued norm is a map $|\bullet| : \mathbb{V} \to \mathcal{V}_0$ which obeys the following condi-

[18] For further discussion of the metaphysics of vector quantities, see Beisbart (2009). The account here is primarily concerned with vector quantities that arise in classical physics; for an illuminating discussion of the metaphysics of quantum-mechanical spin, see Wolff (2015).

tions for any $\mathbf{v}, \mathbf{w} \in \mathbb{V}, \lambda \in \mathbb{R}$:

$$|\mathbf{v} + \mathbf{w}| \leq |\mathbf{v}| + |\mathbf{w}| \tag{18a}$$

$$|\lambda \mathbf{v}| = |\lambda||\mathbf{v}| \tag{18b}$$

$$|\mathbf{v}| = 0 \text{ iff } \mathbf{v} = \mathbf{0} \tag{18c}$$

where $|\lambda|$ is the absolute value of λ. Finally, to say that this norm is *Euclidean* means that it obeys the *parallelogram law*: for any $\mathbf{v}, \mathbf{w} \in \mathbb{V}$,

$$2|\mathbf{v}|^2 + 2|\mathbf{w}|^2 = |\mathbf{v} + \mathbf{w}|^2 + |\mathbf{v} - \mathbf{w}|^2 \tag{19}$$

Note that this definition makes use of the extension of scalar algebra to null completions. This has the useful consequence that $\mathbb{V}$ carries an inner product $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathcal{V} \cdot \mathcal{V}$, defined by the *polarisation identity*

$$\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4} \left(|\mathbf{v} + \mathbf{w}|^2 - |\mathbf{v} - \mathbf{w}|^2 \right) \tag{20}$$

Indeed, an alternative approach would have been to stipulate that $\mathbb{V}$ carries a $(\mathcal{V} \cdot \mathcal{V})$-valued inner product, rather than a $\mathcal{V}$-valued norm.

If we define (say) velocities as taking values in one vector space, that clearly can't be quite the same as the vector space in which forces take values, since velocity and force are different quantities. Yet we do want to permit certain kinds of comparisons between forces and velocities: namely, comparisons of *direction*. For instance, it is physically well-posed to ask whether the force on a ship is in the same direction as the ship's motion, or whether it is at some non-trivial angle, since the answer will affect what happens to the ship subsequently: if the force is directed along the ship's direction of motion, the ship will continue in that direction with increasing speed; if it is angled, the ship will change direction; if it is directly opposed to the ship's motion, the ship will keep the direction but lose speed.

Thus, at certain points in the sequel we will need the notion of a vector quantity's space of directions. We therefore define the *direction* $\vec{\mathbf{v}}$ of a vector $\mathbf{v}$ as the equivalence class of $\mathbf{v}$ under positive rescaling:

$$\vec{\mathbf{v}} := \{\mathbf{w} = \lambda \mathbf{v} : \lambda \in \mathbb{R}^+\} \tag{21}$$

The set of directions of $\mathbb{V}$—i.e., the quotient of $\mathbb{V}$ under this equivalence relation—will be denoted $\vec{\mathbb{V}}$. If $\mathbb{V}$ is n-dimensional, then $\vec{\mathbb{V}} \cong S^{n-1} \cup \{\mathbf{0}\}$, where S^{n-1} is the $(n-1)$-

dimensional unit sphere. Hence, any two vector quantities of the same dimension will have isomorphic spaces of directions; if there is a *canonical* isomorphism between the directions of $\mathbb{V}$ and those of $\mathbb{W}$ (such as in the case of force and velocity), then we will write $\vec{\mathbb{V}} = \vec{\mathbb{W}}$. (We will see in §7 how such canonical isomorphisms can come about.)

Now that we have the notion of direction, we are able to give precise expression to the classic concept of a vectorial quantity as "a quantity which is considered as possessing *direction* as well as *magnitude*."[19] In particular, one can uniquely identify any $\mathbf{v} \in \mathbb{V}$ by specifying $|\mathbf{v}|$ and $\vec{\mathbf{v}}$: if $|\mathbf{w}| = |\mathbf{v}|$ and $\vec{\mathbf{w}} = \vec{\mathbf{v}}$, then $\mathbf{w} = \mathbf{v}$. Furthermore, given any magnitude and direction, there exists some vector with that magnitude and direction. We will use this observation below.

6. Vector algebra

We have seen above how scalar quantities may be combined to form products and ratios. In this section, we consider how scalar and vector quantities may be combined with one another.

Suppose that we wish to combine a scalar quantity $\mathcal{S}$ with a vector quantity $\mathbb{V}$. In this case, we can formulate two further vector quantities: the product quantity $\mathcal{S} \cdot \mathbb{V}$ and the ratio quantity $\mathbb{V}/\mathcal{S}$. These are defined in essentially the same fashion as above. Thus, the product quantity $\mathcal{S} \cdot \mathbb{V}$ consists of equivalence classes of pairs $(a, \mathbf{v})$ where $a \in \mathcal{S}$ and $\mathbf{v} \in \mathbb{V}$, where the equivalence relation $\sim$ is

$$(a, \mathbf{v}) \sim (x * a, x^{-1}\mathbf{v}) \tag{22}$$

for any $x \in \mathbb{R}^+$. As above, we will denote the equivalence class of $(a, \mathbf{v})$ by $a \cdot \mathbf{v}$.

