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Fidelity and mistaken identity for symplectic quantum states

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Abstract The distinguishability between pairs of quantum states, as measured by quantum fidelity, is formulated on phase space. The fidelity is physically interpreted as the probability that the pair are mistaken for each other upon an measurement. The mathematical representation is based on the concept of symplectic capacity in symplectic topology. The fidelity is the absolute square of the complex-valued overlap between the symplectic capacities of the pair of states. The symplectic capacity for a given state, onto any conjugate plane of degrees of freedom, is postulated to be bounded from below by the Gromov width $h/2$. This generalize the Gibbs-Liouville theorem in classical mechanics, which state that the volume of a region of phase space is invariant under the Hamiltonian flow of the system, by constraining the shape of the flow. It is shown that for closed Hamiltonian systems, the Schrödinger equation is the mathematical representation for the conservation of fidelity.

Keywords Indeterminacy relation · Non-squeezing theorem · Symplectic capacity · Quantum fidelity · Born rule · Gibbs-Liouville theorem · Schrödinger equation

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1 Introduction

What is the key character of quantum mechanics which is at the heart of its distinction from classical mechanics? In this article, the subjective point of departure in addressing this question has to do with the ability of the observer to distinguish between the states of pairs of systems. In essence, the problem is to describe how close the states of pairs of quantum systems are on the space of states. There are well-established measures for this distance. Two such measures are trace distance and fidelity. In this article, the focus of attention is the fidelity. The novelty presented is not due to any new insights about the foundations of quantum mechanics. Rather, it lie in the exposition of the theory and the sequence of ideas which has been recast in a different order as compared to the familiar exposition. In the canonical approach [1–6], the space of states is the Hilbert space of complex-valued state vectors where Hermitian operators, representing observables, act on this space. The properties of state vectors and Hermitian operators are summarized in a set of axioms which set the foundation for the mathematical representation of canonical quantum mechanics. The dynamical evolution of the state vector is then encoded in the postulate that it should be unitary, represented in differential form by the Schrödinger equation. The connection between the state vector and experimental reality is then given by postulating the Born rule, which interpret the state vector as a complex-valued probability amplitude whose squared modulus give the probability for the system to occupy the specific state. From this mathematical structure, and the physical postulates, distance measures on Hilbert space, such as fidelity, can be clearly defined and their properties investigated [7]. An equivalent formulation, geometric quantum mechanics [8–21], is obtained by considering the projective Hilbert space, whose elements are the complex-valued rays, as the space of states, where observables are real-valued functions. This space is endowed with a metric, the Fubini-Study metric, which is symplectic, complex-valued and Riemannian, i.e. it is Kählerian. It is worth noting that the projective Hilbert space, unlike Hilbert space, is non-linear and has a symplectic structure, thus sharing the same key features as the classical phase space, but with the key difference that the metric has two additional compatible structures associated with it. The geometry of the projective Hilbert space, as described by the Fubini-Study metric, then allow for a clear discussion on distance between states of pairs of quantum systems [21].

In this article, the space of states is the phase space of generalized coordinates and momenta, just as in classical mechanics, with the key difference that an uncertainty structure, as defined by the indeterminacy relation, is added. This structure can be mathematically represented by concepts in symplectic topology, specifically the notion of symplectic capacity [22]. The mathematics of symplectic capacities thus replace the familiar mathematical axioms of the Hilbert space formulation. The physical postulates presented in this article involve the notion of distinguishability, as measured by fidelity. The first postulate state that the ability of the observer to distinguish between the states of pairs of systems is finite. This is equivalently stated by the indeterminacy relation as expressed in terms of symplectic capacities. The second postulate state that, due to the finite distinguishability, saturated quantum states are complex-valued. This can equivalently be stated in terms of symplectic capacities, by saying that the overlap between pairs of symplectic capacities is complex-valued. The third, and final, postulate state that the distinguishability is conserved in time for closed Hamiltonian systems. This postulate is shown to lead to unitary evolution and the Schrödinger equation for the overlap between pairs of symplectic capacities.

