Formulations of Classical Mechanics

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1. Introduction

Classical mechanics is the physical theory with which we are most familiar, the one we first encounter in school. Philosophers tend to regard classical mechanics as metaphysically unproblematic. At first glance, it does appear straightforward: the theory is fundamentally about particles, with intrinsic features like mass,[1] that move around in three-dimensional space in response to various forces, which arise via interactions between the particles. It seems as though, if any physical theory is metaphysically perspicuous, classical mechanics is. But the theory is not as clear-cut as it initially seems. Our familiarity misleads us.

The reason is not just that classical mechanics ultimately runs into the kind of trouble that presaged quantum mechanics. Even taking it to be the true fundamental theory of a world,[2] classical mechanics does not offer as candid a picture of things as we tend to think. One reason for this is that there are different formulations, which are generally claimed to be equivalent by physics books, but which are at least not *obviously* equivalent—neither in terms of the mathematical structure they use, nor in terms of the physical world they describe.

What I want to do in this chapter is to outline the three leading formulations of classical mechanics, and to raise some questions about them, the chief one being: Are these genuinely equivalent formulations, as usually thought? If so, in what sense are they equivalent? If not, in what way(s) do they differ? Another way to put the focal question of this chapter is by means of a title of Mark Wilson's (2013): "What is 'Classical Mechanics', Anyway?" Indeed, since the terms 'classical mechanics' and 'Newtonian mechanics' "are used virtually

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[1] Also charge, although there is a question of whether electromagnetic features ought to be considered part of the domain of classical mechanics; see for instance note 10.

[2] Of course, because of the previously-mentioned troubles, it is not clear that classical mechanics can be a true fundamental theory of a world, but set that aside here.

synonymously" (Spivak, 2010, 7), one aim of this chapter is to suggest that it is not right to do so. There are different versions of classical mechanics, which might even amount to distinct theories. A related aim is to show that there are interesting philosophical questions that arise in the context of classical mechanics. Classical mechanics merits the attention of philosophers, who often disregard it as either too perspicuous or too outdated to warrant much discussion.[3]

Although this chapter is limited to classical mechanics, a host of general questions in the philosophy of physics and science are touched upon, such as: What is the right notion of theoretical equivalence: when are two scientific theories mere notational variants? How do we interpret a scientific theory: how do we figure out the nature of the world according to a theory? When faced with different theories or formulations, how do we choose which one to adopt? Indeed, must we choose?

2. Three formulations

I will outline the three main formulations of classical mechanics—Newtonian, Lagrangian, and Hamiltonian mechanics—in relatively standard ways, before turning to some questions about them.[4] My focus will be on the dynamical laws and the quantities that appear in them. This is where much of the action lies in comparing and contrasting the different formulations.

2.1. Newtonian mechanics

Newtonian mechanics might be the only formulation one comes across, the others typically not introduced until more advanced college courses. In the Newtonian mechanics of point-particles—point-sized physical objects with intrinsic features like mass[5]—two sets of coordinates specify a system's fundamental state at a time: the positions and velocities (or momenta) of all the particles. Assuming the particles are free to move around in three-dimensional physical space, these coordinates will each have three components, one along each spatial dimension.

For a system consisting of n particles, the total state is specified by means of $6n$ coordinates: three coordinates for the position and three coordinates for

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[3] A recent book-length exception: Sklar (2013).

[4] There are other varieties I don't discuss, such as formulations in terms of Poisson brackets, Hamilton-Jacobi theory, or four-dimensional spacetime geometry.

[5] This is the fundamental ontology assumed here. Wilson (2013) discusses the classical mechanics of rigid bodies and continua and complications involved in trying to encompass all of these within a single theory. See Hall (2007, 5.2); Esfeld et al. (2018); Allori (forthcoming) on the non-standard idea that particles don't have fundamental intrinsic properties.

the velocity of each particle in the system. It turns out to be extremely useful to represent all the possible states of a system in a mathematical space called the *statespace*, each point of which represents a different possible fundamental state of the system. Since we need $6n$ coordinates to specify the state of a system, the statespace will have $6n$ dimensions.

