Pure geometry and geometric cognition

Mario Bacelar Valente[1, 2]

Abstract: The cognitive basis of pure geometry is basically unknown. Even the 'simpler' issue of what kind of representation of geometric object we have. In this work, we set forward a model of the representation of geometric objects at a neurological level for the case of the pure geometry of Euclid. To arrive at the model, we take into account historical aspects of practical and pure geometry together. This enables us to arrive at a coherent model of geometric objects consistent with the historical record.

1. Introduction

The purpose of the present work is to set forward a model of the neural representation of geometric concepts that underline our cognitive activity related to the practice of pure geometry.[3]

From the perspective of a philosophy of mathematical practices (see, e.g., Ferreirós 2016), we should take into account actual practices in their historical context. This is particularly the case with the present endeavor. Our purpose is not to address modern forms of pure geometry but pure geometry at its beginning, or close to it. Our object is the pure geometry of Euclid's *Elements*. To arrive at a coherent model, we found it necessary to take into account earlier forms of geometry.

The models will be based on the hub-and-spoke theory (Ralph, Jefferies, Patterson, and Rogers 2017). According to this theory, the neural representation of concepts is made in terms of spokes, which are modality-specific brain regions that codify modal features of concepts. For example, there are the spokes that encode visual, verbal (speech), and praxis representations. There are also integrative regions – the hub – which blends, in an amodal format, the different aspects codified in the spokes and gives rise to coherent concepts. The hub enables a modality-free codification of further aspects of concepts; accordingly, "[It] allows the formation of modality-invariant multi-dimensional representations that […] code the higher-order statistical structure that is present in our transmodal experience of each entity" (Ralph 2014, 7). We can address a particular concept directly in terms of 'spokes' and a 'hub' not has regions in the brain but as 'parts' of the concept. In this way, "each concept also has a 'hub' – a modality-independent unified representation efficiently integrating our conceptual knowledge" (Eysenck and Keane 2020, 319).

The organization of this work is as follows. In section 2, we will address the practical geometry of ancient Greece. Since not much is known, we take into account also elements of the practical geometry of other ancient cultures. This helps in given a more general characterization of practical geometry and arrive at a model of geometric figure for it. In section 3, we consider the earliest extant form of pure geometry as revealed in Hippocrates of Chios' work. We develop a model for the case of this pure geometry, which has important differences to that of Euclid's *Elements*. In section 4, we present the corresponding model of the neural representation of geometric objects in Euclidean geometry.

[1] Pablo de Olavide University, Seville, Spain. Email: mar.bacelar@gmail.com.

[2] This is a simplified stable draft of a more fully developed work, which is available upon request. Please do not use this version as a reference.

[3] For other works on somewhat related issues, see, e.g., Hohol and Miłkowski (2019) and Ferreirós and García-Pérez (2020).

2. Practical geometry in ancient Greece

Not much is known about Greek practical geometry previous or contemporary to pure geometry (see, e.g., Asper 2003, 109-114). So, in this work, we will consider basic aspects of the Greek practical geometric practice that are common to other practical geometries.

A key aspect of land measurements is to have a common unit of length so that we have a common standard. In this way, different surveyors will arrive at the same measure when using different measuring instruments (e.g., rods or cords). These measuring instruments are 'calibrated' to the adopted standard. For example, in ancient Egypt, the unit of length measure was the *cubit*. This unit corresponds to the common measure of the forearm (Imhausen 2016, 47; Rossi 2007, 59). For land-surveying, it seems that Egyptian used ropes having a standard length of 100 *cubits*, which were divided using knots placed at 1-*cubit* intervals (Imhausen 2016, 18; Rossi 2007, 154). Regarding the Greek length units, a widely used unit during Hellenistic times was the *foot* which was about 30-32 cm (Lewis 2004, xix).

Besides the measurement of the boundaries of fields, it was essential to calculate the areas of these fields. Agricultural plots in the ancient Near East usually had rectangular shapes. This led to the development of the so-called surveyors' formula, which gives good results when a shape approaches that of a rectangle. Let l_1, l_2, l_3, and l_4 be the sides of a quadrilateral field plot that are measured by a surveyor (e.g., l_1 and l_3 are the 'long sides', and l_2 and l_4 are the 'short sides'). The surveyors' formula gives for the area of the field plot the value $(l_1 + l_3)/2 \times (l_2 + l_4)/2$.

Here, we find an important and common feature of practical geometries. The lengths are measured (or taken to be measured), and as such are given in terms of a unit of measure. The area is calculated from these length measures and given in terms of a unit of area. For example, in the Old-Babylonian period, the main unit of length was the *rod* (approximately 6 m), and the unit of area was the *sar*, corresponding to one square *rod* (36 m^2) (Robson 2008, 294).

