World Theory

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In general, a physical theory is based on some fundamental principles concerning space, time, matter, and the laws prescribing how matter is distributed in space and how it evolves in time, and even how spacetime itself changes in response to the changes of matter.

The spectrum of the known theories is very diverse, since they are based on different assumptions. Some theories assume that space and time are totally independent, some that they are warped in a continuum named "spacetime", and some even that there are more dimensions than the usual 3+1. Most theories consider that space and time are continuous, but many recent theories consider them to be discrete. Most dynamical laws of fundamental Physics are deterministic, but apparently Quantum Mechanics shows that they may have an indeterministic component.

Despite these distinctions between various theories, our intuition and experience tells us that they have a lot in common. In this paper I will try to capture the common features in a single mathematical structure. This is not a Theory of Everything, but only a mathematical structure which is common to most physical theories known so far. We will see that these theories can be obtained as particular cases of this structure, which I will call "world". In particular, the worlds described by the Classical Mechanics, the Theory of Relativity and the Quantum Mechanics are examples of worlds according to this definition, but also some theories attempting to unify gravity and QM, like String Theory.

The purpose of this distillation is to provide a mathematical common background to both physical and metaphysical discussions about the various theories of the World.

Keywords: physical law, sheaf theory, fiber bundles, dynamical systems, PDE

CONTENTS

I. INTRODUCTION

What are the most general assumptions one can make about the Physical World? Each theory in Physics and each philosophical system comes with its own vision trying to describe or explain the World, at least partially. In this paper I will try to establish a mathematical framework, at least for the physical world, general enough to keep an essential part from all these visions.

Science tries to discover and understand the rules governing the World. The way of science is to propose hypotheses about what the rules are, and to experimentally test their consequences. Logic is essential in deriving the consequences, in developing the explanations, in checking the logical consistency of each theory. If the logical structure of a theory is flawless, the theory can be put in a mathematical form. Many physicists strongly believe that all the laws of Nature are expressible in a mathematical form ([1, 2]). In fact, the very existence of a science like Physics is a proof that this belief exists.

Maybe Nature can be entirely described mathematically, or maybe only partially. There are important indications that we can use Mathematics to describe Nature, at least partially [3]. Physics is successful in identifying such parts of Nature, and describing them mathematically with an impressive degree of accuracy. It is this part of Nature we will discuss in the following, and I will model it by a mathematical structure here named *world*.

Suppose we know a set of axioms describing (at least partially) the laws of the real world. These axioms represent, mathematically and logically, the relations between various objects of the world. For example, they can tell, considering that the space in the Newtonian Mechanics is Euclidean, that two points determine a unique line. They will tell nothing about what the points are, or what the line is. The formalist point of view says nothing beyond the relations described by the axioms. *Ontic structural realism* even says that the mathematical structure itself is all there is, and it does not even need a *physical interpretation* [4]. My view here will be that, even if the points or lines have a physical meaning, the mathematical description will ignore it, and it will only capture the relations that can be expressed mathematically by the axioms. What can be expressed mathematically becomes part of the axiomatic theory. Any knowledge about *what* the object are is mathematically meaningless, except to the extent that they can be defined in terms of other objects of that theory. Any addition like the nature of things, meanind, *etc.*, even if it may exist, will not be considered as part of the theory. The only possibility to account for the additional objects or meanings will be by extending the theory. Even in this case, there will be in the theory fundamental objects that cannot be defined in terms of more fundamental ones. This is why an axiomatic theory describes only the relations between the objects, and any model (in the sense of the Model Theory [5, 6]) of that theory will not be part of the theory itself. Yet, any theorem deduced in the axiomatic theory will apply to the model too.

The process of abstraction and formalization will allow us to focus on the elements of the theory, and to ignore what is "additional", or the question if there is something "additional". This does not make any implication about the existence of the "additional"; this simply is beyond the scope of the theory.

The article starts by introducing the elements of a mathematical structure, named *world*, which aims to capture the main aspects of a theory in Physics (Section §II).

The world is defined as a mathematical structure containing spacetime, which in general is a topological space, and the physical laws, expressed as a sheaf over the spacetime. The definition of a world is chosen to be very general, to fit the necessities of the most theories in Physics. At the same time, it aims to capture their essence.

Examples of worlds are the various theories in Physics, as shown in Section §III.

In Section §IV I generalize the observation that, to find a description of the world, we need both laws and initial conditions. The resulting generalization aims to capture the way experiments and theory allow us to identify the states of the systems, as well as their laws.

Several possible applications of World Theory are discussed in Section §V: to theoretical models in Physics, to the study and comparison of unified theories, to the modeling of emergent phenomena.

The Appendices contain an illustration of how we select the law sheaf to obtain universal laws (Section §A), and a generalization of World Theory to spacetimes that may not even have points (Section §A).

II. PRINCIPLES OF WORLD THEORY

A. Spacetime

In Newtonian Mechanics, the background of the physical systems is the 3-dimensional Euclidean space. Physical phenomena are considered to unfold on the direct product of the space with the time axis $\mathbb{R}$. Special Relativity merged space with time and obtained a four-dimensional spacetime with Lorentz metric. General Relativity allowed spacetime to be curved, as a response to the distribution of matter, making it into a differentiable manifold with Lorentz metric.

We accept the existence of spacetime – as an arena of all physical phenomena. This arena can play an active role, as in General Relativity. It may be emergent, as in various approaches to Quantum Gravity. In the latter case it may be impossible to decompose it exactly into space and time, and this is the reason why we will take it as a whole, rather than as space and time.

This suggests that spacetime is a topological manifold, that is, spacetime is locally homeomorphic to an Euclidean $n+1$-dimensional space. But there are theories in which this constraint is too strong. In particular, there are theories in which spacetime is considered to be a graph or a foam, which are topological spaces, but not topological manifolds.

These observations suggest the following principle:

Principle 1. There exists a topological space $\mathcal{S}$, which will be named *spacetime*.

Remark 1. We will not constrain the topological dimension of $\mathcal{S}$, to allow the formalism apply to all the theories, no matter how many different spacetime dimensions they need. There is another reason for maintaining the generality: we may find it useful sometimes to apply it to the configuration space and to the phase or state space as well. But the main usage of this Principle will be for the physical space.

Remark 2 (Discrete spacetime)**.** Assuming spacetime to be "continuous" may seem limiting. Maybe it is discrete. We will see that the requirement that spacetime is a topological space does not exclude discreteness, because we can always consider its topology to be finer. At limit it can even be the *discrete topology*, which brings no constrains. However, some definitions that will be subsequently introduced will be greatly simplified if we would consider even the discrete structures as embedded in a background topological manifold, in a background-independent way. This will be the case when we will discuss the symmetries of the physical laws, in Sec. §II D.

Remark 3 (Pointless spacetime)**.** Maybe in reality spacetime is not quite a topological space, but it only seems to be one at a coarse-grained level. Category Theory [7] allows us to accommodate this possibility, as we will show in the Appendix Sec. §B. *Pointless topology*, based on *locales*, which will be defined there, go far beyond the generality needed to describe most, if not all of the known theories in physics. But since for the known theories topological spaces turn out to be sufficient, in the following we will stick to these ones, for simplicity.

