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# A Paradox about Sets of Properties

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# A Paradox about Sets of Properties[1]

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A paradox about sets of properties is presented. The paradox, which invokes an impredicatively defined property, is formalized in a free third-order logic with lambda-abstraction, through a classically proof-theoretically valid deduction of a contradiction from a single premise to the effect that every property has a unit set. Something like a model is offered to establish that the premise is, although classically inconsistent, nevertheless consistent, so that the paradox discredits the logic employed. A resolution through the ramified theory of types is considered. Finally, a general scheme that generates a family of analogous paradoxes and a generally applicable resolution are proposed.

## I. The Paradox of Sets of Properties

I here present a paradox (antinomy) about sets of properties.[2] The paradox is related to Grelling's paradox about 'heterological'. Where the latter concerns properties lacked by an adjective that

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[2] Throughout I use 'property' in the sense of a (singular) 'attribute', 'feature', or 'trait', in their ordinary senses. It is arguable that there are distinct but necessarily co-extensive properties in the relevant sense (e.g., triangularity and trilaterality, or being a valid formula of first-order logic and being a theorem of first-order logic), so that properties are not determined by their metaphysical intensions, i.e., by their associated functions from possible worlds to extensions.

expresses it (e.g., 'monosyllabic' is not itself monosyllabic), *the paradox of sets of properties* instead

concerns properties that are lacked by their unit set (e.g., the unit set of being a penguin is not a

penguin). Like Russell's paradox, the present paradox concerns sets rather than linguistic

expressions, and is thus non-semantic. Like Grelling's, however, it invokes *impredicative definition*

of an attribute, i.e., the introduction ("definition") of an attribute, in this case a property, by

abstraction from an interpreted open formula (a "condition") that quantifies over a totality that

purportedly includes the abstracted attribute.[3] Formal paradoxes have been used to demonstrate

that certain initially attractive theoretic assumptions must be restricted, for example naïve set

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[3] Here by 'attribute' I mean an *n*-ary relation-in-intension for $n \geq 1$ (including a property regarded

as a singular relation-in-intension), an n-ary propositional function for $n \geq 0$ (including a proposition

regarded as a 0-ary propositional function), or any similarly intensional entity, such as a corresponding

concept. (Impredicative definition of an extensional entity, such as a class or a truth value, does not pose the

same difficulty.) This notion of impredicativity is a special case of, but stricter than, the broader notion,

largely based on Henri Poincaré's vicious-circle principle (1906), to wit, that of introducing ("defining") a

particular element of a class by quantifying over the elements of that class. Although it is not impredicatively

defined in the stricter sense, in this broader sense the putative set involved in Russell's paradox (the set of

all and only those sets that are not elements of themselves) is said to be "impredicatively defined." However,

as F. P. Ramsey pointed out (1925, p. 204), so also is the idea of "fixing the reference" (Kripke) of a name by

a superlative definite description, e.g., 'the shortest spy', 'the first child to be born in the 22nd Century', 'the

second shortest spy', etc. (I thank C. Anthony Anderson for supplying this reference.) Compare Gödel 1944.

See note 18 below. The stricter sense of 'impredicative', which is likely what is usually meant in the relevant

literature, is uniformly adhered to throughout the present essay.

The notion of *definition by abstraction* involved in the stricter notion of impredicativity is to be

sharply distinguished from the distinct notion that goes by the same moniker (and which, as Frege showed,

is in fact fictitious) of purportedly defining a function (e.g., the cardinality function) by defining what it is for

arguments to the function to yield the same value. See my (2018).

comprehension or the assumption that a classical language can serve without limitation as its own

metalanguage. Whitehead and Russell (1927) cited paradoxes of impredicativity to argue that

impredicative definition is logically illegitimate. I shall invoke the present paradox to make a

stronger argument for a weaker but still highly significant conclusion: Classical applied higher-

order logic under an intensional interpretation of predicates as standing for attributes is incorrect,

in that (although it is consistent) the classical liberal method of abstraction of an attribute from an

open formula, as typified by classical lambda-expansion, is formal-logically fallacious (a non

sequitur).[4] On the other hand, repair does not require a ban on impredicative definition (let alone

ramified type theory).

There are properties. If there are properties, then there are sets of properties. The

property of primality, for example, conjoins the particular property set {being a natural number,

being greater than one, not being the product of two natural numbers smaller than it}. This would

not normally be subject to serious dispute. Some readers have proposed resolving the present

paradox by embracing properties while rejecting the possibility of a set of properties. This

reaction is scarcely credible. There are sets of a variety of things. There are sets of material

objects, sets of expressions, sets of propositions, sets of colors, sets of real numbers, sets of laws.

There are even sets of sets. Given that there are properties, another kind of thing of which there

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[4] A logician objects that in order to establish that a particular rule of inference is logically fallacious

one must show that it generates actual inconsistency. This objection confuses formal fallaciousness (e.g.,

satisfiability of the negation) and inconsistency (unsatisfiability). The sentence '$\exists x \exists y (x \neq y)$' has models and

also has counter-models, and is thus both consistent and (taken as an axiom) fallacious, indeed classically

invalid. Arguably, derivation of the classically valid '$\forall x Fx \therefore \exists x Fx$' and proofs of certain classical theorems

('$\exists x(Fx \lor \sim Fx)$', '$\exists x(x = a)$', and the like), although consistent, also involve intuitively fallacious inference

rules (classical UI, etc.).

are sets is a property. Sets of properties are employed in a variety of contemporary metaphysical theories: theories of supervenience; theories of plenitude; Meinongian theories of objects; theories of "hylomorphic embodiments"; some theories of propositions; other theories of semantic contents; and more. The present paradox does not present a compelling reason to doubt that there are sets of properties. We here seek a resolution that is neither theoretically disruptive nor offensive to intuition.

