Realism and Model Theory

1 Introduction

Putnam tried to deploy model theory to reach metaphysical conclusions about the legitimacy of realism. In the 'indeterminacy' strand, he levelled three criticisms: first, his permutation argument strove to prove that realist theories have unintended permuted models; second, his Skolemisation argument strove to prove that their theories have unintended countable models; third, his constructivisation argument strove to prove that all empirical evidence is consistent with the Axiom of Constructibility. If successful, Putnam's arguments undermine external realist accounts of truth by way of radical referential indeterminacy between language and world.

I shall defend Putnam's argument as follows. In §2, I lay the background for, and provide an exposition of, Putnam's model-theoretic arguments and how they culminate in the 'just-more-theory' manoeuvre. In §3, I examine the metamathematical challenges posed by Bays against Putnam. In §4-7, I provide counter arguments to 'Bays Dilemma' on behalf of the permutation, Skolemaisation, and constructivisation arguments. I shall therefore conclude that all the arguments are immune to metamathematical objections, and so, that Putnam's model theoretic arguments remain strong against the external realist agenda.

2 Background

2.1 Realism

I will henceforth characterise positions within realism as on a line, with one end at external realism and the other at internal realism. I will characterise the externalist programme as aspiring to reason from a 'God's Eye point of view'[1] from which the realist can evaluate our referential relation with the world. The internal realist in contrast discards this ability to evaluate an objective referential relation, and instead maintains that truth is relative to our conceptual schemes. However, referring through such schemes still determines a referential relation which distinguishes internal realism from relativism.

[1]Putnam 1980, 100

More specifically, following Button,[2] we can illustrate external realism as subscribing to the following three principles:

1. *Independence*: The world consists of a fixed totality of mind-independent objects.

2. *Correspondence*: Truth involves some sort of correspondence relation between words or thought-signs and external things and sets of things.

3. *Cartesianism*: What is epistemically most justifiable to believe may nonetheless be false.

2.2 Models and reality

Putnam strove to disprove the legitimacy of this external realist programme by constructing a model-theoretic analysis of the three principles.

Assume an external realist holds a theory T_0. Assume T_0 is expressed in a formal language as follows:

- Constant symbols: $c_1, c_2, ..., c_n$.
- Predicate symbols: $R_1, R_2, ..., R_n$.
- Function symbols: $f_1, f_2, ..., f_n$.

Let $\mathcal{W}$ denote the external realist's world, and T_0's intended model. $\mathcal{W}$'s domain, W, is constituted by the objects which make up $\mathcal{W}$. The relations between language and world is as follows:

- Constants, 'c', are mapped to the objects $c^{\mathcal{W}}$ in the domain of $\mathcal{W}$.
- Predicates, 'R', are mapped to sets $R^{\mathcal{W}}$ of objects (or pairs, triples, etc. of objects) pulled from W.
- Functions, 'f', map inputs from its domain to outputs in its range pulled from W.

This analysis illustrates the three principles as follows. For *Correspondence* the analysis characterises reference as a mapping between words of the formal language and objects in the world. For *Independence* the analysis characterises the world as a mind-independent model. For *Cartesianism* the analysis renders it a mind-independent case which model is the true model of the world, and there is always a possibility of saying something false about the world.[3]

This model-theoretic treatment aligns with an entrenched three-step approach for performing fundamental metaphysics post-Quine:[4] we first specify the fundamental ontology, then the ideology, and finally its laws. This encourages the use of model theory for performing metaphysics as follows:

[2] Button 2013, 7
[3] ibid., 11
[4] Dorr 2011, 139

- Domain = ontology.
- Language = ideology.
- Axioms = laws.

Whilst this does not entail that any individual external realist has to adopt model theory itself, it does lend credibility to assume that the above analysis is an appropriate instrument for characterising external realism, without necessarily committing the external realist to the existence of any of the models themselves.

