Non-Reflexive Logics
Logics that Derogate the Standard Theory of Identity

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Dedicated to Andrea Loparic, forerunner in exploring non-classical logics.

Abstract

In this paper an outline of a class of heterodox logics is presented. These systems, roughly speaking, question the standard notion of identity as given by classical logic and standard mathematics. Due to the apparent fundamentality of this notion, it is necessary to provide a strong motivation for the elaboration of such systems, and perhaps the main one comes from quantum mechanics. This link is mentioned, and some references are made, but no details are presented; the references provide more detailed works. We just describe some non-reflexive logics and mathematics, ending with a general discussion about the necessity of identity.

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1 Introduction

The XXth century was the period of the raising of non-classical logics. Since the most celebrated principles of classical logic are the principles of identity, non-contradiction and the excluded middle, it has been mainly against these rules that *hererodox* logics start being developed;[1] really, it was most against the two last ones. So, a roughly classification was advanced: *paraconsistent* logics would be those logics in which the principle of non-contradiction is not universally valid, and *paracomplete* logics are those systems where the principle of the excluded middle is derogated. Today there are a plenty of paraconsistent systems, so as of paracomplete logics of several kinds (intuitionistic logic and many-valued logics being the main ones).

Although these three mentioned principles are not the only important ones in classical logic, they become famous by historical reasons. Classical logic is full of other 'fundamental principles', such as *double negation*, *the explosion rule* and *Peirce's law*. But our task here concerns the principle of identity. Surprising enough, it is perhaps the less questioned principle of classical logic. Today, we can say that we have free ourselves from the standard principles of classical logic, with the possible exception of the principle of identity; this has been, even today, a *classical taboo*. Why this is so? Some hints are advanced bellow.

But, first of all, we need to agree that everyone of the mentioned principles admit different and non equivalent formulations, and this is not different with the principle of identity (from now on, simply PI).[2]

For instance, in the standard language of propositional calculus, we may put

$$p \to p, \quad \text{or} \quad p \leftrightarrow p \tag{1}$$

being p a propositional variable. If we permit quantification over such variables, then $\forall p(p \to p)$ would be the case. In first-order languages, being x and individual variable and F a unary predicate, we can write

$$\forall x(F(x) \to F(x)). \tag{2}$$

[1]There are also the (said) non-classical logics that *complement* classical logic with additional operators which provide its language a stronger capacity of expression, such as the standard modal logics, deontic logics, temporal logics, and so on. These systems do not question the classical principles, contrarily to the heterodox systems. Anyway, there are also *mixed* systems, which are both complementary and heterodox, such as the *paraconsistent modal logics*.

[2]The reader can find other formulations in [8].

Higher-order logics also encompass a version of PI, for instance,

$$\forall F \forall x (F(x) \to F(x)), \tag{3}$$

and so we go. But PI is not all that exists about identity. The most relevant thing is what we can call *the standard theory of identity* (STI), which involves more than the PI. Let us sketch STI a little bit bellow. Non-reflexive logics are defined as those heterodox logics which violate STI, and in particular the PI. But, first, it would be adequate to have a glimpse on the involved notion, namely, identity.

1.1 Identity

'Identity' is an apparently simple notion. Googling it, we find things like "it is the relation each thing bears only to itself".[3] Several other 'definitions' can be find easily, so as the discussion whether identity would be a relation or just a property. The notion looks simple, but hides insurmountable difficulties. The discussions about this notion go back to the antiquity, mainly concerning personal identity and identity through time (that is, the question of how can we say that some thing that changes its properties can be said to be the same thing as before). Some philosophers prefer to consider a different concept, yet they continue to call it 'identity'. This is the case of Peter Geach, who said that only *relative identity* could have a sense; according to him, two things can be said to be identical only relative to a certain 'sortal' predicate:[4] x is the same F as y, where F is a sortal predicate. But our account here is concerning *logical identity*, or simply *identity* from now on.

In his *Begriffsschrift* [10], Frege assumed that the symbol of identity (dealt with by the symbol of equality '$=$', introduced by Robert Record in 1557) should be put between two names, but later, he acknowledged in his *On sense and reference* [11] that it must hold between objects. This is our understanding up to now, although we can extend it to cope with properties, relations, and functions as well. When we say that $x = y$ is the case, or that it is true, we mean that x and y denote the same thing, or that they have the same referent. In other words, being this the case, there are no two things, but just one, which can be named either as x or y (among possibly other means).

In nowadays logic, we have at our disposal different kinds of languages, something never dreamed in Frege's time: first-order languages, higher-order languages, and the languages of set theories (I will leave the language of categories out of this discussion). In first-order languages, we have basically two options to introduce STI: either we *define* identity or we take it as a primitive notion. In order to define identity, we need a suitable formula of the language, $\alpha(x, y)$, where x and y are free, and put

$$x = y := \alpha(x, y) \tag{4}$$

[3] https://en.wikipedia.org/wiki/Identity_(philosophy)

[4] Sortal predicates, also called 'count-nouns' distinguish from 'natural nouns'. So, if a sortal predicate applies to some things, they can be named, ordered, counted.