Some sets of properties have (in the sense of 'possess', 'exemplify', or 'instantiate') one of their own elements, for example, {primality, having exactly two prime factors, having exactly three elements}. Others do not. These latter sets have something in common: they each lack at least one of their property elements. Let **R** be their shared property, *being a set at least one element of which is a property the set itself lacks*. Its unit set {**R**} (pronounced 'singleton **R**') either has **R** or else it does not. Which is it? Suppose {**R**} has **R**. Then by definition some element or other of {**R**} is a property that {**R**} itself lacks. Since the only (and hence every) element of {**R**} is **R**, it follows that **R** is a property that {**R**} itself lacks. Therefore, {**R**} lacks **R**. Since **R** is the only (and hence an) element of {**R**}, **R** is a property element of {**R**} that {**R**} itself lacks. In that case, {**R**} is a set at least one element of which is a property that the set itself lacks. By definition, therefore, {**R**} has **R**. This is a paradox.[5]

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[5] The paradox of property sets is not to be confused (as several readers have) with the unit-set variant of Russell's paradox, *to wit*, the paradox of the set **r′** = {{$x$}| {$x$}∉$x$}. Assuming that **r′** exists, {**r′**} is an element of **r′** iff it is not. This variant of Russell's paradox (which has been employed to refute Frege's insufficient weakening of his Basic Law V in response to Russell's original paradox) is a garden-variety set-theoretic paradox that turns on the inconsistency of naïve unrestricted set comprehension. By contrast, the paradox of property sets explicitly invokes sets of properties (not sets of sets), and is independent of naïve unrestricted set comprehension. Significantly, the property-sets paradox invokes no comprehension principle not sanctioned by applied classical logic as based on the simple theory of types.

That there is a Russellian paradox about sets of properties is entirely to be expected. What is notable is how little is involved in its derivation, and therefore how little room there is for a philosophically compelling resolution. This is brought out by formalization. Although **R** is arguably a purely logical property, the paradox is not formalizable in standard *pure higher-order logic*, i.e., in standard higher-order logic without extra-logical constants. For consider how **R** would be represented in pure higher-order logic. Let the logic be a two-sorted logic in which the second-order monadic-predicate variables— say '*X*', '*Y*', etc.—range over properties of individuals rather than classes of individuals, while the third-order monadic-predicate variables—say 'Φ', 'Ψ', etc.— range over classes of properties of individuals rather than properties of properties of individuals. Now one might hope to represent 'Φ has **R**' (in which the third-order variable 'Φ'occurs free) by means of the string of symbols '∃*X*(Φ*X* & ~*X*Φ)'. However, the component string '*X*Φ' involves a clash of logical types ('Φ' is of higher type than '*X*') and is consequently ill-formed. Standard pure higher-order logic thus pre-empts the paradox of sets of properties by having no means to represent the property **R**, which is deemed ill-formed nonsense, a kind of logical mirage. This however is unsatisfactory. It is like purporting to resolve Russell's paradox not by declaring that there is no set of exactly those sets that are not elements of themselves—this would be at least the beginning of a genuine resolution—but instead by banning the term '{*x*| *x*∉*x*}' from the language of set theory, as when in *The Ten Commandments* Pharaoh pretends Moses never existed by prohibiting the utterance of his name. Preempting the formulation of a paradox is not the same thing as resolving it. The English gerund phrase 'being a set at least one element of which is a property that the set itself lacks' evidently designates a definite property **R** of sets of properties. It seems clear, for example, that the set of penguins lacks the property **R** whereas the unit set {being a penguin} has **R**. It is desirable to have some means for representing **R**. Even if the correct resolution of the property-sets paradox is to deny that **R** exists, then that very claim—that there is

no such property as **R**—should be at least expressible. Standard pure higher-order logic is evidently not up to the task.

This is not to say that higher-order logic preempts the paradox of property sets. Rather, the paradox is suitably formalized in *applied logic of third order*, i.e., in third-order logic with extra-logical constants (or on an alternative count, in applied logic of second order), by including sets of properties of individuals in the universe over which the individual variables range and by introducing a dyadic predicate constant for the membership relation to property sets. In particular, including sets in the universe of individuals permits a compound extra-logical expression for the putative property **R**. If **R** exists, then it can be designated. And if it does not, then at least we can say so.

For this purpose, logic of third order with lambda-abstraction is here adopted under an intensional interpretation whereby monadic-predicate variables '*X*', '*Y*', etc. range over properties (alternatively over unary propositional functions) of the individuals over which the individual variables '*x*', '*y*', and '*z*' range, rather than classes (or characteristic functions), and whereby monadic-predicate constants designate such properties. Finite sets of properties are included in the universe over which the individual variables range. We introduce an extra-logical, third-order, dyadic predicate, '$\in_3$', as a term for the binary relation between a property of individuals and a set of which the property is an element. We assume that this relation is governed by axioms exactly analogous to those of a suitable theory of sets of ur-elements. [6] We also include a definite-

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[6] The non-logical predicate '$\in_3$' is not the '$\in$' of standard set theory, which is of altogether different type. It is not uncommon for philosophers to employ set-brace notation in combination with predicate letters to represent a set of properties, as in '{F, G, H}'. This notation implicitly employs '$\in_3$'. The standard membership predicate '$\in$' may be used instead of '$\in_3$' to formalize the property-sets paradox, by postulating that properties of individuals are special individuals

description operator '$\imath$', the logic of which validates the schema $\ulcorner \Pi(\dots \imath\alpha\phi_\alpha \dots) \leftrightarrow \exists\beta[\forall\alpha(\phi_\alpha \leftrightarrow \alpha = \beta) \& \Pi(\dots \beta \dots) \urcorner$ where $\alpha$ and $\beta$ are distinct individual variables and $\Pi$ is a simple monadic or poly-adic predicate. To facilitate the exposition it is assumed contrary to Russell that definite descriptions are designators, and a free logic is employed in connection with them. These assumptions about '$\imath$' are entirely immaterial to the paradox.[7] We let '**R**' be our abbreviation for '$(\lambda y[\exists X(X \in_3 y \ \& \sim Xy)])$'.[8]

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governed by a property-comprehension schema sufficient to generate **R**—insofar as this can be achieved without also generating inconsistency. The present essay investigates the ramifications of the property-sets paradox for the more familiar higher-order intensional logic with $\lambda$-abstraction and is therefore formalized using that apparatus.

[7] The logic of '$\imath$' may be taken to be essentially that of Whitehead and Russell 1927, giving descriptions the narrowest possible scope, with the exception that definite descriptions are taken to be designators and a free logic is employed in connection with them. First-order free-logical UI ($\forall$-Elim) licenses the inference from $\ulcorner \forall\alpha\phi_\alpha \urcorner$ and the supplementary premise $\ulcorner \exists\gamma(\gamma = \beta) \urcorner$ to $\phi_\beta$, where the variable $\gamma$ does not occur free in the singular term $\beta$ and $\phi_\beta$ is the result of uniformly substituting free occurrences of $\beta$ for the free occurrences of the variable $\alpha$ in $\phi_\alpha$. First-order free-logical EG ($\exists$-Intro) licenses the inference from $\phi_\beta$ and the same supplementary premise $\ulcorner \exists\gamma(\gamma = \beta) \urcorner$ to $\ulcorner \exists\alpha\phi_\alpha \urcorner$. (First-order free-logic involves similar modifications of $\forall$-Intro and $\exists$-Elim, but these are not relevant in connection with definite descriptions.) Stricter adherence to Whitehead and Russell would also serve the present purpose but introduces needless complexity.