2 Finite distinguishability

In classical mechanics, it is assumed that the state of the system can be specified with infinite precision. There is no uncertainty in the state. An observer is infinitely able to specify the physical degrees of freedom for the state. Consider any given pair of classical systems. The states of the systems at some time $t = 0$ are given by ψ and ϕ. This define the initial condition for the pair of systems. Due to the infinite ability of the observer to distinguish between states, the pair of systems can either be identified to be identical, i.e. $\psi = \phi$, or, completely distinct, i.e. $\psi \neq \phi$. These are the only two possibilities. Since states in classical mechanics are represented as infinitesimal points on phase space, the systems are identical if the points coincide and distinct if there is a finite distance between them. The Gibbs-Liouville theorem state that the physical distinctions between the pair of systems is conserved in time [23]. In other words, if ψ and ϕ are initially distinguishable, and their distinctions conserved, then their evolutionary paths are not allowed to diverge or converge anywhere on phase space, such that they would become indistinguishable, see Fig.1. The Hamiltonian flow of a classical system is thus incompressible and the volume of a given set of states is conserved in time.

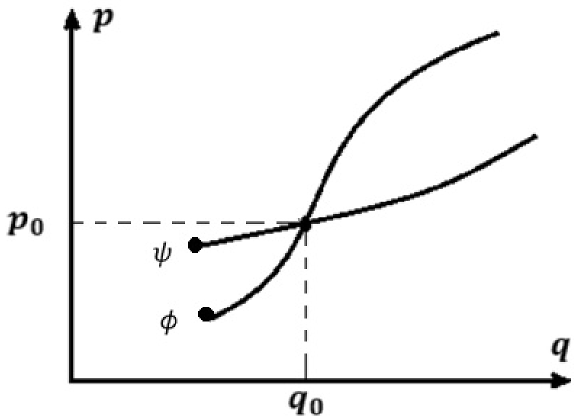

Fig. 1 The pair of initial conditions ψ and ϕ, defining the distinct states of a pair of classical systems, converge at the point (q_0, p_0), thus becoming indistinguishable. This violate the Gibbs-Liouville theorem and is therefore not allowed in classical mechanics.

In statistical mechanics, the observer is not infinitely able to specify the state of the system. This is not due to an inherent property of the system. It is entirely due to the difficulty of the observer to keep perfect track of the large number of degrees of freedom. Due to the uncertainty in the state of the system, the ability of the observer to distinguish between states decrease exponentially over time, as stated by the second law of thermodynamics, until the system has reached statistical equilibrium where all states are indistinguishable [24].

The classical assumption on the infinite ability of the observer to distinguish between pair of states is in this article seen as impossible. Instead, the following postulate is put forth[1]:

Postulate 1: Finite distinguishability
There exist a universal finite upper bound on the ability of the observer to distinguish between the states of any given pair of systems.

To turn this postulate into a more mathematically precise statement, it is necessary to clarify what is meant by the notion of state.

3 Squeezed coherent states

Due to finite distinguishability, it is impossible to physically define, in the sense of observation, the notion of the state as given by an infinitesimal point. In other words, the geometry of phase space is pointless. To obtain a picture of the notion of state on a pointless phase space, consider an $N-$particle system, in d spatial dimensions, at some given time $t = 0$. Let it be assumed that the state of the system, denoted by ψ, is known, at this time, with maximum precision. Such a state is referred to as being saturated. It is further assumed that all conjugate pairs of degrees of freedom for the system, i.e. the coordinate and momenta pairs (q_k, p_k), with $k \in \{1, 2, ..., n\}$ where $n = d \cdot N$, are known to the same level of maximum precision. These symmetric states are the coherent states [27] [28] [29]. The state of the system at time $t = 0$, $\psi(t = 0)$, occupy the $2n$-dimensional ball $B(\epsilon)$ defined by

$$\sum_{k=1}^{n} \left\{ (q_k - a_k)^2 + (p_k - b_k)^2 \right\} = \epsilon^2, \qquad (1)$$

with radius ϵ and origin (a_k, b_k). This define the initial condition of the system. Due to the spherical symmetry in the initial condition, the orthogonally projected area $A^k_\psi(t = 0)$ of the ball onto any given conjugate pair (q_k, p_k), see Fig.2, is given by

$$A^k_\psi(t = 0) = \pi\epsilon^2 \quad \forall k \in \{1, 2, ..., n\}. \qquad (2)$$