For different reasons, in practical geometries, some figures are widely used. These figures are clearly distinguished from all the other possible figures by naming them, even if definitions do not exist (contrary to the case of the pure geometry in Euclid's *Elements*). A good example is the circle. In ancient Mesopotamia, a circle, like other geometrical figures, was conceptualized in terms of its boundary. The circle was the shape enclosed in a circumference. In this case, both had the same name. A translation of the name might be "thing that curves" (Robson 2004, 20). The area of the circle was determined from the measure of the length of the circumference. It was given by the square of the length of the circumference divided by 12 (Robson 2004, 18). The circle was drawn using a specific instrument – the compass (Høyrup 2002, 105; Friberg 2007, 207). The compass made it possible to have a precisely drawn figure.

In terms of a model based on the hub-and-spoke theory, we can conceive of the concept of circle as relying heavily on a 'visual spoke' that represents aspects related to the visual shape of a circle, a 'verbal spoke' that codifies the name of the circle, and a 'praxis spoke' related to the drawing and measurements on the circle. Here, going beyond the spokes taken into account by Ralph and co-workers (see, e.g., Ralph et al. 2017), we propose another spoke related to measure-numbers, i.e., numbers that result directly from measurements in the case of length, or indirectly in the case of areas, and are addressed in terms of abstract symbols in the context of metrological systems (see figure 1).[4]

[4] In this work we departure from the idea that the spokes consist in modality-specific representations. In our case the symbolic number spoke is amodal. Also, it might be best to address the verbal spoke in terms of an amodal representation if we are to include more than simple words in it.

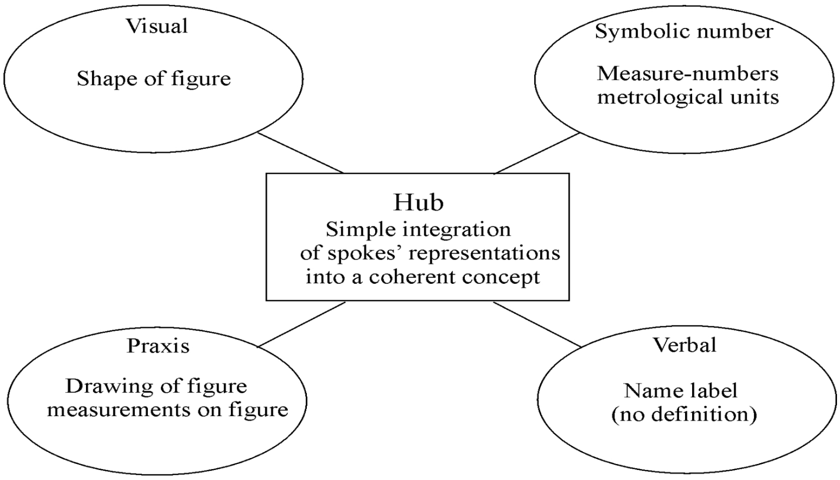

Figure 1. Hub-and-spoke model of the neural representation of geometric figure in practical geometry.

Another aspect shared by different practical geometries is the existence of written geometrical problems. These are couched in the terminology of practical geometry. But they are a somewhat different way of doing practical geometry. Being written problems that are disconnected from an immediate surveying activity, there are no actual measures taken into account in the problems. The measures are putative measures that could have been made according to the practice of practical geometry. This led, for example, to the adoption of conventional lengths in the problems. In Old Babylonian mathematical texts, all circumferences are taken to have the same standard length. This is so independently of the actual length of the circumference drawn with a compass (Friberg 2007, 207). So, while there are references to length measures in problems, these have not been actually measured, and neither corresponds to the actual measures of the drawn figure.

Whatever cognitive processes are at play during actual measurements these are not active during problem solving with conventionalized measures. One still uses whatever conceptual representation is at play when also making measurements, but only part of it. For example, the concept of line segment must include aspects related to the act of measuring. These are the procedures by which one attributes to the line segment a number (its length). In problems, we address these segments taking into account that there is a number associated with them – it is part of the concept –, but we disregard the use of a measuring instrument and the procedure by which the number is obtained. In our view, it might be the case that some aspects of the concept are only loosely taken into account.