[8] If the property **R** is identified with its corresponding propositional function, the lambda-abstract that '**R**' abbreviates may itself be defined by means of the third-order definite description '$\imath Z \forall y[ Zy =_2 \exists X(X \in_3 y \ \& \sim Xy) ]$', where '$=_2$' is a dyadic logical predicate for identity between propositions (as well as between properties of individuals). See (Church 1974), pp. 29-30. Alternatively, the definite-description

It might be objected that sets of properties cannot be included among the individuals because of the special relationship of a set to its property elements, which are of higher logical type than individuals. This, however, is not a good reason not to include sets of properties in the universe of individuals. Logical types (*singular term*, *first-order monadic predicate*, *second-order monadic predicate*, etc.) are semantically-based syntactic categories of expressions. To be sure, these syntactic categories of expressions generate metaphysical categories of designated objects (*individual*, *property of individuals*, *property of properties of individuals*, etc.). First-order monadic predicates are indeed of higher logical type than singular terms. Something that is neither a class, nor an attribute, nor a function, nor a proposition is not the right sort of entity metaphysically to be a suitable value for a higher-order variable. This does not mean, however, that an entity that is an appropriate value of a higher-order variable (e.g., a set of properties) cannot be included in the universe over which the individual variables range. As Alonzo Church remarks (1976, p. 751), "any well-defined domain may be taken as the individuals." If the predicate variables of monadic logic of order ω are interpreted as ranging over sets of ascending ranks rather than properties, the universe of individuals of logic of first order can include those very same sets. When an extra-logical dyadic predicate for set-membership is added to the language—representing what juxtaposition of subject and predicate represents in standard monadic-logic notation—together with axioms governing the predicate, what results is a set theory with ur-elements whose underlying logic is of first order. The "individuals" of first-order set theory with ur-elements includes not only the ur-elements, but sets of them, sets of those sets, and so on for ascending ranks—the same "higher-order" entities of the relevant monadic logic of order ω—with no

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operator '⍳' is definable in terms of lambda-abstraction together with the higher-level function that assigns to any property the only object that has that property if such exists, and is undefined otherwise. Lambda-abstraction underlies all variable binding and is therefore more basic than definite-description formation.

resulting clash of logical types. (Others have also employed individuals that incorporate properties

of individuals as components, for example Fine (2008).)[9]

       With unit sets of properties included in the universe of individuals, they are evidently

subject to the following very weak condition:

*Ex*:     $\forall X \exists y (y = \{X\})$.

The notation '$\{X\}$' is a first-order singular term for a set. The term is defined by the third-order

definite description '$\imath z \forall Y (Y \in_3 z \leftrightarrow X =_2 Y)$', where '$=_2$' is a dyadic logical predicate for identity both

between properties (or propositional functions) of individuals and between propositions.[10]

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[9] I thank C. Anthony Anderson, Saul Kripke, Romina Padro, and Teresa Robertson for discussion of

the issues in this paragraph.

       There is an alternative way of looking at the matter. Like the unit set of a property, which bears a

special relationship to its property element, any meaningful English adjective bears a special relationship to

the property it expresses. Yet adjectives are to be treated as individuals rather than as entities that are of

higher logical type than properties of individuals (whatever that would mean). The relation between an

English adjective and the property it expresses is relevantly analogous to the relation between the unit set of

a property and its element. The analogy is sufficient to support permitting the inclusion of sets of properties

among the entities over which '*x*', '*y*', and '*z*' range. In fact, sets of properties can simply be replaced by their

canonical expressions, in combination with a suitably adjusted reinterpretation of set-theoretic notation

(the predicate '$\in$' for set-membership, set-theoretic braces, etc.)—so that for example '$\{\mathbf{R}\}$' is taken to

designate itself. So interpreted there is no legitimate objection to letting '$\{\mathbf{R}\}$' designate something in the

universe of individuals.

       It is worth noting also that it is critical to the proof of Frege's theorem to treat classes of concepts *X*

under which individuals fall as themselves individuals.

[10] Alternatively, *Ex* may be taken to be '$\forall X \exists y (\forall Z [Z \in_3 y \leftrightarrow X =_2 Z])$'. See note 7.

Despite appearances *Ex* is not a truth of logic.[11] However, *Ex* is a trivial set-theoretic consequence

(*via* Separation) of the apparent truism that every property is an element of some set or other.

Although not a logical truth it is evident that *Ex* is true.

  The deduction of the property-sets paradox proceeds as follows:

1.   $\forall z[\ \exists X(X \in_3 z\ \&\ {\sim}Xz) \leftrightarrow \exists X(X \in_3 z\ \&\ {\sim}Xz)\ ]$      third-order logic

2.   $\forall z[\ \mathbf{R}z \leftrightarrow \exists X(X \in_3 z\ \&\ {\sim}Xz)\ ]$       1, lambda-expansion

3.   $\exists y(y = \{\mathbf{R}\})$             *Ex*, UI/'**R**'

4.   $\mathbf{R}\{\mathbf{R}\} \leftrightarrow \exists X(X \in_3 \{\mathbf{R}\}\ \&\ {\sim}X\{\mathbf{R}\})$     2, 3, UI/'$\{\mathbf{R}\}$'

5.   ${\sim}\mathbf{R}\{\mathbf{R}\}$              4, def. of '$\{X\}$', logic

6.    $\mathbf{R}\{\mathbf{R}\}$             4, 5, *Ex*, def. of '$\{X\}$', logic

The inference at line 4 requires line 3 because the term '$\{\mathbf{R}\}$' might otherwise fail to designate, in

which case instantiation of line 2 to that term would be illegitimate. (See note 7.)

## II. Is the Premise Inconsistent?

  The primary lesson of the property-sets paradox is that there is no unit set of the putative

property of being a set one element of which is a property that the set itself lacks. The supposition

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[11] In particular, '{F} = {F}' fails if '{F}' is an improper description. See note 7. More surprising, as we

shall see the negation of *Ex* is a truth of classical third-order logic with free logic for definite descriptions.

