Symmetry and Symplectic Reduction

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Contents

Abstract

In the last hundred years, Classical Mechanics has been dwarfed by the development of Relativity and Quantum Mechanics. However, significant mathematical developments have been made to Classical Mechanics in the same time frame and this essay aims to explore some of the recent advances. The purpose of this essay is to understand Hamiltonian dynamics, Noether's theorem and symmetry in a geometric framework using Poisson geometry. In particular, we want to analyse the case when the natural configuration space of a system is a Lie group G and transformations on the space correspond to the Lie group acting on itself. In this case, the Lie-Poisson reduction theorem will say that the reduced phase space T^*G/G is a Poisson manifold diffeomorphic to $\mathfrak{g}^*$. We will begin Chapter 1 by outlining the core concepts required to understand symmetry, in particular, we will discuss the notion of a Lie group, its Lie algebra and how each of these objects acts on certain manifolds. In Chapter 2 we will define Poisson and Symplectic manifolds and use them to derive equations of motion and dynamics. Chapter 3 will discuss the geometric understanding of Noether's theorem by looking at momentum maps. Finally, Chapter 4 will discuss the Lie-Poisson reduction theorem. Throughout the essay, we will aim to illustrate the concepts using the physically relevant cases which arise when the configuration space G is a three-dimensional Lie group over the reals.

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Chapter 1

Introduction

1.1 Motivation for Poisson Geometry

In courses on Classical Mechanics such as V.I. Arnold's excellent book [6], we learn that the evolution of a mechanical system with n degrees of freedom can be described in phase space coordinates

$$(q^1(t), ..., q^n(t), p_1(t), ..., p_n(t)), \tag{1.1}$$

where $q^i(t), p_i(t)$ are called the configuration and momentum coordinates respectively, via *Hamilton's equations*

$$\dot{q}^i(t) = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i(t) = \frac{\partial H}{\partial q^i}, \tag{1.2}$$

for a Hamiltonian H of the mechanical system. From here, one can generalise the notion of phase space and Hamilton's equations in terms of symplectic manifolds; however in this essay we will generalise Hamiltonian systems in terms of the more general Poisson manifolds.

To motivate this we recall that we can introduce the idea of a Poisson bracket for the phase space geometry, namely

$$\{f, g\} := \sum_{i=1}^{n} (\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}); \tag{1.3}$$

using this we can reformulate Hamilton's equations as

$$\dot{x}^i(t) = \{H, x^i(t)\}, \tag{1.4}$$

for x^i one of the phase space coordinates and H a Hamiltonian. Moreover we can introduce the idea of a Hamiltonian vector field by defining

$$X_H := \sum_{i=1}^{n} (\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}); \tag{1.5}$$

and so Equation (1.4) can be written in the form

$$\dot{x}^i(t) = X_H[x^i(t)]. \tag{1.6}$$

We call the above development a *canonical* Hamiltonian system.

Our usual first step in creating an abstract framework for Hamiltonian dynamics begins by globally describing the dynamics in terms of a closed, nondegenerate two-form; which is embodied in the theory of Symplectic geometry, where locally the dynamics will look like a canonical Hamiltonian system. However often we are concerned with *non-canonical* Hamiltonian system, which can't be described by Symplectic geometry; and thus Poisson manifolds were introduced to further generalize Hamiltonian dynamics to these systems.

1.2 Motivation for Bianchi Lie Groups

In Chapter 2 we will classify the three-dimensional real Lie algebras and in Chapter 3 we define a Poisson structure on the dual to these Lie algebras $\mathfrak{g}^*$. The problem of classifying three-dimensional real Lie algebras is equivalent to classifying homogeneous and anisotropic 1+3 dimensional cosmologies on spatial slices since the isotropy group is generated by these Lie algebras.

One can show these problems are equivalent by considering a time-slice of spacetime and letting $\bar{g}_{ab}$ be the three-space metric on the time slice. Each time-slice being homogeneous means there exist three Killing vector fields ξ_i, $i = 1, 2, 3$, which are linearly independent at each point. The Killing vector property means that the three-metric is transported under the Lie derivative as

$$\mathfrak{L}_{\xi_i}(\bar{g}_{ab}) = \xi_i^c \partial_c \bar{g}_{ab} + \bar{g}_{cb} \partial_a \xi_i^c + \bar{g}_{ac} \partial_b \xi_i^c = 0; \tag{1.7}$$

in other words, the three-metric is unchanged under transformations along integral curves of the Killing vector fields.

