Towards a New Philosophical Perspective on Hermann Weyl's Turn to Intuitionism

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Abstract

The paper explores Hermann Weyl's turn to intuitionism through a philosophical prism of normative framework transitions. It focuses on three central themes that occupied Weyl's thought: the notion of the continuum, logical existence, and the necessity of intuitionism, constructivism, and formalism to adequately address the foundational crisis of mathematics. The analysis of these themes reveals Weyl's continuous endeavor to deal with such fundamental problems and suggests a view that provides a different perspective concerning Weyl's wavering foundational positions. Building on a philosophical model of scientific framework transitions and the special role that normative indecision or ambivalence plays in the process, the paper examines Weyl's motives for considering such a radical shift in the first place. It concludes by showing that Weyl's shifting stances should be regarded as symptoms of a deep, convoluted intrapersonal process of self-deliberation induced by exposure to external criticism.

1. Introduction

Hermann Weyl's engagement with the foundations of mathematics began in 1910 when he first started to consider constructive methods as a solid opponent to classical mathematics (Beisswanger 1965; Scholz 2000). His philosophical views played a major role in the development of his inclination towards constructive approaches, and later towards Brouwer's intuitionistic ideas (Scholz 2004; Sieroka 2009, 2019). Weyl was deeply influenced by Kant's notion of the primacy of intuition, and even though he cannot be described as a true idealist (since his metaphysical outlook included some realistic aspects, (Bell and Korté 2015)), he did, during 1912, find philosophical enlightenment in Husserl's phenomenology and Fichte's metaphysical idealism. Brouwer was also interested in the philosophy of mathematics and dedicated the second chapter of his dissertation to improving Kant's view of the a priori (Brouwer 1907). The force behind Brouwer's philosophical agenda was Gerrit Mannoury, whose lectures revealed to Brouwer that mathematics was not a realm that existed independently of human beings, but rather a human creation. This was in accordance with Brouwer's idealistic

philosophy and set the ground for Brouwer's choice of research subject (van Dalen 1999; Mannoury 1909).

It should be noted that Weyl's idealism and Brouwer's idealism were quite different: Brouwer's philosophy amounted virtually to solipsism, while Weyl still seemed to have cleaved to phenomenology (Mancosu and Ryckman 2002). Nevertheless, this common philosophical ground (specifically their shared view about the primary intuition of time) have contributed to Weyl's inclination towards intuitionism.

The publication of *Das Kontinuum* (Weyl 1918) marked the emergence of Weyl's commitment to constructivism, but less than three years later, it transformed into an intuitionistic approach that was expressed in his renowned paper "On the New Foundational Crisis in Mathematics" (Weyl 1921). Weyl was critical of Cantor's set-theoretical foundations of mathematics not only on the grounds of its arising paradoxes but mostly because Cantor's set theory addressed the subject matter of mathematics as abstract entities that exist independently of human thought. Weyl, on the other hand, viewed the natural number system as a human conception and insisted on working with explicit definitions of sets and functions (Feferman 1998); Brouwer's intuitionistic program, which was based on mathematical objects as constructions of the human mind, had therefore attracted Weyl's attention.

However, during the 1920s, Weyl gradually shifted his foundational stance away from intuitionism, possibly due to the mathematical sacrifices required in order to achieve the mathematical stability intuitionism had proposed. He found the intuitionistic approach "awkward" (Weyl 1949, 54) and intolerable for practicing mathematicians (van Dalen 1995). In the mid-1920s Weyl retreated from intuitionism and re-approached Hilbert's axiomatic program (Beisswanger 1965; van Dalen 1995). Notwithstanding, Weyl's later works show that he never fully accepted Hilbert's program, and continued to waver between constructive, intuitionistic, and axiomatic approaches to the foundational problem throughout his life.

Solomon Feferman described at least two additional changes of heart vis-à-vis constructivism that emerged in Weyl's later works: one in the late 1930s, when he restated the importance of his early constructive views as essential to the process of forming a solution to the foundational problem, and another in his 1953 lecture where he described himself as being torn between constructivity and axiomatics (Feferman 1998). The lecture, titled "Axiomatic versus constructive procedures in mathematics", ended with the following words:

> "Indeed my own heart draws me to the side of constructivism. Thus it cost me some effort to follow the opposite direction, putting axiomatics before construction, but justice seemed to require this from me." (Weyl 1953, 38)

Historians and philosophers of mathematics have overtly noticed Weyl's frequent changes of heart regarding the foundational crisis. Feferman claimed that "we all change our minds, or most of us do, about things over periods of time and Weyl was certainly no exception" (Feferman 2000, 181). Erhard Scholz suggests that Weyl's "intellectual moves" during the 1920s derived from his philosophical considerations, which were rooted in Fichte's construction of the concept of space and matter (Scholz 2000). Norman Sieroka explains the fluctuation of Weyl's

positions regarding formalism, intuitionism, and constructivism as closely related to the developments in his philosophical views, namely, the separation he made between Husserlian phenomenology and Fichtean constructivism (Sieroka 2009). However, such accounts only address part of the story, as they make no attempt to explain either Weyl's motivations for changing his mind or the roots of his undecidedness.

To do justice to Weyl's story, I will address the broader philosophical issues of normative framework transitions and try to explain how practitioners expound new alternatives. First, I will focus on three central themes that occupied Weyl's thought: the problematic differentiation between the intuitive and the mathematical continuum, the notion of logical existence, and the necessity of intuitionism, constructivism, and formalism to adequately address the foundational crisis of mathematics. The analysis of these themes will reveal Weyl's continuous endeavor to deal with such fundamental problems. Afterwards, building on Menachem Fisch's model of scientific framework transitions and the special role that normative indecision or ambivalence plays in the process (Fisch 2017; Fisch and Benbaji 2011), I will examine Weyl's motives for considering such a radical shift in the first place, and I will show that Weyl's constant changes of heart in regards to the foundations of mathematics were only symptoms of a deeper, intrasubjective process of self-criticism induced by exposure to external normative criticism.

2. The undecided nature of Weyl's thought

2.1 The concept of the continuum

In *Das Kontinuum*, Weyl presented a semi-constructive alternative to the set-theoretic basis of the full real number system $\mathbb{R}$. Though he perceived it as a suitable alternative to those suggested by Dedekind, Cantor, Weierstrass, and Hilbert, Weyl was fully aware of the difficulties arising from his approach,[1] specifically regarding what he saw as the unbridgeable gap between the continuum given by intuition (of space, time and motion) and its mathematical representation as a "discrete" exact concept (that of the real number).