B. Matter

It appears to us that matter is distributed in space and changes in time. Classically, particles and fields live in space and evolve in time. So the natural thing to do is to define matter as mathematical objects in space, evolving in time. Quantum Theory challenges this intuition, and challenged it since its inception, as noticed by Schrödinger, Lorentz, Einstein, and many others [8]. The reason is that, already in Nonrelativistic Quantum Mechanics (NRQM) of $\mathbf{n}$ particles for example, the wavefunction is an object defined on the *configuration space* of the $\mathbf{n}$ particles, $\mathbb{R}^{3\mathbf{n}}$, which consists of the coordinates of the positions of all particles. For this reason, in NRQM it may seem more appropriate to take as spacetime in Principle 1 the direct product between the configuration space and the time dimension, $\mathbb{R}^{3\mathbf{n}} \times \mathbb{R}$. To allow particles to be created or annihilated, as in Quantum Field Theory, we can use the Fock representation of the states on which the creation and annihilation operators act. In this case, the many-particle states can be represented as wave functions on $\bigoplus_{k\in\mathbb{N}} \mathbb{R}^{3k}$, and spacetime can be replaced with $\left(\bigoplus_{k\in\mathbb{N}} \mathbb{R}^{3k}\right) \times \mathbb{R}$.

Remark 4. However, we have shown in [8] that both the NRQM wavefunction, and the states in Quantum Field Theory, can be faithfully replaced with complicated fields defined on the 3D-space. This allows us to understand spacetime from Principle 1 to be $3+1$-dimensional.

In general, the matter fields are scalar, pseudoscalar, vectorial, pseudovectorial, tensorial or spinorial quantities. The natural way to see the matter fields is as sections in vector bundles. The scalar, vector, tensor and spinor fields are in fact functions valued in vector spaces, and the natural mathematical object that allows us to deal with such quantities is the notion of *vector bundle*, or more general, *fiber bundle*.

The classical electromagnetic potentials and fields are described as connections and curvatures of a principal bundle whose symmetry group is U(1). This works for other gauge fields present in the Standard Model of Particle Physics [9–11]. Both connections and curvatures are, in their turn, sections of appropriate bundles [12, 13].

As far as we know, there exist several different kinds of matter fields, so maybe we should have several bundles over spacetime. For classical theories, we simply consider at each point of spacetime the Cartesian product of all the fibers over that point (if the bundles are vector bundles, the Cartesian product becomes the direct sum). Consequently, a single bundle over spacetime is sufficient, without loss of generality. In Quantum Theory, we need to consider tensor products of the vector spaces of sections. But all of these can be incorporated in a unique matter field which is a section of a very large vector bundle, as explained in Remark 4.

Therefore, we will consider that all forms of matter are contained in a section of a bundle over spacetime:

> **Principle 2.** There exists a fiber bundle $\mathcal{M} \stackrel{\sigma}{\to} \mathcal{S}$ over the *spacetime* $\mathcal{S}$, which will be named *matter bundle*.

Here σ is the canonical projection of the bundle on the base manifold $\mathcal{S}$. The bundle $\mathcal{M} \stackrel{\sigma}{\to} \mathcal{S}$ can be a vector bundle, or simply a topological bundle. The sheaf of its local sections is named the *matter sheaf*. The matter field is a global section of the matter sheaf.

An apparently more general approach is to start directly with the matter sheaf:

> **Principle 2′.** There exists a sheaf over *spacetime*, which will be named *matter sheaf*.

But of course, any bundle has associated a sheaf of local sections, and, reciprocally, any sheaf over a topological space can be made into an *étale bundle*, such that the sections in that sheaf are local sections of the étale bundle. So in fact Principle 2′ is not more general than Principle 2. Which one is more convenient depends on the situation. For most classical field theories, Principle 2 may be more intuitive. For discrete approaches in which spacetime itself is part of the discrete structures or it is expected to emerge out of them, Principle 2′ is more appropriate.

One may worry that only fields can be represented like this. In fact, even point-particles and their paths through spacetime can be represented as sections in bundles. A natural idea is to construct sheaves of trajectories. Would it be possible to construct a sheaf that associates to each open set of spacetime a set of curves, so that they describe the paths of point-like particles in spacetime? The sections of the sheaf can be characteristic functions of the sets representing the curves in spacetime. We can maintain continuity, which is important for the very definition of the bundle, and it is general enough, provided that we choose the appropriate topology. But first we need to recall the definition of the Sierpińsky space.

Definition 1. The *Sierpińsky space* $\mathbb{S}$ is a topological space formed by two points, $\{0,1\}$, with the topology $\tau = \{\emptyset, \{1\}, \{0,1\}\}$. The set $\{0\}$ is closed (but not open), and the set $\{1\}$ is open and not closed (its closure is $\overline{\{1\}} = \{0,1\}$).

Example 1 (The Sierpińsky bundle)**.** We can take the spacetime to be $\mathbb{R}^{3+1}$, and the fiber to be the Sierpińsky space $\mathbb{S}$. The continuous functions $f : \mathbb{R}^{3+1} \to \mathbb{S}$ are sections of the trivial bundle $\mathbb{S}\times\mathbb{R}^{3+1} \to \mathbb{R}^{3+1}$ (which we call the *Sierpińsky bundle*). The set $f^{-1}(1)$ is open in $\mathbb{R}^{3+1}$, and $f^{-1}(0)$ is closed. We can see that the continuous sections of the fiber bundle $\mathbb{S} \times \mathbb{R}^{3+1}$ can be identified in a natural way with characteristic functions of open sets of $\mathbb{R}^{3+1}$. We prefer here to identify the continuous sections f with the closed sets of the form $f^{-1}(0)$. Any continuous curve $\gamma : [0,1] \to \mathbb{R}^{3+1}$ is a closed set, therefore it defines sections of this bundle, as restrictions of the characteristic function of its complement. Therefore, the set of all possible trajectories of a particle can be represented by a subsheaf. If we consider a number $\mathbf{n}$ of particles, we simply consider $\mathbf{n}$ trajectories. The particle interactions such as decay and scattering can be expressed as unions of such curves, and they also form a subsheaf of the Sierpińsky sheaf. This example shows that not only the fields, but also the trajectories can be described as sheaves over the spacetime. □

C. Physical laws

All the possible sections of the matter bundle form a sheaf over spacetime, but not all of them are allowed by the physical laws. In the theories whose laws are expressed as *partial differential equations* (PDE) we reduce the sheaf by accepting only the solutions of the equations. The local solutions of the PDE form a *subsheaf* of the sheaf of sections of the matter bundle.

That this can be done is easier to see in the case of fields. After all, sheaf theory itself originates from the study of local solutions of PDE. But the same is true even for the paths of the point-particles through space. Example 1 shows how to define the matter bundle and the matter sheaf in this case. Additional constraints are imposed by the necessity that the paths are differentiable, and that they are solutions of PDE.

Because we consider all the matter fields combined into a single one, the equations can also, at least formally, be combined into a single equation which describes the evolution of the matter fields and their interactions. This unique equation is simply a combination of all the equations that describe the physical laws for all the matter fields. It is not the "Unified Theory of Physics", because any set of equations can be "unified" in this manner. Probably the "Unified Theory" will have this form, but the component "sub-equations" must be parts of the "unified equation" in a more natural way, and not in the trivial way presented here.