Different curves through the statespace represent different possible histories of the system, different sequences of fundamental states over time. (The curves are parameterized by time.[6]) These histories will be given by a theory's dynamical laws, in this case, Newton's second law:[7]

$$\Sigma \mathbf{F}_i = m_i \mathbf{a}_i = m_i \ddot{\mathbf{x}}_i. \tag{1}$$

$\Sigma \mathbf{F}_i$ indicates the sum of the forces—which are vector quantities, written in bold—on a given particle labeled by i (i ranges from 1 to n, for n particles in the system); m_i is the particle's mass; $\mathbf{a}_i$, or $\ddot{\mathbf{x}}_i$, is the particle's acceleration, the second derivative of its position with respect to time, also vector quantities. (A dot over a quantity indicates a derivative with respect to time of that quantity.) In other words: $\Sigma_{j \neq i} \mathbf{F}_{ij} = m_i \mathbf{a}_i$, where $\Sigma_{j \neq i} \mathbf{F}_{ij}$ is the sum of the forces on the given particle due to all the other particles (both in the system and external to it).

The above is a vector equation. There is one such equation for each particle in each component direction—three equations per particle in three-dimensional space. These equations can be grouped together into one master equation, which says how the point representing the state of the entire system moves through the statespace over time. Given the initial state of a system and the total forces acting on it, integrating (twice) yields a unique solution, or history: the laws are deterministic.[8] A solution picks out a trajectory in the statespace, which represents the paths of all the particles through ordinary physical space.[9]

Equation 1 is the fundamental dynamical equation of the theory. Newton's second law, mathematically represented by this equation, predicts the motion of every particle, in any situation. What forces there are will depend on the types of particles involved, and to calculate the forces, we will need additional rules, like the law of gravitation. But this one dynamical law predicts any system's behavior, once given those forces.

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[6] Alternatively, time can be included as an additional dimension of the statespace.

[7] Another familiar version of the law, ordinarily seen as equivalent to the above, is given in terms of momentum: $\Sigma \mathbf{F} = \dot{\mathbf{p}}$. See Hicks and Schaffer (2017) on whether these are equivalent.

[8] Whether the theory really is deterministic is an interesting question. Apparent counterexamples are in Earman (1986) and Norton (2008); further discussion is in Malament (2008) and Wilson (2009).

[9] Standard statespace constructions effectively assume the existence of physical space. See Belot (1999, 2000) on reconstructions that aim to do away with this assumption.

Two other laws of Newtonian mechanics as standardly presented are important to the theory as a whole, but will play a less central role here. Newton's first law says that an object continues with uniform velocity unless acted on by a net external force. This law helps define what it is for an object to not accelerate, or to travel inertially (with the second law saying what happens when an object is subject to a net force that yields an acceleration). Newton's third law specifies the nature of forces. It is often stated in "action-reaction" form: to every action there is an equal and opposite reaction; when one object exerts a force on a second object, the second simultaneously exerts a force equal in magnitude and opposite in direction on the first. This law tells us that forces come in pairs, as the result of interactions between two objects. It "describes the forces to some extent" (Feynman et al., 2006, 9.1), with the particular force laws further indicating that forces do not depend on anything other than the types of particles involved and their spatial separations, and that they are central forces, directed along the line between the particles. (Conservative forces, derivable from a potential.)[10]

2.2. Lagrangian mechanics

In Lagrangian mechanics, two sets of what are called *generalized coordinates* characterize systems' fundamental states at a time: the generalized positions, q_i, and their first time derivatives, the generalized velocities, $\dot{q}_i$ (i from 1 to n, for n particles in the system). As in Newtonian mechanics, we need $6n$ coordinates to completely specify the state of a system of n particles: three generalized position coordinates and three generalized velocity coordinates per particle. But unlike in Newtonian mechanics, these do not have to be ordinary position and velocity coordinates. (They are called generalized positions and velocities by analogy to ordinary positions and velocities.) Generalized coordinates can be any set of independent parameters that completely specify a system's state.[11] Generalized

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[10] There are questions surrounding the further restrictions that forces be central and conservative. It is usually thought that nonconservative forces, like frictional ones depending on velocity, arise from fundamental conservative ones. As Feynman notably put it, "there are no nonconservative forces!" (2006, 14.4). Newton himself did not restrict forces in this way; Feynman suggests it is an additional empirical posit. (Compare Baez: "It is a simplifying assumption that has withstood the test of time and experiment" (2005, 6).) The restrictions are assumed in standard proofs of energy conservation and other theorems. (This is one place the question of electromagnetic features (note 1) comes into play. Consider the magnetic force on a moving charge, which does not satisfy these restrictions.) Concerns over the above have led the odd physicist to doubt the equivalence of the different formulations of classical mechanics: Lanczos (1970, 77 n1); Gallavotti (1983, ch. 3). See also Hertz (1899) and Wilson (2009, 2013, forthcoming) on these and other reasons to doubt their equivalence.