The Killing fields describe how the metric changes from one point in space to another and so we want to consider the relationship between these vector fields. We do this by considering an infinitesimal closed loop by pushing along integral curves of two different vector fields and then returning in the opposite way; this is embodied in the commutator bracket of our Killing vector fields

$$[\xi_i, \xi_j] = f^k{}_{ij} \xi_k, \tag{1.8}$$

giving a Lie algebra structure to the Killing fields. Further details can be found in [7].

We will define the unique connected and simply connected Lie group associated with a three-dimensional Lie algebra as being a *Bianchi group*, from here we will analyse the dynamics on the dual to its Lie algebra $\mathfrak{g}^*$, a space which is diffeomorphic to the reduced phase space T^*G/G as Poisson manifolds.

For the Bianchi Lie groups, the configuration space $\mathfrak{g}^*$ will be three dimensional therefore Symplectic geometry couldn't be used to globally describe the dynamics, since Symplectic manifolds are even-dimensional. Hence we use the theory of Poisson geometry to define Hamiltonian dynamics on this system, illustrating the power and generality of Poisson manifolds.

Chapter 2

Lie Groups and Lie Algebras

In Classical Mechanics, symmetry can be understood as a transformation that leaves the equations of motion for the system invariant. We know from the classical version of Noether's theorem that symmetries imply the existence of conserved quantities; which are of paramount importance to physicists. Lie groups and their Lie algebras will prove to be a useful tool in understanding symmetries arising in a classical mechanical system. We will define how Lie groups act on a manifold and how Lie algebra actions will act as an infinitesimal analogue. We will conclude this chapter by classifying three-dimensional spaces which admit a continuous group of motions. For we will illustrate many of the concepts introduced throughout this essay with these examples in mind.

2.1 Lie Groups

We begin by stating a few definitions about Lie groups and Lie algebras.

Definition 2.1. A *Lie group* G is a group that is also a smooth manifold, such that the map $G \times G \to G$, $(x, y) \mapsto x * y^{-1}$ is smooth. In other words, the multiplication map and inversion are smooth maps on G.

Example 2.2. An example of a Lie group which plays an important role in describing one-dimensional quantum mechanical systems is the Heisenberg Group H, the set of matrices

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

where $a, b, c \in \mathbb{R}$, together with matrix multiplication as the group operation. We will consider the physical implications of H when talking about the Heisenberg Lie algebra. □

Lie groups in typical cases have an action on a manifold M, representing states of the physical system. This provides us with a framework to understand the symmetries of a physical system. The infinitesimal analogue of these actions is understood via Lie algebras.

Definition 2.3. A *Lie algebra* L is a vector space over a field $\mathbb{K}$, equipped with a Lie bracket. That is, bilinear map $L \times L \to L$, $(x, y) \mapsto [x, y]$, $\forall x, y \in L$, such that:

1. $[x, y] = -[y, x]$;
2. $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ (Jacobi identity).

Throughout this essay we will always take $\mathbb{K} = \mathbb{R}$.

Example 2.4. Consider the real vector space with basis:

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is a Lie algebra when equipped with the commutator $[A, B] = AB - BA$, as the Lie bracket. The commutator relations can readily be seen to be $[X, Y] = Z$, $[X, Z] = 0$ and $[Y, Z] = 0$. We will see later that this Lie algebra is Bianchi II. This Lie algebra has an important physical interpretation when considering one-dimensional quantum mechanical systems. Let H be the Hilbert space corresponding to such a system, p, q be the momentum and position respectively, and let f be a wavefunction in the Hilbert space. Moreover consider the position operator $Q(f(q)) = qf(q)$ and the momentum operator $P(f) = i\hbar\frac{\partial f}{\partial q}$, then the commutator of these operators is

$$[P, Q](f) = i\hbar\frac{\partial}{\partial q}(Qf) - Q(i\hbar\frac{\partial f}{\partial q}) = i\hbar f,$$

$$[P, i\hbar] = [Q, i\hbar] = 0. \tag{2.1}$$

Thus the Heisenberg Lie algebra is isomorphic to the Lie algebra consisting of the span of $\{P, Q, i\hbar\}$ with the commutator Lie bracket. □