The projected area $\pi\epsilon^2$ represent the maximum level of precision by which the state of the system can be known for each conjugate pair. In other words, the radius ϵ quantify the greatest resolution available to the observer. Upon the identification of the resolution ϵ with the Planck constant h according to

$$\epsilon \equiv \sqrt{\frac{h}{2\pi}}, \qquad (3)$$

the minimum uncertainty area of the projection of the ball $B(\sqrt{h/2\pi})$ onto the conjugate plane k is given by

$$A^k_\psi(t = 0) = \frac{h}{2} \quad \forall k \in \{1, 2, ..., n\}. \qquad (4)$$

The ball $B(\sqrt{h/2\pi})$ is thus a representation for the coherent state ψ. More generally, the saturated initial condition ψ can have its minimum uncertainty non-symmetrically distributed between the position and momenta. These states are the squeezed coherent states [29] [30] [31] [32]. In the limit that $h \to 0$, the coherent, squeezed or not, state ψ collapse into an infinitesimal point. This is the classical approximation, valid at large scales relative to $h/2$.

At scales lower than $h/2$, i.e. in the interior of the ball, the notion of state loses its physical meaning due to the impossibility of the observer to gain additional information about the physical distinctions characterizing the system. In other words, the position and momenta degrees of freedom cannot be considered as real-valued measurable quantities. This statement is represented by the following postulate:

Postulate 2: Complex-valued coherent states
The interior of the coherent state, squeezed or not, as represented by the ball $B(\sqrt{h/2\pi})$, is complex-valued.

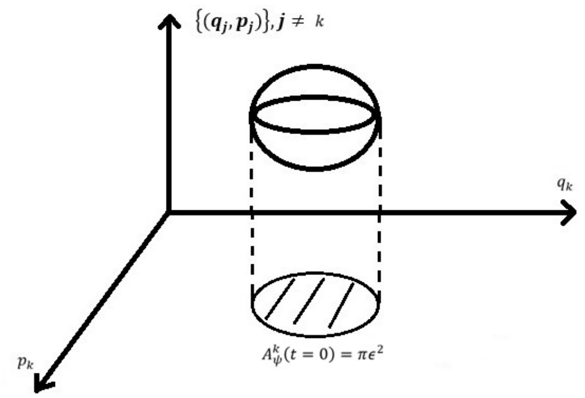

Fig. 2 The projected area A^k_ψ of the phase-space ball $B(\epsilon)$ onto the conjugate pair (q_k, p_k), at time $t = 0$, is given by the minimum uncertainty area $\pi\epsilon^2$.

4 Indeterminacy relation

The coherent states, squeezed or not, are the states which can be distinguished to greatest resolution. Therefore, the projected area $A^k_\xi(t)$ for an arbitrary state ξ,

[1] Related ideas, postulating that the information content of a quantum system is finite, in the sense that the experimenter cannot obtain definite answers to all questions posed about the system, has been proposed by e.g. Zeilinger and Brukner [25] [26], where the aim is to describe quantum physics as an elementary theory of information.

at any given time t, onto the conjugate plane (q_k, p_k), is either equal to, or greater than $h/2$, i.e.

$$A_\xi^k(t) \geq \frac{h}{2} \quad \forall k \in \{1, 2, ..., n\}. \tag{5}$$

This is the indeterminacy relation on phase space. It states that the shape of the state ξ cannot deform during its Hamiltonian flow in such a way that it breach the lower bound as defined by $h/2$. In the language of symplectic topology, the projected area A_ξ^k is referred to as the symplectic capacity c_ξ^k and its minimum value, i.e. $h/2$, as the Gromov width c_G [33]. The arbitrary state ξ is thus mathematically represented by the set of symplectic capacities $\left\{c_\xi^1, ..., c_\xi^k, ..., c_\xi^n\right\}$. The indeterminacy relation thus state that the symplectic capacities of an arbitrary state ξ cannot deform during its Hamiltonian flow in such a way that its value gets smaller than the Gromov width[2], i.e.

$$c_\xi^k(t) \geq c_G \quad \forall k \in \{1, 2, ..., n\}. \tag{6}$$

This indeterminacy relation is mathematically equivalent to the Robertson-Schrödinger indeterminacy relation[3] [34]. The mathematical proof of the impossibility of squeezing the state ξ into a smaller symplectic capacity than $h/2$ at any given time, as the system experience an Hamiltonian flow, was given by Mikhail Gromov in 1985 [35] and is referred to as Gromov's non-squeezing theorem.