We wish to show that this constitutes a vector-valued quantity, with a norm taking values in $\mathcal{S} \cdot \mathcal{V}$. So first, we define addition by

$$a \cdot \mathbf{v} + b \cdot \mathbf{w} := a \cdot (\mathbf{v} + \frac{b}{a} \cdot \mathbf{w}) \tag{23}$$

It immediately follows that $a \cdot (\mathbf{u} + \mathbf{v}) = a \cdot \mathbf{u} + a \cdot \mathbf{v}$. Second, we define "scalar multiplication"—that is, multiplication of elements of $\mathbb{V}$ by elements of $\mathbb{R}$—according to

$$x(a \cdot \mathbf{v}) := a \cdot (x\mathbf{v}) \tag{24}$$

With these definitions, we can prove that $\mathcal{S} \cdot \mathbb{V}$ satisfies the axioms of a vector space.

[19](Gibbs, 1960, p. 1)

(The proofs are straightforward, if a little tedious; they mostly involve simply using the above definitions and applying the fact that $\mathbb{V}$ is a vector space.) The zero vector is $a \cdot \mathbf{0}$, and the inverse of $a \cdot \mathbf{v}$ is $a \cdot (-\mathbf{v})$.

Our second task is to show that $\mathcal{S} \cdot \mathbb{V}$ comes equipped with an $(\mathcal{S} \cdot \mathcal{V})$-valued norm. We define the norm on $\mathcal{S} \cdot \mathbb{V}$ as follows:

$$|s\mathbf{v}| := s \cdot |\mathbf{v}| \tag{25}$$

This takes values in $\mathcal{S} \cdot \mathcal{V}_0$, which is canonically isomorphic to $(\mathcal{S} \cdot \mathcal{V})_0$, i.e. the null completion of $\mathcal{S} \cdot \mathcal{V}$. Showing that it is a Euclidean norm is straightforward.

By a similar process, we can define the ratio quantity $\mathbb{V}/S$. This consists of equivalence classes of pairs $(\mathbf{v}, a)$, subject to the equivalence relation

$$(\mathbf{v}, a) \sim (x\mathbf{v}, x * a) \tag{26}$$

We denote the equivalence class of $(\mathbf{v}, a)$ by $\frac{\mathbf{v}}{a}$. Addition is defined by

$$\frac{\mathbf{v}}{a} + \frac{\mathbf{w}}{b} := \frac{\mathbf{v} + \frac{a}{b}\mathbf{w}}{a}, \tag{27}$$

"scalar multiplication" is defined by

$$x\frac{\mathbf{v}}{a} = \frac{x\mathbf{v}}{a}, \tag{28}$$

and the $\frac{\mathcal{V}}{\mathcal{S}}$-valued norm is defined by

$$\left|\frac{\mathbf{v}}{a}\right| = \frac{|\mathbf{v}|}{a}. \tag{29}$$

Given any $\mathbf{v} \in \mathbb{V}$ and $x \in \mathbb{R}^+$, $\mathbf{v}$ and $x\mathbf{v}$ have (by definition) the same direction. It follows that we can canonically identify the directions of $\mathcal{S} \cdot \mathbb{V}$ and $\mathbb{V}/\mathcal{S}$ with those in $\mathbb{V}$, according to

$$\overrightarrow{a \cdot \mathbf{v}} = \vec{\mathbf{v}} = \frac{\vec{\mathbf{v}}}{a} \tag{30}$$

Hence,

$$\overrightarrow{\mathcal{S} \cdot \mathbb{V}} = \vec{\mathbb{V}} = \overrightarrow{\left(\frac{\mathbb{V}}{\mathcal{S}}\right)} \tag{31}$$

7. Intrinsic Newtonian kinematics

We are now in a position to give an intrinsic theory of Newtonian spacetime; and, correspondingly, to give an intrinsic account of quantities of motion. The standard way to define Newtonian spacetime is to first define spatial displacements as forming a three-dimensional Euclidean vector space, and temporal displacements as forming a one-dimensional Euclidean vector space. Then, we define space and time as the *affine spaces* over these vector spaces. An affine space is the principal homogeneous space for a vector space (regarded as a group, with vector addition as group multiplication): that is, it is a set of points, such that for any two points there is a unique vector defining the "displacement" between them; and if v is the displacement from a to b, and w the displacement from b to c, then $v + w$ is the displacement from a to c.

We will proceed along similar lines, but rather than using Euclidean vector spaces (with norms taking values in $\mathbb{R}^+$), we will use vector quantities as defined above. So first, we presume two scalar quantities: the quantity $\mathcal{L}$ of *length*, and the quantity $\mathcal{D}$ of *duration*. Next, we define the quantity of *spatial displacements* $\mathbb{L}$ to be a three-dimensional $\mathcal{L}$-valued vector quantity, and the quantity of *temporal displacements* $\mathbb{T}$ to be a one-dimensional $\mathcal{D}$-valued vector quantity. And finally, we define *space* to be the affine space X for $\mathbb{L}$, and *time* to be the affine space T for $\mathbb{T}$. Given any two points $x, y \in X$, the spatial displacement between them will be denoted by $y - x$; and given any points $t, s \in T$, the temporal displacement between them will be denoted by $t - s$.