As we alluded to before, the Lie algebra is the infinitesimal analogue of Lie group actions; in fact, to each Lie group we can define a corresponding Lie algebra. To motivate this, we consider a Lie group G and a vector field ξ on G. We say that ξ is a *left-invariant vector field* if its push-forward under the left multiplication map, defined by $g \mapsto gh$, L_g of G, leaves it invariant, i.e.,

$$(L_g)_*\xi = \xi, \quad \forall g \in G. \tag{2.2}$$

It can be shown that any left-invariant vector field can be written as $\xi_g := (L_g)_{*e}X$ for some vector $X \in T_eG$. So we use ξ_X to denote this left-invariant vector field. This allows us to define the Lie algebra of a Lie group.

Definition 2.5. The *Lie algebra of a Lie group* G, denoted $\mathfrak{g}$, is the tangent space at the identity T_eG equipped with the Lie bracket

$$[X, Y] := [\xi_X, \xi_Y]_e, \tag{2.3}$$

where the Lie bracket on the right-hand side (on left-invariant vector fields) is the usual Lie bracket on vector fields restricted to evaluation at the identity, e.

We note that the mapping

$$f : \mathfrak{X}_L(G) \to \mathfrak{g} = T_eG, \quad X \mapsto X(e), \tag{2.4}$$

is a linear isomorphism of vector spaces, where $\mathfrak{X}_L(G)$ denotes the left-invariant vector fields on G. For completeness, we mention without further comment that there is an equivalent formulation of the Lie algebra of a Lie group using right-invariant vector fields.

If $e_1, ..., e_n$ is a basis for a Lie algebra $\mathfrak{g}$ then we can write the commutator relations of the basis in the form

$$[e_i, e_j] = \sum_{k=1}^{n} f^k{}_{ij}e_k, \quad f^k{}_{ij} \in \mathbb{K}; \tag{2.5}$$

and we call $f^k{}_{ij}$ the *structure constants* of $\mathfrak{g}$.
We have seen that given a Lie group we can define its Lie algebra. Conversely, there is also an operation to take us from the Lie algebra to the Lie group, called the *exponential map.*

Definition 2.6. Let ϕ_ξ be the integral curve of the left invariant vector field ξ_X associated with $X \in \mathfrak{g} = T_eG$ such that $\phi_\xi(0) = e \in G$. The *exponential map* is defined as

$$\exp : \mathfrak{g} \to G, \quad X \mapsto \phi_\xi(1). \tag{2.6}$$

For matrix Lie groups (closed subgroups of $GL_n(\mathbb{K})$) the exponential map is the usual matrix exponential.

Let us return to the Heisenberg Lie algebra and see that it exponentiates to give the Heisenberg Lie group.

Example 2.7. We show that the Lie algebra defined in Example 2.4 is the Lie algebra of the Heisenberg group, H. Indeed if we assume that $\exp(tX) \in H$ for all $t \in \mathbb{R}$, $X \in \mathfrak{g}$ then:

$$\exp(tX) = \begin{bmatrix} 1 & x_1(t) & x_3(t) \\ 0 & 1 & x_2(t) \\ 0 & 0 & 1 \end{bmatrix}$$

where each $x_i : \mathbb{R} \to \mathbb{R}$ are smooth functions. Then

$$X = \frac{d}{dt} \exp(tX)\big|_{t=0} = \begin{bmatrix} 0 & \frac{d}{dt}x_1(t) & \frac{d}{dt}x_3(t) \\ 0 & 0 & \frac{d}{dt}x_2(t) \\ 0 & 0 & 0 \end{bmatrix}_{t=0} \tag{2.7}$$

where $\frac{d}{dt}x_i(t)\big|_{t=0}$, $i = 1, 2, 3$ is just a real number. Indeed, if $x_1(t) = e^{at}, x_2(t) = e^{bt}, x_3(t) = e^{ct}$ then Equation (2.7) reduces to $aX + bY + cZ$, with X, Y, Z as in Example 2.4. Thus the vectors X are the span of the basis of the Lie algebra we defined before.
There was an element of cheating involved when defining the Heisenberg Lie group and its Lie algebra and then showing their correspondence. While we can define the group independently as we have done above, in physics we tend to define it as the exponential of the Heisenberg Lie algebra. □

It is natural to ask how much information the Lie algebra contains about the Lie group via the exponential map. When we care about connected and simply connected Lie groups, the answer is quite a lot. This can be summarised in the following theorem due to Lie.