[11] There are some mild constraints on generalized coordinates (José and Saletan, 1998, 2.1.2). Wilson (2009) points out that the idea of generalized coordinates, as well as the requirements

positions can have units of energy, or length squared, or an angle, or can even be dimensionless. We can use any kind of coordinates that are suited to a system, the choice typically guided by the number of degrees of freedom of the system[12] and the topology of the spatial region in which the particles are free to move around. For a pendulum, for example, we might use the angle θ the suspending string makes with respect to the vertical as the generalized position, with $\dot{\theta}$ the generalized velocity (as we will see in section 3).

The Lagrangian statespace is a $6n$-dimensional space with the structure of a tangent bundle. This space comprises a $3n$-dimensional space in which we represent the generalized positions (called the *configuration space*), plus the $3n$-dimensional tangent space at each point (to represent the generalized velocities, which are tangent to the generalized positions). Each point in the statespace picks out a generalized position and generalized velocity for each particle in the system. Standard labels are Q for the configuration space (the "base space" of the tangent bundle), T_qQ for the tangent spaces (the "fibers," one for each q in Q), and TQ for the entire statespace, sometimes referred to as the velocity phase space. Notice the configuration space is what represents the physical space the particles move around in. Given the freedom in generalized coordinates, this representation needn't occur in an obvious way, yet the structure of physical space will still be coded up in the structure of Q.

The dynamical laws, called the Euler-Lagrange, or simply Lagrange, equations, say how the point representing a system's state moves through the statespace over time, given a scalar function called the Lagrangian, L. At each point in the statespace, this function assigns a number, typically equal to the system's kinetic energy, T, minus its potential energy, V.[13] Although this gives the Lagrangian as defined on TQ, we can think of this function as coding up information about particles' ordinary spatial features, those that are relevant to their energies, so that it is ultimately about goings-on in three-dimensional space. The motion of an n-particle system in three-dimensional space is then given by $3n$ second-order equations, one for each particle in each direction—one for each degree of freedom (three per particle in three-dimensional space):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{2}$$

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on them, are not as straightforward as usually assumed.

[12] The number of degrees of freedom is the number of independent parameters "necessary and sufficient for a unique characterization" of the system (Lanczos, 1970, 10).

[13] Standard examples in which it does not have this form come from outside the point-particle mechanics assumed here. See José and Saletan (1998, 2.2.4) and Goldstein et al. (2004, 7.9) for examples from electromagnetism and special relativity.

Given L, these equations uniquely determine the motion for an initial state characterized by the generalized position and generalized velocity of each particle in the system. A solution, found by integrating, gives a function or trajectory on Q, which represents the motions of all the particles through physical space. (Solutions are curves through TQ, which are projected onto Q.)

To get a feel for the Lagrangian statespace, picture the statespace for a particle moving on a one-dimensional circle: figure 1. (Keep in mind that this is "just about the only easily visualized nontrivial TQ" (José and Saletan, 1998, 94); with more degrees of freedom, things quickly become difficult to picture.) This is a two-dimensional space, each point being picked out by two coordinates $(q, \dot{q})$. The circle represents the different possible values of the generalized position coordinate, the lines the different possible values of the generalized velocity. Curves through this space represent different possible histories of the system, different sequences of generalized positions and velocities over time. The figure could represent the statespace of a point-mass pendulum, for instance, with the circle representing the values of θ and the lines the values of $\dot{\theta}$.

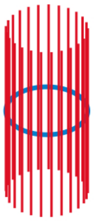

Figure 1: Two-dimensional tangent bundle (image from Wikipedia)

Briefly note three interesting, interrelated differences between the Lagrangian and Newtonian formulations.[14] First, in Lagrangian mechanics, a scalar energy function is what determines a system's motion, whereas in Newtonian mechanics, the motion is given by the forces, which are vector quantities. Second, Lagrangian mechanics takes a more "holistic" approach to describing systems' motions, in terms of the energy of the system as a whole. By contrast, the Newtonian formulation "is intrinsically a particle-by-particle description" (Sussman and Wisdom, 2014, 3), given in terms of the forces on each individual particle due to every other particle. Third, Lagrangian mechanics is a more coordinate-independent formulation of the dynamics, in that we can substitute any kind of coordinates for q and $\dot{q}$ in equation 2. The central equation of Newtonian mechanics, on the other hand, contains an implicit preference for Cartesian coordinates, those