The key character of the quantum Hamiltonian flow, contrasting its classical approximation, is thus the constraint on the shape of the flow as encoded in the indeterminacy relation.

In conclusion, the postulate on finite distinguishability can equivalently be stated as an indeterminacy relation on phase space, in the language of symplectic topology, taking into account the identification of the greatest possible resolution ϵ with the Planck constant, as follows:

Postulate 1': Indeterminacy relation
The symplectic capacity c_ξ^k, for any given state ξ, is bounded from below by the Gromov width $h/2$ for all conjugate planes (q_k, p_k).

This postulate is in direct contradiction with the Gibbs-Liouville theorem. The Gibbs-Liouville theorem state that any initial region on phase space can deform continuously in any conceivable way as long as its volume do not change [36] [37] [38]. Thus, according to the Gibbs-Liouville theorem, it is possible to deform the arbitrary state ξ in such a way that the symplectic capacity onto some given subset of conjugate pairs is smaller than the Gromov width $h/2$, as long as it is balanced by an increase in the symplectic capacity of another subset of conjugate pairs, keeping the volume invariant. Thus, classical mechanics, whose dynamics on phase space is governed by the Gibbs-Liouville theorem, violate the indeterminacy relation and can only be considered a valid approximation when the system is observed at scales much larger than $h/2$.

5 State overlap and distinguishability

Consider any given pair of systems. The pair of saturated states, ψ and ϕ, at some time $t = 0$, define the initial conditions for the pair of systems. The finite size of the pair of states, as represented by their balls B_ψ and B_ϕ, allow for the possibility that they have a non-zero overlap Γ, see Fig.3. This imply that there exist a subset of symplectic capacities, e.g. c_ψ^k and c_ϕ^k, which have a non-zero overlap, $\Omega_k(\psi, \phi)$. In other words, there might be a non-zero degree of indistinguishability between the pair of states ψ and ϕ if they are sufficiently close to each other. Of course, if the pair of states have zero overlap, for all conjugate planes, then they are completely distinguishable. The total area of overlap,

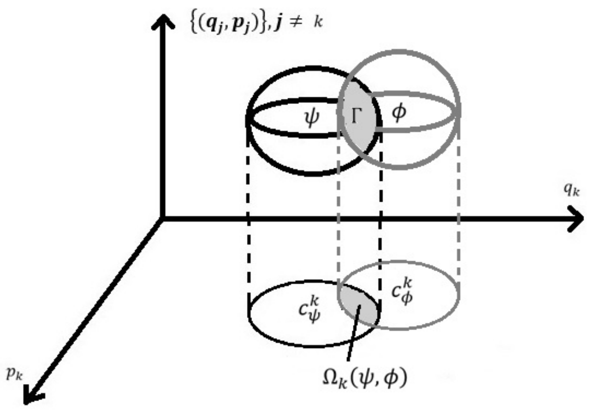

Fig. 3 Given that the state overlap Γ between the saturated balls B_ψ and B_ϕ is complex-valued, the overlap $\Omega_k(\psi, \phi)$ between the symplectic capacities c_ψ^k and c_ϕ^k must also be complex-valued.

$\Omega(\psi, \phi)$, is given by the linear sum of the contributions

[2] It is important to emphasize that there is no restriction on the symplectic capacity of the state onto a non-conjugate pair of degrees of freedom, i.e. the symplectic capacities for (q_i, q_j), (p_i, p_j) or (q_i, p_j), $\forall i \neq j$, can have arbitrarily small sizes.

[3] The Robertson-Schrödinger indeterminacy relation [30, 39–42] generalize the Heisenberg indeterminacy relation [43] [44] due to its inclusion of the covariance between observables.