What about quantities of motion, such as velocity or acceleration? In standard mathematical physics, one can take the space of velocities through an affine space to be the vector space over which that affine space is defined; but this is an artefact of the use of real-valued vector spaces, since the space of displacements has a length-valued norm but the space of velocities should have a speed-valued norm. So, instead, we define velocity to be a *ratio* quantity: specifically, it is the quantity

$$\mathbb{V} := \frac{\mathbb{L}}{\mathcal{D}} \tag{32}$$

i.e., it is the ratio quantity of spatial displacements and durations. As discussed above, this means that $\mathbb{V}$ is a vector quantity, whose set of directions is identifiable with the directions for $\mathbb{L}$, and whose magnitudes take values in (the null completion of) the scalar quantity $\mathcal{V} := \mathcal{L}/\mathcal{D}$; this scalar quantity is, of course, the quantity of *speed*.

Next, we want to know how to define the velocity of some body moving through space over time. To do this, we need to first of all observe that since $\mathbb{T}$ is one-dimensional, its space of directions is simply a three-element set $\{\uparrow, \mathbf{0}, \downarrow\}$ (since $S^0 \cong \{\uparrow, \downarrow\}$). For any $d \in \mathcal{D}$, let $d_\uparrow$ denote the vector in $\mathcal{D}$ with magnitude d and direction $\uparrow$, and let $d_\downarrow$ denote the vector in $\mathcal{D}$ with magnitude d and direction $\downarrow$. The elements $\uparrow$ and $\downarrow$ represent, of course, the two temporal directions; we make no judgment, however, about which of these directions is "past" and which is "future".

We now define a *trajectory* as a smooth curve $x : T \to X$. Then, for any $t \in T$, we define the velocity of this trajectory in direction $\uparrow$ at t to be

$$\dot{\mathbf{x}}_\uparrow(t) := \lim_{\varepsilon \to 0} \frac{x(t + \varepsilon_\uparrow) - x(t)}{\varepsilon} \tag{33}$$

where ε takes values in $\mathcal{D}$. The velocity $\dot{\mathbf{x}}_\downarrow$ of the trajectory in direction $\downarrow$ at is defined similarly; given that x is smooth, it follows that $\dot{\mathbf{x}}_\uparrow = -\dot{\mathbf{x}}_\downarrow$. The fact that the velocity is not determined given the trajectory, but is determined only relative to a direction of time, is just an instance of the more general fact that a derivative is defined only relative to a direction. Without a choice for a direction of time, velocities cannot be uniquely represented by vectors.[20]

Similarly, we define the quantity of *acceleration* to be the ratio quantity

$$\mathbb{A} := \frac{\mathbb{V}}{D} \tag{34}$$

From this, it follows that acceleration is a vector quantity with directions identifiable with the directions for velocity (and hence, with the directions for spatial displacement), and with magnitudes taking values in the scalar quantity $\mathcal{L}/\mathcal{D}^2$. And given a trajectory $x : T \to X$, we define the acceleration at t to be

$$\ddot{\mathbf{x}}(t) := \lim_{\varepsilon \to 0} \frac{\dot{\mathbf{x}}_\uparrow(t + \varepsilon_\uparrow) - \dot{\mathbf{x}}(t)}{\varepsilon} \tag{35}$$

where, again, ε takes values in $\mathcal{D}$. Thus, in effect, we define acceleration as the derivative of $\dot{\mathbf{x}}_\uparrow$ in the $\uparrow$ direction; we could equally well have defined it as the derivative of $\dot{\mathbf{x}}_\downarrow$ in the $\downarrow$ direction, which would have yielded the same association of accelerations to trajectories. (Thus, unlike velocities, accelerations may be uniquely represented by vectors; indeed, it is the fact that Newton's Second Law speaks only of accelerations that means that theories with purely position-dependent forces are time-reversal-invariant.)

[20]See the discussion in (Field, 1980, §8.D) and Malament (2004).

8. Intrinsic Newtonian dynamics

In order to move from kinematics to dynamics—that is, from the pure theory of abstract motion to the theory of forces and causes of motion—we need to introduce a third primitive scalar quantity (to join our two kinematical quantities of length and duration): that of *mass*. We will denote this quantity by $\mathcal{M}$. We assume that for every body A, there is not only an associated trajectory $x_A : T \to X$, but also an associated mass $m_A \in \mathcal{M}$. This mass is taken to be fixed for all time.

We now use this quantity to define the quantity of *force*. In order to do so as economically as possible, we will make use of Newton's Second Law, and define the quantity of force as the product of mass with acceleration:

$$\mathbb{F} := \mathcal{M} \cdot \mathbb{A} \tag{36}$$

Again, this means that force is a vector quantity whose directions live in the space of spatial directions; its magnitudes take values in the scalar quantity $\mathcal{F} = \mathcal{M} \cdot \mathcal{L}/\mathcal{D}^2$.

This definition, however, captures only part of the content of Newton's Second Law. The full assertion of the law states that for any body a, at any time t, there is a quantity associated to it of its *net force*, $\mathbf{F}_A(t)$; and that this force is related to A's acceleration at t via

$$\mathbf{F}_A(t) = m_A \ddot{\mathbf{x}}_A(t) \tag{37}$$

By itself, this assertion contains essentially no dynamical content. Indeed, as is well-known, one could even take this as a definition of the notion of "net force"—and in fact, we will do just that.