For instance, we have Quine's strategy of exhausting the (finitely many) primitive predicates of the language [28]. In ZFC, axiomatized as a first-order theory, we usually put

$$x = y := \forall z (x \in z \leftrightarrow y \in z), \tag{5}$$

and if there are atoms (ur-elements) (ZFU), we write

$$x = y := \forall z (x \in z \leftrightarrow y \in z) \land \forall z (z \in x \leftrightarrow z \in y). \tag{6}$$

Another alternative is to take '=' as a primitive binary predicate subjected to the following postulates:

1. (Reflexivity) $\forall x (x = x)$
2. (Substitutivity) $\forall x \forall y (x = y \rightarrow (\alpha(x) \rightarrow \alpha(y))$, being y a term not encompassing x free.

In a first-order ZFC theory, we still add to these postulates the Axiom of Extensionality in one of the forms bellow, depending whether there are atoms or there are not:

$$\forall x \forall y (\forall z (x \in z \leftrightarrow y \in z) \land \forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y) \tag{7}$$

or (without atoms)

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y) \tag{8}$$

From these postulates, it follows the symmetry and the transitivity of equality. In extensional contexts, such as ZFC, a binary relation is a set of ordered pairs, and if identity is a binary relation, we can define the *identity of the set* D as follows:

Definition 1.1 (The diagonal of a set) *Let D be a non-empty set. The diagonal of D, or the identity of D, termed Δ_D, is so defined:*

$$\Delta_D := \{\langle x, x \rangle : x \in D\}. \tag{9}$$

In semantic terms, let us recall, it is precisely in this set that the primitive symbol '=' is interpreted, being D the domain of the interpretation (but see below).

Theorem 1.1 *Let $\sim$ be an equivalence relation over the set D (in particular, $\sim$ can be a congruence). Then $\Delta_D \subseteq \sim$.*

The proof is trivial, since every equivalence relation is reflexive. It is in this sense that we say that the identity of D is the *thiner* equivalence relation (or congruence) over a set.

In higher-order languages, we can define identity, usually by Leibniz's Law. Let us do it in second-order logic: being F a variable for predicates of individuals and x and y individual variables, we have

Definition 1.2 (Identity in higher-order languages)

$$x = y := \forall F(F(x) \leftrightarrow F(y)). \tag{LL}$$

Informally speaking, the definition says that are equal (the same) those entities that are indistinguishable relative to all their properties. Notice that this theory entails that nothing more than properties (and relations) can be used to characterize an identity. That is, we are in the field of the *bundle theories* concerning identity and individuality (see [12] for a distinction between bundle theories and substratum theories).

Interesting to say that LL is (of course in another way of expressing) the definition given by Whitehead and Russell in their *Principia Mathematica*, and was questioned by F. P. Ramsey, so as by L. Wittgenstein in the *Tractatus*. The first said that there is no logical contradiction to suppose that there may be two things which, despite indiscernible, are not the same thing [29, p.31]; the later was more radical, thinking that the very notion of identity given by LL should be ruled out since we can use one name for each object we are referring to (see [24] and the references therein).

2 Characteristics of the standard theory of identity

The notion of identity characterized above by diverse means (as we have seen) can be called 'classical'. It entails several important facts, such as the following ones:

1) The first-order classical identity doesn't characterize the diagonal of the domain up to elementary equivalence. Let us explain. By the first-order theory of identity, we mean Reflexivity and Substitutivity, with '=' as primitive. Now let $\mathfrak{A} = \langle D, R_i \rangle, (i \in I)$, be a structure, with $D \neq \emptyset$ and the R_i being n-ary relations over D, an interpretation to our language. This is what we call *order-1* structure (see [23]), typical of Model Theory. Following W. Hodges, we call it a *structure with standard identity* if '=' is interpreted in the diagonal Δ_D. But, as Hodges shows, (a similar proof is given by Mendelson [27], who calls *normal* these structures), there are structures which are elementary equivalent to $\mathfrak{A}$ that also model the postulates of identity. This means that from the point of view of the language, we never know of what structures we are speaking about. Consequently, the language doesn't teach us whether we are speaking, say, of objects of D or of subsets (equivalence classes) of elements of D.

This is of course not good for a theory which intends to be a theory of *something*, say, the elements of D. In the informal parlance, we say that (first-order) identity cannot be axiomatized, meaning that Δ_D cannot be uniquely captured from the first-order postulates of STI. The same, of course, holds when the first-order identity is defined.

2) Thus, let us go to higher (second) order languages. Here, LL is mandatory. Once we rule substratum out, we keep with properties and relations (and with formulas in general), and then the collection of properties seems to be enough to individuate an element of the domain, or to provide it its identity (although individuality and identity are distinct concepts [25]). The semantics for our second order language necessitates of a non-empty domain, as in the first-order case, but also a collection of subsets of D where the unary predicate variables of the language range, subsets of $D \times D$ where the binary predicate variables range, and do on. This is called a *frame* for our language. Notice that to individuate a particular element of D, it is enough to have in the frame all unitary sets of the elements of D. This kind of frame if called *principal* by Church [3, pp. 307-8]. The *secondary* interpretations take not all subsets of D, $D \times D$, etc., but just *some* of them. The consequences are huge.

