On the Connection Between Quantum Probability and Geometry

Federico Holik

Instituto de Física La Plata, CONICET, La Plata, Buenos Aires, Argentina. E-mail: holik@fisica.unlp.edu.ar

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We discuss the mathematical structures that underlie quantum probabilities. More specifically, we explore possible connections between logic, geometry and probability theory. We propose an interpretation that generalizes the method developed by R. T. Cox to the quantum logical approach to physical theories. We stress the relevance of developing a geometrical interpretation of quantum mechanics.
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1 Introduction

The need of an axiomatic treatment of probability theory, was sixth in the famous list of problems presented by David Hilbert in a Conference Held in Paris, during the year 1900 (see [1] for the complete list of problems). Hilbert himself dedicated big efforts to solve his sixth problem, and remarkably, he was also involved in the development of quantum mechanics. The formalism of quantum mechanics achieved its rigorous formulation after a series of papers by von Neumann, Jordan, Hilbert and Nordheim [2]. Its final form was condensed by von Neumann [3], after a series of papers (see the discussion in [4] for the influence of P. Jordan in the developments of von Neumann). It is remarkable that the first works presenting an axiomatic treatment of quantum probability appeared before Kolmogorov's masterpiece [5].

In the axiomatic approach of von Neumann, geometry plays a major role: projection operators, that represent closed subspaces of a suitable linear space, are central to the construction. Pure states are represented by points in the projective geometry associated to a Hilbert space, and more general elementary events are represented by higher dimensional linear varieties. Probabilities are defined as measures over the elements of this geometry, in such a way that transition probabilities between pure states are related to the geometrical notion of angle between one dimensional subspaces (also called *rays*). The spectral decomposition theorem [6, 7] allows to associate a projection valued measure to any quantum observable represented by a self adjoint operator [3, 7].

It turns out that the set of projection operators can be endowed with a lattice structure. More specifically, they form an orthomodular lattice [8]. The occurrence of lattices is very natural in probability theory. In Kolmogorov's approach, probabilities are defined as measures over sigma-algebras of subsets of a given set. Sigma-algebras are, in turn, Boolean algebras, which can be defined as complemented distributive lattices. Boolean algebras are naturally related to the algebraic treatment of classical logic. The occurrence of non-Boolean lattices in the axiomatization of quantum probabilities was quickly recognized by Birkhoff and von Neumann as a sort of non-classical logic, which they called *quantum logic* [9]. Later on, the quantum logical approach of Birkhoff and von Neumann was developed further by other researchers, giving rise to different lines of research (see for example [10–21]). For complete expositions of the quantum logical approach see [7, 8, 22–24]. Among the results of this research, it can be shown that a propositional structure associated to a quantum system can be coordinatized in a generalized Hilbert space [11]. A later result by Solèr asserts that, under reasonable conditions, it can only be a Hilbert space over the fields of the real numbers, complex numbers or quaternions [25].

During the 1930s, von Neumann also turned his attention to generalizations of quantum theory in terms of the study of *rings of operators*. The results of this research gave place to the algebraic structures which are known today as *von Neumann algebras* [7]. The theory of von Neumann algebras turns out to be strongly related to lattice theory: in a series of papers, Murray and von Neumann provided a classification of factors (von Neumann algebras whose center is formed by the multiples of the identity operator) in terms of orthomodular lattices [26–29]. The theory of von Neumann algebras is central in the study of the axiomatic formulation of quantum field theory and quantum statistical mechanics (see for example [30–33]). It is also important to remark that the lattices of projection operators play a key role in the study of probabilities associated to von Neumann algebras [34]. In the algebraic approach, states are defined as positive and normalized functionals over the von Neumann algebra, and are said to be normal if they satisfy an additional continuity property (see for example [34] for details). The set of states thus defined is convex, and pure states are defined as its extreme points.

As is well known, there are strong connections between lattice theory and geometry: projective geometry can be described in terms of lattices and related also to vector spaces [35]. The archetypical example is that of vector spaces: each vector space has associated a projective geometry and a lattice of subspaces. In particular, the projection operators associated to the Hilbert spaces used in quantum mechanics form a lattice, and the collections of pure states—which are in one to one correspondence to one dimensional subspaces—define projective geometries. But von Neumann was not only interested in projection lattices coming form Hilbert spaces. He was specially interested in more general geometrical objects, namely, continuous geometries [36, 37]. A remarkable example of this is that of the geometries associated to the type II_1 factors found in the classification theory of Murray–von Neumann. Type II_1 algebras are non-atomic and the type III contains no non-trivial finite projections. In this way, it could be said that the generalization of algebras studied by von Neumann points in the direction of a rather radical generalization of geometry. Using the words of von Neumann:

> "I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more. After all, Hilbert-space (as far as quantum-mechanical things are concerned) was obtained by generalizing Euclidean space, footing on the principle of "conserving the validity of all formal rules". This is very clear, if you consider the axiomatic-geometric definition of Hilbert-space, where one simply takes Weyl's axioms for a unitary-Euclidean-space, drops the condition on the existence of a finite linear basis, and replaces it by a minimum of topological assumptions (completeness + separability). Thus Hilbert-space is the straightforward generalization of Euclidean space, if one considers the vectors as the essential notions.
>
> Now we begin to believe, that it is not the vectors which matter but the lattice of all linear (closed) subspaces. Because:
>
> 1. The vectors ought to represent the physical states, but they do it redundantly, up to a complex factor, only.
> 2. And besides the states are merely a derived notion, the primitive (phenomenologically given) notion being the qualities, which correspond to the linear closed subspaces.
>
> But if we wish to generalize the lattice of all linear closed subspaces from a Euclidean space to infinitely many dimensions, then one does not obtain Hilbert space, but that configuration, which Murray and I called "case II_1". (The lattice of all linear closed subspaces of Hilbert-space is our "case I_∞".) And this is chiefly due to the presence of the rule
>
> $$a \leq c \longrightarrow a \vee (b \wedge c) = (a \vee b) \wedge c \quad \text{[modularity!]}$$
>
> This "formal rule" would be lost, by passing to Hilbert space!" [38]

The approach to quantum probability developed by von Neumann was fundamentally connected to geometry. But also to logic, in the sense of defining quantum probabilities as measures over an algebraic structure which is a variant of the Boolean algebras that underlie the Kolmogorovian approach. In this short article we will discuss quantum probabilities in connection to a *generalized probability theory* [39], emphasizing its geometrical features. We believe that the study of these features allows for a deeper understanding of quantum theory, and can be used as a basis for a geometric interpretation of its formalism.

