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Quantum superpositions and the measurement problem

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Abstract The measurement problem is addressed from the viewpoint that it is the distinguishability between the state preparation and its quantum ensemble, i.e. the set of states with which it has a non-zero overlap, that is at the heart of the difference between classical and quantum measurements. The measure for the degree of distinguishability between pairs of quantum states, i.e. the quantum fidelity, is for this purpose generalized, by the application of the superposition principle, to the setting where there exists an arbitrary-dimensional quantum ensemble.

Keywords Quantum ensemble · Ensemble interpretation · Superposition principle · Measurement problem · Schrödinger's cat · Quantum fidelity · Symplectic capacity

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Contents

1 Introduction

In a previous article by the author, non-relativistic quantum mechanics were recast in a slightly different form, by taking the degree of distinguishability between the states of pairs of systems as the subjective point of departure [1]. The proposed pair of postulates, on the distinguishability, were:

Postulate 1: Finite distinguishability
There exist a universal finite upper bound on the ability of the observer to distinguish between the states of any given pair of systems.

Postulate 2: Conservation of distinguishability
The distinguishability between the states of an arbitrary closed pair of Hamiltonian systems is conserved in time.

These postulates are independent on the specific mathematical formulation of quantum mechanics. However, they were subsequently expressed in the language of symplectic topology, with the notion of symplectic capacity [2] [3] playing the key role in stating the indeterminacy relation [4]. The first postulate led to the introduction of the concept of quantum fidelity, which was physically interpreted as the probability that the pair of states are mistaken for each other[1][2]. The second

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[1] Thus, the quantum fidelity quantify the belief of the observer about the state of the system in between measurements, rather than a description of the state of the system itself. This point of view on the interpretation of probability has been greatly influenced by the works of Cox [5] [6] and, when applied to statistical mechanics, Jaynes [7] [8]. However, the idea that the quantum state is a statement on the knowledge possessed by the observer about the state of the system, rather than an actual description of the state itself, dates back to the early days of quantum mechanics, see e.g. the review [9] and references therein.

[2] The quantum fidelity is also commonly interpreted as the probability associated with the transition between the pair of states [10]. However, in this article, the interpretation is not

postulate were shown to lead straightforwardly to the Schrödinger equation.

In this article, the concept of quantum fidelity is extended to measure the degree of distinguishability between the state preparation and a set of states, of arbitrary dimension, which has a non-zero overlap with the state preparation, referred to as the quantum ensemble. By doing so, the superposition principle is viewed from a new perspective and as such it is applied to the measurement problem and Schrödinger's cat.

2 Fidelity and its conservation

As a quick review of quantum fidelity, as presented in [1], consider an arbitrary pair of closed systems. Their initial conditions, at some given time $t = 0$, are given by the pair of symplectic states[3] ξ and η. If the pair has a non-zero overlap, i.e. $\Omega(\xi, \eta) = \sum_k \Omega_k(\xi, \eta) \neq 0$, where $\Omega_k(\xi, \eta)$ is the overlap between the symplectic capacities c_ξ^k and c_η^k onto the conjugate plane (q_k, p_k), see Fig. 1, then the fidelity between them is defined by the Born rule

$$F(\Omega(\xi, \eta)) = |\Omega(\xi, \eta)|^2 . \tag{1}$$

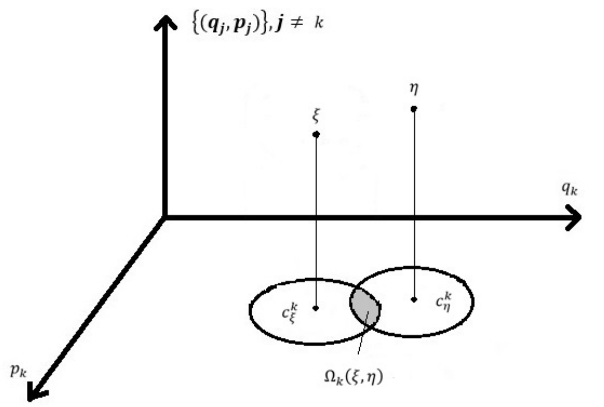

Fig. 1 The pair of systems, with initial conditions ξ and η, has a non-zero overlap $\Omega_k(\xi, \eta)$, onto the conjugate plane (q_k, p_k), between their symplectic capacities c_ξ^k and c_η^k.