Theorem 2.8. *For every finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, there exists a unique connected and simply connected real Lie group G such that $\mathfrak{g} = T_eG$.*

This theorem, known as Lie's third theorem, has important consequences for this essay: it will allow us to study the action of the Lie group via the Lie algebra, which is a much simpler structure to analyse.
The proof of Theorem 2.8 is lengthy and requires a lot of Lie algebra theory and powerful theorems. A (very) rough sketch of the proof is as follows: by Ado's theorem, the Lie algebra $\mathfrak{g}$ is isomorphic to a matrix Lie algebra. From here the exponential of the Lie algebra $\exp(\mathfrak{g})$ can be shown to be a connected Lie group which can be lifted to a unique maximal simply connected covering group. A proof of Ado's theorem can be found in the original paper [9] and a proof of Theorem 2.8 using Ado's theorem can be found in [1, p. 153].

2.2 Representations and Actions

We have now defined the objects used to understand symmetries of a physical system, especially the idea of a Lie group. To understand how symmetries arise, we want to consider how a Lie group will transform the state of a physical system and subsequently inspect the invariants of this operation. A Lie group will act on a manifold M via its *action*.

Definition 2.9. Let G be a Lie group and M be a manifold. A (left) *action* of G on M is a smooth map $\Phi : G \times M \to M$ such that the following two conditions hold:

1. $\Phi(e, x) = x$, $\forall x \in M$.
2. $\Phi(g_1, \Phi(g_2, x)) = \Phi(g_1 g_2, x)$, $\forall g_1, g_2 \in G$ and $x \in M$.

We note that a right action can be defined similarly. Throughout this essay, the action we consider will be a left action and we will often use the notation $gx := \Phi(g, x)$. Moreover, if $g \in G$ then the map $\Phi_g : M \to M$, $x \mapsto \Phi(g, x)$, unless we specify otherwise, is a smooth map. If M is a vector space and Φ_g is a linear transformation for each $g \in G$, then we call Φ a *representation* of G on M. Next, we consider representations of the Lie algebra.

Definition 2.10. A *Lie algebra representation* of a Lie algebra $\mathfrak{g}$ over a vector space V is a Lie algebra homomorphism $\phi : \mathfrak{g} \to End(V)$, where $End(V)$ is the set of endomorphisms on V.

Similarly to before, we will often omit the explicit form of the representation and write the corresponding action as $\xi v := \phi(\xi)v$

We know that there is a Lie algebra-Lie group correspondence in some sense. This correspondence carries over to representations. If we begin by considering an action of a Lie group G on a vector space V, then we can construct an action (representation) on the Lie algebra via

$$\bar{\xi}v = \frac{d}{dt}\exp(t\xi)v\bigg|_{t=0}, \tag{2.8}$$

where $\xi \in \mathfrak{g}$, $v \in V$. This leads us to define the following.

Definition 2.11. Let G be a Lie group and $(g, x) \mapsto gx$ a smooth action on a manifold M. Then the *infinitesimal action* or *infinitesimal generator* of the action corresponding to an element $\xi \in \mathfrak{g}$ is the vector field defined by

$$(\bar{\xi})_x := \frac{d}{dt}\exp(t\xi)x\bigg|_{t=0} \tag{2.9}$$

In general, not all representations of $\mathfrak{g}$ can be obtained this way. However, in the case of a simply connected, connected real Lie group, it can be shown that all representations of the Lie algebra will be of this form. In this essay, we will restrict ourselves to this case.

A specific action which we will consider deeply throughout this essay is the action of a Lie group on the dual space of its Lie algebra via the *coadjoint action*. To motivate this we define the **adjoint action** on $\mathfrak{g}$, written Ad, of a Lie group G on its Lie algebra, as the derivative at the identity of the conjugation map. More explicitly, with $\xi \in \mathfrak{g}$

$$Ad : G \times \mathfrak{g} \to \mathfrak{g}, \quad Ad_g(\xi) = T_e(R_{g^{-1}} \circ L_g)\xi. \tag{2.10}$$

The corresponding Lie algebra representation is

$$ad_\xi(X) = \frac{d}{dt}Ad_{\exp(t\xi)}(X)\bigg|_{t=0} = [\xi, X]. \tag{2.11}$$

Because $\mathfrak{g}$ is a real vector space, the adjoint action $Ad : G \to GL(\mathfrak{g})$, $g \mapsto Ad_g$ is a representation and therefore we can define a dual representation.