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[14] See Lanczos (1970) for discussion of these and other differences. See Butterfield (2004) for extended discussion of Lagrangian mechanics in particular.

in which it has the form of equation 1. We can of course use other kinds of coordinates, but the form of the equation will differ (contrast equation 1 with the form in polar coordinates, e.g.: Taylor (2005, eq. 1.48)). This is not the case in Lagrangian mechanics: "Lagrange's equations, unlike Newton's, take the same form in any coordinate system" (Taylor, 2005, 237). (The form of an equation is the form as a function of its variables, a standard notion in physics.[15])

2.3. Hamiltonian mechanics

Hamiltonian mechanics shares a special kinship with Lagrangian mechanics, more so than with Newtonian mechanics. Here, too, a scalar energy function determines the motion, and the central equations are formulated in terms of generalized coordinates. There are also some notable differences. Hamiltonian mechanics uses a different energy function and a different kind of generalized coordinate, with the result that the dynamical equations and statespace also differ.

The Hamiltonian coordinates are called *canonical coordinates*. These are the generalized positions, q_i, and the generalized momenta, p_i. (Again, i ranges from 1 to n for n particles, three of each coordinate per particle in three-dimensional space.) The Hamiltonian statespace is the cotangent bundle of configuration space, T^*Q: the configuration space, Q, together with the cotangent space, T^* (dual to the tangent space), at each point in Q (to represent the generalized momenta, which are covectors, or one-forms). This is a $6n$-dimensional space, each point of which picks out a generalized position and generalized momentum for each particle in the system. It is often called the momentum phase space, or simply the phase space.[16]

The scalar function that describes a system's motion is called the Hamiltonian, H, (typically[17]) equal to the total energy of the system—the sum of the potential and kinetic energies, instead of the difference between them, as in Lagrangian mechanics. The dynamical laws are a set of $2n$ first-order equations, two for each particle in each direction; two sets of equations for each degree of freedom:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{3}$$

These equations, called the Hamiltonian or canonical equations, uniquely determine a system's motion given an initial state specified by the canonical positions and momenta of all the particles in the system.

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[15] See Brading and Castellani (2007, 1343).

[16] A Hamiltonian statespace can in fact have a more general structure than this: North (2009).

[17] See Goldstein et al. (2004); Taylor (2005, 7.8) for conditions under which this holds.

The Hamiltonian and Lagrangian formulations are both more coordinate-independent than the Newtonian formulation. Each of them is given in terms of generalized coordinates, with the result that the dynamical equations retain their form regardless of which coordinates we use. The reason is that the Lagrangian and Hamiltonian functions, which determine the motion, are scalar functions. In Newtonian mechanics, by contrast, vector quantities—forces—determine the motion. Although vectors are coordinate-independent objects, their components change with the coordinate system. (Vectors can be defined by means of how their components transform under coordinate changes.) And as Feynman puts it, "The general statement of Newton's Second Law for each particle…is true specifically for the *components* of force and momentum [or acceleration] in any given direction," since "any vector equation involves the statement that *each of the components is equal*" (Feynman et al., 2006, 10.3, 11.6; original italics). Scalars are even more coordinate-independent than that, being completely unaffected by coordinate changes, not even "altering component-wise." (The form of a scalar function such as L or H may change with the coordinate system, but not the scalar value, nor the form of the equation in which L or H appear.)

3. Example: plane pendulum

Briefly work through a simple example to get a feel for the different flavor of each formulation. Consider a vertical plane pendulum, which moves through two spatial dimensions, as shown in figure 2. (Assume the usual idealizations: frictionless, rigid suspending string; point-mass bob; negligible air resistance; uniform gravitational field.) Use each formulation to find the equation of motion for the pendulum, the equation that describes the position of the bob as a function of time. We will see that each formulation yields the same equation of motion, but by means of different routes.