First of all, with a principal interpretation, we lose completeness; with secondary interpretation, we may have a weaker form of completeness, called *Henkin completeness*. But notice that with a Henkin semantics, if not all subsets of the domain are chosen, we may have a situation where x and y belong to all the chosen subsets of the frame (that is, they obey all the same properties of the language) and even so they are not necessarily the same object. That is, Leibniz Law can fail. A simple example is enough. Suppose our second order language has two individual constants a and b and three unary predicates P_1, P_2, and P_3. Let us take a frame with $D = \{1, 2, 3, 4, 5\}$, let a be interpreted in 3, b in 5 and P_1, P_2, P_3 respectively in $\{1, 2, 3, , 5\}$, $\{3, 4, 5\}$ and $\{1, 3, 5\}$. Then $P_i(a) \leftrightarrow P_j(b)$ for $i, j = 1, 2, 3$, but even so $3 \neq 5$.

3) Let us suppose, finally, that we are with the STI in ZFC. Let a be any set or atom. By the postulates, we can form the unitary set $\{a\}$ and define the following property we call *the identity of* a, namely, $I_a(x) := x \in \{a\}$. Then a is the only object of the universe that has this property and therefore it provides an *identity criterion* for a. If we call *individual* every object that has an identity criterion, it results that every set or atom is an individual. Once we can build all standard mathematics within ZFC, we can surely say that in standard mathematics there are no *solo numero* indiscernible things, that is, things that differ just for being two things, but without any further differences.

Just a remark: if the reader has a non trivial knowledge of set theory, probably she has heard about *indiscernibles*. There are different kinds of them (see [33], [17] for instance). Apparently, guided by the word 'indistinguishable', these entities are thought of as partaking all their properties (being *indiscernible*) without being identical. But this is not strictly true. By the same argumentation given above, they are *individuals* since obey the postulates of ZFC, hence of STI.

The way of dealing with indiscernibles within a standard framework such as the ZFC set theory (the problem regarding category theory is still an open problem) if to confine them to a *deformable* (non-rigid) structure, that is, a structure encompassing other automorphisms than the identity function (the trivial automorphism), as the 'indiscernible' of the previous paragraph. This is a fake notion of an entity devoid of identity criterion, for it can be proven [7] that every structure can be extended to a rigid structure (with just the trivial automorphism) and so, if not in the original structure, we can discern the elements of the domain in the extended structure. Since the whole universe of sets (seen as a structure) is also rigid [17, p.66], we arrive to our result: standard mathematics (and logic) is a theory of individuals, and in particular some form of the principle of identity holds.

3 Challenging identity

In what sense STI can be questioned? The first way is to follow Ramsey: simply construct a logic where STI doesn't hold. After all, as Hilbert has once suggested, the mathematician should pursue all logically possible theories, and not just those which approach reality (apud [20]). But this would be not necessarily interesting. The better way would be to have some motivation for such a logic. And here enters the more interesting point: there are plenty of reasons to challenge identity in some way. Here we shall explore few, but make reference to many others.

Alfred Korzybski was a Polish-American guy (a real Count) who in a certain period has attended Tarski's seminars in Berkeley. He became famous (apparently, not for Tarski)[5] for having written a book called *Science and Sanity* [19] where, among other things, he claimed that nothing remains identical to itself since it is always changing its properties, and that the lack of being aware of this would be one of the problems with sanity of many people. Consequently, people should to take into account *non-Aristotelian* habits, among then the rejection of the Aristotelian sentence

$$A \text{ is } A, \tag{10}$$

expressing his principle of identity, which should not hold in the real world.

Later, Oliver L. Reiser (1935) discussed what could be called non-Aristotelian logics [30]. According to him, these would be logics violating one of the three basic principles mentioned in the beginnings. Against the principle of non-contradiction, he mentioned Hegelian metaphysics and the 'dynamic' logic of Dewey. As for the excluded middle, he made references to Brouwer's intuitionism and to the many-valued logics of Łukasiewicz, so as to Tarski and C. I. Lewis. But, relatively to identity, Reiser mentions Korzybski. Unfortunately, no logical system was stated, which happened only in 1948.

In this year, in the directions of the lines pointed out by Wittgenstein, O. L. Zich (considered the father of Czech's logic) presented a system without the

[5] In his book *Fads and Fallacies in the Name of Science* [14, Chap.23], Martin Gardner doesn't say much favoring Korzybski and his strange 'semantics'.

symbol of identity but (as Wittgenstein suggested), writting a symbol for each object in the domain of a supposed interpretation. According to Zich, LL of *Principia Mathematica* could not be accepted, since it would be impossible to check in a finite number of steps whether all the properties of two given objects are the same. Zich's paper was published in Polish, but there is a review by Jan Kalicki [18].