Weyl described the discrepancy between an intuitively given continuum (he referred to time as the most fundamental continuum) and the concept of number in §6 of the second chapter in *Das Kontinuum*:

> "For example, I see this pencil lying before me on the table throughout a certain period of time. This observation entitles me to assert that during a certain period this pencil was on the table; and even if my right to do so is not absolute, it is nonetheless reasonable and well grounded. It is obviously absurd to suppose that this right can be undermined by an "expansion of our principles of definition" - as if new moments of time, overlooked by my intuition, could be added to this interval, moments in which the pencil was, perhaps, in the vicinity of Sirius or who knows where. If the temporal

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[1] As Weyl himself stated, "The intuitive continuum and the world of mathematical concepts are so distant from each other that the demand that both coincide has to be rejected as absurd." (Weyl 1918, 83).

continuum can be represented by a variable which "ranges over" the real numbers, then it appears to be determined thereby how narrowly or widely we must understand the concept "real number" and the decision about this must not be entrusted to logical deliberations over principles of definition and the like." (Weyl 1918, 87–88)

Throughout his attempt to analyze the continuous nature of time, Weyl struggled with the transition from what is intuitively present (such as a pencil lying on the table for a certain period), to a description of a time-period consisting of individual time-points (namely, replacing "a certain period" with "every time-point which falls within a certain time span"). The latter is simply not intuitive anymore. Finally, he concluded that exact time points are not the underlying elements of time, as given to us by experience. Within the intuitive continuum,

"the exhibition of a single point is impossible. Further, points are not individuals and, hence, cannot be characterized by their properties. (Whereas the "continuum" of the real numbers consists of genuine individuals, that of the time- or space-points is homogeneous.)" (Weyl 1918, 94)

The section ends with the underlying statement that the intuitive and the mathematical continuum are comprised of fundamentally different elements; thus, the two entities cannot coincide.

Less than two years later, Weyl published "On the New Foundational Crisis in Mathematics" (Weyl 1921), which embodied a perspective on the continuum very different from the one presented in his preceding works. Here, Weyl repudiated his interpretation of the continuum in *Das Kontinuum*, which he referred to as "atomistic" or "discrete", because it derived from a false belief that individual points constitute the continuum and that each point is independent when taken by itself (Weyl 1921, 94).

The new interpretation which he welcomes is Brouwer's intuitive construction of the continuum, as Weyl felt that Brouwer was the only one close enough to bridge the "chasm" between the intuitive and mathematical continua (Bell 2000). As such, it builds upon a completely different conception of the real numbers. Real numbers are no longer represented as individual points; a real number x is represented by a "dual interval", an interval stretching from the number y to the number z and containing x in it.[2] He wrote

"The concept of real number, *as a number that is given only approximatively, yet one for which the degree of approximation can be pushed beyond any limit* is thus to be formulated simply as: a real number is an infinite sequence of dual intervals i, i', i'', … such that every interval of the sequence fully contains in its inside the next following one." (Weyl 1921, 93)

Weyl introduced two additional concepts associated with his new notion of the real number: sequences and laws. Both are used to define the relationship between the new individual real numbers and the new concept of the continuum:

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[2] Weyl presents a formal representation of those numbers in his paper (Weyl 1921, 93), which is not very relevant for the purpose of this argument.

"The individual real number is represented by a law φ that determines a sequence in infinitum, while the continuum is represented by the choice sequence unrestricted by any law in the freedom of its development." (Weyl 1921, 94)

A genuine continuum, according to Weyl, cannot be divided into separate fragments since the true nature of the continuum demands that every part of it can be further divided without limitation (Weyl 1921, 115). The continuum in the traditional analysis was defined as a set of its points, the points being fundamental elements in the set. Each point was a part of the set, such that the traditional continuum could be divided into parts, but this was as far as one could go with such division since a point is considered a part that does not contain any further parts. By redefining primary elements of the construction of the real numbers as intervals rather than points, the mathematical continuum can finally be "in harmony" with the intuitive one. The chasm that represented the unbridgeable difference between the intuitive and the mathematical continuum in *Das Kontinuum* reappears now as the gulf between the new mathematical continuum and a set of discrete elements (Weyl 1921, 95).

Several years later, in "Philosophy of mathematics and natural sciences" Weyl addressed the problem of the continuum as an open question that has not been properly attended yet. In this book, Weyl presented his philosophical approach to mathematics, quoting Anaxagoras to support his claim of the continuous continuum:

"The essential character of the continuum is clearly described in this fragment due to Anaxagoras: 'Among the small there is no smallest, but always something smaller. For what is cannot cease to be no matter how far it is being subdivided'. The continuum is not composed of discrete elements which are 'separated from one another as though chopped off by a hatchet'." (Weyl 1949, 41)

The puzzling nature of the continuum and the problems surrounding its mathematical representation continued to occupy Weyl's thoughts. In his paper "Axiomatic and Constructive Procedures in Mathematics" written in 1954, he still grappled with the "riddle of the continuum" and regarded it as a "serious affair" that has not yet been resolved (Weyl 1954).

2.2 Logical Existence

The first chapter in *Das Kontinuum* engages with core concepts such as "judgement", "category", "proposition", and possible relations between them (Weyl 1918, 5–9). Throughout the chapter Weyl forms six "complex judgment schemes" (Weyl 1918, 10) in order to explain logical notions like "and", "or", logical negations, and the concept of logical existence. For example:

"[…] If F(xy) means "x is the father of y" and accordingly, F(Iy) means "I am the father of y", then F(I*) means "There is someone of whom I am the father", that is, "I am a father."" (Weyl 1918, 11)

Judgement statements such as "Every object has such and such a property" and "There is no object which lacks the relevant property" are deemed equivalent in *Das Kontinuum* (Weyl 1918, 12). It is tacitly pre-supposed that a combination of existence judgement and negation (e.g.,

"there is no object that…") is fully interchangeable with universal statements such as "all" or "every" (Weyl 1918, 12–13).