However, it acquires dynamical content once one postulates laws concerning how the net force on an object may be computed: for example, the Law of Universal Gravitation. This asserts that any two bodies A and B experience a mutual gravitational attractive force, which is proportional to their masses and inversely proportional to the square of their distance. We may therefore express it as consisting of the following two assertions. The first assertion is that there exists an isomorphism of scalar quantities

$$G : \frac{\mathcal{M} \cdot \mathcal{M}}{\mathcal{L}^2} \to \mathcal{F} \tag{38}$$

Using this, for any (distinct) objects A and B, we let the *gravitational force of B on A* be denoted $\mathbf{G}_A^B$, and defined by

$$|\mathbf{G}_A^B| := G\left(\frac{m_A \cdot m_B}{|x_B - x_A|^2}\right) \tag{39a}$$

$$\overrightarrow{\mathbf{G}_A^B} := \overrightarrow{x_B - x_A} \tag{39b}$$

Second, the gravitational forces on objects must be related to the net forces they experience (which, by the Second Law, are linked to the accelerations they undergo). We do this by asserting that the net force on an object is given by the sum of the gravitational forces on it (from all other objects):

$$\mathbf{F}_A = \sum_{B \neq A} \mathbf{G}_A^B \tag{40}$$

9. Evaluation and conclusion

We can now evaluate this theory against the motivations we had for developing it: does it serve its desired purpose? Regarding the first motivation, we can see more or less immediately that it does. This theory is empirically equivalent to standard Newtonian theory, which we can verify by noting that if we make a choice of unit for each of $\mathcal{M}$, $\mathcal{L}$, and $\mathcal{D}$, then the equations governing this theory all reduce to their familiar Newtonian forms. Nor does it appeal to extra-physical entities; only physical magnitudes, equipped with physically significant structure, appear in the theory.

Proving that it satisfies our second motivation—of having a "sophisticated" version of absolutism—is a matter of demonstrating that two models related by a scaling are isomorphic. So, first, suppose that we have a model of intrinsic Newtonian gravitation. That is, we have:

- primitive scalar quantities $\mathcal{M}$, $\mathcal{L}$ and $\mathcal{D}$;
- an isomorphism $G : \frac{\mathcal{M} \cdot \mathcal{M}}{\mathcal{L}^2} \to \mathcal{F}$
- a three dimensional $\mathcal{L}$-valued vector quantity $\mathbb{L}$, and a one-dimensional $\mathcal{D}$-valued quantity $\mathbb{D}$;
- affine spaces X and T, based on $\mathbb{L}$ and $\mathbb{D}$ respectively; and
- a set of "bodies" $A, B, C, \ldots$, where each body A is associated with a certain mass $m_A \in \mathcal{M}$, a smooth trajectory $x_A : T \to X$, and a net force function $\mathbf{F}_A : T \to \mathcal{M} \cdot \mathbb{A}$ defined by (37); and

- for every pair of bodies A, B, a gravitational force function $\mathbf{G}_A^B : T \to \mathcal{M} \cdot \mathbb{A}$ given by (39)

such that equation (40) is obeyed.

Let this model be called K. Suppose that we wish to perform an appropriate joint rescaling of this model, to obtain some rescaled model K': that is, K' will be obtained by rescaling $\mathcal{M}$, $\mathcal{L}$ and $\mathcal{T}$ by factors μ, λ and τ respectively, where $\lambda^3 = \mu\tau^2$. We need to determine what it means to perform such a rescaling, and hence define K'.

It is simple enough to define the rescaling of masses: if a body A has a mass m_A in K, then in K' it has a mass

$$m'_A = \mu * m_A \tag{41}$$

For lengths, we define a new action $\bullet'$ of $\mathbb{L}$ on X, related to the old action $\bullet$ by

$$\mathbf{1} \bullet' y := (\lambda^{-1} * \mathbf{1}) \bullet y \tag{42}$$

Let X' be the affine space which is numerically identical to X, but with the action of $\mathbb{L}$ given by $\bullet'$ rather than $\bullet$. As a result, for any points $x, y \in X$, the displacement between the corresponding points $x', y' \in X'$ is related to the displacement between x and y via

$$y' - x' = \lambda(y - x) \tag{43}$$

We do the same thing for times: times in A' take values in an affine space T', numerically identical to T but equipped with a different action of $\mathbb{D}$, such that for any $t_1, t_2 \in T$ and the corresponding $t'_1, t'_2 \in T'$,

$$t'_2 - t'_1 = \tau(t_2 - t_1) \tag{44}$$

We now stipulate that if a body A has a certain trajectory $x_A : T \to X$ in K, then its trajectory in K' is given by the corresponding trajectory $x'_A : T' \to X'$ (defined by the condition that for all $t \in T$, the value of x'_A at the corresponding point $t' \in T'$ is the point in X' corresponding to $x_A(t) \in X$).