Ω_k, for all $k \in \{1, 2, ..., n\}$, i.e.

$$\Omega(\psi, \phi) = \sum_{k=1}^{n} \Omega_k(\psi, \phi). \tag{7}$$

The summation is linear since the $n-$dimensional set of conjugate planes are linearly independent.

Since the overlap $\Omega_k(\psi, \phi)$ is the projection of the state overlap Γ onto the conjugate plane (q_k, p_k), the postulate on complex-valued coherent states imply the following statement:

Postulate 2': Complex-valued overlap
The overlap $\Omega_k(\psi, \phi)$ between the symplectic capacities c^k_ψ and c^k_ϕ of the pair of saturated states ψ and ϕ is complex-valued.

The direct consequence of this postulate is that the overlap between pairs of non-saturated states, i.e. states whose symplectic capacities are larger than the Gromov width, is also complex-valued. To justify this statement consider the arbitrary pair of non-saturated states ξ and η whose overlap is $\Omega_k(\xi, \eta)$, see Fig. 4. Independent on the size of their symplectic capacities, and their overlap, it is always possible to define a pair of saturated states ψ and ϕ whose overlap $\Omega_k(\psi, \phi)$ lie within the overlap $\Omega_k(\xi, \eta)$. Thus, the postulate that $\Omega_k(\psi, \phi)$ is complex-valued necessarily imply that the overlap $\Omega_k(\xi, \eta)$, of which $\Omega_k(\psi, \phi)$ is a part, is also complex-valued. Thus,

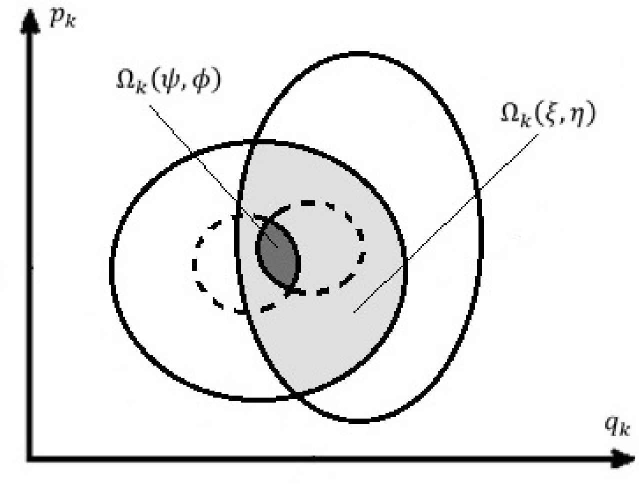

Fig. 4 Given that the overlap $\Omega_k(\psi, \phi)$ between pairs of saturated states ψ and ϕ is complex-valued, the overlap $\Omega_k(\xi, \eta)$ between arbitrary pairs of non-saturated states ξ and η is also complex-valued.

the postulate can be rewritten with respect to arbitrary states ξ and η as follows:

Postulate 2": Complex-valued overlap
The overlap $\Omega_k(\xi, \eta)$ between the symplectic capacities c^k_ξ and c^k_η of the arbitrary pair of states ξ and η is complex-valued.

6 Fidelity as measure of distinguishability

Due to the complex-valuedness of the state overlap, it cannot serve as a physical measure for the degree of distinguishability between arbitrary pairs of systems, whose states are given by ξ and η, at some given time t. For the purpose of constructing a useful physical measure, the function $F(\Omega)$ is introduced, and required to satisfy the following set of conditions:

i It is real-valued.
ii It is non-negative, i.e. $F(\Omega) \geq 0$.
iii It is unitless.
iv $F(\Omega) = 0$ iff $\Omega = 0$. The pair ξ and η are completely distinguishable.
v $F(\Omega)$=1 iff $\Omega_k = c^k_\xi$ and/or[4] $\Omega_k = c^k_\eta$ for all $k \in \{1, 2, ..., n\}$. The pair ξ and η are completely indistinguishable.