  By Cantor's theorem, if the universe of individuals is a set, then there are more sets of individuals

than there are individuals. There are at least as many properties of individuals as sets of individuals, since

for each set there is the unique property of being an element thereof. But *Ex* entails that there are at least as

many individuals as there are properties of individuals, since according to *Ex* for each property of

individuals a unique individual is the unit set thereof. Thus *Ex* has the unsurprising consequence that the

universe of individuals is a proper class.

of the unit set of **R** is inconsistent. In fact, the deduction above is a *reductio ad absurdum* disproof

of the claim that {**R**} exists*.*

Since the deduction is classically proof-theoretically valid and makes no use of any special

(extra-logical) postulates beyond *Ex*, it is extremely tempting to regard the deduction as a disproof

of *Ex*. And indeed, this is the classical resolution. (See note 10.) It appears as if although **R** exists,

its unit set does not. But this overlooks that the puzzle is a genuine paradox. As noted above, to

reject {**R**} is to reject that **R** is an element of any set. To adopt the classical resolution without

further ado is to miss the lesson of the paradox.

The situation is more problematic than so far observed. The deduction constitutes a valid

disproof of *Ex* in applied classical third-order logic with lambda-abstraction (and with free-logical

UI and EG for definite descriptions). Under an extensional interpretation of higher-order logic, on

which first-order monadic predicates are taken as designating classes rather than properties

(hence with suitable axioms of extensionality), *Ex* is at least arguably false. The same logical

apparatus under the present interpretation, on which first-order monadic predicates instead

designate properties, embraces **R** as a genuine property and thereby precludes *Ex*. In an important

sense, this apparatus itself (so interpreted) is the problem. Classical logic is an artificial

idealization. For example, '$\exists x[x = f(a)]$' is classically valid, although if '*a*' designates France and '*f*' is

a symbol for the partial *king-of* function, which assigns to any kingdom its ruling monarch, the

classically valid sentence is indisputably untrue. Classical logic artificially disallows functors for

partial functions. First-order free logic is more realistic than classical first-order logic, hence more

widely applicable. As I use the term here, *free third-order logic* modifies the classical logic of the

third-order universal and existential quantifiers to take account of predicates and other functional

expressions that do not designate any element of their appropriate universe.[12] Just as the presence

of the set-theoretic braces requires a free logic with respect to terms formed by their means, the

presence of lambda-abstraction recommends that a free third-order logic be adopted in

connection with compound predicates. Lambda-abstracts are sufficiently like definite descriptions

that the logic must take account of the possibility that they are improper. (See note 8.)

Despite being inconsistent in classical third-order logic, *Ex* is like '$\sim\exists x[x = f(a)]$' in that

insofar as it is meaningful, it is obviously consistent in some more appropriate sense: consistent in

*real* logic. There is something very much like a model-theoretic proof that insofar as *Ex* is

meaningful it is consistent if ZF set theory is. We consider what I shall call *the pure-set*

*interpretation* of *Ex*. This interpretation is obtained from an intermediate interpretation, *the pure-*

*class interpretation*. On the pure-class interpretation the individual variables range over the pure

sets, while the monadic-predicate variables range over classes of pure sets, including proper

classes. Accordingly, '$\lambda$' is interpreted as an operator for class abstraction. On this interpretation

predicates designate their semantic extensions. A monadic-predication formula $\ulcorner \Pi\alpha \urcorner$ where $\alpha$ is

a singular term and $\Pi$ is a monadic predicate is interpreted in the normal way, so that it is true iff

the designatum of $\alpha$ is an element of the semantic extension of $\Pi$. The predicate '$=_2$' is interpreted

as a term for identity between classes, whereas '$\in_3$' is interpreted as a term for membership

between pure sets and the braces are interpreted as a symbol for the unit set operation. (See note

10.) At this intermediate stage a term $\ulcorner \{\Pi\} \urcorner$ does not designate when the monadic predicate $\Pi$

designates a proper class, since proper classes are not elements. The pure-class interpretation

accommodates classical lambda-expansion, whereas *Ex* fails for any instance in which '*X*' is

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[12] Third-order free logic analogously modifies the classical logic of the quantifiers to take account of

monadic predicates that do not designate any element of the universe over which the monadic-predicate

variables range.

assigned a proper class. In particular, on this intermediate interpretation '**R**' designates the class

of pure sets $y$ that include as an element a pure set of which the set $y$ is not itself an element. Given

Regularity, this is simply the class of non-empty pure sets.

The pure-set interpretation is a modification of the pure-class interpretation. The proper

classes are excised, so that the monadic-predicate variables now range over "small classes" (sets)

of pure sets. Also any lambda-abstract that designates a proper class on the pure-class

interpretation is stripped of its designatum and therewith of its semantic extension. On the pure-

set interpretation both the individual variables and the monadic-predicate variables range over

the pure sets. Accordingly, '$\lambda$' is reinterpreted as an operator for pure-set abstraction. The

predicates '$=_2$' and '$\in_3$' are now taken as synonyms for their un-subscripted counterparts.

On the pure-set interpretation $Ex$ expresses the truism that every pure set has a unit set.

Although $Ex$ is true on the pure-set interpretation, that interpretation is not a model of $Ex$ in the

set-theoretic sense. The deduction of the contradiction establishes that $Ex$ has no classical model;

on every classical model with respect to '$\in_3$', and allowing for improper definite descriptions as

non-designating, the lambda-abstract abbreviated by '**R**' designates while '{**R**}' does not. As with

the intended interpretation of ZF, the universe of the pure-set interpretation is (standardly) a

proper class rather than a set. More germane to the present inquiry, the pure-set interpretation is

also not a classical interpretation (with the exception for improper definite descriptions). In

particular, the interpretation does not conform to the classical requirement that every predicate

have a semantic extension, even if only the empty set. On the pure-set interpretation some

monadic predicates fail to designate anything in the universe over which the monadic-predicate

variables range. Specifically, if the extension of a predicate $\Pi$ on the pure-class interpretation is a

proper class—as for example, '$(\lambda y[y \notin y])$' and '$(\lambda y[y = y])$'—then $\Pi$ does not designate on the pure-

set interpretation and has no semantic extension. In particular, '**R**' does not designate and has no

extension. (The putative corresponding set $\mathbf{r} = \{y|\ \exists z(z \in y\ \&\ y \notin z)\}$ does not exist. See note 5.)

Whereas *Ex* has no classical set-theoretic model, the pure-set interpretation is, in effect, a model of

*Ex* in free third-order logic. The paradox thus discredits the logic employed in the deduction.