The picture presented in "On the New Foundational Crisis in Mathematics" (Weyl 1921) reveals quite a different theory. Building on the new interpretation of the continuum as an entity that can be divided without limitation, the concept of mathematical existence was utterly reformed. In order to deduce from the statement "there is someone of whom I am the father" that "I am a father", one must be able to produce a witness: a child of his. Once the witness is produced, the inference is justified; but if one is unable to produce a witness, existence judgement such as "there is someone" is meaningless. He wrote

> "An existential statement – say, 'there exists an even number' – is not at all a judgement in the strict sense, which claims a state of affairs. Existential states of affairs are empty inventions of logicians. '2 is an even number': This is an actual judgement expressing a state of affairs; 'there is an even number' is merely a judgement abstract gained from this judgement. If knowledge is a precious treasure, then the judgement abstract is a piece of paper indicating the presence of a treasure, yet without revealing at which place. Its only value can be to drive me on to look for the treasure. The piece of paper is worthless as long as it is not realized by an underlying actual judgement like '2 is an even number'." (Weyl 1921, 97–98)

In "Philosophy of mathematics and natural sciences" Weyl differentiated between two types of logic: a finite one and a transfinite one. Finite logic uses logical notions such as "and", "or", and "not", whereas transfinite logic also includes existence judgements such as "all" and "there is". He wrote

> "That part of logic which operates exclusively with the logical connectives 'not,' 'and,' 'or' will be referred to as finite logic, as opposed to transfinite logic, which in addition uses the propositional operators 'some' (or 'there is') and 'all'. The reason for this subdivision is as follows. Suppose several pieces of chalk are lying in front of me; then the statement 'all these pieces of chalk are white' is merely an abbreviation of the statement 'this piece is white & that piece is white & …' (where each piece is being pointed at in turn), Similarly 'there is a red one among them' is an abbreviation of 'this is red v that is red v ... ' But only for a finite set, whose elements can be exhibited individually, is such an interpretation feasible. In the case of infinite sets, the meaning of 'all' and 'some' involves a profound problem which touches upon the core of mathematics, the very secret of the infinite." (Weyl 1949, 13–14)

Within the finite set, Weyl redefined logic as comprised of two logics and identified transfinite logic with existential and universal judgments. In order to deduce existence from an existence statement, it is no longer mandatory to provide a witness within a finite set, since one can eventually be found by pointing out each object. The same procedure applies for universal quantifiers since, within finite sets, each object can be accounted for. However, upon considering infinite sets, the latter no longer holds; existence statements, as well as universal quantifiers, again receive the status of problematic at best and meaningless at worst. This approach to logic, specifically within the delimiter of finite and infinite sets, indicates that Weyl

maintained his ideas from 1921 about logical existence and the problematic aspect of "there is" statements, but also that he accepted the validity of "there is" statements in some realms of logic (namely, those that address finite sets); a relatively moderated approach than the one he presented in 1921.

In later works, the differentiation between finite and transfinite logic within finite sets disappeared, and instead, Weyl used the terms finite and infinite logic to emphasize the differences he saw between two opposing mathematical views: one that consists primarily of construction, and another that subordinates construction to axioms and deduction. He wrote

> "If carried so far, the issue between explicit construction and implicit definition by axioms ties up with the last foundations of mathematics. Evidence based on construction refuses to support the principles of Aristotelian logic when these are applied to existential and general propositions in infinite fields like the sequence of integers or a continuum of points. And if the logic of the infinite is taken into account, it seems impossible to axiomatize adequately even the most primitive process, the transition n -> n' from an integer n to its follower n'. As K. Gödel has shown, there will always be constructively evident arithmetical propositions which can not be deduced from the axioms however you formulate them, while at the same time the axioms, riding roughshod over the subtleties of the constructive infinite, go far beyond what is justifiable by evidence." (Weyl 1940, 446).

Here Weyl described two completely different logics: the Aristotelian logic used by the axiomatic approach and the logic of the infinite that is applied in constructive methods. When considering existence statements in infinite sets, it is only the "logic of the infinite" (i.e., Brouwer's intuitionistic logic) that comes closest to a true mathematical representation of our intuition, according to Weyl.

Weyl had never come to terms with the question of whether mathematics can adequately portray what is given to us by experience (Weyl 1951). The problematic aspects he explicitly pointed out, such as logic existence statements and the concept of the continuum, remained an open question to him, since he felt that neither constructivism nor formalism were able to properly dissolve the tendencies between mathematics and our phenomenological sense of the world.

2.3 Being an intuitionist, a formalist, or a constructivist?

Aside from "On the New Foundational Crisis in Mathematics", Weyl engaged with intuitionism twice more during the 1920s: in 1924 when he presented a constructive proof of the fundamental theorem of algebra, and in 1928 as a response to Hilbert's attack on intuitionism during a seminar talk in Hamburg (van Dalen 1995, 163). The Hamburg symposium note, in which he stated that "in the epistemological evaluation of the new situation thus created, nothing separates me any longer now from Hilbert" (van Dalen 1995, 164), marks Weyl's gradual dissociation from intuitionism which had begun during 1926-1927 (Mancosu and Ryckman 2002).

Weyl was always aware of the problematic aspects of Brouwer's intuitionism. Even in an advocating paper such as "On the New Foundational Crisis in Mathematics", Weyl addressed intuitionism with a grain of salt. He wrote

> "The new conception, as one can see, brings with it far-reaching restrictions with respect to the generality that enthusiastically leads into vagueness and that we have become used to through traditional analysis. We must learn again to be modest. With the intention to storm the heavens, we merely piled up mists upon mists, unable to support anyone who was seriously trying to stand upon them. What remained tangible might, at first glance, appear so insignificant that the possibility of analysis could generally be put into question. This pessimism, however, is ill-founded, as will be shown in the next section. Yet we must firmly and with all energy hold on to the fact that *mathematics is through and through, even as concerns the logical forms in which it moves, dependent on the nature of natural numbers*." (Weyl 1921, 109)

On a less optimistic note, Weyl elaborated on the implications of such restrictions on the mathematical discipline as a whole:

> "Mathematics with Brouwer gains its highest intuitive clarity. He succeeds in developing the beginnings of analysis in a natural manner, all the time preserving the contact with intuition much more closely than had been done before. It cannot be denied, however, that in advancing to higher and more general theories the inapplicability of the simple laws of classical logic eventually results in an almost unbearable awkwardness. And the mathematician watches with pain the larger part of his towering edifice which he believed to be built of concrete blocks dissolve into mist before his eyes." (Weyl 1949, 54)