One example of a geometrical problem is a Hellenistic geometrical problem from a papyrus written in demotic Egyptian in the third-century BCE (here, we take this problem not to have been influenced by the existing pure geometry). The statement of the problem is as follows: "A plot of land that <amounts to> 60 square cubits, [that is rec]tangular, the diagonal (being) 13 cubits. Now how many cubits does it make [to a side]?" (Cuomo 2001, 71). We have a rectangle and are asked to determine the length of its sides. These are calculated to have 12 and 5 *cubits* (Cuomo 2001, 71). This is a problem of practical geometry; however, we do not have an actual shape whose relevant lengths were measured. In fact, in a strictly practical practice, this problem is unfeasible. We can only calculate the area after measuring the lengths of the long side and the short side. In any case, as a didactic problem of practical geometry, the rectangle is conceived in terms of practical measures (in this case, the length of the diagonal) or values that are determined from practical measures (in this case, the area that is calculated from the lengths of the sides). In the problem, we consider a rectangle that is not measured nor even drawn, which has an area of 60 square *cubits*, a diagonal of 13 *cubits*, and sides that have (as calculated) 12 and 5 *cubits*.

These geometrical problems are known by us thanks to written accounts. But we know that many aspects of practical mathematics relied on oral accounts (see, e.g., Asper 2003). Much of this is lost to us. However, we might 'reconstruct' some of these practices.

It has been defended that a well-known passage in Plato's *Meno* about finding a square that doubles the area of another square is an example of Greek practical geometry (Valente 2020). Regarding this passage, it had already been argued by Saito (2018) that it is an example of a written rendering of the oral teaching and discussion of geometry in ancient Greece. It might be the closest we get to the ancient oral communication of geometry. Here, we will address this passage taking into account this two-fold aspect of it. Socrates helps a boy having recollections of his knowledge about a geometric figure – the square. We can re-frame it as an episode of geometrical teaching – a lecture on a particular geometrical subject. We may assume that the teacher begins the lecture by drawing a square on a wax tablet (Saito 2018, 928; Netz 1999, 15): "Tell me now, boy, you know that a square figure is like this? —I do. A square then is a figure in which all these four sides are equal? —Yes indeed." (Plato 1997, 881). The teacher draws two lines perpendiculars to the sides of the square and passing by the center: "And it also has these lines through the middle equal? —Yes." (Plato 1997, 881). The teacher then starts to address issues related to the area of the square:

And such a figure could be larger or smaller? —Certainly. If then this side were two feet, and this other side two feet, how many feet would the whole be? Consider it this way: if it were two feet this way, and only one foot that way, the figure would be once two feet? —Yes. But if it is two feet also that way, it would surely be twice two feet? —Yes. How many feet is twice two feet? Work it out and tell me. — Four, Socrates. (Plato 1997, 882)

For our purpose, we do not need to go further with the lecture. In this part, we see a feature of practical geometry that we have called attention to. We have a drawing of a figure, but we do not measure its sides. We take it to have measures that are useful in a didactic context. The teacher (Socrates) asks the student (the boy) to consider that the sides of the square measure two *feet*, and asks him to calculate the area, which the student does, arriving at the result of four (square *feet*). We have here a clear example of a written rendering of oral teaching of practical geometry. This is an example of the cognitive loosening we mentioned before. We perceive the square to have a particular dimension – it might look as having approximately one *foot*, for example. But we conceive it as having two *feet*. All this is made without an actual measurement procedure. The kind of conceptual representation at play is in some way 'vaguer' than when we pick up a rod and measure the sides of the square and determine that they have one *foot*.

3. Early pure geometry

Hippocrates' quadrature of lunules is taken to be the earliest evidence of Greek pure geometry (see, e.g., Netz 2004; Høyrup 2019). We know of Hippocrates' work by a text of Simplicius from the sixth-century CE. This text is based on two previous accounts, one by Alexander of Aphrodisias and the other by Eudemos. Hippocrates' work is believed to be from the early second half of the fifth century BCE. Written prose was rare; because of this Netz considers that "Hippocrates' treatise on the lunules could well be among the first treatises written in Greek mathematics" (Netz 2004, 247). Regarding Eudemos account, Netz realizes an exercise of reconstruction, trying to determine what in the text is closer to Hippocrates' original. Netz assumes that the text "should have two layers, one closer to Hippocrates' original, and another closer to late fourth century mathematics" (Netz 2004, 259). The main difference between these layers is the adoption or not of lettered diagrams, and the use of letters in the text to refers to parts of these diagrams. Netz's assessment is as follows:

While Eudemus has written his own text, he had before him Hippocrates' text and, even against his will, he would be likely to be influenced by this text. I suggest that he had in front of him an unlettered text, and that he had modernized it in the two more complicated quadratures. Even there, however, he let himself here and there reproduce the original structure of the argument, correlating it however with his own lettered diagram. Thus, some letters are redundant. (Netz 2004, 265)