For generality purposes, we will identify the laws with the subsheaf consisting of local matter fields determined by the physical laws, rather than the equations defining them. The equations are important, but if we want to include more general theories, for example discrete ones, we need to use sheaves rather than PDE.

Principle 3. There exists a subsheaf $\Lambda(\mathcal{S})$ of the matter sheaf, named *law sheaf.*

The set of global sections of the law sheaf is named the *set of solutions*, in analogy to the set of solutions of a PDE.

Remark 5 (The metric tensor)**.** The standard formulations of physics include the metric tensor in the definition of space or spacetime. However, since in General Relativity the metric tensor depends itself on the distribution of matter on spacetime, *via* the Einstein equation, it is a dynamical field. To give a unified treatment as we do here, the metric tensor is included in the matter sheaf itself, along with what we normally call "matter fields", even though it is generally not regarded as matter.

The matter field cannot merely be any allowed section of the matter bundle, but a global section of the law sheaf, because it has to obey the Physical Laws:

Principle 4. The matter field is represented by a global section μ of the law sheaf $\Lambda(\mathcal{S})$, and will be called the *matter section.*

Again, as explained in Remark 5, the metric tensor is considered here to be a component of the matter section.

D. Symmetries of the laws

A condition which lies at the core of the whole idea that the physical laws are knowable is their universality. The physical laws are supposed to be independent on place and time. The actual state of the universe, described by the matter section, can of course depend on time and position.

Principle 5. The law sheaf is independent of the position and the moment of time. In other words, for any two events $p_1, p_2 \in \mathcal{S}$, there is a local isomorphism of the law sheaf mapping p_1 to p_2.

Principle 5 states that the law sheaf and the spacetime itself are *locally homogeneous.* When we will discuss the distinction between space and time, in Section §II E, we will also discuss *local isotropy* (Definition 2).

Question 1. Suppose that the fundamental structure of the world is such that matter itself is different at different points of spacetime precisely because the topological properties at those points are different. This may be the case if spacetime is discrete, if the fundamental structure is graph-like *etc.*. Then how can Principle 5 hold?

Answer 1. For theories based on discrete structures, there are ways to generalize the notion of sheaf over the topological space $\mathcal{S}$ so that Principle 5 holds.

One way is to go along with the generalization in which $\mathcal{S}$ is not a topological space, but a locale, as in Appendix §B, but this would make the formalism too abstract, and no known theory in Physics seems to absolutely need it.

A much simpler way is to consider such discrete structures as embedded in a background topological manifold, which will be taken to be the spacetime $\mathcal{S}$. This can be done even for background-independent theories without loss of background-independence. Consider as an example the cases when the entire theory is based on graphs or foams (*e.g. hypergraphs* which not only have vertices and edges, but also two-dimensional polygonal structures that are determined by more than two vertices or edges in a way similar to how an edge is determined by a pair of vertices in graphs). Then, for each such graph or foam, there is a dimension n so that any two embeddings of the graph or the foam in the topological manifold $\mathbb{R}^n$ are homeomorphic. To see this, consider hypergraphs with $n+1$ vertices. Then, they can be seen as sub-hypergraphs of the total hypergraph, which can be embedded as an *n-simplex* (a higher-dimensional tetrahedron) in $\mathbb{R}^n$. Since any such embedding can be transformed into any other one by homeomorphisms of $\mathbb{R}^n$, background-independence is ensured by the uniqueness of the embedding up to a homeomorphism.

Therefore, even if spacetime itself is emerging from a discrete structure, we can still use sheaves to describe the laws, provided that the "true" spacetime is the emergent one, and $\mathcal{S} = \mathbb{R}^\infty$ is just a convenient representation that does not break background-independence. □

Question 2. We only have access to a matter section, which contains us as observers. We have no access to possible alternative, counterfactual histories of the world, to compare them with our own history. One may therefore think that the laws themselves can change in time, that they can evolve [14]. Then how can Principle 5 hold?

Answer 2. The assumption that the laws are universal is based on the fact that we can reproduce the experiments made in the past, and we can do this at different times and in different places. We can do this because we can concentrate our efforts on subsystems, and we can manipulate them to make the experimental arrangement. Whenever the experiment was not reproduced, assuming that there was no error from our part, we seem to be able to identify a difference. In this sense, even when physical quantities that we assumed to be constant changed in time, we later discovered that in fact they are different as a consequence of more fundamental and more universal laws, which allows them to change, and not because the laws themselves change in time or space.

A more common situation happens when we analyze the empirical data, and choose a formulation that is based on laws that are independent of the reference frame, even though other descriptions are possible. A simple example of this kind is discussed in Appendix §A.

But there is also a triviality argument for the universality of the laws. While we only have access to a particular history of the universe, we can take that history as a section of a sheaf (the matter section), and we can use it to build a sheaf that satisfies Principle 5. This can be done by identifying all local solutions obtained by restricting the matter section at different points in spacetime, and then translating them in spacetime to construct a larger sheaf which satisfies Principle 5. This construction is trivial, and leads to a law sheaf that is universal. □

There will be more to be said about symmetries in Sec. §II E, were we will discuss time.

E. Time

So far we treated spacetime as a whole, without distinguishing space from time. In general, time is a coordinate labeled smoothly with ordered values from $\mathbb{R}$. But in some theories time can be discrete, and in this case it can be labeled with integers from $\mathbb{Z}$. For full generality, the time coordinates can be a totally ordered set $\mathcal{T}$, with a topology consistent with its order relation (also called *order topology*).

Remark 6. Not all continuous maps $\tau : \mathcal{S} \to \mathcal{T}$ from $\mathcal{S}$ to $\mathcal{T}$ is a time coordinate, because space coordinates also may have this form. In standard theories like Classical Mechanics, Relativity, or Quantum Theory, the PDE distinguish between space and time directions. For example, in General Relativity, the metric tensor, which in the formalism presented here is part of the matter field, along with the other physical fields of the theory, specifies which directions are time-like, and this imposes additional conditions. In all these standard theories, in order for a coordinate $\tau : \mathcal{S} \to \mathcal{T}$ to be a time coordinate, the restriction of the law sheaf to each slice $\tau^{-1}(t)$ of $\mathcal{S}$, where $t \in \mathcal{T}$, should have certain symmetries, for example it has to be homogeneous and isotropic, since this expresses the independence of the laws on the spatial reference frame. Only spacelike hypersurfaces of spacetime satisfy isotropy. To generalize this condition, which is essential if we want to distinguish the space and time directions in $\mathcal{S}$, a general notion of direction is needed, which is not present in the mere topology of $\mathcal{S}$, but in general it is implicit in the law sheaf. At least when the law sheaf consists of local solutions of some PDE, partial differentiation requires the existence of directions in spacetime along which the differentiation takes place. If the theory is based on discrete structures, directions may not be definable, and other criteria for distinguishing space and time coordinates are needed. The following definition will help, and it is general enough.