On the free third-order logic underlying *Ex*, besides the classical inference rules governing

quantification, classical lambda-expansion is also invalid and is replaced by a more widely

applicable version. The relevant version of lambda-expansion requires the comprehension schema

$\ulcorner \exists \Pi[\Pi =_2 (\lambda \alpha \phi_\alpha)] \urcorner$ as a supplementary premise (where $\alpha$ is an individual variable and $\phi_\alpha$ is a

formula in which the monadic-predicate variable $\Pi$ does not occur free). The situation is similar to

the first-order free-logical version of EG. (See note 12.) In both cases the required supplementary

premise is valid in classical logic but not in a suitable free logic. Insofar as the $\lambda$-abstract that '**R**'

abbreviates fails to designate a property and has no semantic extension, lines 2 and 3 of the

deduction are both invalid. In particular, each line requires '$\exists X(X =_2 \mathbf{R})$' as a supplementary

premise. In fact, the deduction thus yields a free third-order-logical deduction from *Ex* of the

negation of this needed supplementary premise.

## III. Russellian Resolution of the Paradox

The pure-set interpretation is not isomorphic to the intended interpretation, under which

monadic predicates are terms for properties (or propositional functions) rather than sets (or

characteristic functions). On the intended interpretation the monadic predicates '$(\lambda x[x \notin x])$' and

'$(\lambda x[x = x])$' designate properties despite not having a set as semantic extension. *Ex* is not merely

consistent in free third-order logic; insofar as it is meaningful it is a truism about properties. *Ex*

precludes {**R**} by precluding the putative property **R** itself, not on the ground that the extension is

not a set but on pain of contradiction with *Ex*. Perhaps the most satisfying resolution of the

paradox is to reject **R**, along with lines 2 and 3 of the deduction, while retaining *Ex*.

A full *Ex*-friendly resolution of the property-sets paradox must provide a principled

ground for rejecting **R**. There is a philosophically respectable and principled rationale for rejecting

**R**: The putative property is impredicatively defined, that is, it is abstracted from an open formula (interpreted as intended) that quantifies over a totality of properties that purportedly includes **R** itself. Impredicative definition smacks of circular definition. (See note 3.) For over a century, since Poincaré (1906) advanced his vicious-circle principle, many (and not only mathematical constructivists) have looked upon impredicative definition with profound suspicion.

Whitehead and Russell's ramified theory of types with axioms of reducibility is a logical apparatus tailor-made for theorizing about such things as propositions about propositions and properties that generalize over properties. Ramified type theory repudiates impredicative definition, replacing it with stratification of propositions, properties or propositional functions, and other functions (such as the unit set operation). For example, the impredicative abstraction of a property (or propositional function) F of individuals through quantifying over a plurality of properties including F is replaced with abstraction of a level $n+1$ property $F^{n+1}$ through quantifying over properties $G^n$ of level $n$, $n \geq 1$. Given that the property of being blue in color is level 1, the property of being the same color as the sky is level 2. The property of having some level 2 property or other in common with Napoleon is level 3, and so on.[13]

Ramified type theory requires that *Ex* be replaced with its ramified counterparts:

*Ex^{n+1}*:    $\forall X^n \exists y (y = \{X^n\}^n), n \geq 1$.

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[13] See Church 1976; Russell 1908; and Whitehead and Russell 1927, *12, pp. 161-167. Church's formulation of ramified type theory is followed here. The axioms of reducibility of Whitehead and Russell 1927 entail that every level $n$ property for $n \geq 1$ is co-extensive with a level 1 property. It is often said—following Chwistek (1921), Copi (1950), and Quine's commentary on Russell 1908 (Quine 1967, p. 152)—that the axioms of reducibility defeat the purpose of ramified type theory by reinstating the paradoxes of impredicativity. The claim is incorrect, however, and Russell had explicitly noted as much in 1908 (last paragraph of section V). See also Church 1974b, p. 356; Church 1976, p. 758; and Myhill 1979.

This modification amounts to treating *Ex* as 'typically ambiguous' (Whitehead and Russell 1927,

*65), i.e., as a schema for which it is to be taken that its instances at each level are asserted.

Impredicative λ-expansion sanctions the postulation of **R** as an un-leveled property of sets of

properties. Instead ramified-type theoretic λ-expansion sanctions the postulation of a property

$\mathbf{R}^{n+1}$, for each level $n \geq 1$, which is the level $n+1$ property of being a set one element of which is a

level $n$ property that the set itself lacks. On this conception un-leveled **R** is not a legitimate

property. Ramified type theory blocks the simple-type theoretic deduction of line 4, replacing it

with the following innocuous logical consequence of $Ex^{n+2}$:

$$\mathbf{R}^{n+1}\{\mathbf{R}^{n+1}\}^{n+1} \leftrightarrow \exists X^n(\{X^n\}^n \in_3 \{\mathbf{R}^{n+1}\}^{n+1} \text{ \& } \sim X^n\{\mathbf{R}^{n+1}\}^{n+1}\ ], n \geq 1.$$

where $\mathbf{R}^{n+1}$ abbreviates the level $n+1$ lambda-abstract '$(\lambda y[\exists X^n(\{X^n\}^n \in_3{}^n y \text{ \& } \sim X^n y)])$'. Insofar as

'$\sim\exists X^n(X^n =_2 G^{n+1})$' is a theorem of Russellian ramified type theory, '$\sim\mathbf{R}^{n+1}\{\mathbf{R}^{n+1}\}^{n+1}$' (or rather, what

this abbreviates) is a straightforward consequence of $Ex^{n+2}$. Likely no contradiction ensues.[14]

The ramified-type theoretic modification to *Ex* retains the spirit of the original principle,

whereas the modification to lambda-expansion has the intended effect of banishing

impredicatively defined properties like **R**. As an alternative to ramified type theory, the putative

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[14] Ramified type theory does not provide the only possible resolution of the paradox that employs

stratification. An alternative resolution is to stratify just the set-membership relation. This theory posits a

hierarchy of levels of sets: level 1 sets, whose elements that are properties are restricted to predicatively

defined properties; level 2 sets, whose elements that are properties are restricted to ordinary properties

and level 1 properties; level 3 sets, and so on. Presented with a choice between the two ramified resolutions,

the present author believes that the ramified-type theoretic resolution is decidedly preferable on

philosophical grounds. The rival resolution is part of a more piecemeal approach to paradoxes like that of

(Russell 1903) and others of that ilk. Furthermore, the limitation it imposes on collecting properties into

ordinary sets has little or no intuitive support.

property **R** may be admitted while its unit set is rejected. The principle *Ex* would need to be restricted in such a way as to preclude {**R**}, perhaps by restricting the range of the variable '*X*' to predicatively defined properties. This approach retains impredicative lambda-expansion as a rule of inference. However, the limitation imposed on collecting properties into sets has significantly less intuitive appeal than simply banishing impredicatively defined properties (and with it impredicative λ-expansion) while retaining *Ex* as a typically ambiguous schema. It is arguable that things of a particular sort (e.g., proper classes) cannot be collected into sets. There is no plausible ground for holding that properties cannot. Indeed, sets of properties are commonplace (as bachelorhood conjoins {being a man, not having been married}). In particular, it seems beyond reasonable doubt that if **R** exists, then so does {**R**}. But it is provable that {**R**} does not exist. I do not endorse ramified type theory or a ban on impredicativity. I know of no compelling reason, for example, to disallow the employment of impredicative constructions in class abstraction or in the formalization of mathematics. However, the paradox of property sets indicates that *classical applied third-order logic with property (or propositional-function) abstraction is incorrect*. Contrary to that apparatus, there is no **R**. What I do advocate is a free higher-order intensional logic with λ-abstraction.