In its entirety, Brouwer's intuitionistic theory obliged its adherents to forfeit core mathematical notions while embracing completely different, sometimes even contradicting[3], alternatives. Whereas Brouwer was willing to eschew fundamental classical theorems and accept the invalidity of principle concepts in calculus (Brouwer 1912, 1918; van Stigt 1990), Weyl felt that intuitionistic ideas are of little relevance to mundane mathematical practice, avowing that everyday mathematical theorems are not affected by his new intuitionistic contemplations. In a draft of Weyl's 1921 paper, he remarked that

> "It should be stressed once more that certain individual functions of that kind [functio continua] occur occasionally in mathematics, that general theorems are, however, never asserted about them. The general formulation of these notions is therefore required only if one is giving a justification of the meaning and methods of mathematics; for itself, the subject matter of its theorems, it is never considered at all". (van Dalen 1995, 148–49)

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[3] Aside from refining the classical-set theoretic picture, Brouwer's new intuitionistic mathematics also occasionally contradicted classical mathematics, as in its denial that there are any fully defined discontinuous functions (for details see (Posy 2005)).

Nevertheless, Weyl was reluctant to commit to Hilbert's formalistic approach. From Weyl's point of view, Hilbert's formalization of set theory amounted to reducing mathematics to formal logical systems that can be described objectively without reference to their intended meaning, an approach that Weyl perceived as a degradation of mathematics:

> "Hilbert's mathematics may be a pretty game with formulae, more amusing even than chess; but what bearing does it have on cognition, since its formulae admittedly have no material meaning by virtue of which they could express intuitive truths? The subject of mathematical investigation, according to Hilbert, is the concrete symbols themselves. It is without irony, therefore, when Brouwer says 'To the question where shall mathematical rigor be found, the two parties give different answers. The intuitionist says: in the human intellect, the formalist: on paper'." (Weyl 1949, 61)

Neither Hilbert's formalism nor Brouwer's intuitionism were able to adequately rebuild the foundations of mathematics as Weyl saw fit. In his 1925 paper "The current epistemological situation in mathematics", Weyl claimed that both are necessary for a comprehensive understanding of mathematics. He concluded

> "[…] it is certainly greatly beneficial that Brouwer had strengthened again the sense in mathematics for the intuitively given. His analysis expresses in a pure manner the content of the mathematical basic intuition and is therefore shone through by clarity without mystery. Yet beside Brouwer's way, one will also have to pursue that of Hilbert; for it is undeniable that there is a theoretical need simply incomprehensible from the merely phenomenal point of view, with a creative urge directed upon the symbolic representation of the transcendent, which demands to be satisfied". (Weyl 1925, 141)

Even though the ultimate foundations and meaning of mathematics remained an open problem for Weyl, he saw a mid-way solution between Hilbertian formalism and constructive methods in the form of "dextrous blending": an amalgam of Hilbert's axiomatic approach to mathematics in algebra (championed by Bourbaki) and constructive procedures in other mathematical fields such as topology and geometry (Weyl 1954).

Weyl did not regard intuitionism, constructivism, and formalism as clearly delineated theories. He noted that the borderlines of Brouwer's mathematics are vague, and some of Hilbert's mathematical considerations that appear to be evident, are not always entirely evident (Weyl 1953). Thus, he was not only concerned with questions regarding the subject matter of the theory itself (e.g., which definition of the continuum corresponds better with intuition?). He was also concerned with questions about what it means to be a formalist, a constructivist, or an intuitionist (e.g., What are the implications of building the foundations of mathematics solely on intuitionism\formalism\constructivism? Does such mathematics remain useful?).

Weyl is often described by historians and philosophers of mathematics as a "wanderer" (Scholz 2004, 1), who was traveling amid mathematical approaches and through philosophical fields. His retreat from Brouwer's ideas was portrayed as becoming "disillusioned" (Rosello 2012, 147), and Solomon Feferman argued that when we talk about Weyl's foundational views, we are faced with shifting positions (Feferman 2000). The analysis of Weyl's stances regarding

logical existence, the continuum, and the underlying meaning of being an intuitionist, a formalist, or a constructivist, shows more than a mere change of mind; it depicts Weyl's consistent attempt to reconcile (what he felt was) the intuitive nature of mathematics with the common, everyday use of mathematical practice without losing one to the other.

In the remainder of the paper, I wish to enhance the standard view held by historians and philosophers of mathematics and suggest that Weyl's indecisive acts are symptomatic of profound intrapersonal deliberations. By considering how normative transitions occur in general, and more specifically, by concentrating on the process of exposure to external criticism that is both normative and self-directed, the following sections of this paper seek to shed a different light on Weyl's innate motivations and hesitant stances.

3. Towards a philosophical approach of normative transitions

The question of how can someone deem his own standards of propriety as normatively improper in a rational manner has puzzled the philosopher Menachem Fisch for decades. In his recent works (Fisch 2017; Fisch and Benbaji 2011) Fisch has proposed a theory about the possibility of rationally changing our normative commitments, in which he claims that by intrasubjective deliberations alone we are unable to reach the self-critical lines and cannot rationally change our standards. However, and here lies the novelty of his argument, exposure to the echo chambers of normative criticism coming from outside, from people who are committed to different frameworks, could have a destabilizing impact. Or, to use Fisch's expression, to ambivalate.

Fisch has observed that practitioners of standing are liable, when normatively ambivalated, to produce curiously split and hybrid accounts of the foundations of their field in an attempt to breed a newly found undecidedness. Examples for Fisch's theory are the stories of Tycho Brahe's grafting of a heliocentric planetary system on a basically geocentric cosmology, Galileo's theory of free fall and projectile motion, as well as George Peacock's two-fold account on algebra. Even though the mature works of Brahe, Galileo, and Peacock are confident presentations of the positions to which they were fully committed, it is their halfway positions and intermediary frameworks (often unpublished thus concealed from the public eye) that paved their paths. The fact that Tycho, Galileo, and Peacock were able to form and maintain such halfway positions, negotiating between the prevailing theory on the one hand and their new ideas on the other, shows that they had become sufficiently ambivalent towards their own normative frameworks (Fisch and Benbaji 2011, 295–96).