Definition 2 (Locally isotropic sheaf)**.** Let Ω be a sheaf over a topological space $\mathcal{A}$, and $a \in \mathcal{A}$. Let γ be a continuous path joining a with a distinct point $b \in \mathcal{A}$. Denote by $\mathcal{R}(\gamma)$ the union of all transformations of γ under the homeomorphisms of $\mathcal{A}$ which preserve the point a and the sheaf Ω. The sheaf Ω is said to be *locally isotropic* at the point a if $\mathcal{R}(\gamma)$ is a neighborhood of a.

Definition 3. A *time structure* is a continuous map $\tau : \mathcal{S} \to \mathcal{T}$ from $\mathcal{S}$ to a totally ordered topological space $\mathcal{T}$ satisfying the following conditions:

1. The image of τ has more than one element.

2. For any time $t \in \mathcal{T}$, the restriction of the law sheaf to $\tau^{-1}(t)$ is locally homogeneous and locally isotropic.

3. For any two distinct times $t_1, t_2 \in \mathcal{T}$, the restrictions of the law sheaf to $\tau^{-1}(t_1)$ and $\tau^{-1}(t_2)$ are isomorphic.

Definition 4. For any $t_0 \in \mathcal{T}$ we define the topological subspace $\tau^{-1}(t_0)$, and we name it the *space* at the instant t_0, corresponding to the time coordinate τ.

Principle 6. There exists (at least) a time structure $\tau : \mathcal{S} \to \mathcal{T}$.

It is possible to have more time coordinates that are not equivalent under reparametrizations of time alone, as known from Relativity.

F. Worlds

Let's summarize the principles enumerated so far:

The matter section μ is a global section in a matter fiber bundle $\mathcal{M} \overset{\sigma}{\to} \mathcal{S}$ over a spacetime topological manifold $\mathcal{S}$. There is a locally homogeneous subsheaf Λ of the sheaf of sections of the matter bundle $\Gamma(\mathcal{S}, \mathcal{M})$, named the law sheaf, containing the admissible matter sections. The matter section μ should be a global section of this law sheaf as well. We can identify it with the subsheaf it generates, $\mathcal{F}(\mu)$. All these principles are resumed in the following diagram:

$$
\begin{tikzcd}
\text{Matter field} \arrow[r, hook, "\text{global section}", "\text{subsheaf}"'] & \text{Law sheaf} \arrow[r, hook, "\text{homogeneous}", "\text{subsheaf}"'] & \text{Matter sheaf} \\
& \text{Spacetime} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
\tag{1}
$$

or, using the notations I introduced,

$$
\begin{tikzcd}
\mathcal{F}(\mu) \arrow[r, hook, "\text{global section}", "\text{subsheaf}"'] & \Lambda \arrow[r, hook, "\text{homogeneous}", "\text{subsheaf}"'] & \Gamma(\mathcal{S}, \mathcal{M}) \\
& \mathcal{S} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
\tag{2}
$$

Definition 5. A *world* consists of a topological space[1] $\mathcal{S}$ named the *spacetime*, a locally homogeneous sheaf $\mathcal{M}$ on $\mathcal{S}$ (the *matter sheaf*), a locally homogeneous subsheaf Λ of $\mathcal{M}$ named the *law sheaf*, a global section $\mu \in \Lambda(\mathcal{S})$ (the matter section), and a time structure $\mathcal{T}$, all subject to Principles 1–6.

[1] Or, more general, a locale (see Definition 10).

III. EXAMPLES OF WORLDS IN PHYSICS

Let us start with a simple example.

Example 2 (Classical fields)**.** Let spacetime be $\mathbb{R}^{3+1}$, where space is $\mathbb{R}^3$ and time is $\mathbb{R}$. Let the matter bundle be the trivial vector bundle $\mathcal{M} = \mathbb{R}^k \times \mathbb{R}^{3+1}$ with fiber $\mathbb{R}^k$, where $k \geq 1$. We take as law the *wave equation*

$$
\frac{1}{c^2}\frac{\partial^2 f^a}{\partial t^2} - \frac{\partial^2 f^a}{\partial x^2} - \frac{\partial^2 f^a}{\partial y^2} - \frac{\partial^2 f^a}{\partial z^2} = 0, \tag{3}
$$

where $f = (f_a)_{a \in \{1,\ldots,k\}} \in \Gamma(\mathbb{R}^{3+1}, \mathbb{R}^k \times \mathbb{R}^{3+1})$. Its local solutions form a homogeneous subsheaf (the law sheaf) of the sheaf $\Gamma(\mathbb{R}^{3+1}, \mathbb{R}^k \times \mathbb{R}^{3+1})$. The bundle may include tensors over the tangent space of space or spacetime, connections, and curvatures.

Similarly, we can consider the *heat equation*, *Maxwell's equations* and so on. For classical interacting fields, the bundle is the direct sum bundle of the bundles of the individual fields. The evolution equations include in this case interaction terms, which describe how two fields affect one another. This defines the law sheaf. In this way, a classical theory in which different kinds of classical fields exist can be constructed.

This applies to Special Relativity as well. Topologically speaking, spacetime is in both cases $\mathbb{R}^4$. The difference between Classical Nonrelativistic Mechanics and Special Relativity is due to the different symmetry groups of the laws, which respectively are the *Galilean Group* and the *Poincaré group*. □

Example 3 (Classical gauge theory)**.** Gauge theory can be used to formulate Maxwell's electrodynamics in terms of fiber bundles [9, 10, 12]. Maxwell's equations can be described as in Example 2. But with fiber bundles, the electromagnetic potential gains a geometric interpretation as a *connection*, and the electromagnetic field becomes its *curvature*. The gauge group in this case is U(1). The idea scales to higher-dimensional Lie groups, leading to the *Yang-Mills theory* [15]. This led to the unification of the electromagnetic and weak interactions [16–18], and applies to the strong interactions too [9, 10]. □

Example 4 (Classical point-particles)**.** With the same spacetime $\mathbb{R}^{3+1}$ as in Example 2, we may wonder how can we describe point-particles. The history of a point-particle is a path in $\mathbb{R}^{3+1}$. Continuous fields cannot represent it, if the topology is the manifold topology.

But we can use a different topology – that of the Sierpińsky space from Definition 1. As it was shown in Example 1, this allows us to define the Sierpińsky bundle, which can be used to treat paths in $\mathbb{R}^{3+1}$ similarly to how we treated fields in Example 2. When particles collide and disappear or new particles appear, the paths are connected into a graph. Graphs embedded in $\mathbb{R}^{3+1}$ can also be represented as fields on $\mathbb{R}^{3+1}$, and in the Sierpińsky topology they are continuous sections of the Sierpińsky bundle. To endow the point-particles with properties like mass and charge, which appear in the evolution equations, we need to take the product bundle obtained from the Sierpińsky bundle and the line bundles (with the fiber $\mathbb{R}$) for each type of scalar quantity. In particular, a line bundle for the masses, another one for the electric charges and so on. This way, each point-particle is not merely a path in spacetime, but it also has "attached" charges and masses.

Thus, the matter bundle in this case contains the Sierpińsky bundle, and the law sheaf restricts these "discrete fields" to point-particles. The bundle can be extended to include even fields like the electromagnetic or gravitational fields. The law sheaf is as usual the sheaf of local solutions of the equations describing the evolution and the interactions between fields and point-particles.