2.3 The model-theoretic arguments

This model-theoretic analysis allowed Putnam to deploy results from model theory against external realism. In particular, he presented two model-theoretic arguments: indeterminacy and infallibilism. The indeterminacy argument shows that there are many equally correct correspondence relations, which undermines the *Correspondence* principle. The infallibilism argument shows that ideal theories are infallible, which undermines the *Cartesianism* principle. Jointly these arguments severely undermine the external realist programme.

I will be interested in objections to the *Correspondence* principle, and so, I will constrain my discussion to the argument for indeterminacy. The argument posits the following conditional:

> *The Indeterminacy Conditional*: If there is a way to make a theory true, then there are many ways to make the theory true.

If true, this claim threatens external realism with the notion that it will be entirely indeterminate which correspondence relation of all possible correspondence relations is the true one.

The indeterminacy argument is itself composed of three sub-arguments: permutation, Skolemisation, and constructivisation.

The permutation argument posits the following theorem:

> *The Permutation Theorem*: Let T be a theory with a non-trivial model. T has multiple distinct isomorphic models.

Proof: Let $\mathcal{W}$ model the external realist's theory T_0, and let W be the domain of $\mathcal{W}$. By the Permutation Theorem we can generate a distinct but isomorphic model as follows: a permutation h over the domain W of $\mathcal{W}$ will shuffle the objects, $c_1^{\mathcal{W}}, ..., c_n^{\mathcal{W}}$, in W around, yielding a new model, $\mathcal{P}$. $\mathcal{P}$ has the same domain as the original model, $\mathcal{W}$, but differs on the interpretation of every object. So, we have generated a distinct but isomorphic model. Since $\mathcal{W}$ and $\mathcal{P}$ are isomorphic, they evaluate the truth values of sentences in T_0 identically, and no possible sentence of the object language can be added to T_0 to tell them apart.

So, either of the following schemes can yield T_0's reference relation:

$$\text{`}t\text{' refers to } t^{\mathcal{W}}.$$
$$\text{`}t\text{' refers to } t^{\mathcal{P}}.$$

Either of these schemes can yield the correspondence relation:

$$\text{`}Rt_1, ..., t_n\text{' is true iff } \langle t_1^{\mathcal{W}}, ..., t_n^{\mathcal{W}} \rangle \in R^{\mathcal{W}}$$
$$\text{`}Rt_1, ..., t_n\text{' is true iff } \langle t_1^{\mathcal{P}}, ..., t_n^{\mathcal{P}} \rangle \in R^{\mathcal{P}}$$

□

In sum, the external realist cannot maintain the *Correspondence* principle because evaluating the truth value of all sentences in a language will always be insufficient to determine the reference and correspondence relation.

The Skolemisation argument posits the following theorem:

> *The Completeness Theorem*: Let T be any consistent countable set of sentences of a first-order language. There is a model $\mathcal{N} \models T$ whose domain is a subset of $\mathbb{N}$.

Proof: Assume the external realist holds a theory, T_0, whose intended model $\mathcal{W}$ is uncountable. By the Completeness Theorem of first-order logic, if T_0 has a model, then it has a countable model $\mathcal{N}$. $\mathcal{W}$ and $\mathcal{N}$ are distinct but isomorphic. Since $\mathcal{W}$ and $\mathcal{N}$ are isomorphic, they evaluate the truth values of sentences in T_0 identically. So, as in the case of permutation, there is no definite way to determine the reference and correspondence relation since there is nothing determinate to distinguish $\mathcal{W}$ or $\mathcal{N}$ as the true model.

□

The constructivisation argument posits the following theorem:

> *The Constructivisation Theorem*: For any $s \subseteq \mathbb{N}$, there is an ω-model,[5] $\mathcal{M}$, of $ZF + V = L$,[6] such that s is represented in $\mathcal{M}$.

Proof: Assume the external realist holds a theory, T_0, whose intended model $\mathcal{W}$ contains the sentence '$V \neq L$'. '$V \neq L$' can, for instance, be proven by a subset $s \subseteq \mathbb{N}$, where s is generated by a completely random procedure. If s is generated by such a random procedure, s need not be a definable subset of $\mathbb{N}$. So s could be genuinely non-constructible. By the Constructivisation Theorem, if s is genuinely non-constructible, then it exists in a model $\mathcal{M}$. $\mathcal{W}$ and $\mathcal{M}$ could be distinct but isomorphic. Mutatis mutandis. So, sentences like '$V \neq L$' is simply true in some models and false in others, and the external realist cannot decisively distinguish which is the true one.