Here, we want to suggest that the text might refer to an early written rendering of oral teaching by

Hippocrates. This is not that a bold suggestion. We know that Hippocrates taught about astronomy and geometry (Høyrup 2019, 160). Even if Netz takes Hippocrates to have written a treatise he also mentions the following:

If you wish to convey an argument which relies, among other things, on a diagram, then you must have at least the 'written', i.e., drawn diagram to accompany it […] Briefly, then, some use of writing, in the sense of a physical drawn object, is a necessary aspect of Greek mathematics. (Netz 2004, 246)

This is perfectly compatible with oral teaching of geometry like we have seen in the case of Plato's *Meno*. Hippocrates might have presented his arguments orally to his students accompanying them with the corresponding drawing. Independently that Hippocrates might have made a written rendering of his lectures on geometry we take these to be the main vehicle of his approach to geometry. We suggest that like in the case of practical geometry, in pure geometry there is also an oral practice which might well be the earliest.

In what follows we will consider the passage about the first quadrature as a rewriting of an initially written rendering of oral teaching. It is as follows:

(1) ... Therefore we shall discuss and quote them <=the quadratures> at length. (2) So he made his starting point by assuming, as the first among the things useful to the quadratures, that both the similar segments of the circles, and their bases in square, have the same ratio to each other. ((3) And this he proved by proving that the diameters have the same ratio, in square, as the circles). (4) This being shown to his satisfaction, he first proved by what method a quadrature was possible, of a lunule having a semicircle as its outer circumference. (5) He did this after he circumscribed a semicircle about a right-angled isosceles triangle and, about the base, <he drew> a segment of a circle, similar to those taken away by the joined <lines>. (6) And, the segment about the base being equal to both <segments> about the other <sides>, and adding as common the part of the triangle which is above the segment about the base, the lunule shall be equal to the triangle. (7) So the lunule, having been proved equal to the triangle, could be squared. (8) In this way, taking the outer circumference of a semicircle as the <outer circumference> of the lunule, he readily squared the lunule. (Netz 2004, 248-9)

Our purpose here is to determine what has changed in relation to practical geometry that leads us to say that here we are in the context of a pure geometric practice.

The first quadrature is that of a lunule whose outer circumference can be seen as a semicircle. Thinking in terms of an oral presentation, Hippocrates after mentioning the assumption to his audience might have drawn an isosceles triangle and using a compass drawn a semicircle circumscribing it (see figure 2).

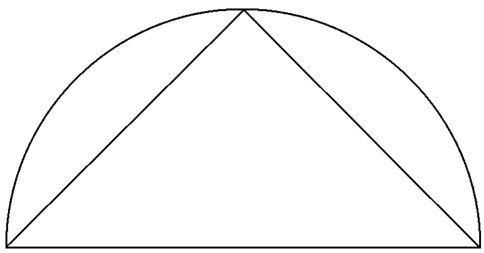

Figure 2. Initial drawing with a semicircle circumscribing an isosceles triangle.

Afterward, he might have completed a square based on the triangle and using a corner of the square as the center of a circle drawn an arc segment that is similar to the two formed previously. According to Netz's rendering of the text in English, "<he drew> a segment of the circle, similar to those taken away by the joined <lines>" (Netz 2004, 249) (see figure 3).

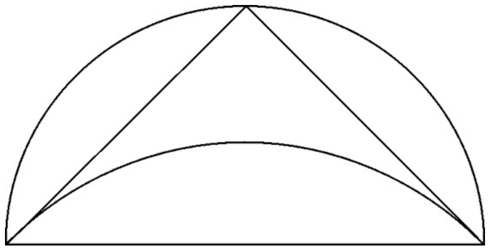

Figure 3. Completion of the drawing of the lunule.

As Høyrup mentions, Hippocrates' arguments have a 'single-level', directly based on the assumptions taken into account (Høyrup 2019, 179). Based on the presupposition mentioned at the beginning, Hippocrates simply mentions that the area of the circular figure about the base is equal to that of both circular figures about the other sides of the triangle. He then proceeds to add the area not included in either of these to each one of them. We have what we might call a visual operation in which we alternatively imagine each of the area addition operations (see figure 4, left and right).