Probability measures can be defined in general von Neumann algebras [15, 34]. A generalized non-Kolmogorovian probability calculus can be developed including Kolmogorovian probabilities as a particular case (i.e., when the algebra is commutative) [34]. Thus, the approach developed by von Neumann and others leads to interesting connections between logic, geometry and probability theory, that we will discuss here.

After reviewing the standard approaches to Kolmogorovian, quantum, and generalized probabilities in Section 2, we continue in Section 3 with a discussion of a problem posed by von Neumann regarding the foundations of logic and probability theory. In Section 4 we review an approach to generalized probability theory presented in [40], and then, in Section 5, we discuss its consequences for the problem posed by von Neumann. Finally, in Section 6, we draw our conclusions.

2 Formal aspects of probabilistic models

In this section we describe classical probabilities (following Kolmogorov's axiomatics) and quantum probabilities. Finally, we review a generalized version of probability theory, that includes classical and quantum probabilities as particular cases.

2.1 Kolmogorovian probabilities

The final form of Kolmogorov's axiomatization of probability theory was presented in [5], during the 1930s. It is based on measure theory, and it synthesized the previous efforts of mathematicians in the field of probability theory [41]. In order to motivate the definition, let us consider a very simple example. Suppose that we have a dice. In any throw of the dice, any member of the set $\Omega = \{1, 2, 3, 4, 5, 6\}$ can occur as a possible outcome. Given that Ω represents all possible outcomes of the experiment, it is called the *outcome set*. A probabilistic state of the dice is determined by assigning probabilities p_i, $i = 1, ..., 6$, to each element of Ω. These numbers must satisfy $p_i \geq 0$ for all i (probabilities are positive), and $\sum_i p_i = 1$ (probabilities are normalized). If the dice is not loaded, then $p_i = \frac{1}{6}$ for all i. But a realistic description of a dice must assume that it can be loaded, and then, all possible probabilistic assignments must be considered.

One is not only interested in the probabilities of occurrence of each element of Ω, but also in the probabilities assigned to subsets of it. The reason is simple: one may wonder which is the probability of occurrence of an even result, and this event is represented by the subset $\{2, 4, 6\}$. A simple argument shows that the probability of this event is given by $p_{Even} = p(\{2, 4, 6\}) = p_2 + p_4 + p_6$. The event associated to the proposition "the outcome is an even number" is then represented by the set $\{2, 4, 6\}$. Its negation, "the outcome is an odd number", is represented by its set theoretical complement, which is the set $\{1, 3, 5\}$. We obtain that the probability $p_{Odd} = p(\{1, 3, 5\}) = p_1 + p_3 + p_5 = 1 - p_{Even}$, yielding a functional relation between the probability of a given event, and that of its negation. In a similar way, the disjunction of two given events is represented by the set theoretical union, and the conjunction by the set theoretical intersection. Let us consider examples of this, based on the propositions "the outcome is (strictly) greater than 3" and "the outcome is even". These propositions are associated to the events represented by the sets $\{4, 5, 6\}$ and $\{2, 4, 6\}$, respectively. It is easy to check that the conjunction of those propositions is given by the set $\{4, 6\} = \{4, 5, 6\} \cap \{2, 4, 6\}$. Similarly, the disjunction is given by $\{2, 4, 5, 6\} = \{4, 5, 6\} \cup \{2, 4, 6\}$.

Thus, one needs to consider all possible events of interest in the description, which are represented by subsets of Ω. And these are endowed with the set theoretical operations of complement "$(...)^c$" (representing the logical negation "$\neg$"), intersection "$\cap$" (representing conjunction "$\wedge$") and union "$\cup$" (representing disjunction "$\vee$"). In this way, each logical connective has a set theoretical counterpart. The collection of all possible events forms what is known as a σ-algebra (or a $\sigma-$field) Σ, which is a collection of subsets of Ω that is closed under complements and countable unions, and includes Ω itself. A probability measure assigns a probability to each member of Σ. The axioms that we give below are a natural generalization of the simple rules described in the dice example.

Given an outcome set Ω, consider a σ-algebra Σ of subsets of Ω. A probability measure will be a function μ such that

$$\mu : \Sigma \to [0, 1] \tag{1a}$$

satisfying

$$\mu(\Omega) = 1\,, \tag{1b}$$

and for any pairwise disjoint denumerable family $\{A_i\}_{i\in I}$

$$\mu\left(\bigcup_{i\in I} A_i\right) = \sum_i \mu(A_i) \tag{1c}$$

Conditions (1) are known as *Kolmogorov's axioms* [5]. The triad (Ω, Σ, μ) is called a *probability space*. Since

σ-algebras are also *Boolean algebras*, probability spaces obeying Eqs. (1) are usually referred to as Kolmogorovian (also: classical, commutative, or Boolean) [15, 39]. One of the most important features of the Boolean algebra Σ, is that the connectives obey what is known as the distributive law:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad \forall\ A, B, C \in \Sigma \quad (2)$$

A similar distributivity law is obtained by interchanging the roles of "$\cap$" and "$\cup$" in 2 (see [8] for details). In terms of the set theoretical complement, these imply that:

$$A = A \cap (B \cup (B)^c) = (A \cap B) \cup (A \cap (B)^c), \quad \forall\ A, B \in \Sigma \quad (3)$$

Equations 2 and 3 have their counterparts in logical expressions, by exploiting the connection between the set theoretical and the classical logic connectives: by putting "$\vee$" instead of "$\cup$" and "$\cap$" instead of "$\wedge$". These are given by

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C), \quad \forall\ A, B, C \in \Sigma \quad (4)$$

Also, by replacing "$(\ldots)$" by "$\neg$", we obtain

$$A \wedge (B \vee \neg B) = (A \wedge B) \vee (A \wedge \neg B), \quad \forall\ A, B \in \Sigma \quad (5)$$

It is easy to find examples of (4) and (5) using propositions about the outcomes of the dice example.