□

[5] An ω-model is a model of set theory whose numbers are the set of natural numbers $\mathbb{N}$.

[6] 'V = L' is the Axiom of Constructibility where 'V' denotes the set-theoretic hierarchy and 'L' denotes the constructable set-theoretic hierarchy

In sum, Putnam's model-theoretic arguments threaten the external realist's conception of truth by way of radical indeterminacy in the correspondence between language and world. So, the external realist is forced to provide an account of what makes a particular correspondence relation the true one. Or, in other words, the external realist must describe what makes an interpretation intended.[7]

2.4 Preferable models and just-more-theory

The immediate response to Putnam's indeterminacy argument is to hold that some models are more preferable than others for explaining reference. This 'preferability' can be glossed in numerous ways, but the common argument is that some notion of preferability can distinguish isomorphic models, and this preferability can determine reference. Such a preference hierarchy would counteract Putnam's indeterminacy argument since the unintended models could be distinguished from the intended model.

For instance, let us follow Kripke and gloss preferability in terms of causation.[8] The Permutation Theorem noted that the instantiation of every object in $\mathcal{W}$ and $\mathcal{P}$ differ. Say that in $\mathcal{W}$, the predicate C: '... is Chanel No. 5', picks out two perfumes: $c_1^{\mathcal{W}}, c_2^{\mathcal{W}}$; but in $\mathcal{P}$, C picks out a mixture of perfumes and non-perfumes: $c_1^{\mathcal{P}}, \neg c_2^{\mathcal{P}}$. The permuted model, $\mathcal{P}$, thus seems indifferent to the causal relationships that map the word 'Chanel No. 5' to the perfumes in the domain. So, a preferable model seemingly has to respect certain causal constraints on the reference relationship between language and objects.

In response, Putnam maintains that any account of preferability must characterise the preferability relation and describe how preferable models better clarify reference,[9] i.e. external realists have to commit to a theory of preferability. However, the permutation argument says that any theory has several models. So, the theory of preferability is 'just-more-theory' – more fodder for the permutation grinder. So preferability cannot rescue the reference relation. This can be iterated into an infinite regress. So, radical referential indeterminism seems hard to circumvent for the external realist.

3 Bays Dilemma

Instead of diving further into the debate surrounding the external realist's claim that Putnam's 'just-more-theory' argument is or is not avoidable, I instead intend to focus on a more fundamental issue raised by Bays – namely, whether the metamathematical underpinnings of permutation, Skolemisation, and constructivisation are legitimate. This, I regard, may be the only adequate way to

[7]Even if the external realist attempts to forfeit correspondence for a more generic truthmaking theory, their attempts will similarly fail since a more generic theory is easier to generate objections for, see Button 2013, 19

[8]Kripke 1972, 91

[9]Putnam 1980, 477

oppose Putnam's argument, since it is the only alternative which could avoid the 'just-more-theory' manoeuvre and undermine Putnam from within rather than appealing to 'magic'. Bays' argument pertains most closely to constructivisation. However, he intends his objection to raise an intrinsic erroneous feature in all of Putnam's arguments, and so it should apply equally well against Skolemisation and permutation.[10] I shall argue this is unfounded on all points.

In essence, the model-theoretic arguments as laid out in §2.3 have the following outline. The external realist posits a theory, T_0, posing to describe the world $\mathcal{W}$ as its intended model $\mathcal{M}_0$. Putnam uses some model theory, T_1, to generate an additional model, $\mathcal{M}_1$. $\mathcal{M}_0$ and $\mathcal{M}_1$ are distinct but isomorphic. So, the external realist must provide some theory as to why $\mathcal{M}_1$ is unintended. Any new theory will be 'just-more-theory'.