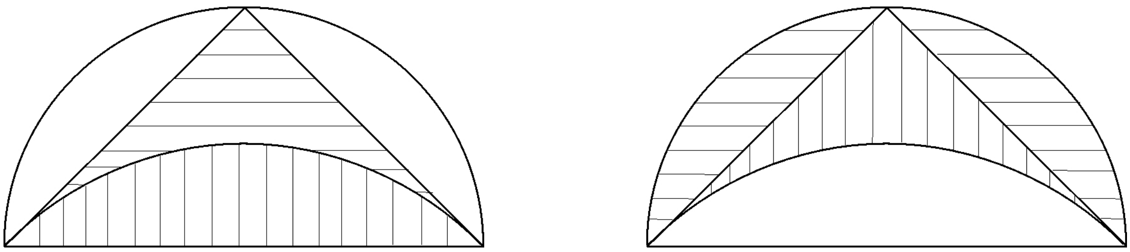

Figure 4. Two ways of doing the 'visual operation' of adding areas of figures.

Evidently, the area of the two figures is the same and so "the lunule, having been proved equal to the triangle, could be squared" (Netz 2004, 249). This addition of areas is something that we find in practical geometry. In fact, it is one of the basic elements of ancient Near Eastern geometry (Damerow 2016, 115). We have what we might call the principle of conservation of area.

What is it then that makes this example a case of early Greek pure geometry and not of Greek practical geometry? The key evidence that we are not engaged in a practical geometrical practice is the lack of reference to metrological units. They are completely absent. This corresponds to a crucial conceptual change that is at the crux of the reformulation of practical geometry as pure geometry. We assist to a perfectioning of the geometrical figures that leads to what we might call the exactification of lengths.

We have seen that within practical geometry we assist to a loosening of the conception of geometric figure by not taking into account directly the measurement practices. As we have seen, we can have geometric figures that we do not measure but conceive as having lengths that cannot be even approximately like those of the figure. We mentioned the case of circles in Old Babylonian problems that adopt a conventional length for the circumference or the case of the passage in Plato's *Meno*, in which the sides of the square are taken to have a particular measure not related to the actual measure of the drawing. In Hippocrates' pure geometry we have what we might call perfect figures. These are figures that look as having no irregularities and that if we were to measure them, we would find measure-numbers that are the same. That is, whatever small differences there are, they are invisible to the eye even when using the available measuring tools. This perfection of the geometric figure does not correspond to doing a more precise practical geometry. It is the opposite; not measuring the figures with better measuring techniques we take them to be perfect. In this way, e.g., the sides of a square are taken to be exactly equal. This is what we mean by the exactification of lengths. In this context, the lengths are exact and 'belong' to the figures. A length as a measure-number is the result of a measurement procedure in which, e.g., we put a measuring rod side by side with the side of a square and check that they are congruent. The length as a measure-number arises from this measurement procedure. In Hippocrates' case, we do not have this anymore. The sides of the square have lengths 'of their own', independently of whatever measurement we might make, and it is senseless to mention a metrological unit in this context.

As it is, early Greek pure geometry is the geometry of perfect figures (not yet of geometric objects). Like in the case of practical geometry, these are not explicitly defined. As mentioned by Høyrup, "there is not the slightest reference to a definition in the Eudemos text" (Høyrup 2019, 179). This has important consequences that we will address in the next section when comparing Hippocrates' pure geometry to that of the *Elements*. The main difference with the previous practical geometry is a further loosening of the concepts in relation to its more practical aspects related to measurements.

In terms of the very simple hub-and-spoke model of geometrical concept that we are using in this work, the main difference in relation to the concept of geometric figure of practical geometry is in the praxis and symbolic number spokes. In the praxis spoke the main features represented are related to the drawing of figures; however, there is still a representation not so much of particular measurement procedures as of the possibility of making measurements on the figure – there are 'traces' of the praxis of measuring. Regarding the symbolic number-magnitude spoke there is no encoding of measure-numbers or metrological units. Instead, we have a representation of length, which, taking into account the praxis spoke, is dissociated from any particular measurement procedure. The visual spoke is basically the same; it encodes the visual shape of a geometric figure. The verbal spoke is also the same. It encodes the 'label' for the figure. A 'higher order' change can be taken to occur in the hub that would enable to encode a conceptualization of figure as a perfect figure (see figure 5).