There is a position worthy of consideration that broadens the combination of ramified type theory with lambda-abstraction (with axioms of reducibility), through a significant concession to simple type theory. As with ramified type theory, on this intermediate position, properties are stratified and impredicative definition of propositions and properties is prohibited. On the other hand, univocal quantification over propositions and properties of all levels is permitted, with un-superscripted propositional variables and monadic-predicate variables. The un-superscripted monadic predicate 'F' is to be understood as meaning F-*at-some-level or-other*, and the un-superscripted propositional letter '*p*' is to be understood as meaning *p-at-some-level*

*or-other*.[15] Accordingly, abstraction of properties or propositions from constructions involving

bound un-superscripted variables is not permitted. Thus, while it is possible to assert that

Napoleon had all of the properties, at each level, of a great general, this assertion is not to be

regarded as generating a new, un-leveled property of a great general: that of having all of the

properties at each level of a great general. This variant of ramified type theory supports the

intuition (for those who harbor it) that if Napoleon and George Patton each had some but not all of

the properties of a great general—including level 1 properties, level 2 properties, and so on—it

does not logically follow that they had some property of a great general in common: that of having

some property at some level or other of a great general. This intermediate position has the virtue

that it can retain *Ex* intact (not merely as typically ambiguous) while still disallowing **R**, and

therewith {**R**}.[16]

## IV. A Family of Paradoxes

The paradox of property sets belongs to a broad Russellian family of paradoxes, which

includes both set-theoretic paradoxes and semantic paradoxes. The family can be characterized in

terms of classes rather than properties, but the latter approach better reveals significant features

of the role of impredicativity. The paradoxes exemplify a pattern or scheme having four

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[15] Thus un-superscripted 'F' could be taken as shorthand for '$(\lambda x[\exists n F^n x])$' and un-superscripted '*p*'

as shorthand for '$\exists n p^n$', except that the meanings of such expressions in ramified type theory with '$\lambda$' involve

vacuous quantification and are not what is intended here.

[16] The intermediate position has the further advantage that neither the liar sentence *l* = '$\exists p(l$

expresses *p* & $\sim p$)' nor the truth-teller sentence *t* = '$\exists p(t$ expresses *p* & *p*)' can be said to express any

proposition, since neither sentence expresses any proposition of any particular level. The position also has

the significant disadvantage that, although *Ex* can be maintained, it cannot be said that *Ex* expresses any

proposition, since it too does not express any proposition of any particular level.

components. Let us call it *the encoding scheme*. First is a particular kind $K$, e.g., the kind *set*. Second

is a particular binary relation between entities of kind $K$ and properties of such entities. We shall

use the generic term 'encode' for this relation. Third is the particular putative property, *being*

*something of kind K that encodes some property that it itself lacks*. Where the universe is the class of

entities of kind $K$, this paradox-generating property is designated by '$(\lambda y[\exists X(y$ encodes $X$ & $\sim Xy)])$'.

We shall use '$\Re$' as an abbreviation for this lambda-abstract. Finally is a particular putative entity

$\rho$ of kind $K$ that encodes $\Re$ and does not encode any property not co-extensive with $\Re$. The

paradoxical conclusion generated by the encoding scheme is that $\rho$ has $\Re$ if and only if $\rho$ lacks $\Re$.

The encoding scheme generates Grelling's paradox as follows: let $K$ be the linguistic

category *meaningful English adjective*; let the encoding relation be that of semantically expressing;

let $\rho$ be the word 'heterological'. In this case $\Re$ is heterologicality. (See note 9.) Structurally

analogous paradoxes are similarly generated. The paradox of property sets is obtained as follows:

Let $K$ be the kind *set of properties*, and let the encoding relation be that between a set of properties

and any one of its property elements. Let $\rho$ be $\{\Re\}$. The Russell-Myhill paradox in Russell 1903

receives an analogous analysis. Let $K$ be the kind *proposition*. Say that a proposition $q$ encodes a

property $\Phi$ of propositions iff $q$ is the proposition (about $\Phi$) that $\forall p(\Phi p \rightarrow p)$. Then $\Re$ is the

impredicatively defined putative property $(\lambda q[\exists \Phi([q = \forall p(\Phi p \rightarrow p)]$ & $\sim \Phi q)])$ , i.e., *being, for some*

*property or other, the proposition that every proposition with that property is true, while itself*

*lacking that same property*. Finally, let $\rho$ be the paradoxical proposition that $\forall p(\Re p \rightarrow p)$.[17]

Interestingly, Russell's paradox about sets can also be regarded as an instance of the

encoding scheme: Let $K$ be the kind *set*; let the encoding relation be that between a set and any

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[17] This analysis reveals that the paradox need not be regarded as invoking the notion of logical

product (conjunction). *Cf.* Salmón forthcoming and Robertson Ishii and Salmón 2019. The two "stone caster"

paradoxes discussed there have an analogous analysis in terms of the encoding scheme.

property shared by all and only its elements; let $\rho$ be the Russell putative set. Assuming

(erroneously) that the Russell putative set exists, it encodes the property $(\lambda y[y \notin y])$. This paradox-

generating property is predicatively defined. However, $\Re$ is not this property. It is $(\lambda y[\exists X(y =$

$\{z|\ Xz\}\ \&\ \sim\!Xy)])$, i.e., *being, for some property or other, the set of things with that property, while*

*itself lacking that same property*. This impredicatively defined property is equivalent to $(\lambda y[y \notin y])$.