Related to the above distributivity relations, it is possible to show that in any probability space (Ω, Σ, μ), the *inclusion-exclusion principle* holds:

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B), \text{ for all A and B} \quad (6)$$

Again, these can be expressed using the logical connectives:

$$\mu(A \vee B) = \mu(A) + \mu(B) - \mu(A \wedge B), \text{ for all A and B} \quad (7)$$

Notice that Eq. (7) implies that $\mu(A \vee B) \leq \mu(A) + \mu(B)$, for all A and B (something that will no longer be valid when we deal with events associated to quantum systems).

The *Borel sets* ($B(\mathbb{R})$) are defined as the smallest family of subsets of $\mathbb{R}$ such that (a) it is closed under set theoretical complements, (b) it is closed under denumerable unions, and (c) it includes all open intervals [6]. A random variable f can be defined as a measurable function $f : \Omega \longrightarrow \mathbb{R}$ satisfying that, for every Boreal subset B of the real line, we have that $f^{-1}(B) \in \Sigma$ (i.e., the pre-image of every Borel set B under f belongs to Σ, and thus it has a definite probability measure given by $\mu(f^{-1}(B))$). Any random variable f defines a pre-image map satisfying:

$$f^{-1} : B(\mathbb{R}) \longrightarrow \Sigma \quad (8a)$$

satisfying

$$f^{-1}(\emptyset) = \emptyset \quad (8b)$$

$$f^{-1}(\mathbb{R}) = \Sigma \quad (8c)$$

$$f^{-1}\left(\bigvee_j B_j\right) = \bigvee_j f^{-1}(B_j) \quad (8d)$$

for any disjoint denumerable family B_j. Also,

$$f^{-1}(B^c) = (f^{-1}(B))^c \quad (8e)$$

The function f^{-1} is based in the pre-image of Borel sets, and should not be confused with the inverse function, that has a different domain.

Eqs. (8) are important for us, because we will use them to compare quantum vs classical observables in the following section.

Summarizing: we have seen that a Kolmogorovian probabilistic model can be described using measures defined over subsets of a given set, which are endowed with algebraic operations that correspond to the connectives of classical logic. Connectives can be defined in such a way that events satisfy the distributivity laws 4 and 5, and probabilities satisfy the inclusion-exclusion principle 7.

2.2 Quantum probabilities

Quantum mechanics is a very special theory, since it is essentially probabilistic. And this is so, independently of the chosen interpretation: any empirically *testable* prediction of the theory has a probabilistic character. In most axiomatizations, the theory of Hilbert spaces plays a key role, since to any physical system a separable complex Hilbert space $\mathcal{H}$ is assigned. Physically meaningful observables are then represented by linear (self-adjoint) operators acting on $\mathcal{H}$, and states are mathematically described by linear functionals over them. As we explain below, the closed linear subspaces of $\mathcal{H}$ are central to the formalism, since they represent elementary experiments and are the building blocks out of which any other observable can be constructed. Furthermore, the values that states assign to the elementary experiments represented by those subspaces fully determine the probabilistic information that one has to deal with in experiments.

In order to understand why closed linear subspaces represent elementary experiments, let us consider an example. Suppose that we design a devise to test wether the energy of a given quantum system is ϵ_0 or not. This can be called a YES-NO experiment, given that it only has two possible outomes: we answer YES if the measured energy is ϵ_0, and NO if we obtain any other result. If we prepare the system in an eigenstate $|\psi\rangle$ of the Hamiltonian with eigenvalue ϵ_0, then, the outcome of the experiment will be YES with certainty. Any other eigenstate (with the same eigenvalue) will yield the same outcome with certainty. And also any linear combination of them. Thus, the collection of pure states that yield the outcome YES with certainty forms a subspace of the Hilbert space. Thus, to the proposition "the energy of the system is ϵ_0", we can naturally assign a subspace $\mathbb{S}_{\epsilon_0}$. Equivalently, since closed subspaces and orthogonal projections are in a one to one correspondence, we can also assign a projection to the proposition (namely, the one that projects into the linear subspace). If we compare this feature of the quantum formalism with the dice example discussed in the previous Section, we find that, in quantum mechanics, *events* are represented by linear subspaces instead of the simple subsets of a given outcome set. Similarly, each closed subspace (or orthogonal projection) of the Hilbert space will have an associated YES-NO experiment. As we will see below, linear subspaces are very specific geometrical objects.

Let $\mathcal{P}(\mathcal{H})$ be the collection of closed subspaces associated to a complex separable Hilbert space. Given $\mathbb{S}, \mathbb{T} \in \mathcal{P}(\mathcal{H})$, define $\mathbb{S} \vee \mathbb{T} := \overline{\mathbb{S} \oplus \mathbb{T}}$ (the closure of the direct sum), $\mathbb{S} \wedge \mathbb{T} := \mathbb{S} \cap \mathbb{T}$ and $\neg\mathbb{S} := \mathbb{S}^{\perp}$. With these operations, $\mathcal{P}(\mathcal{H})$ has the structure of an *orthomodular lattice* (see [8] for details). The connectives "$\vee$", "$\wedge$" and "$\neg$" are the quantum logical analogues of the Boolean connectives. It is important to remark that, with these connectives, the lattice of closed subspaces does not satisfies the distributivity laws (4 and 5), and then, it is non-Boolean [9]. This implies that the new connectives cannot be interpreted in a classical way (i.e., as the logical connectives of classical logic). Notwithstanding, the new connectives can find a very natural operational interpretation (see for example the discussion in [11]). As an example, consider the proposition: "the energy of the system *is not* ϵ_0" (which is just the negation of the proposition considered in the above example). It is easy to check that the pure states that make this proposition true are all contained in the subspace $(\mathbb{S}_{\epsilon_0})^{\perp}$ (the orthogonal complement of $\mathbb{S}_{\epsilon_0}$). Similarly, if we consider the pure states that make true the propositions "the system has energy ϵ_0 *or* ϵ_1" and "the system has energy ϵ_0 *and* ϵ_1", we can easily find that they are contained in the subspaces $\mathbb{S}_{\epsilon_0} \vee \mathbb{S}_{\epsilon_1}$ and $\mathbb{S}_{\epsilon_0} \wedge \mathbb{S}_{\epsilon_1}$, respectively.