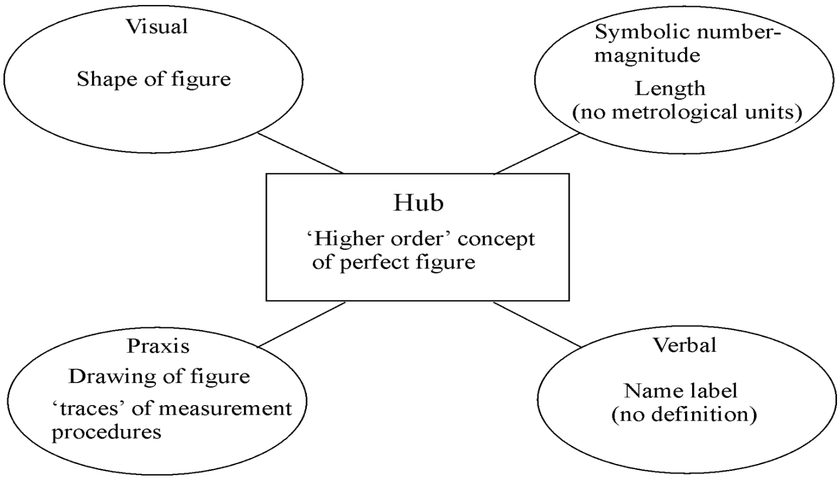

Figure 5. Hub-and-spoke model of the neural representation of geometric figure in Hippocrates' pure geometry.

3. The pure geometry in Euclid's *Elements*

We think that we can conceive the early stages of the geometry of the *Elements* in terms of lectures in which the teacher helps the student to memorize the enunciation of the proposition (the protasis). According to Saito the role of the protasis was to enable to memorize and refer to the propositions (Saito 2018, 928-9). It suffices to memorize the protasis and the corresponding diagram. Accordingly, "with the [unlettered] diagram in your memory, you can surely understand the protasis" (Saito 2018, 929). With these two elements, a practitioner can recover the whole of the content related to a proposition as manifested in the *Elements*. In Saito's view, "mathematical teaching was very probably performed directly and orally by drawing a diagram in front of pupils, and explaining it" (Saito 2018, 924). We can conceive of a lecture as starting with the teacher reciting an enunciation (protasis) of a proposition. He would then proceed, e.g., to construct a particular figure by starting to draw a diagram and pointing to the correctness of the successive steps until the final figure is constructed.

The diagram was probably drawn "on sand or a wax tablet" (Saito 2018, 928), and, importantly, the teacher would indicate the "points and other geometrical objects by finger". (Saito 2018, 928). In the context of an oral presentation, it is very unlikely that the teacher assigned labels to the diagram. As Netz shows, there is a close relationship between lettered diagrams and the type of written text adopted in the demonstrations of pure geometry (Netz 1999). According to him, "the introduction of letters as tools is a reflective use of literacy". (Netz 1999, 62). In this way, with the written form, "the lettered diagram is the tool which […] was made more central" (Netz 1999, 66). By contrast, the oral teaching as rendered in Plato's *Meno* reveals that "the diagram is not lettered, and the geometrical objects in it are referred to by the word 'this'" (Saito 2018, 932). According to Saito, "the written text of the *Elements* with lettered diagrams shows certainly a more developed stage" (Saito 2018, 932).

In a more organized lecture, the teacher would first present the necessary definitions, postulates, and common notions, or he would refer to them as needed. This would be the counterpart of Hippocrates' practice, where we would start by mentioning the presuppositions that are used in the demonstration.

Where do we find then a cognitively relevant difference between Hippocrates' pure geometry (PG_1) and Euclidean pure geometry (PG_2)? In our view, the difference is in the way we learn to look at the diagrams and use them. That there is a crucial change from the Hippocratic practice can be seen in the fact that in the Euclidean practice we have explicit linguistic definitions at play. For example, a point is defined as "that which has no part" (Euclid 1956, 153). Regarding lines, according to definitions 2 to 4, "A line is breadthless length. The extremities of a line are points. A straight line is a line which lies evenly with the points on itself" (Euclid 1956, 153).

According to Harari, the definition of point makes reference to the idea of measurement by 'contraposition': "a point is characterized as a non-measurable entity, as it has no parts that can measure it" (Harari 2003, 18). In the models of geometrical concept that we tentatively suggest in this work, when going from practical geometry to PG_1 there is a loosening in the praxis spoke of aspects related to measurements. The change from PG_1 to PG_2 might be seen as the 'overwriting' using the verbal spoke of any encoding related to a measurement praxis still existing in PG_2 (and 'inherited' from PG_1). This is achieved by extending the content of the verbal spoke that would consist now of a definition and not just a label. The changes in the spokes go hand and hand with changes in the hub that gives rise to a coherent concept. That this might be so can be seen, e.g., in the definitions related to lines. In PG_1 we already have a perfect figure but one that has a breadth; in PG_2 we move beyond this and conceive of something that is breadthless. This linguistic term points to something that is not even visualizable – it is an abstract object.

Taking this into account together with the definition of point we see that this definition goes beyond the "non-measurability" mentioned by Harari. The point is also an entity that goes beyond a visualized figure – as perfect as it might be. The verbal spoke helps to recreate the concept of perfect figure of PG_1 as the concept of abstract object of PG_2.