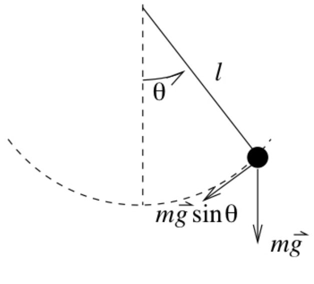

Figure 2: Plane pendulum (MIT OCW)

To use Newton's law, equation 1, first choose a rectangular coordinate system. Let y be in the radial direction, with x in the direction tangential to the path of the bob. Resolve the forces on the bob into their components in this coordinate system. There are two forces on the bob: the tension directed along the string, and the downward-directed gravitational force. The component of the gravitational force in the direction of the acceleration along the path—the tangential force—is $mg\sin\theta$, where θ is the angle the string makes with respect to the vertical, as shown in the figure.

There are two component equations of Newton's law, one for each direction of our coordinate coordinate system: $F_x = ma_x$ and $F_y = ma_y$. Plugging in the relevant force components yields $F_x = -mg\sin\theta = ma_x$ (the negative sign because the gravitational force points downward) and $F_y = T - mg\cos\theta = ma_y$, with T the tension in the string. Note that $a_y = 0$; as a result, we effectively ignore this second equation when solving for the equation of motion. (T has no component in the direction of nonzero acceleration: it is merely a "constraint force.")

The arclength, which measures the distance traveled by the bob along the curved path, is given by $s = l\theta$. The second derivative of this quantity, $\ddot{s} = l\ddot{\theta}$, is the acceleration along the path. Plug into the x-component equation of Newton's law, and we get the following equation of motion for the pendulum:

$$-g\sin\theta = l\ddot{\theta}. \tag{4}$$

We get the same equation of motion, in a different way, using Lagrangian mechanics. We could use rectangular coordinates as we did above; but things are simpler if we instead use generalized coordinate θ, with $\dot{\theta}$ the generalized velocity. We can plug these coordinates directly into equation 2 to get the solution. We can effectively treat θ and $\dot{\theta}$ as ordinary position and velocity coordinates, and, perhaps surprisingly, this yields the right answer.

First calculate the Lagrangian, $L = T - V$. The kinetic energy $T = \frac{1}{2}mv^2 = \frac{1}{2}m(l\dot{\theta})^2$. (The arclength is $s = l\theta$, the velocity its first time derivative.) The potential energy $V = -mgl\cos\theta$, setting the zero at the height of the pivot point where $\theta = \frac{\pi}{2}$. (Gravitational potential energy $= mgy$, with y the vertical distance from a chosen zero.) Thus, $L = \frac{1}{2}m(l\dot{\theta})^2 + mgl\cos\theta$. Calculate the following derivatives (in effect treating θ and $\dot{\theta}$ as independent variables, even though one is really defined as the time derivative of the other): $\frac{\partial L}{\partial q} = \frac{\partial L}{\partial \theta} = -mgl\sin\theta$ and $\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$, so that $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$. Finally, plug into equation 2: $ml^2\ddot{\theta} - (-mgl\sin\theta) = 0$, i.e. $l\ddot{\theta} + g\sin\theta = 0$, which, rearranged, is equation 4.

In Hamiltonian mechanics, we first find the Hamiltonian, $H = T + V$. Given L above, we can see that $H = \frac{1}{2}m(l\dot{\theta})^2 - mgl\cos\theta$, but we need to rewrite this in terms of canonical coordinates. To find the generalized momentum, p_θ, which is "conjugate" to the position variable θ, use this equation: $p = \frac{\partial L}{\partial \dot{q}}$, often taken to be the definition of the generalized momentum.[18] Using the equation $p_\theta = \frac{\partial L}{\partial \dot{\theta}}$, we find that $p_\theta = ml^2\dot{\theta}$, so that $\dot{\theta} = \frac{p_\theta}{ml^2}$, which we can use to eliminate $\dot{\theta}$ from the expression for H. Thus, $H = \frac{1}{2}m(l\frac{p_\theta}{ml^2})^2 - mgl\cos\theta = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta$. Now we can find the equation of motion for the pendulum using the Hamiltonian equation $\dot{p} = -\frac{\partial H}{\partial q}$; that is, $\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta$. Differentiate $p_\theta = ml^2\dot{\theta}$ to get $\dot{p}_\theta = ml^2\ddot{\theta}$, and plug into the equation for $\dot{p}_\theta$ to get $ml^2\ddot{\theta} = -mgl\sin\theta$; i.e. $l\ddot{\theta} = -g\sin\theta$, which again yields equation 4.