Again, this applies to both Classical Nonrelativistic Mechanics and Special Relativity, for the reasons given in Examples 2 and 3. □

Example 5 (General Relativity)**.** In General Relativity, spacetime is a four-dimensional manifold. This means that it is only locally homeomorphic to $\mathbb{R}^{3+1}$. It actually is more than this, it is locally diffeomorphic to $\mathbb{R}^{3+1}$, which ensures the existence of the *tangent bundle*, whose fiber is the vector space $\mathbb{R}^4$. To this bundle, we associate the bundle of symmetric nondegenerate bilinear forms. The metric tensor is a section of this bundle, and it changes from point to point in a way that depends on how matter is distributed on spacetime, according to the *Einstein field equation*.

This bundle is then combined in a direct product with a bundle whose fibers are the same as in Special Relativity, which may include fields and point-particles as in Examples 2–4. The metric tensor field interacts with the other fields, and this happens in both directions, since on the one hand the differential operators in the field equations depend on the metric tensor, and on the other hand the metric tensor depends on the curvature, which is related by Einstein's equation to the stress-energy tensor of the other matter fields. As we see, in this case the metric tensor is included as a component of the matter field. This is natural, since the metric tensor is itself a dynamical field that interacts with other fields.

Other generalizations or modifications of General Relativity, for example theories obtained by replacing the Einstein-Hilbert Lagrangian with other Lagrangians like $f(R)$ [19], conformal gravity [20], Einstein-Cartan theory [21] *etc.*, can be described in a similar way.

The occurrence of singularities in General Relativity suggests that we should release the constraint that the metric is nondegenerate. For example, *Singular General Relativity* [22] is equivalent to General Relativity outside the singularities, but in addition it can describe a large class of singularities without running into infinities. □

As already mentioned, by using the Sierpińsky bundle we can define graphs on manifolds. Also, we can define graphs with labeled vertices or edges, by considering a bundle with fibers $\mathbb{S} \times \mathbb{K}$, where $\mathbb{S}$ is the Sierpińsky space, and $\mathbb{K}$ is a set, for example a field. We can also define triangulations into simplices, and associate to the manifold simplicial complexes. To the k-simplices we can associate values in the same manner.

As explained in Answer 1, the spacetime can be in this case the topological manifold $\mathbb{R}^\infty$, in which hypergraphs (including simplicial complexes) can be embedded uniquely up to homeomorphisms. This uniqueness ensures that spacetime does not break background-independence.

Example 6. *Regge calculus* is based on discretizing the Lorentz manifold in General Relativity by replacing it with a simplicial complex, which is a hypergraph. The metric is replaced with numbers associated to the edges (their lengths), and the curvature with angles of the 2-simplexes. Tullio Regge showed how we can translate in this setup a discretized version of the Einstein equation as constraints on these angles [23]. His work led to significant applications in numerical relativity and Quantum Gravity. □

Example 7. *Spin networks*, initiated by Roger Penrose [24–27], are graphs having all vertices of order 3, and the edges labeled with integers satisfying a set of rules (the *triangle inequality* and the *fermion conservation*). Later, the spin networks were generalized, by replacing the numbers on the edges with group representations, the vertices with intertwining operators, and by allowing the order of each vertex to be greater than 3 [28–30]. *Spin foams* are similar with spin networks, but with two-dimensional facets. They are used in Loop Quantum Gravity [31]. □

Example 8. Let us employ instead of a continuous spacetime, a lattice of points in the Euclidean version of the Minkowski spacetime, obtained by a Wick rotation. Define fermionic fields at the vertices of the lattice, and gauge fields (elements of a Lie group) on the edges linking them or on the loops they form. This is how we can construct a *lattice gauge theory* [32]. □

Example 9. Starting from the observation that in General Relativity the causal structure contains, up to a conformal transformation, all the information about the geometry, Sorkin [33–35] initiated the idea of *causal sets*. In this theory, we keep only a discrete set of points of the Lorentz manifold of the General Relativity, and a partial order relation encoding the causal structure. A causal set has an order relation which is irreflexive, transitive, and between any two points there is a finite number of intermediate points. The number of points in each region we define the 4-density, which together with the causal structure recovers a discrete version of the spacetime geometry. □

Example 10. In *string theory*, the bundle can have as fiber the product of the Sierpińsky space and a space of extra dimensions, for example a *Calabi-Yau manifold*. The Sierpińsky bundle is needed to define submanifolds like strings and d-branes. □

All Examples 6–10 are particular cases of worlds in which the Sierpińsky bundle is used.

Example 11 (Quantum Theory)**.** Examples 2–10 are classical, they need to be quantized. This, largely speaking, consists of taking a classical Hamiltonian formulation and promoting the Poisson brackets to commutators on an appropriate Hilbert space. This can be achieved in different ways.

In general, the Hilbert space is a tensor product of smaller Hilbert spaces, corresponding to elementary particles. The Hilbert space of an elementary particle consists of wavefunctions on space, and they can be seen, when evolving in time, as fields on spacetime. But when at least two elementary particles are involved, this is not enough. The wavefunction becomes a field on the configuration space, which means that spacetime representations seem to be insufficient. As explained in Sec. §II B, this problem was known since the beginning, in particular by Schrödinger, Lorentz, Einstein, and others. In [8] a representation completely equivalent to that of Quantum Theory, but in terms of (very complicated) fields on spacetime, was given. This works for both NRQM and Quantum Field Theory. □

Example 12. Another class of examples (also of the PDE type) can be defined if we take the base manifold as only consisting of the time dimension, and the fiber as being the *phase space*. We require the sections to respect the *Hamilton's equations*:

$$\begin{cases} \dot{p} = - & \frac{\partial H}{\partial q} \\ \dot{q} = & \frac{\partial H}{\partial p}, \end{cases} \tag{4}$$

where q and p are the generalized coordinates and momenta, and H the Hamiltonian. Hamilton's equations can be used to formulate Classical Physics. □

Example 12 is a particular case of *dynamical system*.

Definition 6. A *dynamical system* is a partial action of a monoid $(T,+)$ on a set M, (T,M,α):

1. $\alpha : U \subseteq T \times M \to M$,
2. $\alpha(0,x) = x$, and
3. $\alpha(t_1,\alpha(t_2,x)) = \alpha(t_1+t_2,x)$ for $(t_1,x),(t_2,x),(t_1+t_2,x) \in U$,

where α is named the *evolution function* and M the *phase space* or *state space*.

Example 13. To a dynamical system (T,M,α) we can associate a world, by taking T with the order topology as spacetime, and the matter bundle as $T \times M$. The law sheaf consists in the restrictions of the partial functions $\alpha_x : I(x) \to T \times M$, $\alpha_x(t) = (t,\alpha(t,x))$, where $I(x) = \{t \in T | (t,x) \in U\}$. □

IV. DETERMINING THE MATTER SECTION

A. Determining a section

If the law sheaf comes from PDE, a solution can be determined by a set of initial and/or boundary conditions. But this is idealized approach has two problems:

1. it only works if the law sheaf comes from PDE, and
2. it is not what we do in practice, when we determine the matter section through experiments spread at various points of spacetime.