In order to present quantum probabilities in a measure-theoretic fashion, one can use the following axioms on a function s [7]:

$$s : \mathcal{P}(\mathcal{H}) \to [0; 1] \tag{9a}$$

such that:

$$s(\mathbf{1}) = 1 \text{ } (\mathbf{1} \text{ is the whole Hilbert space}) \tag{9b}$$

and, for a denumerable and pairwise orthogonal family of closed subspaces $\mathbb{S}_j$

$$s\Big(\bigvee \mathbb{S}_j\Big) = \sum_j s(\mathbb{S}_j). \tag{9c}$$

Gleason's theorem [42] grants that if $dim(\mathcal{H}) \geq 3$, for any measure s satisfying (9), there exists a positive Hermitian trace class operator (of trace one) ρ_s, such that

$$s(\mathbb{P}) := \mathrm{tr}(\rho_s P) \tag{10}$$

where P is the projection operator associated to the subspace $\mathbb{P}$. And also the converse is true; using Eq. (10), any positive trace-class Hermitian operator of trace one defines a measure satisfying (9). Thus, Eqs. (9) can be used as an equivalent way of defining the set of all possible quantum states.

One of the main differences between the axioms (1) and (9) is that the σ-algebra in (1) is Boolean, while $\mathcal{P}(\mathcal{H})$ is not. In this sense, the measures defined by Eqs. (9) are called *non-Kolmogorovian (or non-Boolean) probability measures*.

It is also very important to mention that, even if $\mathcal{P}(\mathcal{H})$ is not Boolean, it has Boolean subalgebras. Each *maximal* Boolean subalgebra of $\mathcal{P}(\mathcal{H})$ is intended to represent a *measurement context*. In other words, the collection of all possible events associated to a given quantum mechanical empirical context C—defined by an experiment that measures a complete set of commuting observables—forms a maximal Boolean subalgebra Σ_C of $\mathcal{P}(\mathcal{H})$. When the function representing a state $s(...)$ (i.e., a measure satisfying (9)) is restricted to Σ_C, we obtain a classical probability space (satisfying Eqs. (1)). Thus, a quantum state can be considered as a collection of classical probability distributions, one for each context. But the remarkable fact is that there is no *classical* joint probability distribution for all of them (see, for example, the discussion in [43]). Furthermore, $\mathcal{P}(\mathcal{H})$ can be described as a pasting of its maximal Boolean subalgebras [8, 44]. And this pasting is such that two different maximal Boolean subalgebras may share elements between them. In other words, contexts are *intertwined* in a complex way [45]. This means that a given observable may belong to very different and incompatible measurement contexts. The intertwining of the Boolean subalgebras of $\mathcal{P}(\mathcal{H})$ is behind the Kochen–Specker contextuality (see [46–48]). According to the Kochen–Specker theorem, there is no algebra homomorphism between $\mathcal{P}(\mathcal{H})$ and the two valued Boolean algebra $\{0, 1\}$ (see, for example, [49] for details).

In quantum theory, physical quantities are represented by self-adjoint operators acting on a separable Hilbert space. These are the non-commutative analogs of the random variables of Kolmogorov's theory (see for example, the discussion in [39]). Due to the *spectral theorem* any self-adjoint operator A can be written as [6]:

$$A = \int_{\mathbb{R}} \lambda\, dP_A(\lambda) = \int_{\mathbb{R}} \lambda P_A(d\lambda) \tag{11}$$

where $P_A(...)$ is the *spectral measure* associated to A. More specifically, given an observable A, a *projection valued measure* (PVM) is a map P_A satisfying:

$$P_A : B(\mathbb{R}) \to \mathcal{P}(\mathcal{H}), \tag{12a}$$

such that

$$P_A(\emptyset) = \mathbf{0} \ \ (\mathbf{0} := \text{null subspace}) \tag{12b}$$

$$P_A(\mathbb{R}) = \mathbf{1} \tag{12c}$$

$$P_A\left(\bigvee_j (B_j)\right) = \bigvee_j P_A(B_j), \tag{12d}$$

for any disjoint denumerable family B_j. Also,

$$P_A(B^c) = \mathbf{1} - P_A(B) = (P_A(B))^{\perp} \tag{12e}$$

There are important facts related to PVMs. First, notice that the image of $P_A(...)$ (namely, $P_A(B(\mathbb{R}))$), is a Boolean subalgebra of $\mathcal{P}(\mathcal{H})$. Second, the spectral theorem allows us to associate a PVM to each observable, so that they are in a one to one correspondence. With this in mind, it is now important to compare Eqs. (12) with Eqs. (8). Both maps (i.e., $P_A(...)$ and $f^{-1}(...)$) send the collection of Borel sets into a Boolean algebra: a subalgebra of $\mathcal{P}(\mathcal{H})$ in the first case, and a subalgebra of Σ in the second. Furthermore, Eqs. (12) and (8) tell us that both $P_A(...)$ and $f^{-1}(...)$ define an algebra homomorphism between $B(\mathbb{R})$ and their respective images. These similarities show that quantum observables can be considered as the non-Kolmogorovian (or non-commutative) version of classical observables.

In finite dimensions, the spectral theorem acquires a simple form:

$$A = \sum_i a_i P_i \tag{13}$$

where the a_i's are the eigenvalues of A, and P_i is the projection operator that projects onto the subspace associated to the eigenvalue a_i.