Definition 2.12. The *coadjoint representation* of a Lie group G on the dual of its Lie algebra $\mathfrak{g}^*$ is defined as follows. Let $Ad_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$ be the map such that $\langle Ad_g^* f, X\rangle = \langle f, Ad_g X\rangle$ for $g \in G$, $X \in \mathfrak{g}$, $f \in \mathfrak{g}^*$, then the *coadjoint action* is the map

$$G \times \mathfrak{g}^* \to \mathfrak{g}^*, \quad (g, f) \mapsto Ad^*_{g^{-1}} f, \tag{2.12}$$
$$Ad^*_{g^{-1}} = (T_e(R_g \circ L_{g^{-1}}))^*,$$

where the right hand side asterisks denotes the usual dual operator on a linear function.
The corresponding Lie algebra representation is

$$ad^*_\xi(f) = \frac{d}{dt}Ad^*_{\exp(t\xi)}(f)\bigg|_{t=0} = -f \circ ad_\xi. \tag{2.13}$$

For a Lie group action Φ on a manifold M it can be shown that the orbit of an element $x \in M$, $O_x = \{\Phi_g(x) : g \in G\}$ is an immersed submanifold. We finish this section by looking at our familiar case of the Heisenberg Group.

Example 2.13. For all matrix Lie subgroups of $GL_n(\mathbb{R})$ the adjoint representation takes the form

$$Ad_g(X) = gXg^{-1}$$

In particular, returning to the Heisenberg group, let

$$g = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \in H, \quad \xi = \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{h},$$

where $\mathfrak{h}$ is the Lie algebra of H. We have:

$$Ad_g(\xi) = g\xi g^{-1} = \begin{bmatrix} 0 & x & ax+y-bx \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix}.$$

To find the coadjoint action note that

$$Ad_{g^{-1}}\xi = \begin{bmatrix} 0 & x & z-ay+bx \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix}.$$

Letting $\langle X, Y \rangle = X(Y)$ and X^1, X^2, X^3 be a basis of $\mathfrak{g}^*$ with respect to the evaluation, and $f = \alpha_i X^i \in \mathfrak{g}^*$, we have

$$Ad_g^* f = \beta_i X^i$$

where

$$\beta_1 = \alpha_1 + b\alpha_3, \beta_2 = \alpha_2 - a\alpha_3, \beta_3 = \alpha_3.$$

We will return later to analyzing the coadjoint orbits of this group. □

2.3 Classification of Three-Dimensional Lie Algebras over the Reals

In cosmology, a *Bianchi Universe* is a cosmological model which assumes homogeneity of space and no additional symmetry. By considering the metric at a given time slice, homogeneity means the metric will be the same under change of coordinates; such a transformation we call a *motion.* The set of motions forms a group and we say that a space is homogeneous if it admits a *continuous group of motions*, that is the group of motions is a finite-dimensional Lie group. Any continuous group of motions of a finite-dimensional space will be generated by infinitesimal transformations corresponding to elements of the Lie algebra. From this starting point, in [3], Luigi Bianchi classified all three-dimensional spaces which admit a continuous group of motions by classifying three-dimensional Lie algebras over $\mathbb{R}$.

We aim to provide a different classification of all three-dimensional Lie algebras over $\mathbb{R}$ up to an isomorphism. Einstein summation convention will be assumed and repeated indices will sum from 1 to 3.
We begin by fixing a basis (e_1, e_2, e_3) of $\mathfrak{g}$. In this basis the commutator relations are:

$$[e_i, e_j] = f^k{}_{ij} e_k, \quad f^k{}_{ij} \in \mathbb{R}; \tag{2.14}$$

where any other basis will be related to $\{e_i\}$ by an isomorphism of the Lie algebra. In particular, consider a change of basis:

$$x_i = A^j{}_i e_j. \tag{2.15}$$

We want to see how the structure constants in the $\{x_i\}$ basis are related to those in the original basis. Let

$$[x_i, x_j] = C^k{}_{ij} x_k, \quad C^k{}_{ij} \in \mathbb{R}. \tag{2.16}$$

Then using (2.15) this becomes:

$$[A^n{}_i e_n, A^m{}_j e_m] = C^k{}_{ij} A^l{}_k e_l \tag{2.17}$$

By the bilinearity of the commutator bracket and (2.14) the left-hand side becomes:

$$A^n{}_i A^m{}_j [e_n, e_m] = A^n{}_i A^m{}_j f^l{}_{nm} e_l. \tag{2.18}$$

Using this and the right hand side of (2.17) we find that, under the change of basis, the structure constants transform as:

$$(A^{-1})^k{}_l A^n{}_i A^m{}_j f^l{}_{nm} = C^k{}_{ij}, \quad k = 1, 2, 3; \tag{2.19}$$

where A^{-1} exists because the change of basis is an isomorphism. In particular, the structure constants transform as a $(1, 2)$ tensor.

A priori, there are $3 \times 3 \times 3 = 27$ components of the tensor $f^k{}_{ij}$. Not all of these components are independent: using the antisymmetry of the Lie bracket, we find that for each k we have $f^k{}_{ij} + f^k{}_{ji} = 0$, this reduces the number of independent components of the tensor by $3 \times (3 + 2 + 1) = 18$, leaving us with 9 independent components of the tensor $f^k{}_{ij}$. We note that we can represent $f^k{}_{ij}$ with a 3×3 matrix F^{lk} via the identification:

$$f^k{}_{ij} = \epsilon_{ijl} F^{lk}; \tag{2.20}$$

where ϵ_{ijl} is the totally antisymmetric three-dimensional Levi-Civita symbol with $\epsilon_{123} = 1$. From standard linear algebra, we can split the matric F^{lk} into a symmetric and antisymmetric part. We call η^{lk} the symmetric part and α^{lk} the antisymmetric part. Similar to the above identification, we can write the antisymmetric matrix as $\alpha^{kl} = \epsilon^{kln} a_n$.

The above decomposition shows us that:

$$\begin{aligned} f^k{}_{ij} &= \epsilon_{ijl} F^{lk} \\ &= \epsilon_{ijl}(\eta^{lk} + \epsilon^{kln} a_n) \\ &= \epsilon_{ijl}\eta^{lk} + \epsilon_{ijl}\epsilon^{kln} a_n \\ &= \epsilon_{ijl}\eta^{lk} - \delta^{kn}_{ij} a_n \end{aligned} \tag{2.21}$$

where $\delta^{kn}_{ij} = \delta^k_i \delta^n_j - \delta^n_i \delta^k_j$ is called the generalised Kronecker delta.

Thus far, we have only used the antisymmetry and the bilinearity of the Lie bracket. But the Lie bracket has one more important property, namely, the Jacobi identity. The Jacobi identity says: $[e_i, [e_j, e_k]] + [e_k, [e_i, e_j]] + [e_j, [e_k, e_i]] = 0$. So, using (2.14) this becomes:

$$f^l{}_{jk} f^m{}_{il} + f^l{}_{ij} f^m{}_{kl} + f^l{}_{ki} f^m{}_{jl} = 0. \tag{2.22}$$

Combining this with (2.21) we see that the Jacobi equation reduces to:

$$\eta^{ab} a_b = 0. \tag{2.23}$$

We note that (2.21) and (2.23) are tensor equations, so that they are true in any basis. This has an important consequence: namely that η^{lk} is a symmetric matrix over $\mathbb{R}$ and thus there is a basis of $\mathfrak{g}$ such that η^{kl} is a diagonal matrix with eigenvalues n_1, n_2, n_3 taking up values $0, 1, -1$. Moreover, (2.23) implies that the vector a_b lies in the direction of one of the eigenvectors and, without loss of generality (we could just permute the basis), we can assume $a_b = (a, 0, 0)$ implying that (2.23) reduces to $an_1 = 0$. Thus, either $a = 0$ or $n_1 = 0$.