Putnam's model-theoretic arguments are all made from different theorems of set theory, and can be constructed by different set theories. However, they all share a common idea: the external realist's T_0, and Putnam's T_1, will be distinct theories. Bays then argues that if T_1 is stronger than T_0, then Putnam is confronted by a dilemma: either (i), if Putnam argues against external realists who only approve of weaker set theory than him, then they will reject him because they reject the set theory Putnam abides by; or (ii), if Putnam argues against external realists who approve of similarly strong set theory as him, then his argument fails to undermine their theories.

In essence, Bays states two methods for undermining Putnam:

(a) *Rejection*: An external realist who approves of T_0 is not committed to T_1. So he can dismiss T_1 and deny that there are any non-standard models of T_0.

(b) *Ignorance*: An external realist who approves of T_1 as well as T_0 can ignore the existence of non-standard models of T_0. He is only concerned about the existence of non-standard models of $T_1 + T_0$. Putnam has not given any account of such models.

Bays holds that any external realist will abide by either method to counteract Putnam. The dilemma would demonstrate that Putnam cannot definitively generate an unintended model for every external realist theory.

4 Argument Statement

In the case of the permutation and Skolemisation arguments, I shall follow Button[11] and argue that the external realist must accept that there are unintended models for the theories he posits. In the case of the constructivisation argument, I shall follow Kanamori[12] and argue that the logic of the Constructivisation Theorem can be entirely supported by the indeterminacy conditional. My argument

[10] Bays 2001, 331
[11] Button 2011, 332
[12] Kanamori 2018, 240

will run as follows. To pose a permutation, Skolemisation, or constructivisation argument against T_0, it is perfectly adequate to assume weaker theories than T_0. So, the external realist cannot deploy either *rejection* or *ignorance* to avoid Putnam's arguments.

5 Skolemisation

Let us first investigate how Bays dilemma applies to the Skolemisation argument. I will then detail how the dilemma fails to threaten Skolemisation because the argument adopts a theorem which merely relies on sufficiently weak model theories.

5.1 Bays' dilemma and Skolemisation

The Completeness Theorem stems from WKL_0.[13] WKL_0 is a mathematical theory weaker than Z. So, we can assume the external realist's T_0 will be stronger than WKL_0 if he wants use of any rudimentary set theory. So, he will inevitably have to admit the Completeness Theorem. Moreover, let us assume that T_0 is effectively axiomatisable, since this would be a basic requirement for any complete realist theory.[14]

Since T_0 is stronger than WKL_0, it encompasses sufficient arithmetic to formalise language about all effectively axiomatisable theories. So T_0 can formalise language about T_0. Take $\mathscr{L}$ to be the language of T_0. So, there exists an $\mathscr{L}$-sentence which formalises an English statement like 'T_0 is a consistent countable set of sentences'.[15] In light of this, consider:

> T_0-*conditional*: If T_0 is a consistent countable set of sentences, then there is a model $\mathcal{N}_0 \models T_0$ whose domain is a subset of $\mathbb{N}$.

The T_0-conditional can be seen as a formal sentence in $\mathscr{L}$. This conditional is therefore a theorem of T_0. The Completeness Theorem is also a theorem of T_0, since it is a theorem of WKL_0. We can thus receive the T_0-conditional by instantiating T_0 in the Completeness Theorem. So, the T_0-conditional must be true for the external realist.

However, the T_0-conditional does not guarantee that T_0 contains a non-standard model $\mathcal{N}_0$. This can only be proved by discharging the conditional's antecedent. However, due to Gödel's Second Incompleteness Theorems, the external realist cannot prove in T_0 that T_0 has both a consistent and countable set of sentences. For consistency, the external realist must ascend up the hierarchy of theories to a new theory, T_1, such that $T_1 = T_0 + Con(T_0)$.

[13] WKL_0 is a subset of second-order arithmetic with the addition of an axiom: Weak König's Lemma. The Lemma holds that every infinite subtree has an infinite path. Otherwise WKL_0 abides by the axioms of RCA_0, Simpson 1999, 139-141.