These stipulations determine the values of the remaining kinematical quantities in A'. For any body a, its velocities and acceleration in A' are related to its velocities and acceleration in A via

$$\dot{\mathbf{x}}_{\uparrow}^{A'}(t') = \frac{\lambda}{\tau}\dot{\mathbf{x}}_{\uparrow}^{A}(t), \quad \dot{\mathbf{x}}_{\downarrow}^{A'}(t') = \frac{\lambda}{\tau}\dot{\mathbf{x}}_{\downarrow}^{A}(t) \tag{45}$$

and

$$\ddot{\mathbf{x}}^{A'}(t') = \frac{\lambda}{\tau^2}\ddot{\mathbf{x}}^{A}(t) \tag{46}$$

Next, we can use equations (37) and (39) to calculate the net force and gravitational force upon any body in A'. For any body A, it follows from equations (37), (41) and (46) that the net force upon it is given by

$$\mathbf{F}'_A = \frac{\mu\lambda}{\tau^2} \tag{47}$$

As for the gravitational force, let A and B be any two bodies. We observe first that

$$\overrightarrow{x'_B - x'_A} = \overrightarrow{x_B - x_A} \tag{48}$$

and so $\overrightarrow{\mathbf{G}_A^{B\prime}} = \overrightarrow{\mathbf{G}_A^B}$. Second, since G is an isomorphism (of scalar quantities), it commutes with the action of $\mathbb{R}^+$, and so

$$|\mathbf{G}_A^{B\prime}| = G\left(\frac{m'_A \cdot m'_B}{(|x_B - x_A|')^2}\right) \tag{49}$$

$$= \frac{\mu^2}{\lambda^2} G\left(\frac{m_A \cdot m_B}{(|x_B - x_A|)^2}\right) \tag{50}$$

$$= \frac{\mu^2}{\lambda^2}|\mathbf{G}_A^B| \tag{51}$$

It follows that

$$\mathbf{G}_A^{B\prime} = \frac{\mu^2}{\lambda^2}\mathbf{G}_A^B \tag{52}$$

Hence, given that $\lambda^3 = \mu\tau^2$, we obtain that

$$\mathbf{F}'_A = \sum_{b \neq a} \mathbf{G}_A^{B\prime} \tag{53}$$

That is, A' satisfies (40), and hence is a model.[21]

It remains only to show that K' is isomorphic to K. For the sake of uniformity in notation, let us denote the "copy" of $\mathcal{M}$ in which bodies in K' take their mass-values as $\mathcal{M}'$. Then our task is to give isomorphisms $f : \mathcal{M} \to \mathcal{M}'$, $g : X \to X'$ and $h : T \to T'$, such that for any body a, $m'_A = f(m_A)$ and $x'_A(t') = g(x_A(h^{-1}(t)))$. For mass, this is simple enough: we let $f : m \mapsto \mu m$, and can see immediately that f is an isomorphism. For g and h, we take them to be the identity-maps on the underlying point-sets; we now need to show that these are isomorphisms of (metric) affine spaces, which in turn requires showing that there is an isomorphism $p : \mathbb{L} \to \mathbb{L}$ such that for any $\mathbf{l} \in \mathbb{L}$ and

[21]This also makes clear that if $\lambda^3 \neq \mu\tau^2$, then K' is not (in general) a model—i.e., that *arbitrary* rescalings are not symmetries, as desired.

$x \in X$,[22]

$$(\mathbf{1} \bullet x)' = p(\mathbf{1}) \bullet' x' \tag{54}$$

We let p be given by $p : \mathbf{l} \mapsto \lambda\mathbf{l}$; the fact that p is an isomorphism of $\mathbb{L}$ then follows from the fact that it is induced by the isomorphism $l \in \mathcal{L} \mapsto \lambda l \in \mathcal{L}$. Hence, $g : X \to X'$ is an isomorphism; by similar reasoning, we can show that $h : T \to T'$ is an isomorphism. Hence, K is isomorphic to K'.

Hence, if we are willing to grant the legitimacy of the sophisticationist strategy, then the way is opened up for us to regard models related by a rescaling as representing the same possible world. Whether we should grant that strategy's legitimacy, of course, is a different question. The main purpose of this paper has been to exhibit the formal apparatus above, and to recommend it as a means of giving an intrinsic treatment of physical theories. Although I have only discussed the simple case of Newtonian gravitation in the above, the framework can be straightforwardly extended to at least some other theories, as shows in Appendix B. The extension to more complex theories (e.g. relativity theory, quantum mechanics, or theories set on curved spacetimes) appears less straightforward; I leave that project for future work.[23]

A. Complete ordered positive structures

In this Appendix, I describe some simple axioms that can be used to characterise scalar and numerical quantities (in that order, since that will simplify the presentation, despite the fact that this is the opposite of the order in which these structures are discussed in the main text).[24]

First, scalar structures. A *complete dense ordered positive semigroup* $\mathcal{S}$ consists of a set

[22] Recall that an isomorphism between metric affine spaces A and B, over normed vector spaces $\mathbb{V}$ and $\mathbb{W}$ respectively, consists of an bijection $j : A \to B$ and an isomorphism $k : \mathbb{V} \to \mathbb{W}$ such that for any $\mathbf{v} \in \mathbb{V}$ and $a \in A$,

$$j(v \bullet a) = k(v) \bullet j(a);$$

equivalently, such that for any $a, b \in A$,

$$k(b - a) = j(b) - j(a)$$

[23] For intrinsic treatments of relativistic spacetime, see Mundy (1983) and Babic and Cocco (nd); for quantum mechanics, see Balaguer (1996) and Chen (2018); and for curved geometry, see Arntzenius and Dorr (2012).