It thus emerges on the present analysis that Russell's paradox may legitimately be regarded as

invoking an impredicatively defined property equivalent to $(\lambda y[y \notin y])$, albeit perhaps only

implicitly.[18]

The paradox about time and thought in Kripke 2011, although more complicated, similarly

exemplifies the encoding scheme. We assume first that to think of a set *s* is exactly to invoke some

property *X* had by all and only the elements of *s* and to entertain the particular set-concept *the set*

*of things y such that Xy*, which directly concerns the property *X*. As Kripke formulates this

assumption, "one must think of a set of instants by virtue of thinking of it through a defining

property" (2011, p. 377). This may be taken as a consequence of a definition of 'thinks of'. We also

assume that Kripke thinks of no more than one set at a time. Now let *K* be the kind *instant of time*

*at which Kripke thinks of a set of times*; and let the encoding relation be that between such a time *t*

and any property *X* shared by all and only the elements of the time-set Kripke thinks of at *t*. Then

as in the previous case, although it is impredicatively-defined $\Re$ is equivalent to a predicatively-

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[18] See note 13. The axioms of reducibility entail that in each case the property $\Re$ is co-extensive with

a level 1 property.

See note 3. The dyadic predicate '$\in$' for set-membership is a primitive of the language of set theory.

It is arguable, however, that set-abstraction (*the set of individuals z such that*) is conceptually prior to set-

membership. If it is, then set-membership is properly analyzed as $(\lambda xy[\exists Z(Zx\ \&\ y = \{w|\ Zw\})])$. This would

have the result that $(\lambda y[y \notin y])$ is itself impredicatively defined (in the stricter sense used here).

defined property: *being a time at which Kripke thinks of a set of times, which time-set excludes the very time of his thinking*, i.e., ($\lambda t'[t' \notin$ the time-set that Kripke thinks of at $t']$). Finally, let $\rho$ be a particular time $t_0$ at which Kripke purportedly thinks of $\{t \mid \Re t\}$.

The primary lesson to be drawn from paradoxes of the relevant family is in each case that there is no such entity of kind $K$ as $\rho$. Specifically, there can be no such adjective as Grelling's is supposed to be, no time at which Kripke thinks of the relevant putative set of times, no such set as Russell's, not even such a set as $\{\mathbf{R}\}$. It is often said that Russell's putative set does not exist because the class of non-self-membered sets is "too large" (to be a set). But Russell's putative set of sets, Kripke's putative set of times, and the unit set $\{\mathbf{R}\}$ all fail to exist for a common reason. By any standard the class $\{\mathbf{R}\}$ is sufficiently small to be a set. Even the class of times at which Kripke thinks of a set is a "small class" and, in fact, a relatively small set.

One correct explanation for the non-existence of $\rho$ that is applicable to each of the paradoxes is a simple but remarkably munificent theorem of first-order logic, which I (forthcoming) call 'Russell's law':[19]

$$\sim\exists x \forall y[R(xy) \leftrightarrow \sim R(yy)].$$

This law holds for any universe of individuals and for any relation $R$ on that universe. To see how it applies to the relevant family of paradoxes we let the universe of individuals be *entity of kind K*, and where $x$ and $y$ are both of kind $K$ we say that $x$ *encompasses* $y$ iff $x$ encodes some feature of $y$. The following rock-hard result is then a schematic instance of Russell's law:

> *No individual of kind K encompasses all and only those individuals of kind K that do not encompass themselves.*

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[19] For a contrasting view of the matter see Martin 1977.

The exact sense of 'encompass' depends on the chosen sense for 'encode'. In the sense relevant to Russell's paradox, a set "encompasses" exactly those sets that are its elements. In the sense relevant to Grelling's paradox, an English adjective "encompasses" exactly those English adjectives to which it correctly applies. In the sense relevant to the property-sets paradox, a property set $s$ "encompasses" exactly those property sets that have one or more of $s$'s property elements. In the sense relevant to Kripke's paradox, an instant $t$ "encompasses" exactly those instants that are elements of the time-set Kripke thinks of at $t$.

It is to be noted that Russell's law does not itself invoke impredicativity. It does, however, preclude the existence of any individual that encodes the particular impredicatively defined putative property $\Re$ and does not encode any property not co-extensive with $\Re$. A compelling explanation that $\rho$ does not exist is simply that the supposition of such an entity is logically inconsistent. Just as the supposition of $\{\mathbf{R}\}$ is precluded by logic, so too is the supposition of Russell's putative set. More surprising, so too is the supposition that 'heterological' means in English what it is supposed to mean. More surprising still, so too is the supposition of a time at which Kripke thinks of the set of times at which he thinks of a time-set that excludes the very time of his thinking. It may be that though $\Re$ exists nothing of kind $K$ encodes it; it may be instead that there is no such property to be encoded.[20] The matter of which of the possible explanations obtains is part of a full resolution of the paradox, and in fact varies from one paradox to the next. Grelling's and Kripke's paradoxes are of the former sort. Heterologicality in German is expressible in English. If at a particular time $t_0$ Kripke entertains the concept *{t | $\Re$t}*, he does not thereby think

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[20] Typically (not always), the third logical possibility, that whatever encodes $\Re$ also encodes some property not co-extensive with it, can be stipulated not to obtain.

of $\{t \mid \Re t\}$. For in that case *{t | ℜt}* is not a concept of any set, and therefore there is in that case no

such set for Kripke to think of. (*Cf.* Salmón 2013.) The property-sets paradox is of the latter sort.[21]

Whatever is defective with circular definition, it is not contradiction. Impredicativity is a

kind of circularity; it is not in itself a kind of inconsistency. Russell's law suggests that the ultimate

source of contradiction in the relevant impredicativity-invoking paradoxes is not the

impredicativity *per se*, but rather the tempting supposition that something encompasses all and

only those things that do not encompass themselves. Impredicativity is a genus of a particular

species of purported properties which are posited by certain unrestricted forms of property

comprehension, and the encoding of which, along with that of some properties not of the same

genus, is typically precluded by Russell's law.[22]

## Appendix: Russell's Alternative Scheme

Russell (1907, p. 35; reprinting p. 142) provides the following scheme that is even broader

in scope than the encoding scheme: (in addition to a particular kind *K*) (*i*) a putative property $\phi$;

(*ii*) its corresponding putative set $w =_{df} \{y \mid \phi y\}$; and (*iii*) a function *f* purportedly from the power

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[21] Meinongian theories face a problem, discovered by Romaine Clark, analogous to the property-sets

paradox. See (Rapaport 1978). A resolution proposed by Alan McMichael and adopted by Edward Zalta,

when adapted to the present paradox, bans the abstraction of any property that involves the membership

relation $\in_3$ between a property and a set. See (Zalta 1983), p. 160. The purported resolution allows one to

assert that the property of primality itself has specific properties and is an element of {primality}, while

leaving no means for expressing (let alone inferring) that primality has the particular property of being an

element of {primality}. This move is an evasion of the paradox rather than a genuine resolution.