I think that these are really the forerunners of the ideas of logics questioning the standard intuitive notion of identity as grounded in our metaphysical tradition. As said before, the *metaphysical* notion that identity is something an object share just to itself and with nothing more was impregnated in our metaphysical pantheon. It would be difficult to leave this realm.

But, as we shall see below, perhaps the great inspiration for a logic without identity (at least for some entities) comes from a quite reasonable interpretation of the quantum mechanics formalism.

4 Non-reflexive logics

In his book *Ensaio sobre os Fundamentos da Lógica* [4], Newton da Costa intended to show that every principle of classical logic can be questioned or, as he prefers to say following Bachelard, *dialetized*. As for the principle of identity, without any reference to these previous mentioned works, he found a quite strong motivation in some philosophical ideas of the great physicist Erwin Schrödinger, one of the father founders of quantum mechanics and Nobel winner of 1933. In his book *Science and Humanism* [31], Schrödinger affirmed that the notion of identity ('sameness') doesn't apply to elementary particles in the quantum realm. According to him, there is no sense in saying that a particle here and now *is the same* as the particle there and after, despite everything may suggest the opposite. Thus he runs in the same direction as David Hume, who attributed to the re-identification of a thing as something due to the habit, since according to him there is no causal connection between the first and the second appearances of the object [16, *passim*].

Da Costa showed how to develop a first-order two sorted system he called 'Schrödinger Logic', denoted by 'S', as follows. The language has individual variables and individual constants of two kinds. These are the terms of the language. To the terms of the first kind, expressions of the kind $s = t$ are not formulas (the same for their negations). Thus, identity doesn't apply to the objects denoted by the terms of the first species. The axioms are immediate, respecting that the theory of identity (really, STI), holds only for the objects of the second kind (see [4, pp.117ff]).

We can repute S as the real first logical system where the principle of identity in the first-order language (namely, $\forall x(x = x)$) is violated. Notice that da Costa didn't show that the negation of this sentence holds, for this would imply the existence of an object that is not identical with itself (identity understood in the standard sense). What he did was to suspend the judgment about that: the notion of identity, as suggested by Schrödinger, simply fails to apply to some objects: $x = y$ is not always a well formed formula.

As for semantics, da Costa has just indicated a possible direction. He assumed a set D as the domain, such that $D = D_1 \cup D_2$ and interpreted the terms of the first kind in D_1 and the terms of the second kind in D_2. But, as he advanced, there are difficulties:

> ... D_1 cannot be considered as a set in the standard sense of set theories, since for its elements the relation of identity should have no sense; only for the elements of D_2 we can say that they are equal [identical] or distinct. (*op.cit.*, p. 119)

Then, and fundamentally, he suggests that

> [i]n order to surpass this difficulty, there are two open ways: 1. to look for a generalization of the notion of set, for instance building a theory of *quasi-sets* which would contain the standard sets as particular cases, and in such a theory to edify a semantics for S. [the second point concerns an informal semantics and shall be not considered here].

This is really amazing. Not only a system of non-classical logic was sketched, its semantics indicated, but a strong connection with quantum physics was indicated. To explore these connections was the Ph.D. program developed by this author under the supervision of da Costa in the late 1980s at USP (University of São Paulo). In 1990, my Ph.D. thesis was approved [20]. There, I extended da Costa system S to a higher-order logic S^ω (simple theory of types) and provided it a Henkin semantics according to which the system resulted Henkin-complete (see also [5]) — this semantics was built in a standard set theory encompassing STI. Furthermore, I went further in the ideas of Schrödinger and of quantum mechanics in general, and developed a first version of the suggested theory of quasi-sets [21].

But the remarks made above about the set D_1 remained in my mind. It would be necessary fo find a *quasi-set semantics* for S^ω, so making semantics in agreement with the claims of the logic. This was achieved in 1995 in my thesis for full professor of Foundations of Mathematics at the Department of Mathematics of the Federal University of Paraná, where I worked that time. But things changed also with the logic. The used system was not the original S^ω, but a different one, a modal higher-order logic in the sense of Daniel Gallin [13]. The reasons for choosing such a modal version was to consider, as M. L. Dalla Chiara and G. Toraldo di Francia, that the quantum world seems to be a "world of intensions" (see [9], [22], [6]), and then a purely extensional semantics would not be well suited for this case.

In the quasi-set semantics, we may have (say) unary predicate constants which are semantically associated not to a specific sets as in the extensional semantics, but to *some* quasi-set (see below) of a collection of indiscernible quasi-sets. So, we could give a formal account to the idea (expressed also by Dalla Chiara and Toraldo di Francia, but see the above references) that, contrary to standard semantics, in quantum mechanics one and the same intension may have different 'extensions'; as they exemplify, the 'intension' that characterize electrons (a certain mass, electric charge, magnetic momentum, etc.) may have different 'extensions', namely, any collection of electrons.