[24] The results cited here are originally due to Hölder (1901) (translated in Michell and Ernst (1996)); for further discussion, see Clifford (1958), Fuchs (1963), Krantz et al. (1971), Hofmann and Lawson (1996), Bourbaki (1998, §5.2), (Wolff, 2020, chap. 5) and references therein.

S equipped with a binary relation $<$ and a binary operation $+$, obeying the following axioms:

1. Total Order. $<$ is both transitive (for all $a, b, c \in S$, if $a < b$ and $b < c$ then $a < c$) and trichotomous (for all $a, b \in S$, exactly one of $a < b$, $b < a$, or $a = b$ is true).
2. Density. For any $a, c \in S$ such that $a < c$, there is some $b \in S$ such that $a < b < c$.
3. (Dedekind) Completeness. Any non-empty subset $P \subseteq S$ that has an upper bound has a least upper bound.[25]
4. Associativity. For any $a, b, c \in S$,
$$(a + b) + c = a + (b + c) \tag{55}$$
5. Commutativity. For any $a, b \in S$,
$$a + b = b + a \tag{56}$$
6. (Strict) Monotonicity. For any $a, b, c \in S$, if $a < b$ then $a + c < b + c$.
7. Solvability. For any $a, b \in S$ such that $a < b$, there is some $c \in S$ such that
$$a + c = b \tag{57}$$
8. Positivity. For any $a, b \in S$,
$$a < a + b \tag{58}$$

For ease of comparison with the literature, I note here some simple consequences of these axioms.

Proposition 2. $\mathcal{S}$ is weakly monotone: if $a \leq b$ then $a + c \leq b + c$.

Proof. Immediate from Strict Monotonicity. □

Proposition 3. $\mathcal{S}$ is cancellative: that is, if $a + c = b + c$ then $a = b$.

Proof. This follows from Strict Monotonicity and Total Order: if $a \neq b$ then $a < b$ or $b < a$; hence $a + c < b + c$ or $b + c < a + c$; in either case, $a + c \neq b + c$. □

[25] $a \in R$ is an *upper bound* of S if $a \geq b$ for all $b \in S$; and an upper bound a is a *least upper bound* of S if, for any upper bound c of S, $a \leq c$. (Where $\leq$ is defined, as per usual, by the condition that $a \leq b$ iff $a < b$ or $a = b$.)

Proposition 4. If $a < b$, then there is a *unique* c such that $a + c = b$; we may therefore introduce the notion $(b - a)$ to denote this unique c.

Proof. This follows from Solvability and Proposition 3. □

Proposition 5. $\mathcal{S}$ has no identity element: that is, for any $a, b \in \mathcal{S}$, $a + b \neq a$.

Proof. Immediate from Positivity and Total Order: since $a + b > a$, $a + b \neq a$. □

Proposition 6. $\mathcal{S}$ has no least element: that is, for any $a \in \mathcal{S}$, there is some $b \in \mathcal{S}$ such that $b < a$.

Proof. Consider any $a \in \mathcal{S}$. By Density, let c be such that $a < c < a + a$; it follows by Strict Monotonicity that $a + a < a + c$. Now consider $(a + a) - c$. If $(a + a) - c \geq a$, then $a + a \geq a + c$ (by Propostion 2); so by contradiction, $(a + a) - c < a$. □

Proposition 7. $\mathcal{S}$ has the Archimedean property. That is, for any $a \in S$, let $\mathcal{N}_a$ be the subset of $\mathcal{S}$ defined inductively as the smallest set satisfying the following conditions:

- $a \in \mathcal{N}_a$
- for any $n \in \mathcal{N}_a$, $n + a \in \mathcal{N}_a$

Then: $\mathcal{N}_a$ has no upper bound (for any choice of a).

Proof. Suppose that $\mathcal{N}_a$ did have such an upper bound. Then by Completeness, it would have a *least* upper bound; call that least upper bound b. Since $a + a \in \mathcal{N}_a$, $b \geq a + a$; and since $a + a > a$ (by Positivity), $b > a$. Therefore, $(b - a)$ exists.

Now, for all $n \in \mathcal{N}_a$, $n + a \in \mathcal{N}_a$, and so $b \geq n + a$. Suppose for *reductio* that for some n, $(b - a) < n$. Then $b < n + a$ (by Strict Monotonicity). So we have a contradiction, from which it follows that $(b - a) \geq n$ for all $n \in \mathcal{N}_a$; i.e., that $(b - a)$ is an upper bound on $\mathcal{N}_a$. But then, since $(b - a) < b$, b is not the least upper bound after all. So we have a contradiction, from which the theorem follows. □

It is straightforward to verify that $\langle \mathbb{R}^+, <, + \rangle$, where $<$ and $+$ are the usual order and addition operations, is a complete dense ordered positive semigroup. Hölder (1901) demonstrated that, up to isomorphism, this is the *only* complete dense ordered positive semigroup.