4. Equivalent formulations?

We find the same equation of motion for the pendulum regardless of which formulation we use. This turns out to be true in general. It is often simpler to use Lagrangian or Hamiltonian mechanics rather than Newtonian mechanics, since we do not have to calculate the various component forces on each particle. Nonetheless, it is generally agreed that each formulation suffices for describing the motion of any classical mechanical system.[19] The difference seems to be merely a matter of calculational convenience.

Indeed, physics books typically state, and go on to prove, an equivalence among the three formulations, by showing that their dynamical equations are all inter-derivable.[20] A typical route is to begin with Newton's laws, derive the Lagrangian and Hamiltonian equations from them, and then show that the derivation can go the other way. Thus José and Saletan, at the beginning of their chapter on Lagrangian mechanics, following the one on Newtonian mechanics, write, "In this chapter we show how the equations of motion can be rewritten....We should emphasize that the physical content of Lagrange's equations is the same as that of Newton's" (1998, 48). They then show that Hamilton's equations, in turn, can be derived from Lagrange's, and vice versa, concluding that they all "contain the same information" (1998, 207). Another book concludes that, "From the point of view of the physicist this division [into

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[18] The above is an instance of a Legendre transformation, which can be used to change back and forth between Hamiltonian and Lagrangian coordinates, energy functions, and statespaces: see Lanczos (1970, ch. 6); Arnold (1989, 3.14); José and Saletan (1998, ch. 5).

[19] Or so I assume here, setting aside reasons for hesitation on this point (note 10).

[20] Examples: Arnold (1989); Marion and Thornton (1995); Hand and Finch (1998); José and Saletan (1998); Talman (2000); Goldstein et al. (2004); Baez (2005); Taylor (2005); see also Feynman (1965, ch. 2).

the three formulations] is rather artificial.…The segregation is based entirely on the mathematical methods used" (Talman, 2000, 163). It certainly seems like these are simply "alternative statements of the laws" (Marion and Thornton, 1995, 213), with "nothing new…added to the physics involved" (Goldstein et al., 2004, 334) as we pass from one formulation to another. That is the standard view: the three formulations are completely equivalent, mere notational variants; they say all the same things, just in different ways.

I want to urge caution in adopting the standard view. The alleged equivalence is not as straightforward as the above statements would have us believe. The reason is that there *are some* differences among the formulations, and it is not obvious that they are as superficial as usually thought. Draw a rough distinction between two kinds of differences: mathematical and metaphysical. I won't go into these in detail, but will point to places where there is a case to be made that the differences go deeper than ordinarily claimed.

4.1. Mathematical differences

It is important to keep in mind that two things can be similar or equivalent in some ways while differing in other ways. Two objects can share a shape yet have different colors or patterns. Two spaces can share a distance structure yet differ in whether they have a privileged location. In mathematics more generally, two mathematical objects are considered equivalent when there is the relevant structure-preserving mapping between them, in which case they are said to be equivalent with respect to that structure. Two such objects can still differ with respect to other kinds of structure.

All of which is to say that, even if the three formulations of classical mechanics are equivalent in all the ways that physics books suggest, the formulations could still be inequivalent in other ways. The question is whether they are equivalent, full stop. The answer depends on whether what differences there are *matter* in any way.

There is one patent mathematical difference among them: the formulations use different symbols, in equations that do not "look" the same. The standard view is that this difference does not matter. Consider the change from Cartesian to polar coordinates to describe a Euclidean plane, or from one set of Cartesian coordinates to another that is rotated or translated with respect to the first. Some things will be different when we switch to the other coordinate system—the points will get different numerical labels, for example—but we know that nothing has *really* changed. The plane remains the same; we have simply used a different, equally legitimate way of describing it. The standard view is that the differences among the three formulations of classical mechanics are just like the differences among the coordinate-based descriptions of the plane: just a change in the coordinates or variables being used to describe the very same physics.