The system is a modal higher-order logic with a semantics given in the theory of quasi-sets. Thus, we could vindicate the idea that some individual constants can be associated to *one* element of a collection of indistinguishable objects, but without the means to specify the very identity of this element (by the way, it this kind of thing that happens in quantum mechanics when we say that one of the two electrons of a Helium atom in the fundamental state has spin UP in a given direction, while the another one has spin DOWN — nothing can tell us which electron is this one, although it is described perfectly well by a definite description : '*the* electron that has spin up in the chosen direction'). In the same vein, we can grant that (say) a unary predicate might be associated to 'everyone' among a certain collection of indiscernible quasi-sets, precisely in the direction advanced by Dalla Chiara and Toraldo di Francia. The system results also Henkin-complete relative to such a semantics.

By the way, it seems to me that this was the first time that a semantics was provided for a logical system using a non-standard theory of 'sets'. This system was also presented in [6].

4.1 Quasi-set theory

As we see, the theory of quasi-sets plays an important role in all of this. The theory is a set-theoretical version of a non-reflexive logic, and roughly speaking runs as follows, and surely is the strongest non-reflexive system we have till now. The main target, as it were, is to deal with collections of indiscernible elements, but without any standard trick of confining them to deformable (non-rigid) structures or something similar. As suggested by the philosopher of physics Heinz Post, at least in the quantum realm, indiscernibility should be looked for "right from the start" (see [12] for all the discussion).

Let us call $\mathfrak{Q}$ the theory of quasi-sets. Indiscernibility is a primitive concept, dealt with by a binary relation '$\equiv$' satisfying the properties of an equivalence relation, but not full substitutivity.[6] In this notation, '$x \equiv y$' means 'x is indiscernible from y'. This binary relation is a partial congruence in the following sense: for most relations, if $R(x, y)$ and $x \equiv x'$, then $R(x', y)$ as well (the same holds for the second variable). The only relation to which this result does not hold is membership: $x \in y$ and $x' \equiv x$ does not entail that $x' \in y$ (for the poof, see [12]).

Quasi-sets can have as elements other quasi-sets; particular quasi-sets (qsets), termed *sets*, are copies of the sets in a standard theory (in the case, the Zermelo-Fraenkel set theory with the Axiom of Choice).The theory admits also two kinds of atoms (entities which are not sets), termed M-atoms (representing objects of

[6]If we add substitutivity to the postulates, then no differences between indiscernibility and logical first-order identity would be achieved.

classical physics), which are copies of a standard set theory with atoms (ZFA) and m-atoms (for quantum objects), which have quantum objects as their intended interpretation, to whom it is supposed that the logical notion identity (STI) does not hold. If we eliminate the m-atoms, we are left with a copy of ZFU, the Zermelo-Fraenkel set theory with atoms. Hence, we can reconstruct all standard mathematics within $\mathfrak{Q}$ in such a 'classical part' of the theory.

Functions cannot be defined in the standard way. When m-atoms are present, a standard function would not be able to distinguish between indiscernible arguments and values. Therefore, the theory generalizes the concept to 'quasi-functions' ('q-functions'), which map indiscernible elements into indiscernible elements.

Cardinals (termed 'quasi-cardinals') are also taken as primitive, although they can be proven to exist for finite qsets (finite in the usual sense). The concept of quasi-cardinals can be used to speak of 'several objects', even without identity. So, when we say that we have two indiscernible q-functions, according to the above definition, we are saying that we have a qset whose elements are indiscernible q-functions and whose q-cardinal is two.[7] The same happens in other situations.

An interesting fact is that qsets composed of several indistinguishable m-atoms do not have an associated ordinal. This lack of an ordinal means that these elements cannot be counted by standard means, since they cannot be ordered. However, we can still speak of a collection's cardinal, its *quasi-cardinal*. This existence of a cardinal but not of an ordinal is similar to what we have in quantum physics when we say that we have some quantity of systems of the same kind but cannot individuate or count them, e.g., the six electrons in the level $2p$ of a Sodium atom.[8]

Identity (termed *extensional identity*) "$=_E$" is defined for qsets having the same elements (in the sense that if an element belongs to one of them, then it belongs to the another) or for M-objects belonging to the same qsets.[9] However, one can hypothesize that *if* a specific object belongs to a qset, then so and so. This is similar to Russell's use of the axioms of infinite (I) and choice (C) in his theory of types, which assume the existence of certain classes that cannot be constructed, so going against Russell's constructibility thesis.

What was Russell's answer? He transformed all sentences α whose proofs depend on these axioms into conditionals of the form $I \to \alpha$ and $C \to \alpha$. Hence, *if* the axioms hold, *then* we can get α. In $\mathfrak{Q}$ we are applying a similar reasoning: *if* the objects of a qset belong to the other and vice-versa, *then* they are extensionally identical. It should be noted that the definition of extensional

[7]Quasi-cardinals turn to be *sets*, so we can use the equality symbol among them. We use the notation $qc(x) = n$ for 'the quasi-cardinal of (the qset) x'.

[8]To count a finite number of elements, say 4, is to define a bijection from the set with these elements to the ordinal $4 = \{0, 1, 2, 3\}$. This counting requires that we identify the elements of the first set.