In practice, we only learn new additional information from experiments, and this information is only local, and it is subject to the experimental error. Experiments and observations, including quantum measurements, are local operations, whose results are never 100% precise. They tell us that the measured properties have values within this or that interval. We are not Laplace's daemon who completely knows the initial state of the world.

In this section, I will define the *sheaf selections*, which generalize the initial/boundary conditions from the case of the PDE to the sheaf language, and for the kinds of observations we are actually making in practice.

Definition 7. Let μ be a section of a sheaf $\mathcal{F}$. A A *sheaf selector* is a subsheaf of the law sheaf. A *sheaf selection* is a set of selectors, whose intersection contains at least a global section.

The matter field is supposed to be a global section of a sheaf selection obtained from observations and experiments. In particular, assuming we know from experiment the initial conditions of a system that evolves deterministically, there is a unique matter field satisfying those initial conditions. To the matter section we associate a sheaf, consisting of the restrictions of the matter field to each open subset of $\mathcal{S}$. Let us define in general such sheaves associated to a global section.

Definition 8. Let $\mathcal{F}$ be a sheaf. A subsheaf $\mathcal{M} \leq \mathcal{F}$ is said to be *section-like* if for any $U \in \mathcal{O}(\mathcal{M})$, $\mathcal{M}(U)$ has exactly one element.

Remark 7. Any global section of $\mathcal{F}$ defines canonically a section-like subsheaf. Reciprocally, each section-like subsheaf of $\mathcal{F}$ admits a unique global section, which is, of course, also a section of $\mathcal{F}$. Thus, there is a one-to-one correspondence between the global sections and the section-like subsheaves of a sheaf of sets, $\mathcal{F}$.

The matter field is therefore equivalent to a section-like subsheaf of the law sheaf.

B. Determining the matter section

Since we are not Laplace's daemon, we can only know partially the initial conditions. In fact, we can only know the values of certain properties of the matter section in certain regions of spacetime, and even for those values we can only know a probability distribution, due to measurement error. These conditions are not *stricto sensu* "initial", because they are spread in various regions of spacetime where observations occur, so I will call them *delayed initial conditions*.

Classically, we think that the matter field is predetermined. In other words, that its initial conditions are already fixed long time ago, likely at the Big Bang. By this view, our measurements partially reveal preexisting information about the matter field, information already encoded in the fixed initial conditions.

But when quantum measurements are performed, this description seems to be contradicted. By making a succession of two incompatible measurements, the two resulting eigenstates are inconsistent with the Schrödinger equation. For example, if we measure the spin of a spin $\frac{1}{2}$ particle along an axis, we find it to be along the positive or the negative direction of the axis. If we then measure the spin again, but along a different axis, we find it to be along the positive or the negative direction of the second axis. Hence, the initial conditions established by the first measurement are not consistent with those established by the second one, even if we take into account that the particle evolves in time according to the Schrödinger equation. Then, the Projection Postulate is invoked, stating that a wavefunction collapse occurred. The previously determined initial conditions are reset and replaced with the new ones. Consequently, it seems that there is no global matter sheaf consistent with more measurements.

There is another possibility, described by Wheeler in terms of the *game of twenty questions*, in which the players try to guess a word by asking yes/no questions. Collecting observations and measurements can be seem as questioning the Universe about its state. But Wheeler mentions another version of the game, in which the word to be guessed is not fixed since the beginning of the game. The person who chooses the word just makes sure to give answers that are consistent with the existence of a nonempty set of words. So eventually the set may be restricted to only one word, which was not previously known to the person who chooses it. Wheeler uses this idea to interpret the universe as being *participatory* [36, 37]. This is his interpretation of Bohr's views on Quantum Mechanics, encoded in the saying "no phenomenon is a phenomenon until it is an observed phenomenon". He illustrates this with the *delayed-choice experiment*, which seems to be most naturally interpreted in terms of choices that can affect the past history of a particle. It can in fact be interpreted in all interpretations of Quantum Mechanics, but this particular type of experiment seems to be more naturally interpreted by assuming that the past is not predetermined.

In terms of sheaf selections, this idea gains a natural interpretation. There is no predetermined matter field, only a sheaf selection that is updated in time. Every new measurement adds a new selector to the sheaf selection, restricting the number of global sections.

But how can this solve the inconsistency between successive measurements, like in the example of two spin measurements along different axes? This seems to indicate that there is no global section, unless we allow the sections to be discontinuous in time. However, if the matter section is not predetermined, then the resulting freedom can allow apparently incompatible measurements to be accommodated by global sections. To understand this, one should recall that not only the observed system is in an undetermined state, but also the measuring apparatus. The measuring apparatus may have a well defined state at the macro level, but its actual micro state can be one of infinitely many possible. This remains undetermined both by construction and functionality, by the way in which it is manipulated by the experimentalist. The actual state of the observed system is also not determined. What is determined is what the measuring device indicates it to be, and this is a macro state. For this reason, measurements only determine a selector with many possible sections, but the actual section remains undetermined. With every measurement, the possible matter states are reduced, but they remain still infinitely many. The possibility that incompatible measurements are actually compatible is discussed in [38–42].

C. Determining the physical laws

The law sheaf may result from the conjunction of more physical principles. In this case, each principle determines a locally homogeneous subsheaf of the matter sheaf. Their intersection is the law sheaf. Let us see some examples showing how pinching can reduce a locally homogeneous sheaf to a locally homogeneous subsheaf.

Example 14. Let M be an analytical manifold. Its structure can be specified by a pseudogroup of analytical transformations, composed by the transition maps. We can consider a hierarchy $\mathcal{T}^{\omega}(M) \subset \mathcal{T}^{\infty}(M) \subset \ldots \subset \mathcal{T}^{k}(M) \subset \ldots \subset \mathcal{T}^{1}(M) \subset \mathcal{T}^{0}(M)$ of pseudogroups of transformations, from analytical transformations to differentiable transformations of finite degree, ending with continuous ones. These layers reflect the differentiable structures and the topological structure of an analytical manifold. Considering the real-valued functions on M, we obtain the following hierarchy of sheaves: $\mathcal{C}^{\omega}(M) \subset \mathcal{C}^{\infty}(M) \subset \ldots \subset \mathcal{C}^{k}(M) \subset \ldots \subset \mathcal{C}^{1}(M) \subset \mathcal{C}^{0}(M)$. □

In the previous example the selection was a sequence or a hierarchy. The following example provides us a latticeal selection which is not sequential.

Example 15. Let V be an topological vector space. The continuous linear forms on V define a sheaf which can be obtained by intersecting the sheaf obtained from the linear forms on V with the sheaf of continuous functions from V to $\mathbb{R}$. If the dimension is finite, the linear forms on V are also continuous, and the selection is sequential.

In the infinite dimensional case, the linear forms on V are not necessarily continuous, therefore the selection is latticeal but not sequential. $\square$

Example 16. As an example of totally ordered selection which is not sequential, because it is continuous, we can take the Sobolev spaces $\{W^s(X,E)\}_{s\in\mathbb{R}}$, of a Hermitian differentiable vector bundle $E \to X$. Sobolev spaces $\{W^s(X,E)\}_{s\in\mathbb{R}}$ satisfy $W^s \subset W^t$ for any $s > t$, and it is a selection of $W^\infty(X,E)$. $\square$

While these examples are mathematical, they apply to fields in Physics too.