Fisch's concept of ambivalence refers to a delicate moment in the history of scientific transformations, a moment that can be easily overlooked or dismissed as mere confusion or indecision when analyzed without proper attention. He writes:

> "What makes such figures indispensable to the deeply transformative moments in which they partook, and, hence, to the latter-day historian, is the way their initial ambivalence was captured in their Solomonic attempts to split their subject matter in the hybrid manner described. Carefully analyzed, pried apart at their (usually) rough

seams, and properly reenacted, such doubles and splits offer historians much toward recapitulating the destabilized mindset of their authors. But only if read and analyzed prospectively, as history is lived and should be studied, as acts of anxious and urgent engagement, rather than retrospectively, as mere intermediate stations between one stable framework and the next; as fossilized normative dilemmas, as it were, preserving for posterity frozen, ossified snapshots of tortured, creative indecision. For they are too often written off as unimaginative first tries, or as the work of confused reactionary rearguarding." (Fisch 2010, 536–37)

Notably important is the difference between ambivalence and persuasion. To render someone ambivalent towards one or more of his normative standards does not necessarily mean to convince him of their unseemliness. Practitioners can become ambivalent towards a theory without being convinced that the prevailing theory is improper. Once exposed to the normative criticism of others, the stage of ambivalence can last months, years, or even a lifetime, during which practitioners are on the lookout for new alternatives without dismissing the old ones. Moreover, ambivalated practitioners often feel torn between the two stances, since

> "normative ambivalence amounts to being of two minds with respect to certain elements of one's normative framework; a form of indecisive dithering with respect to those elements. To the remainder of their framework (the part of their I-part to which they have not become ambivalent) such "ambivalated" individuals remain wholeheartedly committed." (Fisch 2010, 537)

Fisch's model of normative transformations introduces two insights about the concept of "becoming ambivalent" that are of significance to Weyl's story:

(1) Ambivalence can sometimes be misinterpreted as indecision or confusion.
(2) Ambivalence does not necessarily imply abandonment of old commitments, nor a complete acceptance of new ones.

The story so far gives rise to the question of how does a practitioner deal with his own ambivalence. Does he attempt to settle his indecision by turning his back on one of the theories he is torn between? Must the ambivalence be eventually resolved, or can one self-deliberate with his own indecision throughout his life? The following section will examine how Weyl addressed his own ambivalence, as well as how mathematicians, philosophers, and historians of mathematics viewed it.

4. Ambivalated from within *(How did Weyl deal with his own ambivalence?)*

Weyl's changing views about the foundations of mathematics were shaped, to a large extent, by his exposure to the echo chambers of several practitioners and groups of mathematicians. The French semi-intuitionists' criticism of Cantor's set theory and Zermelo's axiom of choice (Hesseling 2003; Moore 1982) have influenced Weyl's foundational viewpoint as can be seen both in Weyl's 1910 paper on the definition of fundamental mathematical concepts (namely,

definable relations) and in *Das Kontinuum* (Feferman 1998). In the 1918 preface to *Das Kontinuum*, Weyl describes his then-promising half-way solution: a constructive approach to the foundations of mathematics that will avoid the "vicious circle" of Russell's paradoxes and Cantor's set theory and its dangerous implications. He wrote:

> "It is not the purpose of this work to cover the "firm rock" on which the house of analysis is founded with a fake wooden structure of formalism – a structure which can fool the reader and, ultimately, the author into believing that it is the true foundation. Rather, I shall show that this house is to a large degree built on sand. I believe that I can replace this shifting foundation with pillars of enduring strength. They will not, however, support everything which today is generally considered to be securely grounded. I give up the rest, since I see no other possibility." (Weyl 1918, 1)

Even though Weyl was not a professional philosopher, he was attracted to idealist philosophy, and the latter played a significant role in his foundational thought. When Weyl was a student of mathematics in Gottingen, he attended lectures by Husserl, and during his Zurich years, he was introduced to Fichte's philosophical views, much due to the encouragement of his friend and colleague, Fritz Medicus, who was also a professor of idealist philosophy at ETH (Scholz 2004; Sieroka 2007, 2009). The influence of both Husserl and Fichte on Weyl presents itself in *Das Kontinuum*, as Weyl specifically addressed Husserl's philosophical stance in the book's preface:

> "Although this is primarily a mathematical treatise, I did not avoid philosophical questions and did not attempt to dispose of them by means of that crude and superficial amalgamation of empiricism and formalism which still enjoys considerable prestige among mathematicians (even though it is attacked with gratifying clarity in Frege (1893). Concerning the epistemological side of logic, I agree with the conceptions which underlie Husserl (1913a). The reader should also consult the deepened presentation in Husserl (1913b) which places the logical within the framework of a comprehensive philosophy. Our examination of the continuum problem contributes to critical epistemology's investigation into the relations between what is immediately (intuitively) given and the formal (mathematical) concepts through which we seek to construct the given in geometry and physics." (Weyl 1918, 2)

At the outset of the discussion on judgement and property in the first chapter of *Das Kontinuum*, Weyl acknowledged the expertise of metaphysical philosophers, such as Fichte, that is essential in order to achieve a clear definition of what judgement is. He wrote:

> "We cannot set out here in search of a definitive elucidation of what it is to be a state of affairs, a judgment, an object, or a property. This task leads into metaphysical depths. And concerning it one must consult men, such as Fichte, whose names may not be mentioned among mathematicians without eliciting an indulgent smile[4]." (Weyl 1918, 7)

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[4] According to Norman Sieroka, the mathematician's "indulgent smile" refers to the fact that Fichte was a gifted philosopher but not quite as good a mathematician (Sieroka 2009).

Much like his foundational stances, Weyl's philosophical views were also subject to changes. In a lecture delivered in 1954 titled "Insight and Reflection", Weyl portrayed his long philosophical voyage that begun with Kant, then went on to idealist phenomenology, and ended with Weyl's recognition that the latter entails several problematic aspects (Weyl 1955). Considering these issues, Weyl modified his philosophical approach, which became closer to Cassirer in some aspects[5], and in other aspects to theologists such as Meister Eckhart (Bell 2004). According to Erhard Scholz, even the deep inspiration of Fichte's ideas on Weyl's thoughts had faded over the years, and Weyl's late reflections indicate that he became closer to Heidegger's existentialist or Gonseth's dialectical philosophies in the late 1940s (Scholz 2004).