[9]There are subtleties that require us to provide further explanations. In $\mathfrak{Q}$, you cannot do the maths and decide either a certain m-object belongs or not to a qset; this requires identity, as you need to identify the object you are referring to. So, the theory employs some artifices to deal with these situations.

identity holds only for sets and M-objects. It can be proven that the extensional identity has all the properties of classical logical identity for the objects to which it applies. However, it does not make sense for q-objects. That is, $x =_E y$ does not have any meaning in the theory if x or y are m-objects. It is similar to speak of categories in the Zermelo-Fraenkel set theory (supposed consistent). The theory cannot capture the concept, yet it can be expressed in its language. From now on, we shall abbreviate "$=_E$" by "$=$," as usual.

The postulates of $\mathfrak{Q}$ are similar to those of ZFA, but by considering that now we may have m-objects. The notion of indistinguishability is extended to qsets through an axiom that says that two qsets with the same q-cardinal and having the same 'quantity' (we use q-cardinals to express that) of elements of the same kind (indistinguishable among them) are also indiscernible. As an example, consider two sulfuric acid molecules H_2SO_4. They can be seen as two indistinguishable qsets, for both contain q-cardinal equals to 7 (counting the atoms as basic elements), and the elements of the sub-collections of elements of the same kind are also of the same q-cardinal (2, 1, and 4 respectively). Then we can state that "$H_2SO_4 \equiv H_2SO_4$," but of course, we cannot say that "$H_2SO_4 = H_2SO_4$," as for in the latter, the two molecules would not be two at all, but just the same molecule (supposing, of course, that "$=$" stands for classical logical identity). In the first case, notwithstanding, they count them as two, yet we cannot say which is which (and *this* is the core idea).

Let us speak a little bit more about quasi-functions. Since physicists and mathematicians may want to talk about random variables over qsets as a way to model physical processes, it is important to define functions between qsets. This can be done straightforwardly, and here we consider binary relations and unary functions only. Such definitions can easily be extended to more complicated multi-valued functions. A (binary) q-relation between the qsets A and B is a qset of pairs of elements (sub-collections with q-cardinal equals 2), one in A, the other in B.[10] Quasi-functions (q-functions) from A to B are binary relations between A and B such that if the pairs (qsets) with a and b and with a' and b' belong to it and if $a \equiv a'$, then $b \equiv b'$ (with a's belonging to A and the b's to B). In other words, a q-function maps indistinguishable elements into indistinguishable elements. When there are no m-objects involved, the indistinguishability relation collapses in the extensional identity, and the definition turns to be equivalent to the classical one. In particular, a q-function from a "classical" set such as $\{1, -1\}$ to a qset of indiscernible q-objects with q-cardinal 2 can be defined so that we cannot know which q-object is associated with each number (this example will be used below).

To summarize, in this section, we showed that the concept of indistinguishability, which conflicts with Leibniz's Principle of the Identity of Indiscernibles, can be incorporated as a metaphysical principle in a modified set theory with indistinguishable elements. This theory contains "copies" of the Zermelo-Fraenkel axioms with *Urelemente* as a particular case when no indistinguishable q-objects

[10] We are avoiding the long and boring definitions, as, for instance, the definition of ordered pairs, which presuppose lots of preliminary concepts, just to focus on the basic ideas. For details, the interested reader can see the indicated references.

are involved. This theory will provide us the mathematical basis for formally talking about indistinguishable properties, which we will show can be used in a theory of quantum properties. We will see in the next section how we can use those indistinguishable properties to avoid contradictions in quantum contextual settings such as KS.

5 Other non-reflexive logics

To derogate the quantificational principle of identity is not the only way to build non-reflexive logics. It can be done also at the propositional level, as the following case illustrates.

In the propositional level, as we have seen, the principle of identity can be written as $p \to p$, where p is a propositional variable, and this is one of the thesis of classical propositional logic. But, as shown by R. Sylvan and da Costa, this can also be challenged [32]. In their logic, termed *logic of causal implication*, a kind of conditional $\ni$ is used, so that $A \ni B$ means A *causes* B, being A and B not sentences, but certain terms which "are sometimes propositional or fact-like or rendered such by happening or occurrence functions" (op.cit.). The authors claim that they do not intend to explain causation, a difficult topic which perhaps cannot be covered by a sentential connective, but just to deal with the idea that something may *implies* something, as (their example) 'that flood caused a famine', and leave open the possibility of the introduction of other ideas such as 'reverse causation', 'proximate causation', etc.

The fact is that the new connective is not reflexive, that is, $A \not\ni A$ holds. According to our classification, this system counts as a non-reflexive logic. Newton da Costa has other accounts on non-reflexive logics and identity, which can be seen in [1], many of them in superposition with that was presented above.