For applications in Physics, pinching the law allows us to think about theories at different levels of abstraction. The most abstract level within World Theory is to think of the world as the mathematical object "world" in Definition 5. Then we can introduce various principles and define less general worlds, for example quantum worlds, relativistic worlds *etc.*.

V. WHAT WORLD THEORY CAN DO?

The mathematical object named world can be used to mathematically model the laws of Nature. We have seen that a very large class of theories in Physics can be expressed in this language. This is because it abstracts the most general properties of the known theories. World Theory's main purpose is to derive logically general conclusions that can be applied to a large class of physical theories. This means that it does not compete with particular theories. It is not likely to be falsifiable, but this is not a handicap, because this theory does not try to eliminate from competition other theories. Its purpose is merely to extract the common assumptions in an abstract manner, and to derive logical and mathematical consequences.

Another possible application is to provide a framework to mathematically formulate theories from other domains of Science, such as Chemistry, Biology, Psychology, Sociology, Economics. World Theory being a mathematical one, it can be applied to any system that satisfies its definition. These theories study *emergent phenomena*, which cannot be directly inferred from the laws of Physics. Although it is assumed that we can describe these emergent objects and laws in terms of physical objects and laws, in practice it is not always possible to reduce everything to the fundamental level, due to complexity. It is also preferable to emphasize the higher level behavior. We can compare these emergent laws with a computer program, which is a computer program without regard of its physical support (hardware).

At a higher level or coarse grained level many details of the fundamental level are ignored. A higher level object can be constructed in more ways at the fundamental level. Macro states are obtained by coarse graining from the fundamental states in the state or phase space, as in Statistical Mechanics.

Another possible application is to offer less ambiguous ways to discuss and think about concepts that usually fit in areas of Philosophy, such as Metaphysics and Epistemology. A philosophically neutral, unified and unambiguous language provided by Mathematics can bring more clarity in such fields.

Appendix A: Symmetries and universality of the laws. An example

For the particular kind of physical law defined by PDE, the law provides just the evolution equation, having an infinity of solutions. In order to specify a solution, we can use a condition at a given moment $t_0 = 0$. In the case of most equations of the Physics, specifying the evolution equation and the initial conditions gives a unique solution. In the following, we will exemplify the way we choose the law sheaf by applying considerations of symmetry, which are comprised in the condition of local homogeneity of the law sheaf (Section §II D).

Example 17. For example, let us consider a point-particle moving with a given constant acceleration $\mathbf{a}$ in the Euclidean plane $\mathbb{R}^2$. Suppose that the evolution is described by the equation

$$\frac{\mathrm{d}^2\,\mathbf{x}}{\mathrm{d}\,t^2} = \mathbf{a} \tag{A1}$$

which admits solutions of the form

$$\mathbf{x}(t) = \frac{1}{2}\mathbf{a}t^2 + \mathbf{v}_0 t + \mathbf{x}_0, \tag{A2}$$

where $\mathbf{a}$, $\mathbf{v}_0$, $\mathbf{x}_0$ and $\mathbf{x}$ are vectors in $\mathbb{R}^2$, $\mathbf{a}$ is given, $\mathbf{v}_0$ and $\mathbf{x}_0$ are free parameters. Let us denote the subsheaf of sections of this form by Λ. Spacetime is $\mathbb{R}^2 \times \mathbb{R}$, and the matter bundle is defined as in Example 4 using the Sierpińsky bundle.

Knowing

$$\mathbf{x}(t_0) = \mathbf{x}_0 \tag{A3}$$

for an initial moment $t_0 = 0$ gives us an initial condition. This condition does not determine a unique solution, but a subspace of the general solution space, of the form:

$$\mathbf{x}(t) = \frac{1}{2}\mathbf{a}t^2 + \mathbf{v}_0 t + \mathbf{x}_0,$$

where the only free parameter is now $\mathbf{v}_0$. These solutions define a subsheaf Λ' of the sheaf Λ of all solutions. We also need to know $\mathbf{v}_0$, the initial speed – the first derivative of $\mathbf{x}(t)$ at the moment $t_0 = 0$. Let's write this second initial condition:

$$\frac{\mathrm{d}\,\mathbf{x}}{\mathrm{d}\,t}(t_0) = \mathbf{v}_0. \tag{A4}$$

The two conditions (A3) and (A4) reduce the solution space to a space containing a unique element, which is just the solution, and defines a subsheaf Λ''. Then the solution is given in term of three equations – the evolution equation and the two initial conditions (A3) and (A4). It seems that the law sheaf for this case is Λ.

But the condition for the initial position defines a subsheaf Λ' in Λ, isn't it possible to consider Λ' instead? The answer is yes, but our common sense tells us that Λ is more appropriate, because it is independent of the value $\mathbf{x}$ may take at a particular moment $t_0 = 0$. Of course, Λ depends on the acceleration $\mathbf{a}$, but $\mathbf{a}$ is presumed independent of t.

This example shows that when we consider the physical laws expressed as equations of evolution, it appears to exist a natural way to distinguish the law sheaf from the subsheaves determined by particular initial conditions. The distinction is made by the condition that all the coefficients in the equation are independent of time and space. But $\mathbf{v}_0$ and $\mathbf{x}_0$ are constants as well, hence it may seems that all sheaves Λ, Λ' and Λ'' can be equally considered as the law sheaf. Still, why does it seem more natural to choose Λ? Let's apply a transform to the time, $t \mapsto t + \theta$. The equation (A2) is transformed into

$$\mathbf{x}(t) = \frac{1}{2}\mathbf{a}t^2 + \mathbf{v}_1 t + \mathbf{x}_1, \tag{A5}$$

where $\mathbf{x}_1 = \frac{1}{2}\mathbf{a}\theta^2 + \mathbf{v}_0\theta + \mathbf{x}_0$ and $\mathbf{v}_1 = \frac{1}{2}\mathbf{a}\theta + \mathbf{v}_0$. We see that only $\mathbf{a}$, the coefficient of t^2, is independent of time, while the coefficients of t^1 and t^0 are not. We can extract the following prescription: if the law sheaf contains the function $\mathbf{x}(t)$, it should also contain $\mathbf{x}(t + \theta)$. This makes the law sheaf invariant to time translations. More generally, we want the physical laws to be independent of space and time, and this is why we prefer a locally homogeneous law sheaf. □

The invariance properties are used to characterize the properties of the law sheaves. In general the law sheaf can be defined in more than one way. We want the law sheaf to describe best the systems we are studying. We want the description to be general – for example to be independent of the particular time and position, or of the initial conditions. At the same time, we do not want the description to be too generic, otherwise it will not be useful. For instance, in the same Example 17 we could define the law sheaf as the solutions of the equation $\mathrm{d}^3\,\mathbf{x}/\mathrm{d}\,t^3 = 0$. We could even define it as the sheaf of all continuous functions $x : \mathbb{R}^2 \to \mathbb{R}$. But, while these definitions give locally homogeneous sheaves, they are too generic.