The mathematician Henri Poincare also played a significant role in forming Weyl's foundational as well as philosophical viewpoint. The impact of Poincare's philosophy of mathematics, specifically his position regarding impredicative definitions and the concept of existence and potential infinity (Poincare 1963), is present in Weyl's 1918 works. Solomon Feferman argues that in *Das Kontinuum*, Weyl accepted a significant part of Poincare's definitionist philosophy, including Poincare's conception of the natural number and the idea that there are no completed infinite totalities (Feferman 1998, 2000). According to Richard Feist's reading of Weyl's *Das Kontinuum* and *Space-Time-Matter*, Poincare's predicativist philosophy led Weyl to modify Husserl's semantics and ontology; concerning the concept of the natural number, Feist describes Weyl's definition of it as an amalgam between Poincare's view and Husserl's approach (Feist 2002).

Two years after the appearance of *Das Kontinuum* and *Space-Time-Matter*, the publication of "On the new foundational crisis in mathematics" (1921) suggested that yet another mathematician had shaped Weyl's thought. Weyl was familiar with Brouwer's topological work, at least from the early 1910s, as he referred to Brouwer's "fundamental papers on topology" from 1909 in his paper on Riemann surfaces from 1913 (van Dalen 2013). There is historical evidence that the two have met in 1912, during Brouwer's visit in Gottingen, but it remains unclear whether they discussed foundational issues at that time[6].

Weyl's exposure to the echo chambers of Brouwer's normative criticism of the foundations of mathematics dates back to 1919 when the two had met for the second time during a summer

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[5] It should be noted that the developments in Weyl's philosophical stances have influenced his philosophy of physics as well (Ryckman 2005; Scholz 2001; Sigurdsson 1991). Norman Sieroka argues that some of the ideas Weyl developed in physics, such as the "agens theory" of matter, were influenced by his reading of Fichte and Husserl (Sieroka 2007), and Christophe Eckes points out that there exists a relation between Weyl's conception of apriorism and holism to the philosophical views of Husserl and Cassirer (Eckes 2018). Weyl's ambivalence in regards to the philosophy of physics is an intriguing topic that goes beyond the scope of the current paper but hopefully will be pursued in further research.

[6] There are opposing views in regards to whether Brouwer's work in topology has roots in his philosophical views on intuitionistic mathematics. Some historians and mathematicians argue that Brouwer's topological work is utterly separated from his intuitionistic endeavor, while others such as Arend Heyting and Dirk van Dalen remain unsure as to the unity or disunity in Brouwer's work. The historians and mathematicians Jan van Mill and Teun Koetsier claim that the dividing line in Brouwer's work does not run through topology and intuitionism, but separates his research to pre-1917 and post-1917 works (Koetsier and Van Mill 1997). They show that there is more that links Brouwer's intuitionistic work to his topology in the years up to 1917 than separates it, and conclude that Brouwer's topological work derive naturally from the same basic principles.

vacation in Engadin, and Weyl got to hear the story of Brouwer's intuitionism first hand (van Dalen 1995, 2013; Hesseling 2003). Deeply influenced by Brouwer's ideas, Weyl wrote the most advocative paper of Brouwer's intuitionistic program, "On the new foundational crisis in mathematics" (Weyl 1921), which displayed a considerable deviation from the formalist framework to which Weyl was firmly committed as well as from the constructive approach Weyl embraced in *Das Kontinuum*, as an attempt to solve the difficulties Brouwer had raised.

In the 1932 preface to *Das Kontinuum*, Weyl's difficulty in properly addressing the foundational problem prevails, but he is less optimistic about the ability of constructive or intuitionistic methods to solve the conundrum. As he put it:

> "The point of view adopted in this monograph continues to strike me as a natural transitional stage in the development of foundational research. However, in the period since its appearance, my work has been superseded by two trends identified by the catchwords Intuitionism and Formalism. Still, this deeper grounding of the foundation has not led to an even moderately satisfying or defensible conclusion; things remain in a state of flux. [...] It would not be possible, without radical rebuilding, to bring the content of this monograph into harmony with my current beliefs." (Weyl 1918, 2–3)

The indecisive nature of Weyl's foundational perspective remains omnipresent in his later writings, as observed in his 1946 review paper, where he maintains his skepticism about our ability to confidently build up mathematics from solid foundations. Thus:

> "This history should make one thing clear: we are less certain than ever about the ultimate foundations of (logic and) mathematics, like everybody and everything in the world today, we have our "crisis". We have had it for nearly fifty years. Outwardly it does not seem to hamper our daily work, and yet I for one confess that it has had a considerable practical influence on my mathematical life: it directed my interests to fields I considered relatively "safe", and it has been a constant drain on the enthusiasm and determination with which I pursued my research work. The experience is probably shared by other mathematicians who are not indifferent to what their scientific endeavors mean in the context of man's whole caring and knowing, suffering and creative existence in the world." (Weyl 1946, 11)

Moreover, Weyl's inclination towards constructive methods for rebuilding the foundations of mathematics did not disappear entirely over the years, as can be observed in one of his last papers "Axiomatic and Constructive Procedures in Mathematics," written in 1954:

> "...the constructive transition to the continuum of real numbers is a serious affair... and I am bold enough to say that not even to this day are the logical issues involved in that constructive concept completely clarified and settled." (Weyl 1954, 17)

These citations from Weyl's scattered works share a commonality: each embodies some aspect of self-reflection about his own shifting stances up to that point. In his 1932 note he acknowledged that a comprehensive endeavor is required in order to build a bridge between his old beliefs (presented in 1918) and his new ones; in 1946 he confessed that his uncertainty had influenced his own scientific interests; and in 1954 he referred to himself as "bold enough",

since it requires fortitude to maintain an undecided position after so many years, and to still remain unsure. These are the words of a self-conscious person who is fully aware of his undecidedness and its problematic aspects as a practitioner in a scientific community. The frequent changes in Weyl's mathematical views mark the trail he blazed in his fifty-year-long process of becoming ambivalent, living with his indecision, and attempting to resolve it.

5. Confused from without *(How do practitioners in the scientific community deal with Weyl's ambivalence?)*

Mathematicians, philosophers, and historians of mathematics have tried to account for Weyl's wavering foundational stances over the years. The mathematician and philosopher Solomon Feferman saw in Weyl's *Das Kontinuum* a "substantial advance in the predicativist program" (Feferman 1988, 16). He ascribed to Weyl a predicativist position which derived, to a large extent, from Weyl's criticism against the set-theoretical approach (that involved vicious circles) as well as his views about the irreducibility of the concept of the natural number and the necessity of explicit definitions for mathematical concepts such as sets and functions (Feferman 1998). Even though only two years later Weyl had repudiated his ideas from *Das Kontinuum* in favor of Brouwer's intuitionistic ideas, Feferman maintained that Weyl continued to regard the predicative approach as being of genuine value and that he "never really gave up the achievements of his 1918 monograph" (Feferman 2000, 181).