The postulate that the fidelity is conserved in time is mathematically represented by the Schrödinger equation, i.e.

$$i \frac{h}{2\pi} \frac{\partial \Omega(\xi, \eta)}{\partial t} = \mathcal{H} \Omega(\xi, \eta). \tag{2}$$

Thus, given the initial conditions, the Schrödinger equation predict exactly the overlap at any other time during the Hamiltonian flow of the pair of systems.

3 Ensemble of similar quantum states

Consider an ensemble of closed systems. Each member has been submitted to the same, arbitrary, state preparation ξ at the same time $t = 0$. This define the initial conditions for the members of the ensemble. Alternatively, a single system could be considered. The requirement is that it is observed in many successive trials and before each new measurement it is resubmitted to the same state preparation ξ[4]. In classical mechanics, the initial condition can, by assumption, be prepared with infinite precision. The members of the ensemble are thus identical copies of each other. Identical measurements on the identical members will yield the same experimental outcomes. In quantum mechanics, due to the non-zero overlap between ξ and η, this is no longer the situation. The state preparation ξ might be mistaken for the state η by the observer upon a measurement. By this, it is meant that eventhough the system is prepared in the state ξ, it might occupy the state η, due to their non-zero overlap. In other words, while the observer thought the system was prepared in ξ it might have been prepared in η. Therefore, when the measurement is performed, the system might be found in the state η rather than the state preparation ξ. In this sense, the two states are mistaken for each other, from the perspective of the observer. Thus, the members are not necessarily identical. Identical measurements on the ensemble will not necessarily yield the same results. More generally, consider the situation when the state preparation ξ has a non-zero overlap with each member of the $M-$ dimensional set of states $\{\eta_1, ..., \eta_j, ..., \eta_M\}$, i.e. $\Omega_k(\xi, \eta_j) \neq 0, \ \forall j \in \{1, 2, ..., M\}$, see Fig.2. Such a set will be referred to as the quantum ensemble associated with the state preparation ξ. Then, the initial condition ξ might be mistaken for any given state η_j in the quantum ensemble. The members of the ensemble of systems, all of which have been submitted to the same state preparation ξ at the same time, are thus not necessarily identical. However, they are similar, in the sense that they all have a non-zero overlap with ξ. Upon measurement, there will be a statistical distribution for the states in which the systems are found, depending on the size of the overlap between ξ and the members of the quantum ensemble. If e.g. the overlaps are all equal,

that there is an actual transition between the pair of states. Rather it is a question of mistaken identity.

[3] By this, it is meant the representation of quantum states in terms of their symplectic capacities, see [1] and references therein.

[4] The typical single-system experiment is the double-slit experiment, where e.g. individual electrons are subsequently, and independently from each other, submitted to the same initial condition.

i.e. $\Omega_k(\xi,\eta_j)=\Omega_k(\xi,\eta_i)$, $\forall i\neq j\in\{1,2,...,M\}$, then it is expected, in the limit of a very large ensemble of systems, that all states in the quantum ensemble will appear an equal number of times.

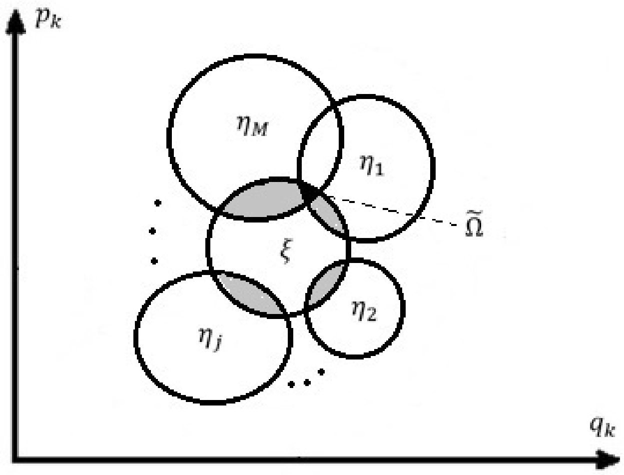

Fig. 2 The $M-$dimensional quantum ensemble associated with the state preparation ξ is defined by the set of all states $\{\eta_1,...,\eta_j,...,\eta_M\}$ which have a non-zero overlap with ξ. The black shaded area, denoted by $\tilde{\Omega}$, i.e. the mutual overlap between ξ, η_1 and η_M, is physically constrained to be zero.