What does this imply regarding the diagrams that are drawn during the lectures? These are not conceived anymore as perfect figures. They are representations of geometrical objects as defined and instantiated following the postulates in the *Elements* (Valente 2020). This has consequences concerning how we address the diagrams during the oral lectures (or in the written treatises). According to Ferreirós:

The first definitions indeed suggest a *way of reading diagrams*, a perspective for seeing or conceiving what is implied by a diagram, and what is not. And this way of reading is not at all evident, especially if one previously knows only practical geometry. For the definitions and the reading that comes with them lead the practitioner to certain crucial idealizations. More importantly, the definitions suggest certain forms of response (and of indifference) to some aspects of the diagram: thus, the crossing of two drawn lines will be a (very small) planar region, but we are taught to disregard this and consider in the argumentation that one and only one point has thus been determined. (Ferreirós 2016, 144).

We suggest that the conceptual change leading from PG_1 to PG_2 might have arisen in a practice based on oral lectures with PG_1 where the above-mentioned way of looking into and reading diagrams arose.

That is, PG_2 does not lead to this particular way of attending to the diagrams; it would be the other way around. This way of attending to the diagrams (or a very similar precursor) would give rise to an early oral version of PG_2. This change would be made more explicit and stable by developing explicit linguistic definitions that help to stabilize the concepts, and further 'sedimented' by written treatises and a teaching practice that would rely more and more on these.

In terms of the simple model of geometrical concept that we are working with in the present paper, the verbal definition might be encoded in the verbal spoke that would be much more developed than in the cases of practical geometry and Hippocrates' pure geometry. In these cases, the content of the verbal spoke consisted only of the label used to name a geometric figure. Now we have a definition. The main difference would occur in how the hub re-represents the encoding in the spokes (see figure 6).

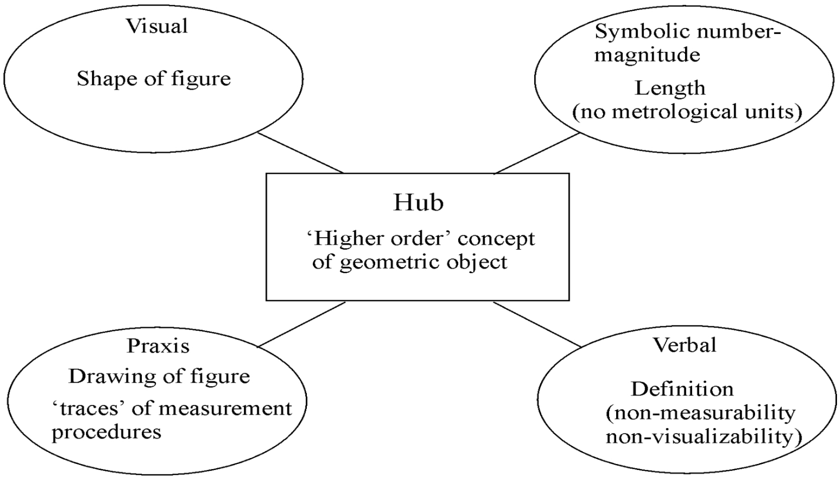

Figure 6. Hub-and-spoke model of the neural representation of geometric object in Euclidean pure geometry.

The representations in the visual, praxis and symbolic number-magnitude spokes are represented in a highly abstract way by taking into account the encoding of the verbal spoke. When we look at a figure there is a particular indifference and responsiveness to its features related to the verbal definition such that we re-conceive the figure in terms of an abstract geometric object and not as a perfect figure anymore (as mentioned, the figure becomes for us a representation of the geometric object). In this way, the verbal spoke becomes decisive in how the representations in the visual, praxis and symbolic number-magnitude spokes are interpreted and recombined in the hub giving rise to a 'higher order' representation of geometric abstract objects.

In our view, the hub-and-spoke models of the neural representation of geometric figure/object make more intelligible what a geometric object might be since we can relate its neural representation to that of a geometric figure in Hippocrates' pure geometry and in practical geometry.

5. Conclusions

The purpose of this work is to set forward a tentative model of the neural representation of abstract geometric objects. This model might be useful in relation to the development of a future theory of the semantic cognition underpinning pure geometry.

To develop the model consistent with the previous geometrical practices, we have considered a historically informed account of practical geometry. The objective was to provide a basic characterization of practical geometry. Taking into account these basic 'characteristics' we build a model of the neural concept representation of geometric figure in practical geometry using in a very simple way the hub-and-spoke theory of neural concept representation.

We then address pure geometry. Previous to the geometrical practice related to Euclid's *Elements*, there was a development of a pure geometry that was an intermediary stage between practical geometry and the pure geometry of the *Elements*.