However, there are some reasons to question this idea. Take Newtonian mechanics, on the one hand, and Lagrangian and Hamiltonian mechanics, on the other. The latter are comparatively coordinate-independent formulations of classical mechanics. This suggests that they more directly get at the nature of classical mechanical reality, apart from our descriptions of it—just as the metric tensor on the Euclidean plane, rather than any coordinate-dependent distance formula, more directly captures the intrinsic structure of the plane. (The familiar form of the distance formula stemming from the pythagorean theorem, $d = \sqrt{\Delta x^2 + \Delta y^2}$, for instance, assumes Cartesian coordinates and won't work in other types of coordinates, even though the distance between any two points is the same regardless of the coordinate system.) This, in turn, suggests that we have reason to prefer these formulations. Physics prizes coordinate-independence, and with good reason.[21] Since there is freedom in which coordinate system to use, any choice we do make will be arbitrary—a conventional choice made from among equally good descriptions. (Recall the different coordinate systems for the plane.) We can be misled into thinking that coordinate-dependent features, which rest on an arbitrary choice in description, reflect genuine features of reality.[22] A formulation that is independent of coordinates is then preferable, other things being equal, when it comes to figuring out what physics says about the world. So that even if the equations of the three formulations are inter-derivable in some sense, there is also a sense in which the formulations are not mathematically on a par, a sense in which they are not *completely* equivalent. Some of them may more directly represent physical reality than others.[23]

We can go further. For the way in which the formulations differ in their reliance on coordinates suggests particular physical differences among them. Newtonian mechanics contains an implicit preference for Cartesian coordinates, the kind of coordinates in which its core equation takes the standard form. A preference for Cartesian coordinates, in turn, is indicative of a Euclidean metric structure. This suggests that the spatial structure of a Newtonian world is

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[21] Lanczos notes of the Lagrangian equations that they "stand out as the first example of that 'principle of invariance' [a kind of coordinate-independence] which was one of the leading ideas of 19th century mathematics, and which has become of dominant importance in contemporary physics" (1970, 117).

[22] Einstein once said that the main reason it took him so long to develop general relativity is that "it is not so easy to free oneself from the idea that co-ordinates must have an immediate metrical meaning" (Schilpp, 1970, 67).

[23] All that said, the role of coordinates in physics is more subtle and complicated than the above discussion might suggest: see North (forthcoming).

Euclidean. (Newton himself, of course, assumed such a structure.) Lagrangian mechanics, which allows for a wider range of coordinates in describing classical systems, does not constrain the spatial structure in the same way. This suggests that the physical space of a Lagrangian world has a "looser" metric structure. (I explore this difference, which will be reflected in the theories' statespace structures, in North (forthcoming, ch. 4).) Hamiltonian mechanics allows for even greater freedom of coordinates than that. (In particular, it allows for coordinate changes that mix up the p's and q's, whereas in Lagrangian mechanics, since $\dot{q}$ is defined as the time derivative of q, there is no allowable transformation in which these coordinates "get intermingled" (Taylor, 2005, 538).[24]) As a result, the Hamiltonian formulation does not require a metric structure, but only a lesser type of structure akin to a volume measure. (I explore this difference in North (2009).)

I'd go so far as to suggest that there is a hierarchy, in order of increasing mathematical structure, from Hamiltonian to Lagrangian to Newtonian mechanics—a mathematical inequivalence among the three. (In the above-mentioned writings, I argue that less such structure is in general a reason to prefer a theory.) If we take a theory's mathematical structure seriously in telling us about the nature of the physical world, then this mathematical difference should reflect a similar hierarchy in the physical structure of the world(s) each theory describes—a physical inequivalence among them. In other words, these may not be wholly equivalent formulations, neither mathematically nor physically, contrary to the standard view.[25]

4.2. Metaphysical differences

Since the dynamical equations and basic quantities of the three formulations are inter-derivable in ways that physics books claim, you might want to conclude that the different formulations are simply "mutually supporting, compatible perspectives on the phenomena of mechanical motions" (Wilson, 2007, 179). That, once again, is the standard view.[26] But there are other differences among

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[24] There is a mathematical transformation between them (note 18), but even it "leads one to suspect that there actually is a nontrivial difference between L and $\dot{q}$ on the one hand and H and p on the other" (José and Saletan, 1998, 217).

[25] Opposition to this conclusion, for different reasons, can be found in Swanson and Halvorson (2012); Curiel (2014); Barrett (2015). Barrett (2019) points out how our judgments about the relationship between the theories will depend on what we take to be their core structures, with different views on their structures leading to different such judgments.

[26] Following Coffey (2014), the standard view may more accurately be put as that Newtonian mechanics accurately represents classical mechanical reality, with Lagrangian and Hamiltonian mechanics being mere reformulations of it.

the formulations, what I call here "metaphysical" ones, that could lead to a different conclusion. (Don't let the term mislead you: these differences arguably matter to *physics*.) Although no theory wears its metaphysics on its sleeves, on a natural way of interpreting the formulations, they differ from one another in potentially significant ways. All assume a fundamental ontology of point-mass particles with relative positions. Beyond that, each one offers a fairly different picture of the world, given the different quantities that appear in their respective dynamical equations. (What follows are some initial suggestions; the metaphysics of the three formulations has not been much explored in the literature.)