Theorem 1 (Hölder's Theorem). *Let $\mathcal{S} = \langle S, \prec, \oplus \rangle$ be a complete dense ordered positive semigroup. Then $\mathcal{S}$ is isomorphic (as an ordered semigroup) to $\langle \mathbb{R}^+, <, + \rangle$. That is, there is a bijection $f : \mathcal{S} \to \mathbb{R}^+$, such that*

$$a \prec b \Leftrightarrow f(a) < f(b) \tag{59}$$

$$f(a \oplus b) = f(a) + f(b) \tag{60}$$

The isomorphism is unique up to a positive rescaling factor: that is, if $f, g : \mathcal{S} \to \mathbb{R}^+$ are two such isomorphisms, then there exists some $x \in \mathbb{R}^+$ such that for any $a \in \mathcal{S}$, $g(a) = x \cdot f(a)$.

To connect this to the characterisation of scalar structures in the main text, we desire to show that an ordered semigroup is a complete dense ordered positive semigroup just in case it is a principal homogeneous space for the multiplicative group of $\mathbb{R}^+$ (with the order and semigroup operations "imported" from $\mathbb{R}^+$ by the procedure described in the main text). So, first, let S be a principal homogeneous space for the group $\langle \mathbb{R}^+, \cdot \rangle$. Define an order relation $\prec$ and an addition operation $\oplus$ on S as described in the main text. Then using the fact that $\langle \mathbb{R}^+, <, + \rangle$ is a complete dense ordered positive semigroup, we can easily show that $\langle S, \prec, \oplus \rangle$ is a complete dense ordered positive semigroup. For example, to prove the Density axiom, we suppose that $a \prec c$, i.e. that

$$\frac{a}{k} < \frac{c}{k} \tag{61}$$

for some arbitrarily chosen $k \in S$. Then by the fact that $\mathbb{R}^+$ is dense, there is some $x \in \mathbb{R}^+$ such that

$$\frac{a}{k} < x < \frac{c}{k} \tag{62}$$

Since $x = \frac{x*k}{k}$, it follows that

$$a < x * k < c \tag{63}$$

and so, $\mathcal{S}$ is dense. The proofs for the other axioms are similar.

Second, suppose that $\mathcal{S}$ is a complete dense ordered positive semigroup. By Theorem 1, let f be an isomorphism from $\mathcal{S}$ to $\langle \mathbb{R}^+, <, + \rangle$. Now define an action of $\langle \mathbb{R}^+, \cdot \rangle$ on $\mathcal{S}$ by

$$x * a := f^{-1}(x \cdot f(a)) \tag{64}$$

This definition is independent of the choice of f, given that any two isomorphisms are related by a positive rescaling factor. Further, it does indeed define a group action, since

$$\begin{aligned}(xy) * a &= f^{-1}(x \cdot y \cdot f(a)) \\ &= f^{-1}(x \cdot f(y * a)) \\ &= x * (y * a)\end{aligned}$$

It remains only to show that this action is regular, i.e. that for any $a, b \in S$, there is a unique $x \in \mathbb{R}^+$ such that $x * a = b$. For existence, observe that

$$\frac{f(b)}{f(a)} * a = f^{-1}\left(\frac{f(b)}{f(a)} \cdot f(a)\right) = b$$

For uniqueness, suppose that $x * a = b$; then,

$$\begin{aligned}f^{-1}(x \cdot f(a)) &= b \\ x \cdot f(a) &= f(b) \\ x &= \frac{f(b)}{f(a)}\end{aligned}$$

This suffices to axiomatically characterise the scalar quantities. I now turn to an axiomatic characterisation of the numerical quantities (in case, as discussed in the text, one does not wish to simply define them as having the structure of the strictly positive real numbers).

Thus, a *complete ordered positive semifield*[26] $\mathcal{R}$ consists of a set R, equipped with a binary relation $<$ and binary operations $+$ and $\cdot$, such that:

1. CDOPS. $\langle R, <, + \rangle$ is a complete dense ordered positive semigroup.
2. Associativity (of multiplication). For any $x, y, z \in S$,

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \tag{65}$$

3. Commutativity (of multiplication). For any $x, y \in S$,

$$x \cdot y = y \cdot x \tag{66}$$

[26]The terminology here follows Hebisch and Weinert (1996), according to which a semifield is an algebraic structure in which addition is a semigroup and multiplication is a group, with addition distributing over multiplication (i.e. a field, but without the requirement of additive inverses). Note, in particular, that we do *not* require that a semifield should contain an additive identity.

4. Identity. There exists a privileged element $1 \in R$ such that for any $x \in R$,

$$1 \cdot x = x \tag{67}$$

5. Inverse. For any $x \in R$, there is a unique element $x^{-1} \in R$ such that

$$x^{-1} \cdot x = 1 \tag{68}$$

6. Distributivity. For any $x, y, z \in R$,

$$x \cdot (y + z) = x \cdot y + x \cdot z \tag{69}$$

7. Compatibility. For any $x, y, z \in R$,

$$x \cdot z < y \cdot z \tag{70}$$

By Theorem 1, we know that given any complete ordered positive semifield $\langle R, \prec, \oplus, \odot \rangle$, there is an isomorphism of ordered semigroups $f : \langle R, \prec, \oplus \rangle \to \langle \mathbb{R}^+, <, + \rangle$. Taking any such f, let $i : R \to \mathbb{R}^+$ be defined by the condition that for any $a \in R$,

$$i(a) = f(1)^{-1} \cdot f(a) \tag{71}$$

This yields the following result:

Theorem 2. *i is an isomorphism of ordered semifields: that is, for any $a, b \in R$,*

$$a \prec b \Leftrightarrow i(a) < i(b) \tag{72}$$

$$i(a \oplus b) = i(a) + i(b) \tag{73}$$

$$i(a \odot b) = i(a) \cdot i(b) \tag{74}$$

The isomorphism is unique: that is, if $h : \mathcal{R} \to \mathbb{R}^+$ is an isomorphism (of ordered semifields), then $h = i$.