In conclusion, the state preparation ξ, due to the possibility that it is mistaken for any other state in its quantum ensemble $\{\eta_1,...,\eta_j,...,\eta_M\}$, cannot be understood, from the observational point of view, to describe a unique and specific state of an individual system. It describes an ensemble of systems which are similar, in the sense that their states have non-zero overlaps with the state preparation. This is eventhough they have been prepared, from the viewpoint of the observer, in an identical manner. This is the quantum ensemble interpretation of the quantum state as advocated in this article. There have been many other variants of ensemble interpretations for the quantum state, see e.g. the excellent reviews [9] [11] and references therein.

4 Superposition and ensemble fidelity

Consider the linear combination, or, superposition, of overlaps between the state preparation and its quantum ensemble, denoted by $\omega(\xi|\eta_1,...,\eta_M)$, i.e.

$$\omega(\xi|\eta_1,...,\eta_M)\equiv\sum_{j=1}^{M}a_j\Omega(\xi,\eta_j)\,, \tag{3}$$

where the coefficients a_j are in general complex-valued. For the quantum ensemble depicted in Fig.2, the overlaps $\Omega(\xi,\eta_1)$ and $\Omega(\xi,\eta_M)$ have a part which is mutual, denoted by $\tilde{\Omega}$. Therefore, it is necessary to consider the subtraction $\omega-\tilde{\Omega}$ in order to not count the same area twice. Since the fidelity between the state preparation and any given member of the quantum ensemble is postulated to be conserved in time, any given overlap $\Omega(\xi,\eta_j)$ is a solution to the Schrödinger equation. Therefore, due to the linearity of the Schrödinger equation, the linear combination $\omega(\xi|\eta_1,...,\eta_M)$ is also a solution. This is not the case for the mutual overlap $\tilde{\Omega}$, despite that it seemingly satisfy the Schrödinger equation. The Schrödinger equation originate from the postulate that the quantum fidelity between pairs of quantum states is conserved in time. The mutual overlap $\tilde{\Omega}$ is not an overlap between pairs of states and hence cannot be incorporated into the postulate, i.e. there is no physical reason which suggest that it should satisfy the Schrödinger equation. Thus, that the mutual overlap is a mathematical solution to the Schrödinger equation has no physical meaning. It should be excluded from any physical discussions on the distinguishability between the state preparation and its quantum ensemble. Therefore, the quantum ensemble is physically constrained by the requirement that there exist no mutual overlaps between the state preparation and two, or more, members of the quantum ensemble. Put differently, the set of overlaps between the state preparation and the members of the quantum ensemble are linearly independent from each other. However, it should be noted that any given pair of members of the quantum ensemble are allowed to have non-zero overlaps with each other, i.e. $\Omega(\eta_i,\eta_j)\neq 0$, $\forall i\neq j\in\{1,2,...,M\}$, as long as this overlap do no coincide partially with the state preparation.

The superposition of overlaps, Eq.3, can be used to generalize the notion of quantum fidelity to measure the distinguishability between the state preparation and the quantum ensemble. Consider the situation when $M=2$. The fidelity $F(\omega(\xi|\eta_1,\eta_2))$ for the linear combination $\omega(\xi|\eta_1,\eta_2)=a_1\Omega(\xi,\eta_1)+a_2\Omega(\xi,\eta_2)$ becomes, using the Born rule,

$$\begin{aligned}F&=|a_1\Omega(\xi,\eta_1)+a_2\Omega(\xi,\eta_2)|^2\\&=|a_1|^2F(\Omega(\xi,\eta_1))+|a_2|^2F(\Omega(\xi,\eta_2))+\\&+a_1^*a_2\Omega^*(\xi,\eta_1)\Omega(\xi,\eta_2)+a_2^*a_1\Omega^*(\xi,\eta_2)\Omega(\xi,\eta_1)\\&=\sum_{j=1}^{M=2}|a_j|^2F(\Omega(\xi,\eta_j))\\&+\sum_{j=1}^{M=2}\sum_{i\neq j}^{M=2}a_j^*a_i\Omega^*(\xi,\eta_j)\Omega(\xi,\eta_i).\end{aligned} \tag{4}$$