The conditions (ii) and (v) correspond to the first and second, respectively, Kolmogorov axioms of a probability measure [45]. The physical interpretation[5] of $F(\Omega)$ is thus that it give the probability that the pair of systems, occupying states ξ and η, are mistaken for each other by the observer upon a measurement at the given time t. It is a quantitative measure for the belief of the observer about the state of the system, rather than a description of the state of the system itself. This point of view on the character of probability originate from the works of Cox [47] [48] and, when applied to statistical mechanics, Jaynes [49] [50]. The probability presented here is the symplectic representation[6] of the quantity known in quantum information theory as the quantum fidelity between pairs of pure states [51] [21]. The overlap lacks a physical interpretation due to it being non-observable. However, its modulus is the symplectic representation of the square root quantum fidelity [7].

The most obvious candidate for the fidelity, satisfying all the imposed conditions, is given by the squared modulus of the overlap, i.e.

$$F(\Omega) = |\Omega(\xi, \eta)|^2. \tag{8}$$

[4] For saturated states, ψ and ϕ, $\Omega_k = c^k_\psi = c^k_\phi = h/2$. For arbitrary non-saturated states, ξ and η, the possibilities are that $c^k_\xi = c^k_\eta$ or that one of the symplectic capacities are enclosed by the other.

[5] It also has been interpreted as the probability associated with the process that the states transition into each other, and thus referred to as the transition probability [46].

[6] That is, a representation in terms of symplectic capacities.

This is interpreted as the symplectic representation of the Born rule [52] [53].

7 Conservation of fidelity

Considering that the Gibbs-Liouville theorem is a statement on the conservation of distinguishability between pairs of classical states [23], its generalization to the pointless geometry of phase space in quantum mechanics is proposed to be given by the following statement:

Postulate 3: Conservation of quantum fidelity
The distinguishability between an arbitrary pair of quantum states, as measured by quantum fidelity, is conserved in time.

Thus, the fidelities between the pair of states evaluated at arbitrary different times, e.g. t_0 and t, are equal, i.e.

$$F(\Omega)|_t = F(\Omega)|_{t_0}, \tag{9}$$

or, alternatively,

$$\frac{F(\Omega)|_t}{F(\Omega)|_{t_0}} = 1. \tag{10}$$

Due to the Born rule, the conservation of fidelity can equivalently be stated in terms of the overlaps as

$$\frac{\Omega^*\Omega|_t}{\Omega^*\Omega|_{t_0}} = 1. \tag{11}$$

The infinitesimal flow of the overlap, from the initial time t_0 to the final time $t = t_0 + \delta t$, to first order in the infinitesimal time step δt, is given by

$$\Omega|_{t_0} \to \Omega|_t = \Omega|_{t_0} - \delta\Omega|_{t_0,t} \cdot \Omega|_{t_0}, \tag{12}$$

or, alternatively,

$$\frac{\Omega|_t}{\Omega|_{t_0}} = 1 - \delta\Omega|_{t_0,t}, \tag{13}$$

where $\delta\Omega|_{t_0,t}$ represent the infinitesimal change in the overlap, during the time δt, relative to the initial overlap $\Omega|_{t_0}$. The flow of the complex-conjugated overlap is given by

$$\frac{\Omega^*|_t}{\Omega^*|_{t_0}} = 1 - \delta\Omega^*|_{t_0,t} \tag{14}$$

which thus gives that

$$\begin{aligned}\frac{\Omega^*\Omega|_t}{\Omega^*\Omega|_{t_0}} &= (1 - \delta\Omega^*|_{t_0,t})\,(1 - \delta\Omega|_{t_0,t}) \\ &= 1 - \delta\Omega^*|_{t_0,t} - \delta\Omega|_{t_0,t} + \delta\Omega^*|_{t_0,t} \cdot \delta\Omega|_{t_0,t} \\ &\approx 1 - \delta\Omega^*|_{t_0,t} - \delta\Omega|_{t_0,t},\end{aligned} \tag{15}$$

where the second-order term has been dropped. If quantum fidelity is conserved, then it must be that

$$\delta\Omega^*|_{t_0,t} + \delta\Omega|_{t_0,t} = 0. \tag{16}$$

This is only possible if $\delta\Omega|_{t_0,t}$ is imaginary-valued. Furthermore, since the pair of systems is assumed to be closed, it has no explicit dependence on time, i.e.