[14] It is possible to prove the same results for negation-complete theories, see Button 2011, 330

[15] Franzen 2004, 172–176, Button 2011, 328

This is where Bays' dilemma enters. (i) the external realist can reject T_1 – he approves of the conditional, but dismisses the consequent. (ii) he can approve of T_1 and acknowledge an unintended model of T_0, but ignore this result since he is merely concerned with unintended models of T_1.

5.2 Skolemisation survives

(i) is strictly untenable. Following Bellotti[16] we observe that, if an external realist holds T_0 to be true, then he must admit both T_0 and $Con(T_0)$, because truth entails consistency. However, this is simply to admit T_1, since $T_1 = T_0 + Con(T_0)$.

So the external realist is left with (ii), ignorance. Bays now has to demonstrate that Putnam's argument does not apply to T_1. However, it is entirely possible to apply the T_0-conditional argument on T_1:

> T_1-*conditional*: If T_1 is a consistent countable set of sentences, then there is a model $\mathcal{N}_1 \models T_1$ whose domain is a subset of $\mathbb{N}$.

The T_1-conditional is a formal sentence in $\mathscr{L}$, and is therefore a theorem of T_0. The external realist holds T_1 to be true, so he holds T_1 to be consistent. So, he has to retreat to a stronger theory, T_2, such that $T_2 = T_1 + Con(T_1)$. He must now continue being ignorant. However this same story repeats in an infinite regress where Putnam can always generate a new model-theoretic argument. So, for whatever theory the external realist posits, Putnam can generate a countable model. This is all the Skolemisation argument requires.

In essence, the only necessary requirement for Putnam's argument is the *Indeterminacy Conditional*, and since the external realist will always have to assume T_0 to have a model, he will inevitably be pushed into the grinder.

In sum, the Skolemisation argument survives Bays' dilemma because of two metamathematical properties inherent in the Completeness Theorem:

1. The Completeness Theorem states that any consistent set of sentences has an 'unintended' model.

2. The Completeness Theorem is provable in significantly weak model theory.

By (1), all well developed theories have a countable model. By (2), the external realist has to accept (1) because it is proven within a model theory weaker than any he already accepts. So, whilst Putnam cannot wield a conclusive formal proof against the external realist, he can nevertheless ensure that any admissible realist theory is captured by his argument.

6 Permutation

I will now demonstrate that the two metamathematical properties just mentioned apply to the Permutation Theorem as well, and so makes it immune to Bays' dilemma.

[16]Bellotti 2005, 405

6.1 Bays dilemma and permutation

The issue runs as before, either permutation can or cannot be proven in a weaker theory than T_0.

The Permutation Theorem stems from Z-I.[17] Z-I is a mathematical theory trivially weaker than Z. So, as before, we can assume the external realist's T_0 will be stronger than Z-I if he wants use of any rudimentary set theory. So, he will inevitably accept the Permutation Theorem.

So, the previous metamathematical properties likewise apply to the permutation argument:

1. The Permutation Theorem states that any consistent set of sentences has an 'unintended' model.

2. The Permutation Theorem is provable in significantly weak model theory.

We can now run the same reasoning as in §5.2 against Bays. By (1), all well developed theories have a permuted model. By (2), the external realist has to accept (1) because it is proven within a model theory weaker than any he already accepts. So the Permutation Theorem survives Bays' dilemma.

Having been granted both the Permutation Theorem and the Completeness Theorem, Putnam can severely undermine the external realist. The Completeness Theorem breeds an unintended model from a theory. A classic response by an external realist is to evade the force of the Completeness Theorem by dismissing the method of 'fixed' theories and models. However, the Permutation Theorem breeds an unintended model from a given model. This is strictly more problematic. The external realist necessarily perceives the world to be some intended model of some all encompassing theory. But now the Permutation Theorem states that any world will be accompanied by a permuted world. So, the external realist has to accept that his theory will inevitably come with an unintended model.