6 Is identity really fundamental?

At least as present in every logical system, I guess no. The formation of our theories and conceptions (so as our metaphysics) vary in time, from culture to culture, and from different ways of conceptualization the surrounding world. I mean 'to conceptualize' rather than 'to perceive', for a reasonable theory should be the product of reasoning, and not only of just feelings. And, fundamentally, we should take into account the difficulties in putting our claims in agreement with those of the scientific community, mainly when it departs from the standards. Anyway, we need also to be aware of Martin Gardner's warning of not entering in the rol of the pseudo-scientists who think that they (and only they) are right and all the others are wrong (see the first chapter of his book [14]). So, in questioning a notion that is impregnated in our logic, our mathematics, and our standard physics, for not speaking of our standard metaphysics, such as identity, we need to be quite sure that the venture is a worthwhile endeavor. And we are quite sure that this is the case with STI, mainly in considering quantum mechanics.[11]

But here we need to be succinct. So, let us just make reference to an interesting paper advanced by Otavio Bueno [2] where he try to safeguard the fundamentality of the notion of identity. Unnecessary to say that he is thinking of STI. An answer to Bueno was given in [24], so it not necessary to repeat the discussion here. Thus, let me take this opportunity to advance some particular further reflections on the reluctance some people has in questioning standard identity.

One of the arguments is that we need identity to elaborate any theory, in particular a theory where identity doesn't hold for some entities. So, identity could not be dispensed with. This rests in a clear confusion between levels of languages. For instance, think of paraconsistent logics. These are logics where the *explosion rule* doesn't work unreservedly; there may be situations where A and $\neg A$ do not conduce to a trivialization (every well formed formula is a theorem). In particular, by considering the paraconsistent negation (there are several), the principle of non-contradiction $\neg(A \wedge \neg A)$ is not universally valid. Anyway, in building these systems,[12] we do assume that the principle holds in the metalanguage (or, if we would be more precise, in the metalogic). Really, no one would suggest that some expression is a formula and simultaneously that it is not, or that a certain sentence is a theorem and that it is not.

The same could be said about mathematics. Don't we need to know what is two in order to try to find a definition of 'two'? Here, the differences among levels of languages show their importance. We start reasoning in a rather informal way, almost in a constructive way, where informal ideas are articulated, combined and developed. With the progress of reasoning, we may abstract some ideas and, in a limit, to arrive at strong systems (such as the ZFC set theory) where we can, so to say, 'reconstruct' the notions we have used informally, such as of the number two, so as the basic logical rules. This was called the *Principle of Constructivity* by N. da Costa (see [4, p.57] and [23, p.43] for a discussion). That is, we start with a free capacity of creative heuristics, but step by step we will conforming our stuff to our ways of conceptualization. Thus, there is no reason to say that we really can elaborate systems where the standard notion of identity is questioned, as the principle of non-contradiction is in some systems.

7 Philosophy of non-reflexive logics

There are other interesting philosophical questions related to non-reflexive logics. Let us consider those logics (like Schrödinger logics) that restrict the applicability of the notion of identity, so accepting that identity doesn't apply to some objects of the domain of the discourse. Da Costa and Bueno [8] point to

[11] There are also in some Eastern philosophies strong motivations for looking for alternative notions than standard identity. It is remarkable that Schrödinger and H. Weyl based some of their claims about identity in such views (again, see [12]).

[12] As said before there are plenty of paraconsistent systems, all of them with one sole finality: to break the explosion rule.

the some questions that apparently provide puzzles to this kind of logic. In this section, we provide a way to answer them into the scope of the non-reflexive philosophy.

The first question concerns quantization. As they remark, in 'reflexive logics', such as classical logic, there is an equivalence between 'all Fs are Gs' and 'each F is a G', both symbolized by $\forall x(F(x) \rightarrow G(x))$. Then, they note that in order to quantify over *each* object of the domain, these objects need to be identified, so identity needs to apply to them. Due to the identification between 'each' and 'all', according to these authors, we would be in trouble in trying to quantify over objects that don't have identity, for the standard definition of satisfaction (so as denotation and truth) does require that identity holds unreservedly. So, they say, a non-reflexive semantics cannot be given to a non-reflexive logic, since for a faithful semantics, identity should not be available in the metalanguage where the semantics is to be developed. They call this question the *vicious circle of non-reflexivity*.

Secondly, they say that non-reflexive logics distinguish between objects that do have identity from those that do not, so "[i]t seems reasonable to take any object of the first category from those of the second [...] this is precisely the sort of thing that *cannot* be said in a non-reflexive logic". They call this the *paradox on non-reflexivity*.

Let us address to these questions, starting with the last one. Within the scope of non-reflexive ideas, we see that identity is not strictly necessary to strongly distinguish among classes of objects. Suppose we have a molecule of sulfuric acid, H_2SO_4. Notice that we have *four* Oxygen atoms, but cannot distinguish them. We also cannot say that they are *different* under STI, for this would require a property which applies to one of them but not to the others, and of course there are none. So, it is enough to work with a weaker notion of *discernibility* (and *indiscernibility*), which is not necessarily equivalent to identity. The four Oxygen atoms are indiscernible, and that fits better what physics assume. As we have seen, this is achieved in the theory of quasi-sets and in the higher-order Schrödinger logics.