The last two terms clearly illustrate the key difference between the notion of probability in statistical and quantum mechanics. In classical probability theory, any disjoint pair of events satisfy Kolmogorov's third axiom [12]. Thus, the classical prediction would be that if the state preparation ξ were mistaken for e.g. the state η_1, then that would exclude the possibility that ξ were mistaken for the state η_2, with the consequence that the fidelity for the linear combination would be given by

$$F\left(\omega\left(\xi|\eta_1,\eta_2\right)\right) = F\left(\Omega(\xi,\eta_1)\right) + F\left(\Omega(\xi,\eta_2)\right). \qquad (5)$$

In quantum mechanics, on the other hand, there are additional terms which mix the states η_1 and η_2, despite the fact that the the members of the ensemble of systems are all supposed to be closed. The conclusion is thus that the mistaking of identity for the state preparation with the states η_1 and η_2 are not mutually exclusive[5]. This type of non-exclusivity between members of the quantum ensemble is referred to as quantum interference. It is the key distinction between the theories of statistical and quantum mechanics[6].

For an arbitrary M-dimensional quantum ensemble, the fidelity for the ensemble is given by

$$\begin{aligned} F\left(\omega(\xi|\eta_1,...,\eta_M)\right) &= \sum_{j=1}^{M} |a_j|^2 F\left(\Omega(\xi,\eta_j)\right) \qquad (6) \\ &+ \sum_{j=1}^{M}\sum_{i\neq j}^{M} a_j^* a_i \Omega^*(\xi,\eta_j)\Omega(\xi,\eta_i). \end{aligned}$$

The physical interpretation of the fidelity is that it give the probability associated with the event that the state preparation is mistaken for any given state in the quantum ensemble upon measurement by an observer. Given this interpretation, the ensemble fidelity is required to satisfy the condition $0 \leq F(\omega) \leq 1$. Clearly, $F(\omega) = 0$ when $\Omega(\xi,\eta_j) = 0$, $\forall j \in \{1,...,M\}$, at which the state preparation is completely distinguishable from the quantum ensemble. It is furthermore real-valued for arbitrary non-zero overlaps, for all possible complex-valued coefficients. The requirement that the ensemble fidelity is bounded from above by unity, i.e. $F(\omega) \leq 1$, is the problem of normalization in quantum mechanics. It amounts to the statement that, in the limit $M \to \infty$, it is guaranteed that the state preparation will be mistaken by the observer upon measurement. In other words, the normalization condition is given by

$$\lim_{M\to\infty} F\left(\omega(\xi|\eta_1,...,\eta_M)\right) = 1. \qquad (7)$$

It is important to emphasize that the ensemble fidelity can be calculated without the need to perform multiple measurements on an ensemble of systems. For any given system, even an individual particle, the task is to guess the Hamiltonian of the system. Once that has been achieved, and that is the truly difficult part, the complex-valued overlap as a function of time is obtained by solving the Schrödinger equation. The application of the Born rule then defines the ensemble fidelity of the system. Thus, the definition of the ensemble fidelity is not dependent on the frequency with which outcomes appear. However, if such measurements on an ensemble of systems were performed, there will be a statistical distribution for the states in which the members of the ensemble are found. In the case of a single particle, the interpretation is as follows. The preparation of the particle in the initial condition of the experiment is uncertain due to the indeterminacy relation. Thus, even if a bunch of particles, independent from each other, are submitted to the same state preparation ξ, the actual state of any given particle can be either ξ or any of the states in its quantum ensemble. Therefore, due to the statistical distribution in the initial conditions of the particels, there must be a statistical distribution in their states at any later times[7]. This is the origin for the interference fringes in the double-slit experiment.