One of the expressions of the fact that quantum and classical probabilities are different, is that Eq. (6) is no longer valid in quantum mechanics. Indeed, in quantum mechanics it may happen that

$$s(\mathbb{P}) + s(\mathbb{Q}) < s(\mathbb{P} \vee \mathbb{Q}) \tag{14}$$

for suitably chosen $\mathbb{P}$, $\mathbb{Q}$, and s. In the quantum domain, the sum rule holds whenever $\mathbb{P}$ and $\mathbb{Q}$ are compatible (i.e., if they commute). When this is the case, then, they can be shown to be contained in a Boolean subalgebra representing a measurement context. But, if $\mathbb{P}$ and $\mathbb{Q}$ are taken to be non-compatible (i.e., non-commutative), the sum rule will no longer be valid in general, and an inequality such as (14) might be obtained.

It is of major importance for us the fact that the elements of $\mathcal{P}(\mathcal{H})$ are the linear varieties associated to the projective geometry of a Hilbert space. As Varadarajan clearly expressed at the beginning of his book on the geometry of quantum mechanics:

> "It must be pointed out, however, that the precise mathematical nature of the superposition principle was only implicit in the discussions of Dirac; we are indebted to John von Neumann for explicit formulation. In his characteristic way, he discovered that the experimental statements of a quantum mechanical system formed a projective geometry—the projective geometry of subspaces of a complex, separable, infinite dimensional Hilbert space. With this as a point of departure, he carried out a mathematical analysis of the axiomatic foundations of quantum mechanics which must certainly rank among its greatest achievements.
>
> Once the geometric point of view is accepted, impressive consequences follow..." (see the Introduction of Ref. [13]).

A physical interpretation of this geometry could be of great help for a better understanding of quantum phenomena, and it is a very interesting point of departure for an interpretation of quantum probabilities.

Let us finish this section with a discussion about the geometrical interpretation of probabilities in quantum mechanics. A complex projective space (as is the case in quantum mechanics) is a compact Kähler manifold. In that space, it is possible to give a notion of geodesic distance that satisfies the following identity (see for example, [50] and [51]):

$$|\langle\psi|\phi\rangle|^2 = \cos^2\left(\frac{\sigma(|\psi\rangle, |\phi\rangle)}{\sqrt{2}}\right) \tag{15}$$

where $\sigma(|\psi\rangle, |\phi\rangle)$ is the geodesic distance separating the rays defined by $|\psi\rangle$ and $|\psi\rangle$. Notice also that the distance is naturally connected with the notion of angles between two given rays. If the system is prepared in state $|\psi\rangle$, the transition probability to the state $|\psi_i\rangle$ is then given by

$$p_i(\psi) = \cos^2\left(\frac{\sigma(|\psi_i\rangle, |\psi\rangle)}{\sqrt{2}}\right) \tag{16}$$

If a_i is a degenerate eigenvalue of an observable, there is a projection operator P_i associated to that eigenvalue, that defines a submanifold of the projective space. In that case, the transition probability $p_i(|\psi\rangle)$ of obtaining the outcome a_i of a given observable, given that the system is prepared in state $|\psi\rangle$, is given by

$$p_{a_i}(|\psi\rangle) = \cos^2\left(\frac{\sigma(P_i, |\psi\rangle)}{\sqrt{2}}\right) \tag{17}$$

where $\sigma(P_i, |\psi\rangle)$ is the length of the shortest curve with initial point in the ray defined by ψ and terminal point in the subset of projective space defined by the projection operator P_i. These formulas can be used to provide a natural geometrical interpretation for transition probabilities: given a quantum system prepared in a state $|\psi\rangle$, the probability of obtaining a result of a particular experiment is ultimately determined by the shortest geodesic distance between the point in projective space defined by the initial state, and those points defined by the projection operator.

2.3 Generalized probabilistic models

In the algebraic formulation of relativistic quantum theory there appear algebras which are different from the Type I factors used in non-relativistic quantum mechanics. Normal states over these algebras define measures which obey axioms similar to those of the classical (Eqs. (1)) and quantum (Eqs. (9)) cases. Indeed, the orthogonal projections associated to Factor von Neumann algebras form orthomodular lattices [7]. This suggests that, in principle, one could conceive more general probabilistic models than those of standard quantum mechanics. We describe here a possible generalization, based in orthomodular lattices. Let $\mathcal{L}$ be a σ-complete orthomodular lattice (standing for the lattice of all possible empirical events of a given model). Then, define

$$s : \mathcal{L} \to [0; 1], \tag{18a}$$

such that:

$$s(\mathbf{1}) = 1. \tag{18b}$$

and, for a denumerable and pairwise orthogonal family of events E_j

$$s\left(\bigvee_j E_j\right) = \sum_j s(E_j). \tag{18c}$$

If we put $\mathcal{L} = \Sigma$ and $\mathcal{L} = \mathcal{P}(\mathcal{H})$, we recover the Kolmogorovian and quantum cases, respectively. For a discussion on the conditions under which measures as those defined in Eqs. (18) are well defined see [24, Chapter 11]. The fact that projection operators of arbitrary von Neumann algebras define orthomodular lattices [34] shows that the above generalization includes many examples of interest. In particular, quantum systems involving infinitely many degrees of freedom.

The states defined in Eqs. (18) define Kolmogorovian probabilities when restricted to maximal Boolean subalgebras of $\mathcal{L}$. Every orthomodular lattice $\mathcal{L}$ can be described as a pasting of its maximal Boolean subalgebras (see for example [8] and [44]). This implies that a state defined as a measure over an orthomodular lattice can be considered as collection of Kolmogorovian probabilities for which their σ-algebras are intertwined. If there is only one maximal Boolean subalgebra, then the whole $\mathcal{L}$ has to be Boolean. Thus, we recover a Kolmogorovian model. For theories which display contextuality, such as standard quantum mechanics [39, 52], there will be more than one intertwined empirical context, and then, the above decomposition will not be trivial.