The proof is merely an adaptation of the proof that there exists a unique isomorphism (of ordered fields) from any complete ordered field to $\mathbb{R}$.[27] Thus, any complete ordered positive semifield may be (canonically) identified with $\mathbb{R}^+$; this demonstrates that by

[27] For example, that given in (Spivak, 1994, chap. 30).

axiomatically defining numerical quantities as complete ordered positive semifields, we are fully entitled to treat such quantities as having the structure of $\mathbb{R}^+$.

B. Some other intrinsic theories

Here, I give some further elementary illustrations of the general apparatus developed above, by showing how to define Galilean spacetime, Maxwell spacetime, and the basic laws of electrostatics in an intrinsic fashion.

B.1. Galilean and Maxwell spacetime

As is well-known, Newtonian spacetime structure is, in some ways, not the most appropriate setting for Newtonian theories: a better setting is *Galilean spacetime* (aka neo-Newtonian spacetime) or *Maxwell spacetime* (aka Huygensian spacetime). By following the constructions in Saunders (2013), we can easily specify these spacetimes in intrinsic terms. In both cases we begin—as is the case for Newtonian spacetime—by assuming two primitive scalar quantities of length, $\mathcal{L}$, and duration, $\mathcal{D}$.

First, to define Galilean spacetime, we consider a four-dimensional vector space $\mathbb{G}$, with a privileged three-dimensional subspace $\mathbb{L}$ equipped with an $\mathcal{L}$-valued norm. We then equip the quotient space ${}^{\mathbb{V}}\!/_{\mathbb{L}}$ with a $\mathcal{D}$-valued norm, so that it is a (one-dimensional) vector quantity. Finally, we take Galilean spacetime to be an affine space G over $\mathbb{G}$.

Second, to define Maxwell spacetime, we consider an $\mathcal{L}$-valued three-dimensional vector quantity $\mathbb{L}$ and a $\mathcal{D}$-valued vector quantity $\mathbb{D}$. We then take Maxwell spacetime to be a set H equipped with

- A free but not transitive action of $\mathbb{L}$ on H, and
- A free, transitive action of $\mathbb{D}$ on the orbits of H under $\mathbb{L}$.

B.2. Electrotatics

In order to treat electrostatics, we need a fourth primitive scalar quantity: *electrical charge magnitude*, $\mathcal{Q}$. We then introduce *electrical charge* $\mathbb{Q}$, a $\mathcal{Q}$-valued one-dimensional vector quantity. Note that as a one-dimensional vector quantity, its space of directions consists of three points: $\vec{\mathbb{Q}} = \{\blacktriangle, \mathbf{0}, \blacktriangledown\}$. (Note that I use different symbols to the case of $\mathbb{D}$, in order to make clear that there is no canonical isomorphism between $\vec{\mathbb{Q}}$ and $\vec{\mathbb{D}}$.)

For the dynamics, we again assume Newton's Second Law (37). To state how electrostatic forces come about, we first introduce Coulomb's constant in the guise of an isomorphism

$$k_e : \frac{\mathcal{Q} \cdot \mathcal{Q}}{\mathcal{L} \cdot \mathcal{L}} \to \mathcal{M} \cdot \mathcal{A} \tag{75}$$

We now assert that to any two bodies A and B, there are associated electrical charges $\mathbf{Q}_A$ and $\mathbf{Q}_B$. The *electrostatic force of B on A* is denoted by $\mathbf{E}_A^B$. Its magnitude is defined by

$$|\mathbf{E}_A^B| = k_e \left(\frac{|\mathbf{Q}_A| \cdot |\mathbf{Q}_B|}{|x_B - x_A|^2} \right) \tag{76}$$

Its direction is given by the familiar rule: opposite charges attract, like charges repel. To enable a compact statement of this, let us define the following map $\circ : \vec{\mathbb{Q}} \times \vec{\mathbb{Q}} \to \{-1, 1\}$:

$$\blacktriangle \circ \blacktriangle = \blacktriangledown \circ \blacktriangledown = -1 \tag{77}$$

$$\blacktriangle \circ \blacktriangledown = \blacktriangledown \circ \blacktriangle = 1 \tag{78}$$

Then we can state that the direction of the electrostatic force is given by

$$\overrightarrow{\mathbf{E}_A^B} = (\overrightarrow{\mathbf{Q}_A} \circ \overrightarrow{\mathbf{Q}_B}) \cdot \overrightarrow{x_B - x_A} \tag{79}$$

Finally, as with the gravitational theory, we assert that the net force is just the sum of the electrostatic forces: for any bodies A and B,

$$\mathbf{F}_A = \sum_{B \neq A} \mathbf{E}_A^B \tag{80}$$

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