[22] Although no such characterization has been attempted here, it would be useful to have an

independent specification of exactly which subclass of impredicatively defined attributes must be rejected

to avoid inconsistency. (Feferman 2005) provides a comprehensive survey of impredicativity.

set P($w$) to $w$ and such that $\forall z[z \subseteq w \rightarrow f(z) \notin z]$. (As stipulated, $f$ is surjective, i.e., onto $w$, since

every element $x$ of $w$ is the value of $f$ for $w$–$\{x\}$.) The paradox is that $f(w)$ "both has and has not

the property $\phi$." The resolution is that $f(w)$ does not exist; hence either $w$ does not exist, or else it

does but $f$ does not exist as stipulated (e.g., the stipulation that $f$ is into $w$ must be weakened so

that $\forall z[z \subset w \rightarrow f(z) \in w]$). Some cases go one way, some the other.[23]

Russell asserts that "this generalization is important, because it covers all the

contradictions that have hitherto emerged in this subject."[24] Russell's paradox emerges on this

scheme as follows: Let $K$ be the kind *set*. Let $\phi$ be the property ($\lambda y[y \notin y]$), so that $w$ is the putative

Russell set. Let $f$ be the identity function. Then $f(w)$ is also the putative Russell set.[25] Though

Russell does not mention it, the Russell-Myhill paradox is obtained on this scheme as follows: Let

$K$ be the kind *proposition*; let $\phi$ be the putative property ($\lambda q[\exists \Phi(q = \forall p(\Phi p \rightarrow p)\ \&\ \sim\Phi q)]$); and let

$f(x) = \forall p(p \in x \rightarrow p)$. The putative proposition that $\forall p(p \in w \rightarrow p)$ both has $\phi$ and lacks $\phi$. The

property-sets paradox does not fit as neatly into Russell's scheme, but a close relative emerges as

follows: Let $K$ be the kind *set of properties*. Let $\phi$ be the putative property **R**. Let $f$ be the putative

function that assigns to any set $x$ of sets having **R** the particular unit set $\{(\lambda y[y \in_3 x])\}$. Then $f(w)$ =

$\{(\lambda y[y \in_3 w])\}$, whose sole element is an equivalent surrogate for **R**. Like $\{$**R**$\}$, assuming it exists

$\{(\lambda y[y \in_3 w])\}$ both has and lacks **R**.

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[23] Russell says that either $\phi$ is impredicative or $f$ does not exist, although "it may often be difficult to

decide which of them to choose" (p. 35; p. 143).

[24] Priest 1994 and 1995 demonstrates, through judicious selections to fill the roles of $\phi$ and $f$—and

in some cases with some finesse—that the range of paradoxes exemplifying generalizations of Russell's

scheme is remarkably broad. (I thank C. Anthony Anderson and Graham Priest for alerting me to this.)

[25] Since the identity function exists, Russell concludes that ($\lambda y[y \notin y]$) is impredicative. See notes 3,

18, and 23.

Many paradoxes that exemplify Russell's scheme, although not all, also exemplify the

encoding scheme. This is generally accomplished through the following definitions:

$D_{encode^*}$:          $y \ encodes^* \ X =_{df} \forall z(Xz \rightarrow \phi z) \ \& \ y = f(\{z| \ Xz\})$.

$D_{\Re^*}$:          $\Re^* =_{df} (\lambda y[\exists X(y \ encodes^* \ X \ \& \sim Xy)])$.

$D_{\rho^*}$:          $\rho^* =_{df} f(w)$

$D_{encomp^*}$:          $x \ encompasses^* \ y =_{df} \exists Z(x \ encodes^* \ Z \ \& \ Zy)$.[26]

By $D_{enc}$, an entity of kind $K$ encodes* every property co-extensive with any property it encodes*.

Also by $D_{enc}$, $f(w)$ encodes* $\phi$ (assuming both exist). Furthermore, $f(w)$ does not encode* any

property not co-extensive with $\phi$. For let F be a property such that $\forall x(Fx \rightarrow \phi x)$ but $\sim \forall x(\phi x \rightarrow Fx)$.

By the stipulations on $f$, $f(\{x| \ Fx\}) \in w$ whereas $f(w) \notin w$. (This remains true even if the stipulations

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[26] An interesting exception is Berry's paradox, presented in Russell 1906. It may be set out as

follows. Say that something is *concisely English-definable* iff it is designated in unaltered English by a non-

indexical definite description consisting of twelve words or less (e.g., as three is designated in English by

'the third positive integer'). Since the English lexicon is finite, there are finitely many concisely English-

definable positive integers. The paradox is that the smallest positive integer not concisely English-definable

is evidently concisely English-definable by $d$ = 'the smallest positive integer indefinable in twelve English

words or less'. See note 24. Priest 1994 (p. 29) shows how Berry's paradox may be regarded as exemplifying

Russell's scheme, and asserts (p. 33) that Russell's law is inapplicable. It should be noted, however, that a

variant of Grelling's paradox shows that there is a problem forming English descriptions that, like $d$, invoke

designating in English. *Cf.* Church 1976, pp. 756-757. Russell's law solves paradoxes of the relevant family by

precluding $\rho$. By contrast, insofar as the English lexicon is finite, then (as may be expressed in a suitable

metalanguage for English) there does exist a smallest positive integer not concisely English-definable.

However, it cannot be correctly said of that integer in English that it is designated in English in the manner

proposed.

on $f$ are made consistent as long as $\forall z[z \subset w \to f(z) \in w]$ and $\forall z[z \subseteq w \to f(z) \notin z]$.) It follows that

$f(w) \neq f(\{x|\ Fx\})$, so that $f(w)$ does not encode* F. Given the stipulations on $f$, $\mathfrak{R}^*$ is co-extensive

with $(\lambda y[\exists z(y = f[z])])$, which is co-extensive with $\phi$ (assuming all three properties exist). It

follows that $\rho^*$ encodes* $\mathfrak{R}^*$ and does not encode* any property not co-extensive with $\mathfrak{R}^*$.

On Russell's scheme a variant of encoding is explained in terms of a variant of $\mathfrak{R}$, whereas

on the encoding scheme $\mathfrak{R}$ itself is directly defined in terms of encoding. For example, on Russell's

scheme a broad semantic or semantic-like relation of expressing* between heterological adjectives

and properties is explained in terms of heterologicality, whereas on the encoding scheme

heterologicality is defined in terms of semantically expressing. Unlike Russell's scheme, the

encoding scheme reflects how paradoxes of the relevant family arise.


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