In this work, we present a basic characterization of this geometrical practice as revealed in Hippocrates' work on the quadrature of lunules. In our view, this pure geometry deals not with abstract objects, but still with geometric figures – perfect figures. We provide a model of the neural concept representation of perfect figures again relying on the hub-and-spoke theory.

We then address what kind of neural representation of geometric concept we have in Euclidian pure geometry. For that, we reconstruct some aspects of the Euclidean practice taking into account how these are different from the corresponding aspects in the Hippocratic practice. Taking these differences into account together with the models of neural concept representation in practical geometry and Hippocrates' pure geometry we proposed a simple model of abstract geometric object. Comparing this model with the ones related to geometric figures makes more intelligible what it is to have a concept of abstract object and how this concept may underline the Euclidean practice.

References

Asper, M. (2003). The two cultures of mathematics in ancient Greece. In E. Robson, and J. Stedall (eds.), *The Oxford handbook of the history of mathematics* (pp. 107-132). Oxford: Oxford University Press.

Cuomo, S. (2001). *Ancient mathematics*. London: Routledge.

Euclid (1956). *The thirteen Books of the Elements* (second edition, Vols. I-III). Translated with introduction and commentary by Sir Thomas L. Heath, from the critical edition of Heiberg. New York: Dover Publications.

Eysenck, M. W., and Keane, M. T. (2020). *Cognitive psychology: a student's handbook*. London: Psychology Press.

Damerow, P. (2016) The impact of notation systems: from the practical knowledge of surveyors to Babylonian geometry. In M. Schemmel (ed.) *Spatial thinking and external representation: towards a historical epistemology of space* (pp. 93-119). Berlin: Edition Open Access.

Ferreirós, J. (2016). *Mathematical knowledge and the interplay of practices*. Princeton: Princeton University Press.

Ferreirós, J., and García-Pérez, M. J. (2020). Beyond natural geometry: on the nature of proto-geometry. *Philosophical Psychology* 33, 181-205.

Friberg, J. (2007). *A remarkable collection of Babylonian mathematical texts: manuscripts in the Schøyen collection cuneiform texts I*. New York: Springer.

Harari, O. (2003). The concept of Existence and the role of constructions in Euclid's *Elements*. *Archive for History of Exact Sciences* 57, 1-23.

Hohol, M., and Miłkowski, M. (2019). Cognitive artifacts for geometric reasoning. *Foundations of Science* 24, 657-680.

Høyrup, J. (2002). *Lengths, widths, surfaces: a portrait of Old Babylonian algebra and its kin*. New York: Springer.

Høyrup, J. (2019). Hippocrates of Chios – his *Elements* and his lunes: A critique of circular reasoning. *AIMS mathematics* 5, 158-184.

Imhausen, A. (2016). *Mathematics in ancient Egypt: a contextual history*. Princeton: Princeton University Press.

Lewis, M. J. T. (2004). *Surveying instruments of Greece and Rome*. Cambridge: Cambridge University Press.

Netz, R. (1999). *The shaping of deduction in Greek mathematics: a study in cognitive history*. Cambridge: Cambridge University Press.

Netz, R. (2004). Eudemus of Rhodes, Hippocrates of Chios and the earliest form of a Greek mathematical text. *Centaurus* 2004, 243-286.

Plato (1997). *Complete works*. Edited with introduction and notes by J. M. Cooper. Indianapolis: Hackett Publishing Company.

Ralph, M. A. L. (2014). Neurocognitive insights on conceptual knowledge and its breakdown. *Philosophical Transactions of the Royal Society B: Biological Sciences* 369, 20120392

Ralph, M. A. L., Jefferies, E., Patterson, K., and Rogers, T. T. (2017). The neural and computational bases of semantic cognition. *Nature Reviews Neuroscience* 18, 42-55.

Robson, E. (2004). Words and pictures: new light on Plimpton 322. In M. Anderson, V. Katz, and R. Wilson (eds.) *Sherlock Holmes in Babylon and other tales of mathematical history* (pp. 14-26). Washington: Mathematical Association of America.

Robson, E. (2008). *Mathematics in ancient Iraq*. Princeton: Princeton University Press.

Rossi, C. (2007). *Architecture and mathematics in ancient Egypt*. Cambridge: Cambridge University Press.

Saito, K. (2018). Diagrams and traces of oral teaching in Euclid's *Elements*: labels and references. Zentralblatt für Didaktik der Mathematik (ZDM) 50, 921-936.

Valente, M. B. (2020). From practical to pure geometry and back. *Revista Brasileira de História da Matemática* 20, 13-33.