3 Birkhoff and von Neumann's quantum logical approach to quantum probabilities

As is well known, any Boolean algebra can be represented in a set theoretical framework (as subsets of a given set) [53]. With regard to this relationship, von Neumann asserted that

> "And one also has the parallelism that logics corresponds to set theory and probability theory corresponds to measure theory and that a given system of logics, so given a system of sets, if all is right, you can introduce measures, you can introduce probability and you can always do it in very many different ways."(unpublished work reproduced in [54, pp. 244]).

In this way, we can see that, in the classical setting, there is a close connection between logic, set theory, and probability. What does this means? The definition of Cantor of a set reads

> "A set is a gathering together into a whole of definite, distinct objects of our perception [Anschauung] or of our thought—which are called elements of the set." [55]

A set is a collection of *objects*, and the internal logic governing them is *classical*. This also applies to the classical picture for *things in space-time*. In the usual approach to classical theories, material bodies fill space, and relate between themselves defining different trajectories in space-time. They are identifiable, and can be considered in collections. The situation of fields is similar, and a full description is given by telling how their associated magnitudes vary in space-time. Space-time itself is considered as a collection of space-time points, satisfying definite topological and geometrical requirements. In this sense, one can say that the ultimate level of the classical organization of experience in an Euclidean space-time—as well as in the curved background of General Relativity—is strongly related to classical logic.

As discussed above, the classical set-theoretical description is not only applied to material bodies or points in space-time, but also to processes and events. As explained in Section 2.1, the different events associated to a probabilistic description of a given system are organized as a σ-algebra of subsets associated to an outcome set. The singletons formed by the elements of Ω are the elementary events (or processes), and all the other events are formed by applying the set-theoretical operations to them. We have seen that these operations are naturally related to the connectives of classical logic. But the things change radically in the quantum formalism, as von Neumann pointed out

> "In the quantum mechanical machinery the situation is quite different. Namely instead of the sets use the linear sub-sets of a suitable space, say of a Hilbert space. The set theoretical situation of logics is replaced by the machinery of projective geometry, which is in itself quite simple.
>
> However, all quantum mechanical probabilities are defined by inner products of vectors. Essentially if a state of a system is given by one vector, the transition probability in another state is the inner product of the two which is the square of the cosine of the angle between them. In other words, probability corresponds precisely to introducing the angles geometrically. Furthermore, there is only one way to introduce it. The more so because in the quantum mechanical machinery the negation of a statement, so the negation of a statement which is represented by a linear set of vectors, corresponds to the orthogonal complement of this linear space." (unpublished work reproduced in [54, pp. 244]).

And von Neumann continues:

> "And therefore, as soon as you have introduced into the projective geometry the ordinary machinery of logics, you must have introduced the concept of orthogonality. This actually is rigorously true and any axiomatic elaboration of the subject bears it out. So in order to have logics you need in this set a projective geometry with a concept of orthogonality in it.
>
> In order to have probability all you need is a concept of all angles, I mean angles other than 90°. Now it is perfectly quite true that in geometry, as soon as you can define the right angle, you can define all angles. Another way to put it is that if you take the case of an orthogonal space, those mappings of this space on itself, which leave orthogonality intact, leave all the angles intact, in other words, in those systems which can be used as models of the logical background for quantum theory, it is true that as soon as all the ordinary concepts of logics are fixed under some isomorphic transformation, all of probability theory is already fixed." (unpublished work reproduced in [54, pp. 244]).

As von Neumann clearly indicates, the set-theoretical description is replaced by the geometrical machinery given by the collection of closed subspaces of a suitably chosen linear space. In other words: in the probabilistic calculus of quantum theory events are not organized set-theoretically, but geometrically. The peculiar algebraic properties of the geometries associated to quantum systems cannot be identified with the set-theoretical operations that come from classical logic, and then, the new connectives cannot have the same interpretation as those of classical logic. This analogy is the origin of the term "quantum logic" (which should not be considered as a logic *strictu sensu*). We believe that the fact that events (or observable processes) associated to quantum systems can be organized in a geometric way, is a fundamental feature of quantum theory. But it is important to remark that this notion of geometry, is not that of classical space-time. Quite on the contrary, *is the geometrical form in which quantum processes are organized*, which is of a very different nature. In analogy with the set-theoretical description of classical events, this geometrical form *has an internal logical structure, which is that of being a quantum logic, and is algebraically described as an orthomodular lattice*.

It is important to remark that this logic does not necessarily deny the classical logic that we use when we think. The word *logic* in this context refers to the *organization of processes (phenomena) in the quantum domain*. But what is the connection of all this with probability theory? J. von Neumann suggested a clue as follows

> "This means, however, that one has a formal mechanism, in which logics and probability theory arise simultaneously and are derived simultaneously." (unpublished work reproduced in [54, pp. 245]).

The above quotation clearly underlines a concrete research program: up to which extent one can build a "formal mechanism" in which logics and probability theory "arise and are derived simultaneously"? Digging into this possibility might shed new light in the understanding of quantum probability theory. In the following section we will discuss the implications of a derivation of quantum probabilities using the algebraic properties of the propositional lattice of quantum mechanics [40].

4 Cox' approach to quantum probabilities

R. T. Cox showed that, if it is assumed that a rational agent deals with an event structure that conforms with the rules of classical logic (i.e., the algebra of events is represented by a Boolean algebra), then a plausibility calculus—which is formally equivalent to that of Kolmogorov—can be derived in a natural way [56, 57]. This means that if one wants to define a degree of belief function which is compatible with the algebraic properties of the classical logical connectives, it must be equivalent to a Kolmogorovian probability (Eqs. (1)).

In a previous work [40] it was shown that the approach to probability theory of R. T. Cox [56, 57] can be applied to lattices much more general than Boolean (see also [58]). And in particular, that quantum probabilities and the generalized probability theory can be obtained by applying a variant of this method. We do not have place here to introduce all the details (for which we refer to [40]) and just limit ourselves to describe the general method:

- Our starting point is a *complete atomic orthomodular lattice* $\mathcal{L}$. We assume that the elements of $\mathcal{L}$ represent the events of a given system. This is a natural assumption, given that many relevant—classical or quantum—probabilistic models in physics fall into this class.