$$\delta\Omega|_{t_0,t} \sim i\delta t \cdot \mathcal{H}, \tag{17}$$

where the phase-space function $\mathcal{H}$ is explicitly time-independent and real-valued with the units of energy. It is the Hamiltonian. Thus, in quantum mechanics, the Hamiltonian generate the flow in time of the overlap between pairs of systems. Furthermore, due to the indeterminacy relation, the Hamiltonian cannot quantify changes in the overlap with infinite precision. The Hamiltonian can therefore only be defined in units of the greatest possible resolution $\epsilon = \sqrt{h/2\pi}$. However, since the infinitesimal change in the overlap must be unitless, the measure of resolution that enter into its definition must be $\epsilon^2 = h/2\pi$. Thus, in conclusion, the infinitesimal flow of the overlap is given by

$$\frac{\Omega|_t}{\Omega|_{t_0}} = 1 - i\frac{\delta t \cdot \mathcal{H}}{h/2\pi}. \tag{18}$$

Extending over arbitrarily many time-steps m, such that $m \cdot \delta t = t - t_0$, the flow in time of the overlap is determined by

$$\begin{aligned}\frac{\Omega|_t}{\Omega|_{t_0}} &= \lim_{m\to\infty}\left(1 - i\frac{(t-t_0)}{m}\frac{\mathcal{H}}{h/2\pi}\right)^m & (19)\\ &= e^{2\pi i\mathcal{H}\cdot(t-t_0)/h}. & (20)\end{aligned}$$

The relation between overlaps at different times is commonly denoted by $U(t, t_0)$, i.e.

$$U(t,t_0) \equiv \frac{\Omega|_t}{\Omega|_{t_0}} = e^{2\pi i\mathcal{H}\cdot(t-t_0)/h}, \tag{21}$$

and referred to as the time-evolution operator. It is unitary, i.e.

$$U^*U = 1. \tag{22}$$

The notion of unitarity is thus just a restatement, by the application of the Born rule, of the conservation of quantum fidelity.

8 Schrödinger's equation

Eq. 18 can be rewritten as a differential equation, i.e.

$$i\frac{h}{2\pi}\frac{\Omega|_t - \Omega|_{t_0}}{\delta t} = \mathcal{H}\Omega|_{t_0}, \tag{23}$$

which becomes

$$i\frac{h}{2\pi}\frac{\partial\Omega(t)}{\partial t} = \mathcal{H}\Omega(t). \tag{24}$$

This is the Schrödinger equation for the overlap in the symplectic representation. It is a direct consequence of conservation of quantum fidelity. This is the analog of the situation in classical mechanics, where the Hamilton equations are the direct consequences of the Gibbs-Liouville theorem [23]. Thus, the Schrödinger equation is a representation for the quantum generalization of the Gibbs-Liouville theorem.

The Schrödinger equation predict exactly the value of the overlap at some given time, if the initial condition on the overlap is known. This displays the key difference between the notion of determinism in classical and quantum mechanics. In classical mechanics, the exact state of the system is predictable at any given time, given the initial condition. In quantum mechanics, the state cannot be predicted with absolute certainty. It is only the overlap between the symplectic capacities of pairs of states which is exactly predictable, given the initial overlap.

9 Conclusion

Quantum mechanics is a probabilistic theory which generalize the notion of distinguishability between pairs of states. In classical mechanics, the pair can either be completely distinct or identical. In quantum mechanics, the two classical possibilities are the extremum values of the quantum fidelity, F, which is a physical measure for distinguishability, i.e. $F = 0$ when they are completely distinct and $F = 1$ when identical. The key difference is thus that quantum mechanics allow for the possibility that the distinction between pairs of states is given by any value in-between $F = 0$ and $F = 1$. The quantum fidelity can be physically interpreted as the probability that pairs of states, upon measurement, are mistaken for each other. This type of mistaken identity is impossible in classical mechanics[7].

[7] The case of mistaken identity in statistical mechanics is due to the ignorance of the observer, whereas in quantum mechanics it is due to the constraint of finite distinguishability imposed by Nature on the observer.

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