In what concerns their exemple, that of discerning between entities that have identity from those that do not, let us remark that we can discern between a group of electrons and a group of chairs (which, by hypothesis, obey the standard rules of identity), yet we cannot discern the electrons from one each other. No problem here. In a non-reflexive setting, such as in the theory of quasi-sets, electrons, protons and other quantum objects can be classified among the m-objects, while chairs can be represented by M-objects. Quasi-set theory can indeed be used as the metamathematics for a non-reflexive semantics for non-reflexive logics, as we have seen before. So, with an adequate metamathematics, there is neither paradox of non-reflexivity nor the vicious circle of non-reflexivity; suffice to consider a non-reflexive mathematics acting in the metalanguage. Notice that these authors consider just classical logic and standard semantics, although we clearly are in a situation where these frameworks are being questioned.

Of course da Costa and Bueno could argue agains my argument by saying that in proposing a quasi-set semantics for a non-reflexive logic, I am begging the question, since I need to 'create' the theory of quasi-sets first and this cannot be done without identity. This, again, is not fair. As we have seen above, using their own Principle of Constructivity we can build such a system first assuming identity in the 'constructive stage' and later bypassing it, as paraconsistent logics did with the principle of non-contradiction (for more details on this, see [23, Chap. 3]).

But the most interesting question is of course the problem of quantification. Again, the authors are grounded on classical logic, assuming the equivalence between 'each' and 'all' in the sense seem above. This is *one* way to understand quantification. In a non-reflexive setting, of course, we are not obligated to assume this 'classical' equivalence. It makes sense to speak of 'all' objects (even m-objects) of a kind, but this doesn't imply that we need to identity each of them one by one. This can be seen, again, in the quantum contexts. Suppose a Helium atom in its fundamental state (less energy). It has two electrons in an entangled state, which means that there is no way to distinguish between them or to take the states as separated. Important to remark that this is not a purely epistemological impediment, but as far as quantum mechanic is believable, it is an ontological problem; we really are dealing with entities that depart from the standard objects of our surroundings. So, in the quantum realm there is no the alleged equivalence between 'all' and 'each'. Non-reflexive semantics gives sense to that.

Notice that although we cannot discern electrons by pointing which is which, we surely can say that *all of them* (of a certain class) have such and such characteristics, like to share an entangled state. It is not necessary (by the way, it is impossible) to speak of 'this' electron in distinction to 'that' electron. The most we can say, let me emphasize, is that *one* of them has, say spin UP in a given direction while the other one has spin DOWN in the same direction, but never to determine which is which; by the way, this question doesn't have any sense at all.

So, how could we say that 'some' electrons (or protons, or neutrons, or an H atom, or a water molecule, whatever quantum system you wish to consider) are so and so? Without identity, we can ground our language on the notion of cardinal (better, quasi-cardinals), as in quasi-set theory. Remember that the notion of quasi-cardinal is primitive in $\mathfrak{Q}$, although it can be defined for finite qsets. For instance, take again the He atom as above and the property P 'x is an electron (of He) and has spin UP in the z-direction'. As said before, we know that just one of them has this property, but (according to quantum mechanics) we cannot say which one. But, in $\mathfrak{Q}$, let e stand for 'electron'. We can form the *strong singleton* of e, termed $[\![e]\!]_{He}$ meaning *one* electron of the He atom,[13] but we cannot give him (significantly) a proper name, for we need identity to make sense of that. This qset has q-cardinal 1 (as it follows from $\mathfrak{Q}$) and its intension act as a definite description, but it cannot be eliminated by a contextual definition (in Russell's sense) due to the lack of identity. Really, there

[13]Of course we have the same problem in identifying the He atom, which is also a quantum object, but we leave this point out.

will be just two of such strong singletons, and they are perfectly indiscernible; they count as two, but we cannot point to any difference.

The same can be done with more than one m-object, say n of them. So, having a qset with q-cardinal $m > n$, using a similar device in terms of q-cardinals, we really *can* speak of n of them satisfying a certain characteristics, in the same vein that we say that in the 2p shell of a Na atom, there are six electrons, even without being able to say which of the 11 electrons of the atom are those of the level 2p.[14] Notice that the sentence 'those electrons among the 11 that belong to the 2p shell' doesn't identity a sub-collection, for we really cannot identify these six electrons from the remaining ones, despite their peculiar characteristics.

This is a key point we need to insist. Suppose you have a box with 11 indiscernible balls and that you are in a completely dark room. You need to separate 6 of them. You can do it, but you don't know which ones you are taking. Only later, with light or other devices (you can make a physical mark on the six) you can identify them. But this cannot be done with electrons or other quantum systems! So, you can say that the Na atom has six electrons in the 2p shell, but there are no means to say which ones are there. I strongly hope that this shows you that you are not discerning them by a property, since you don't know which ones are your six.

So, these criticisms don't stick to non-reflexive logics. Furthermore, as we see, non-reflexive logics are really a class of well characterized logical systems, with adequate syntactical and semantical counterparts, so they can be legitimately classified in the class of heterodox logics.

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