- It is reasonable to assume that there is a definite state of affairs determined by the preparation of the system. This is a precondition of physical science, something which is not necessarily true in any field of experience. A similar remark holds for the existence of—at least—statistical regularities: if such regularities were not present, the mathematical description of phenomena would be untenable. We are not asserting that any phenomena could be subsumed into this condition, but that it is a precondition of physical science. It is not relevant here whether the preparation process is natural or artificial. What is important for us is that the system has its own definite history, which is the collection of circumstances that give place to a concrete state of affairs.

- Define a function $s : \mathcal{L} \longrightarrow \mathbb{R}$ such that $s(E) \geq 0$ $\forall E \in \mathcal{L}$ and it is order preserving ($E_1 \leq E_2 \longrightarrow s(E_1) \leq s(E_2)$). This function is intended to represent the degree of likelihood about what would happen in the different future situations. But it is important to remark that this measure is a manifestation of a structured state of affairs.

It can be shown that under the above rather general assumptions, a probability theory can be developed [40] following a variant of R. T. Cox approach [56, 57]. In other words, it is possible to show that:

$$s\left(\bigvee_i^{\infty} E_i\right) = \sum_{i=1}^{\infty} s(E_i) \tag{19a}$$

$$s(\neg E) = 1 - s(E) \tag{19b}$$

$$s(\mathbf{0}) = 0 \tag{19c}$$

(where the $\{E_i\}$ in Eq. (19a) form a denumerable and orthogonal family). Let us see an example of how the Cox's machinery works. If $E_1, E_2 \in \mathcal{L}$ and $E_1 \perp E_2$, it is reasonable to assume that $s(E_1 \vee E_2)$ can only be a function of $s(E_1)$ and $s(E_2)$. In this way, $s(E_1 \vee E_2) = f(s(E_1), s(E_2))$, with f an unknown function to determine. Due to associativity of "$\vee$", $s((E_1 \vee E_2) \vee E_3) = s(E_1 \vee (E_2 \vee E_3))$ for any $E_1, E_2, E_3 \in \mathcal{L}$. If E_1, E_2 and E_3 are orthogonal, we will have $s((E_1 \vee E_2) \vee E_3) = f(f(s(E_1), s(E_2)), s(E_3))$ and $s(E_1 \vee (E_2 \vee E_3)) = f(s(E_1), f(s(E_2), s(E_3)))$. But then $f(f(s(E_1), s(E_2)), s(E_3)) = f(s(E_1), f(s(E_2), s(E_3)))$. Or put in a more simple form, we are looking for a function f (assumed to be continuous and strictly increasing in both arguments) such that

$$f(f(x, y), z) = f(x, f(y, z)) \tag{20}$$

But Eq. (20) is a functional equation [59] whose solution—*up to rescaling*—is $f(x, y) = x + y$. For a discussion about the rescaling we refer to [56]. In this way we arrive at $s(E_1 \vee E_2) = s(E_1) + s(E_2)$ (whenever $E_1 \perp E_2$). We refer to [40] for the complete derivation.

As explained above, it is possible to show that the probability theory defined by Eqs. (19) is non classical in the general case. If $\mathcal{L}$ is not Boolean, it may happen that $s((E_1 \wedge \neg E_2) \vee (E_1 \wedge E_2)) = s(E_1 \wedge \neg E_2) + s(E_1 \wedge E_2) < s(E_1)$ (for suitably chosen E_1, E_2 and s), while any Kolmogorovian probability satisfies $s(E_1) = s(E_1 \wedge E_2) + s(E_1 \wedge \neg E_2)$ [40].

5 Quantum probability and structured processes

Quantum mechanics seems to pose a problem in the interpretation of space-time as is expressed, for example, in the difficulties in defining trajectories for the particles without appealing to non-local hidden variables. This seems to suggest that a new kind of structure underlies the quantum mechanical description of natural processes.

Space-time—as considered by modern physics—is not a naturally given structure: it was a great achievement of mankind—that took many years—to develop geometry as an axiomatic theory. This geometrical approach to the world we experience underlies the mathematical description of reality provided, for example, by classical mechanics. And it turns out that our description of the geometry of space-time changes in history: general relativity assumes a different organization of space-time events. Our experience of natural phenomena is not a complete chaos. Quite the contrary, it can be structured in a geometrical way. But we must never forget that the fact that we can organize our experience using a space-time description is just an assumption, whose consistency is to be tested empirically. General relativity shows us that one can use a more elegant and more powerfully predictive description of experience than the one provided by the Euclidean flat space-time of classical physics. The limits and success of these descriptions are not granted a priori: they must be confronted with their capability of defining a consistent connection between theory and experiment.

We are somehow committed to a classical description in the following sense: we need definite and objective things to happen in order to even *speak* about an experiment. An example of this is a pointer of an instrument yielding a value in a given outcome set (which could be, for example, the set of real numbers, but it could also be more general, like the set formed by $\{+, -\}$). As we saw in Section 2.1, the fact that the outcomes of a experiment always form a set (called the *outcome set*) and the events will be represented by its subsets (forming a σ-algebra), ties us to a very specific kind of algebraic structure (namely, a Boolean algebra) which is closely related to classical logic. For certain observables, the outcome sets can also be endowed with very specific geometries (for example, Euclidean geometry, or a curved space-time). This perspective could be understood as a more accurate explanation of the observations of N. Bohr: the very possibility of exerting experiments ties us to classical logic and a set theoretical organization of experience. The space-time description is just a particular case of this more general regulative logical machinery.