7 Constructivisation

Bays' attack on constructivisation is a slightly harder case since it pertains to realism about sets. I shall first explicate his objection, and then provide a counter argument following Kanamori that the underlying conditional which constructivisation turns upon can be provided in ZFC. This, I shall maintain, is everything needed for the argument to run. So, like previously argued, Bay's dilemma will be rebutted.

[17] Z-I is a subsystem of Z. It contains the same axioms with the Axiom of Infinity subtracted. Indeed, the Permutation Theorem could be proven in an even weaker set theory, ZF-IPF, where Powersets and Replacement are further removed, Button 2011, 342.

7.1 Bays dilemma and constructivisation

First, let us make a distinction between set-models and class-models. Let T_0 be the external realist's object theory. Denote Putnam's model theory as T_1. T_1 is a set theory, so if there is a model of T_0 in T_1, then T_1 implies a set with specific properties.

However, we can infuse set theories with the notion of classes, as long as we do not reify them.[18] This enables us to state in T_1 that $\mathcal{Q}$ is a class-model of T_0. In essence, this entails that for all sentences ϕ of T_0:

$$T_1 \models \phi^{\mathcal{Q}}$$

Here, $\phi^{\mathcal{Q}}$ is a relativisation of ϕ to $\mathcal{Q}$. Relativisation is specified recursively;[19] if T_0 is some pure set theory, we merely note a domain Q and define the following schema:

$$\begin{aligned}(x = y)^{\mathcal{Q}} &= (x = y)\\ (x \in y)^{\mathcal{Q}} &= (x \in y)\\ (\phi \wedge \psi)^{\mathcal{Q}} &= (\phi^{\mathcal{Q}} \wedge \psi^{\mathcal{Q}})\\ (\neg\phi)^{\mathcal{Q}} &= \neg(\phi^{\mathcal{Q}})\\ ((\exists x)\phi)^{\mathcal{Q}} &= (\exists x \in Q)\phi^{\mathcal{Q}}\end{aligned}$$

In essence, where ϕ is a formula in set theory, then $\phi^{\mathcal{Q}}$ is yielded by ϕ from restricting all quantifiers of ϕ to Q.

We now have two sorts of models: set-models and class-models. Let us now turn to Putnam's proof for the Constructivisation Theorem.[20]

Proof: The Constructivisation Theorem can be expressed as the Π_2-sentence:

$$\phi := (\forall s)(\exists M)(s \subseteq \mathbb{N} \rightarrow (s \in M \wedge M \models ZF + V = L))$$

By Shoenfield's Absoluteness Lemma[21] we get $\psi^L \leftrightarrow \psi$, for any Π_2-sentence ψ. Proving ϕ, we merely prove its relativisation to L:

$$\phi^L := (\forall s \epsilon L)(\exists M \epsilon L)(s \subseteq \mathbb{N} \rightarrow (s \in M \wedge M \models ZF + V = L))$$

For all $s \epsilon L$, there exists a model (L), that satisfies '$V = L$', and has s as a member. From the Skolem Hull Theorem,[22] there exists a countable submodel $\mathcal{M}$, which is equivalent to L and has s as a member. From Gödel's Condensation Lemma,[23] $\mathcal{M}$ is in L.[24] So, ϕ^L is relativised to L.

□

[18] Kunen 1980, 24

[19] ibid., 112

[20] This exposition follows Button 2011, 326

[21] If Π_2 predicates over ω are relativised to L, then Π_2 subsets of ω are constructible

[22] For any model $\mathcal{A}$, there is a countable submodel, $\mathcal{B}$, which is elementarily equivalent to $\mathcal{A}$. The theorem is an elaboration on the previously used Completeness Theorem by Skolem, for a proof see Button 2011

[23] If $\mathcal{M} \models ZF + V = L$ is a transitive pure set-structure, then $M = L_\gamma$, for an ordinal γ.

[24] Putnam 1980, 468

However, it is now evident that L is a proper class instead of a set, and the Skolem Hull Theorem merely pertains to sets. This undermines the proof.