The philosopher Norman Sieroka connects the shifts in Weyl's foundational stances with the developments in his philosophical thought: Weyl's objection to reduce mathematics to logic and set theory in *Das Kontinuum* derived from his leaning both on Fichte and Husserl, but during the 1920s he distinguished between the two, connecting Husserl's phenomenology to Brouwer's intuitionism and Fichte's constructivism to formalism. After a brief affiliation with the "intuitionistic-phenomenological" approach, from 1925, Weyl came to believe that Fichte's "formalistic-constructivist" approach "was on the right track" (Sieroka 2009, 90). Sieroka finds analogues between Weyl's interdisciplinary lines of thought (mathematics, physics, and theory of subjectivity) and suggests that his analysis of Weyl's "interdisciplinary intellectual neighborhoods" set the ground for "a more systematic elaboration of the role of transformations and invariances in the context of historiographical issues" (Sieroka 2019, 120).

Biographical notes about Weyl written by mathematicians often dismiss his intuitionistic deliberations or ascribe them to his philosophical interests, which they consider as entirely separate from his mathematical work. Michael Atiyah's short essay (Atiyah 2003) analyzes Weyl's contributions to group theory and quantum mechanics while overtly ignoring his engagement with intuitionism and constructivism, and his ongoing ambivalence towards the foundational crisis. Max Newman dedicated a section he called "mathematical logic" to Weyl's intuitionistic contemplations, but his piece ends with a functional tone regarding Weyl's turn to intuitionism; as if Weyl's only motivation was to make Brouwer's mathematics accessible to other practitioners (Newman 1957, 323).

On the other hand, the mathematician and historian Dirk van Dalen claims that Weyl's commitment to intuitionism was not a mere infatuation or transitional stage. Throughout his

paper on Weyl's intuitionistic mathematics, van Dalen portrays Weyl as a life-long admirer of Brouwer's foundational approach that could never quite break with his earlier intuitionistic past (van Dalen 1995). According to van Dalen, the story of Weyl and Brouwer ends with their Zurich meeting, not long before Weyl died, where Weyl remarked sadly: "Brouwer, everything is unsteady again."[7] The philosopher John Bell (Bell 2000) tells a similar story of Weyl's lasting fascination with intuitionism, specifically with the notion of the mathematical continuum.

The historian of mathematics Erhard Scholz ascribes "historical rationality" to the moves of Hermann Weyl by taking a "non-systematic" perspective that attempts to elucidate Weyl's changes of mind (Scholz 2000). Scholz describes Weyl's inclination towards intuitionism as rooted in his philosophical considerations that were deeply influenced by Fichte's approach to the concept of continuum and space. Even though the connection Weyl saw between intuitionism and the established structures of purely infinitesimal geometry may never exist outside his mind, Scholz maintains that "Illusions have often been historical driving forces, and if such closeness may be considered in retrospect as an illusion, we nevertheless have to take it into account, if we want to understand how Weyl became a protagonist in the spread of intuitionistic analysis" (Scholz 2000, 5). Scholz's attempt to take into account Weyl's inner "Besinnung" (translated as "reflection" or "contemplation", see: (Scholz 2004, 14)) as part of his motivation to consider different foundational ideas is a perspective that I wish to broaden using Fisch's concept of ambivalence.

The retelling of Weyl's story in light of Fisch's notion of ambivalence suggests a view that enhances Bell's, van Dalen's, and Scholz's readings; it adds to their interpretations a deeper philosophical aspect that is currently missing from their accounts and should not be overlooked if one wants to gain a better understating of Weyl's shifting positions. Weyl's ambivalence regarding the foundations of mathematics is a continuous thread that is woven into his writings throughout his life; thus, his motivation for changing his mind and the roots of his indecision play a significant role within the contours of the whole story.

By addressing Weyl's undecidedness as ambivalence in a Fischian manner, the rationale behind his constant change of mind unravels: as a practitioner that has been exposed to external criticism (not only in the form of Brouwer's intuitionism, but also through the works of philosophers and mathematicians such as Kant, Fichte, Husserl, Poincare, the French intuitionists, and from 1927 Hilbert again), Weyl found himself engaged in normative self-critical deliberations not only about whether mathematics can be built upon intuitionistic, constructive, or formalistic foundations but also about what it means to be an intuitionist, a formalist, or a constructivist.

Developments in Brouwer's intuitionism and Hilbert's formalism have also contributed to Weyl's foundational considerations. Neither theory remained stagnant over the years: Brouwer's intuitionistic views had changed as he went from addressing the intuition of the continuum as an unanalyzable entity (Brouwer 1907) to introducing new concepts such as spreads and choice sequences in order to show how points on the continuum are identified with specific choice sequences (van Atten 2017; Brouwer 1918). A later development in

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[7] The original quote appears in (van Dalen 1995, 166).

Brouwer's intuitionism occurred when his proof of fan theorem appeared in 1924, followed by the introduction of continuity theorem dealing with intuitionistic continuous functions (Brouwer 1924, 1927).

Hilbert's views on the foundations of analysis have also transformed during the first decades of the 20ᵗʰ century (Corry 2004), specifically his perspective regarding the logical basis required for rebuilding the foundations of mathematics. Hilbert's quest for consistency proof in the early 1900s had led him to abandon Dedekind's logicism (Ferreirós 2009) and turn to Schröder's conception of logic, even though it was not particularly suitable for Hilbert's formalistic purposes (Zach 2019). In 1914, a few years after the publication of Russell and Whitehead's *Principia Mathematica*, Hilbert became intensely involved with this work. He believed that the point of view developed there could lead to laying down the necessary logical foundations for his axiomatic treatment of mathematics (Mancosu 1999). Later on, however, he came to realize that other fundamental problems of axiomatics still remained unsolved. Therefore, he devoted the following years (1917-1921) to develop first-order logic and presented his new foundational approach in 1922 as a response to Brouwer's intuitionism (Hilbert 1922; Zach 2007). During 1926-1928 Hilbert's program was further developed, as he introduced a distinction between real and ideal formulas, followed by his notions of ideal propositions (Hilbert 1926) and real propositions (Hilbert 1927).