5 The measurement problem

If no measurements are made on the system, such that it can be considered as closed from the environment, the Schrödinger equation state that the belief of the observer about the state of the system has not changed in between measurements. It does not imply that the

[5] Put differently, in the jargon of transition probability, the transitions $\xi \to \eta_1$ and $\xi \to \eta_2$ cannot be considered as mutually exclusive events.

[6] Of course, the origin for this distinction is due to the fact that in statistical mechanics, eventhough there is an uncertainty associated with the determination of the state of the system, it is still assumed that the state exist as a physically real entity at all scales, i.e. that it can be described with infinite precision by real-valued degrees of freedom which are, in principle, measurable by a non-ignorant observer. Due to the postulate on finite distinguishability, i.e. the indeterminacy relation, this is not the situation in quantum mechanics. There, the notion of state, from the observer point of view, do not physically exist beyond the scale set by the Gromov width since the degrees of freedom can no longer be considered as real-valued measurables.

[7] Of course, the effect on the state of the particle by the act of observation, as Heisenberg emphasized [13] [14], is non-negligable. However, it is not the sole origin for the uncertainty in the determination of the state for quantum systems. This type of observer effect on the state is present for all systems, albeit the smaller the system, the more pronounced is the effect. Most importantly, the observer effect is not suggested to characterize the fundamental distinction between the type of uncertainties that appear in classical and quantum mechanics. That distinction is due to the postulate on finite distinguishability.

system, in between measurements, simultaneously exist in all possible superposed states in the quantum ensemble. When the measurement is made, the system is found to occupy a definite state, either it being the state preparation or any of the members of the quantum ensemble. At this instant, the quantum fidelity is updated to unity for the definite state and to zero for the other states in the superposition. This is no different as compared to e.g. the throwing of a dice. Before the observer has looked on the dice, i.e. measured the outcome of the throw, equal probabilities are assigned to all possibilities[8]. But, when the measurement is made, the dice is found in a definite state, e.g. 2, and the probabilities associated with the other five alternatives are immediately updated to zero and the probability for the event 2 is updated to unity. However, if the dice is thrown many times over, it is expected that all possibilities will be realized an equal number of times. But in no situation does the dice simultaneously exist as a linear combination of all sides before the measurement. The same is true for the superposition of states in quantum mechanics. The problem of wave-function collapse, i.e. how a definite state of the system can be realized upon measurement when the system supposedly 'exist' in a superposition of states before measurement, is thus seen to not constitute a problem at all. The observer is ignorant before the measurement is made and hence assign weighted probabilities, depending on the overlap between the state preparation and the members of the quantum ensemble, to the possible outcomes. The 'collapse' simply indicate that the observer has gained some amount of knowledge about the state of the system.

The thought experiment put forth by Schrödinger involving a cat [15] aim to illustrate the absurdity of the idea that the superposition principle suggest that the system, before measurement, physically exist simultaneously in all possible states in the quantum ensemble. The set of superpositioned states before measurement, i.e. the cat being alive or dead, do not indicate any situation where the cat in fact is both dead and alive at the same time. It merely indicates that the observer does not possess enough information about the system to conclude which of these two possible states the cat exist in. To put it differently, the observer might not know the state of the cat, and therefore assign weighted probabilities to the situation that the cat is dead or alive, but the cat knows. If the cat is alive, the cat knows.

[8] Assuming, of course, that the observer do not possess information which indicate that the dice is not perfectly symmetric.

6 Conclusion

The quantum fidelity can be generalized to the setting where the state preparation has a non-zero overlap with an arbitrary-dimensional quantum ensemble. The key distinction between classical and quantum probability measures is the appearance of quantum interference, i.e. the non-exclusivity in the mistaking of identity between the state preparation and members of its quantum ensemble. The origin for this interference is the linearity of the Schrödinger equation. With quantum fidelity being interpreted as the probability associated with the mistaking of identity, it is clear that the linear superposition of overlaps, between the state preparation and members of its quantum ensemble, in between measurements, should not be understood to imply the simultaneous existence of quantum states.

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