First compare Newtonian mechanics, on the one hand, with Lagrangian and Hamiltonian mechanics, on the other. Newtonian mechanics "describes the world in terms of forces and accelerations (as related by the second law)" (Taylor, 2005, 521), where "force is something primitive and irreducible" (Lanczos, 1970, 27). Lagrangian and Hamiltonian mechanics describe systems in terms of energy, with force being "a secondary quantity" derivable from the energy (Lanczos, 1970, 27). According to Newtonian mechanics, the world is fundamentally made up of particles that move around in response to the various forces between them. According to Lagrangian and Hamiltonian mechanics, particles move around and interact as a result of their energies. Although energy and force functions are inter-derivable in ways that physics books will show (albeit under certain contestable assumptions: note 10), these are nonetheless prima facie different pictures of the world, built up out of different fundamental quantities, with correspondingly different explanations of the phenomena. Compare: the Schrödinger and Heisenberg formulations of quantum mechanics are generally considered inter-derivable, yet you might not want to regard them as wholly metaphysically equivalent even so; many philosophers take only the former to directly or perspicuously represent what is going on physically, for instance. (You might think that Lagrangian and Hamiltonian mechanics can be seen as fundamentally forced-based, given in terms of "generalized forces." However, generalized forces are so-called by analogy to ordinary forces. It isn't clear that they count as regular forces of the Newtonian kind.)

There are potential metaphysical differences between the two energy-based approaches as well. In Lagrangian mechanics, generalized velocities are defined as the first time derivatives of the generalized positions. This suggests that positions are the only truly fundamental dynamical features of the particles, the velocities being defined in terms of them. In Hamiltonian mechanics, on the other hand, the canonical positions and momenta are both independent variables, neither being defined in terms of the other: both seem to be fundamental. (This, in turn, may amount to an "impetus" view in the medieval tradition, with further metaphysical repercussions: Arntzenius (2000, sec. 4). This assumes that the second law of Hamiltonian mechanics is not a definition of the generalized momentum, as often claimed, but a further fundamental dynamical law.) Another difference is that the Hamiltonian is typically equal to the total energy of a system, whereas the Lagrangian is the difference between the kinetic and potential energy. Perhaps this, too, amounts to a genuine difference.[27]

In fact, there is a range of potential views on what's fundamental to each of the formulations, and it is not clear which is correct. It is an open question whether, on any of them, ordinary three-dimensional space is fundamental, or whether what we usually think of as the merely abstract, high-dimensional statespace (or the configuration space) is. Relatedly, it is open whether particle features like positions and momenta are fundamentally defined on the low- or high-dimensional space. (Compare the debate in quantum mechanics over the fundamentality of the high-dimensional space of the wavefunction versus ordinary three-dimensional space.) Within energy-based approaches, it is open whether the energy function, L or H, is fundamental, or whether instead the potential and kinetic energies are; or indeed whether any energy quantity is fundamental, rather than the particle positions and velocities in terms of which the energy is standardly defined; or whether all of these might be fundamental. Analogous questions arise for Newtonian mechanics: are total forces or component forces fundamental?[28] For that matter, can Newtonian mechanics be seen as a fundamentally energy-based theory, given the inter-derivability of the different quantities?[29] Finally: are any of these genuinely distinct possibilities, or are they all equivalent—just different, equally legitimate ways of describing the same physical reality, analogous to the different coordinate-based descriptions of the plane? Although physics books generally assume the latter, certain metaphysical views will say that only one description gets at the real or fundamental properties (Lewis, 1983; Sider, 2011).

In all, it seems very much an open question whether the three formulations of classical mechanics are genuinely equivalent, mere notational variants of a single theory, as usually thought. There is a case to be made that the differences are significant enough to render them more like distinct theories, with different accounts of what the physical world is like. All of this warrants further investigation.[30]

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[27] Baez (2005, ch. 1) tries to distinguish these physically.

[28] Cartwright (1983, ch. 3) argues against the reality of component forces.

[29] Wilson (2007) defends the existence of Newtonian forces against various objections.

[30] Some further investigation is in North (forthcoming), especially chs. 4 and 7.

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