In 1925, as the debate gained increasing attention from practitioners inside and outside the field (Hesseling 2003), Weyl had published a paper in response to Hilbert's reaction to Brouwer's intuitionism. Despite the paper's conciliatory tone, Weyl criticized Hilbert for his intention to "secure not the *truth*, but the *consistency* of the old analysis" (Weyl 1925, 136). Brouwer's intuitionism offered a very substantial and attractive idea of mathematical truth (Dummett 1973, 1977); but when Brouwer's fan theorem proof was published in 1924, it revealed a problematic aspect in the epistemological standards of Brouwer's intuitionism, threatening the validity and justifiability of the intuitionistic view of mathematical truth (Epple 2000). The realization that Brouwer's intuitionism may not be able to adequately address the notion of mathematical truth as it promised, alongside with the developments in Hilbert's program[8] (Hilbert 1926; Sieg 1999), might have weakened Weyl's confidence in Brouwer's revisionist program and its core concepts, thereby contributing to his feelings of ambivalence.

The feeling of ambivalence in Fisch's model accounts for more than merely explaining the persistence of practitioners' hesitant stances or indecisive acts; it primarily motivates for action. Driven by their ambivalence, practitioners are motivated to try and reconcile the stances they are torn between, creating new hybrid solutions. Even though these split accounts often

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[8] As can be read from Sieg's citation of Hilbert from 1923, Hilbert stated that the consistency problem is "no longer a question of proving that a system of infinitely many things is logically possible, but only of recognizing that it is impossible to derive a pair of formulas like A and ~A from the axioms extant in the formulas according to the rules of the logical calculus" (Sieg 1999, 40). Otherwise stated, Hilbert no longer searched for a proof of the logical possibility of an axiomatic system, but rather for a formal proof that an axiomatic system is non-contradicting. Moreover, it shows that Hilbert did not regard consistency proofs as indicating anything about mathematical existence, a view that coincides with the intuitionistic approach to consistency.

preserve the practitioner's indecision, they serve to ambivalate leading practitioners within the discipline who attempt, in turn, to formulate less jarring accounts (Fisch 2017).

Within the contours of Weyl's story, I can point to at least two attempts to formulate similar hybridic solutions: one in Weyl's 1918 monograph *Das Kontinuum* and another one three years later in "On the New Foundational Crisis in Mathematics". In *Das Kontinuum* Weyl offered to subject the epsilon relation to type restriction in order to avoid the paradoxes of set theory, and to restrict the use of quantifiers in the set-theoretic comprehension scheme in order to avoid the vicious circle of impredicative definitions (Weyl 1918). The solution he proposed in "On the New Foundational Crisis in Mathematics" was a more radical one, since here Weyl embraced Brouwer's choice sequences and completely forfeited arithmetically definable sequences, and the principle of excluded middle. Both attempts were to some extent misread, and even though *Das Kontinuum* is held these days in high esteem, it took the scientific community almost 60 years to properly read it (van Dalen 2013).

The question of whether Weyl's attempts are endowed with the same characteristics of hybridity and potential influence as Peacock's case of "A Treatise on Algebra" is an intriguing question worth perusing in a study of its own. Nevertheless, the opportunity to address Weyl's indecision as ambivalence sheds a different light not only on his hesitant stances but also on his motivations for developing new solutions to the foundational problem.

Weyl's frequent changes of heart might also be attributed, at least partially, to a wider historical context of cultural shifts. In his monograph, *Plato's Ghost: The Modernist Transformation of Mathematics*, the mathematician and historian of mathematics Jeremy Gray portrays a connection between individuals' works and a broader social perspective on transitions in mathematics. Gray argues that between 1890 and 1930, the whole discipline of mathematics went through a cultural shift he refers to as "modernist transformation". Individual's works that convey genuine intellectual concerns are promoted by "significant groups of people with the right opportunities" who are able to spread them and thereby contribute to the overall process of cultural change (Gray 2008, 5). Cultural and political events such as the First World War also played a substantial role in shaping the mathematical landscape (Gray 2008, 406–7). Even though Gray's analysis mainly focuses on one direction, namely, how individual's works affect the process of transition (and not the other way around), it would be shortsighted to dismiss the way cultural and historical changes affected Weyl's views regarding certain theories, concepts, and ideas. As Dirk van Dalen points out, Weyl himself had characterized the tone of his 1921 paper as influenced by "the mood of excited times - the times immediately following the First World War" (van Dalen 1995; Weyl 1956). It goes beyond the scope of the current paper to comprehensively characterize the impact of cultural transitions on Weyl's changes of mind and vice versa, but hopefully the perspective presented here on Weyl's ambivalence provides prolific ground for further research on this topic.

6. Concluding Remarks

Throughout this paper, I have attempted to present an alternative view to Hermann Weyl's turn to intuitionism. My central argument is that a philosophical perspective of normative framework transitions is fundamentally missing from the current historical accounts of Weyl's story. Specifically, I claim that the key to understanding Weyl's undecidedness lies in Menachem Fisch's notion of ambivalence.

Fisch maintains that we are capable of becoming ambivalent, but not ambivalating ourselves towards our norms. Only a trusted criticism coming from without can change our whole point of view and induce a rational, justified transformation of our normative framework. The process of becoming and being ambivalent has no time restrictions and can last throughout a practitioner's professional career. Moreover, there is no guarantee that ambivalence must eventually resolve in a clear decisive stance; a practitioner might remain ambivalent during his whole life, and Weyl is a case in point.

Unlike other practitioners who may have experienced the same difficulties regarding the foundational crisis, Weyl expressed his prolonged search for a solution in his writings, explicitly articulating the problematic aspects of every mathematical theory he had considered. The continued tone of ambivalence present in his papers is not a common phenomenon within the mathematical landscape, and it was often attributed to his philosophical inclinations that also changed over the years. The philosophical perspective presented in this paper suggests ways of providing a different analysis of Weyl's continuous undecidedness, taking into consideration the intrasubjective process of changing one's normative framework. Weyl's so-called shifting positions should be regarded only as symptoms of a much deeper, convoluted intrapersonal process of self-deliberation, in his attempt to find a solid ground on which mathematics can be built.

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