Evidently the proof fails in ZF, so in order to rescue the proof we would have to give L in a set theory where it can be accommodated as a set. For instance, we could deliver the same proof in ZFK.[25] However, retreating to ZFK makes the constructivisation argument prey for Bays' Dilemma. The external realist can reject ZFK since it is stronger than ZF; or he can approve of ZFK and welcome non-standard models of ZF, but ignore such models since he is merely concerned with non-standard models of ZFK.

7.2 Constructivisation survives

We can observe that Putnam's proof crucially turns on Shoenfield's Absoluteness Lemma. Bays' critique attacks its ability to do so: the Constructivisation Theorem cannot be a theorem of ZF because it asserts the existence of a model of ZF, and hence the consistency of ZF. However, were one to demand that the Constructivisation Theorem should be stipulated in ZFC to appeal to Shoenfield, then it is possible to do so. Consider the following conditional:

1. If for any real s there is an $\in$-model of ZF containing s, then for any real s there is an ω-model of $ZF + V = L$ containing s.

Assuming a constructible real s, there exists by hypothesis an $\in$-model $\mathcal{M}$ of ZF with s as a member, so there exists a model in the form $(L_\gamma, \in)$, i.e. $\forall s\rho \to \forall s\psi$. So, the Π_2 sentence ϕ is satisfied in L, and Putnam's conclusion ensues by Shoenfield.

Putnam's indeterminacy arguments all turn on the *Indeterminacy Conditional* at root, and they can all be borne by its logic. This conditional was initially supported in ZFC by Putnam's Downward-Löwenheim Skolem Theorem, but even stronger ZFC results can be deployed to the same end.[26] As Putnam himself notes,[27] consider Barwise's Theorem:

2. Every countable model of ZF has a proper end extension which is a model of $ZF + V = L$.

If (i) $\langle \mathcal{A}, \mathcal{B} \rangle$ and (ii) $\langle \mathcal{A}, \mathcal{B}' \rangle$ are models of ZF, then (ii) is an extension of (i) if $\mathcal{A} \subseteq \mathcal{B}$, and the membership relation M' extends the membership relation M. Additionally, (ii) is an end extension if for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $M'ba$, implies that $b \in \mathcal{A}$. The theorem is a strong upward Löwenheim-Skolem Theorem which yields an end extension that satisfies $V = L$.

Finally, the Constructivisation Theorem uses ω-models. These can be supported by a corollary of Barwise's theorem:

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[25] ZFK is ZF with the additional conjecture that there is some unattainable cardinal κ, see Bellotti 2005, 396, for an attempted proof.

[26] Kanamori 2018, 242

[27] Putnam 1980, 468

3. If there is a countable $\in$-model of ZF containing a real s, then there is an ω-model of $ZF + V = L$ containing s.

In essence, if (1) is schematised as $\forall s\rho \rightarrow \forall s\psi$, then (3) is a stronger version $\forall x(\rho \rightarrow \psi)$.

Rendering the *Indeterminacy Conditional* in ZFC by the above, we can apply the aforementioned mathematical properties (1-2) as before. By (1), all well developed theories have an ω-model of $ZF + V = L$ containing s. By (2), the external realist has to accept (1) because it is proven within a model theory weaker than any he already accepts. So, the constructivisation argument evades Bays' dilemma.

8 Conclusion

Bays has sought to undermine Putnam's model-theoretic arguments by a dilemma. Following Button, I have aimed to demonstrate that his dilemma is unfounded whilst two metamethematical properties remain attached to Putnam's arguments: (1), the theorem must state that any consistent set of sentences has an unintended model; and (2), the theorem is provable in a significantly weak model theory. The application of these properties to Putnam's arguments have been shown unthreatened by Bays' objections.

Whenever Putnam's theorems have properties (1) and (2), the external realist is sucked into Putnam's theory grinder. He must therefore admit that his theory has unintended models and provide some theory as to why they are unintended rather than intended. The metamathematical legitimacy of this result is what I have sought to demonstrate. It still remains open whether the external realist can undermine Putnam's arguments by providing some general account of what creates an unintended model. But it should now be clear that this is the structure of discourse within which the external realist is confined, and being so confined his methods for recourse are hitherto severely limited.

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