The choice of the law sheaf depends on the intended level of abstraction – in the example above, if we just want to study the continuous motions, the sheaf of continuous functions is good enough. If we want to study the generic accelerated motions, we can consider a sheaf of accelerated motions with all the possible accelerations. If we want to study a particular example, we can consider the sheaf Λ'' which contains a unique solution.

In general in Physics, the laws are independent of the initial conditions, but there are theories which study precisely the initial conditions – such as the theories of the origin of the physical world. Also there exist laws of Physics which currently are the same at any moment and position, but are dependent on the initial conditions of the Universe. For example the Second Law of Thermodynamics is valid everywhere (although its validity is only statistical), but it depends on the initial low level of entropy of the Universe. Other theories explain some properties of the particles and their interactions by a spontaneous symmetry breaking occurred in the early ages of the Universe, *e.g.* the electroweak theory [16–18].

In most cases, the physical laws expressed by the law sheaf have some invariance properties. Sometimes the law sheaf is invariant to groups or pseudogroups of transformations of the underlying topological manifold which is the spacetime. This is the case of the Newtonian Physics, whose laws are invariant to global transformations which form the Galilei group, and with the Special Theory of Relativity, whose invariance group is the Poincaré group. Some equations, like Maxwell's, are invariants of a larger group – the conformal group $O(2, 4)$. Spinor field equations in Special Relativity are invariants of the group $SL(2, \mathbb{C})$. General Relativity states that the laws should be diffeomorphism-invariant, *i.e.* invariant to the transition functions of the differentiable manifold representing the spacetime. Transformations of the fibers also can leave the form of physical equations unchanged – this is the case of the gauge transformations.

Appendix B: Generalized worlds

Throughout this article, spacetime $\mathcal{S}$ has been taken to be a homogeneous manifold, the matter bundle $\mathcal{M} \stackrel{\sigma}{\to} \mathcal{S}$ was a bundle over $\mathcal{S}$, the law sheaf Λ was a subsheaf of $\Gamma(\mathcal{S}, \mathcal{M})$, and the matter section μ a global section of Λ.

Theories of Physics, like the Classical Mechanics, Electromagnetism, Special Relativity, General Relativity, Quantum Mechanics, Dirac's Relativistic Quantum Mechanics, Gauge Theory, Quantum Field Theory etc. can be viewed as special cases of the model described above. Now we will make the natural generalization of the world defined before.

Any topological space can be seen as a category $\mathrm{Op}(\mathcal{S})$ whose objects are the open subset of $\mathcal{S}$, $\mathrm{Ob}(\mathrm{Op}(\mathcal{S})) = \{U \subset \mathcal{S} | \mathring{U} = U\}$, and whose morphisms (or arrows) are the inclusion maps, $\mathrm{Hom}(\mathrm{Op}(\mathcal{S})) = \{i : U \hookrightarrow V | U, V \in \mathrm{Ob}(\mathcal{O}(\mathcal{S})), U \subset V\}$. The category $\mathrm{Op}(\mathcal{S})$ defines the topology of $\mathcal{S}$ (more precisely, the topology of $\mathcal{S}$ can be recovered if $\mathcal{S}$ is a *sober topological space*). The occurrences of the notation $\mathcal{S}$ in this article can then be understood as denoting the category $\mathrm{Op}(\mathcal{S})$.

Now we can generalize spacetime to a *locale*, a generalization of a topological space which may or may not have enough points to distinguish any pair of open subspaces. It may even have no points at all (*pointless topology*). For the following definitions see [43, 44].

Definition 9. A *frame* is a complete lattice with all finite meets and all arbitrary (finite or infinite) joins satisfying the infinite distributive law:

$$U \wedge \left(\bigvee_i U_i \right) = \bigvee_i (U \wedge U_i)$$

for any element U and any family of elements U_i.

Definition 10. Frames are objects in a category named (Frames), having as morphisms maps of partially ordered sets which preserve the frame structure. This means that they preserves the finite meets and arbitrary joins. The objects of the dual category $(\text{Locales}) := (\text{Frames})^{\text{op}}$ are named *locales*. We denote the corresponding frame of a locale X by $\mathcal{O}(X)$.

For example, a topology is a locale. This fact allows us to define a functor from the category of topological spaces (Spaces) to the category of locales (Locales),

$$\text{Loc} : (\text{Spaces}) \to (\text{Locales}) \tag{B1}$$

which associates to each topological space T the locale dual to the frame of its open subsets, by

$$\mathcal{O}\left(\text{Loc}(T)\right) := \mathcal{O}(T). \tag{B2}$$

Definition 11. A *point* of a locale X is a map of locales $1 \to X$ from the terminal object 1 of the category (Locales) to X. We denote by $pt(X)$ the set of all points of the locale X. It is a topological space in a canonical manner, with the open sets $pt(U) = \{p \in pt(X) | p^{-1}(U) = 1\} \subseteq pt(X)$.

Principle 1′. There exists a locale $\mathcal{S}$, which will be named *spacetime*.

Locales and the more general Principle 1′ are provided for full generality. The structures proposed in this article being sheaves over topological spaces, they can be easily extended to sheaves over locales. As such, the matter sheaf becomes a sheaf over the spacetime locale, the law sheaf a subsheaf of the matter sheaf, and the matter field a section of the law sheaf.

In Section §II E, time was considered to be a set $\mathcal{T}$ endowed with an order topology. Now we have to replace this with an order locale. For time intervals that do not contain points, as it can happen if time has a locale structure, there will be no space at a given moment of time $t_0 \in \mathcal{T}$ as in Definition 4, because there will be no definite moment of time t_0. There can only be slices of spacetime corresponding to time intervals that are open in $\mathcal{T}$.

In the same spirit of considering the locale structure of $\mathcal{S}$ as its defining structure, time can be introduced as a subcategory of $\mathcal{S}$. Start with the case when spacetime $\mathcal{S}$ is a topological space, and let $\tau : \mathcal{S} \to \mathbb{R}$ be a time coordinate on $\mathcal{S}$. The subsets of $\mathcal{S}$ defined for $t \in \mathbb{R}$ by

$$\mathcal{S}_t := \{p \in \mathcal{S} | \tau(p) < t\} \tag{B3}$$

are open, and they form a subcategory of $\text{Op}(\mathcal{S})$. Since the time coordinate is supposed to be continuous (with respect to the topology of $\mathcal{S}$), the *inclusion functor* defining the subcategory whose objects are the open sets (B3) has to be *continuous*. Recall that a functor is continuous if it preserves the limits. When the inclusion functor is continuous, we say that the subcategory it defines is continuous. This captures the idea that time is continuous with respect to the topology of spacetime.

Since time is totally ordered, we should also transfer this order to categories. A category is said to be *totally ordered* if its morphisms define a total order "$\to$" on its set of objects.

Due to the way we formulated them, these observations generalize immediately to locales. Hence, we can now define time.

Definition 12. A *time structure* is a totally ordered continuous subcategory $\mathcal{T}$ of $\mathcal{S}$. The time structure $\mathcal{T}$ is called *global time structure* if it includes among its objects the *initial* and *terminal* objects of the category $\mathcal{S}$.

When $\mathcal{S}$ is a topological space, its initial object is the empty set $\emptyset$, and its terminal object is the set $\mathcal{S}$.

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