But there is absolutely nothing granting us that the Boolean description (thought necessary to exert experiments) will exhaust the scenario in which phenomena occur. And this lies at the heart of the existence of complementary (and incompatible) contexts in quantum mechanics: in order to determine the state of a quantum system, a quantum tomography must be performed, and thus, we are obliged to study the system in different incompatible measurement contexts. While in classical mechanics the description of phenomena can be based in a purely set-theoretical approach (and thus, we can describe probabilities by appealing to a Boolean algebra associated to an outcome set), in quantum mechanics, this is no longer possible. Indeed, as explained in Section 2.2, we are interested in how the different Boolean algebras associated to the different measurement contexts are *intertwined*. The remarkable fact is that, even if the quantum description of all possible events cannot be reduced to that of a Boolean algebra, it can be endowed with a very precise *geometrical structure*. In quantum mechanics the geometry associated to the experimental propositions becomes of the essence, and then, we need more structure than the one given by the simple collection of subsets of a given set. In classical mechanics, the description of an object can be equated with its space-time representation: from the point of view of classical mechanics, the main goal is to describe continuous motion of material bodies (or the variation of fields) in space-time. That is why motion (and change) can be naturally described as the solutions of deterministic differential equations. And this feature is much more general than the usual description of a particle moving through space under the action of forces. Any quantity of interest taking continuous values, if it is classical, will have associated a time derivative, and thus the description reduces to the motion of a system in an abstract phase space obeying deterministic differential equations. The classical probabilistic description, appealing to stochastic equations, can be fully described using Kolmogorov's framework, and one can always assume an ignorance interpretation. Quite contrarily, quantum mechanics is (at least, empirically) characterized by *jumps*, by *discontinuous* and *unpredictable* behavior. That is why the organization of experience in quantum mechanics comes inherently endowed with a probabilistic description: it is impossible to predict the future events with complete certainty, and thus, the *actual* state of affairs has to be unavoidably described by appealing to a *probability distribution*. This is what led many authors to suspect that, in the quantum realm, probabilities have an ontological status, and that they do not accept an ignorance interpretation (see, for example, the discussion in [60]).

In the above sense, quantum mechanics fails to give a spatio-temporal description of phenomena. In other words, *it shows us that the spatio-temporal description is just a part (or perspective) of the whole scenario in which phenomena are structured*. One can only set up a measurement context aimed to measure the localization of a quantum system. But in order to give a complete description, other incompatible contexts must be considered. One of the most important consequences of quantum mechanics is that space-time can no longer be considered as an exhaustive scenario in which physical events take place. Quite on the contrary, experience can be structured as a sort of logic and, at the same time, as a specific kind of geometry, as von Neumann explained. Different models of event structures represent different possible ways in which phenomena can be structured.

The results presented in [40] show that once the logic-algebraic properties of a structured domain of phenomena are determined, to great extent, the whole probability theory is determined. In this way, we gave a concrete step in the solution of the problem posed by von Neumann discussed in Section 3. Experience is not complete chaos: quite on the contrary, it can be structured. This organization of phenomena has a definite *logical form* (as is the case in the classical or the quantum descriptions), and this *form* is expressed in a *geometric* way.

But in quantum theory this geometry must not be confused with the geometrical background of space-time (Euclidean space or the curved background of general relativity); the space-time description is just an aspect or perspective of a more general state of affairs. In the general case, physical events can be organized as lattices, which are non-Boolean in general. Or even more generally, as σ-orthocomplemented orthomodular posets [15], which are not even lattices in the general case.

Thus, once the structure of experience is determined as a logic (or as a geometry), a probability calculus follows. If the description is Boolean, then probabilities will be Kolmogorovian. But if the logic is non-Boolean, any description based on an ignorance interpretation of probabilities will be difficult to sustain (at least, without appealing to non-local hidden variables). In the standard formulation of quantum mechanics, deterministic equations of motion (as the Schrödinger equation) must be complemented with "jumps" (as the quantum jumps) and the concomitant *processes* that they trigger.

In the quantum mechanical description objects appear as a partial aspect of a particular description. As an example, think about a laboratory. Each object (the door, the walls, the chairs, a source, a photon counter, the computers, etc) has a definite position and is situated in a definite relationship with respect to the others. We can describe any concrete measurement context using the tools of classical physics (in terms of event structures represented by Boolean algebras). But the laboratory itself, as it presents to us, is more than that. It comes into being as an *organized structure* of objects and possibilities: *everything is correlated in some way, and the situation is open to different empirical setups*. The openness to the different experimental arrangements that we can set is an essential aspect of the lab. And, as explained in Section 2.2, *there is no joint classical probability distribution for the events of all the possible contexts*. The totality of results associated to all possible experimental setups, gives place to a description based in intertwined Boolean algebras which is globally non-Boolean and is related to a projective geometry. Quantum mechanics tells us that the possible outcomes associated to the different experimental arrangements *are organized in a geometrical way*, and that the probabilities associated to the different processes that we can observe are constitutive coordinates of the state of affairs produced by a state preparation. The nature of a quantum system is closely related to the *place* in which events and processes occur: *it is related to how phenomena are structured*. After a preparation, there is an actual state of affairs in the lab, which has its own history, and the set-theoretical-spatio-temporal description as a collection of objects in space is only an *aspect* of it. This structure can be considered as logical form expressed as a particular form of geometry. The experiments and processes associated to a quantum system in a laboratory cannot be reduced to a classical description: this is at the heart of the complementarity principle. In this way the quantum mechanical description manifests itself as the study of probability distributions, which can be considered objective and (at least in principle), experimentally controllable.

6 Conclusions

Throughout this work, we have discussed the connection between quantum probabilities and geometry. Our main conclusions can be summarized as follows:

- The processes that take place at the fundamental level of nature are structured in a logical way. This is, at the same time, related to a very specific form of geometry.
- States are determined as measures over the linear varieties associated to the above mentioned geometry.
- A formalism can be obtained [40] in which logic, geometry and probability theory appear articulated. We consider this as a step forward in the research program initiated by von Neumann that we have discussed in Section 3.
- The geometry of space-time appears as a substructure of the more general geometry associated to the elementary processes. In other words: physical phenomena seem to have a geometrical structure that is not exhausted by the one that we use to describe space-time.
- The discussion in this work points naturally to a geometrical interpretation of quantum mechanics, that can be framed in similar approaches that have as the main goal the development of a quantum theory of gravity.

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