We can thus find a basis X_1, X_2, X_3 of $\mathfrak{g}$ such that the commutator relations are:

$$\begin{gathered} [X_1, X_2] = aX_2 + n_3 X_3 \\ [X_2, X_3] = n_1 X_1 \\ [X_3, X_1] = n_2 X_2 - aX_3. \end{gathered} \tag{2.24}$$

To finish the classification we need only consider the possibilities for a, n_1, n_2, n_3, noting that permuting n_1, n_2, n_3 or multiplying them by a constant will give isomorphic Lie algebras. Firstly, $an_1 = 0$ implies we either have $a = 0$ or $n_1 = 0$. If $a = 0$ then n_1, n_2, n_3 can take up any values in $0, 1, -1$, giving us six non-isomorphic Lie algebras. On the other hand, if $n_1 = 0$, we have either $a = 1$ or $a \in \mathbb{R} - \{1, 0\}$ and considering the possibilities for n_2, n_3 we get five further non-isomorphic Lie algebras, two with a continuous parameter.

We call this the *Bianchi Classification*, summarized in the table below.

Type	a	n_1	n_2	n_3
I	0	0	0	0
II	0	1	0	0
VII_0	0	1	1	0
VI_0	0	1	-1	0
IX	0	1	1	1
VIII	0	1	1	-1
V	1	0	0	0
IV	1	0	0	1
VII_a	a	0	1	1
III	1	0	1	-1
VI_a	a	0	1	-1

Chapter 3

Poisson Geometry and Dynamics

Our next goal is to provide a geometric framework for understanding Hamiltonian mechanics in terms of symplectic and Poisson manifolds. In particular, we will use these objects to study geometric analogues of Dynamics and Equations of Motions; an interpretation of Noether's theorem using momentum mappings will be developed in the next chapter. In this essay, we will explore symplectic manifolds as a complementary structure to the Poisson manifold. Not only will we see that Poisson geometry will allow us to generalise further the idea of Hamiltonian dynamics on a symplectic, but we will also see that a Symplectic structure will induce a Poisson structure which will be of paramount importance since we can define a natural symplectic structure on the cotangent bundle T^*G of a Lie group G. In Chapter 4 we will use the induced Poisson structure to show that the reduced phase space T^*G/G is isomorphic as a Poisson manifold to the Lie-Poisson structure on $\mathfrak{g}^*$.

3.1 Symplectic Manifolds

Definition 3.1. A *symplectic manifold* (M, Ω) is a manifold M equipped with a closed nondegenerate two-form Ω on M.

The non-degeneracy of Ω means that symplectic manifolds are necessarily even-dimensional; this makes sense why we traditionally use symplectic manifolds to generalise Hamiltonian systems as Hamiltonians necessarily operate on even dimensional phase space. Presenting dynamics in terms of symplectic manifolds is the first step of generalising Hamiltonian systems and presents dynamics in terms of the two-form Ω; however, in this essay, we will develop dynamics in terms of Poisson manifolds.

Given two symplectic manifolds $(M_1, \Omega_1), (M_2, \Omega_2)$ we say that a smooth mapping $f : M_1 \to M_2$ is *symplectic* if

$$f^*\Omega_2 = \Omega_1, \tag{3.1}$$

and if f is also a diffeomorphism we call it a *symplectomorphism.*

Example 3.2 (Cotangent Bundle)**.** We now wish to show that the cotangent bundle of a manifold naturally inherits a symplectic structure. Let M be a manifold, then we can define a one-form Φ on the cotangent bundle T^*M as

$$\Phi_\alpha(f) := \langle \alpha, d\pi_M x \rangle, \tag{3.2}$$

where $\alpha \in T^*M$, $x \in T_\alpha(T^*M)$, $\pi_M : T^*M \to M$ is the natural projection and $d\pi_M : T(T^*M) \to TM$ is the tangent map of π_M. After some algebra one can verify that $(T^*M, \Omega := -\mathbf{d}\Phi)$ is a symplectic manifold. If we treat our manifold M as the configuration space of a mechanical system then the cotangent bundle can be thought of as the phase space of the system, on which we expect to be able to define Hamiltonian dynamics, and so it makes sense for it to inherit a symplectic structure. □

We conclude this brief section by stating a powerful theorem (Darboux's) for symplectic manifolds, the proof of which can be found in [2, p.149]. Darboux's theorem tells us that there is a coordinate system for a symplectic manifold in which locally the manifold looks like the phase space of a mechanical system.

Theorem 3.3 (Darboux's Theorem)**.** *If (M, Ω) is a finite-dimensional symplectic manifold then M necessarily has dimension $\dim M = 2n$ for $n \in \mathbb{N}$. Moreover, for each $x \in M$ there exist local coordinates $(q^1, ..., q^n, p_1, ..., p_n)$ such that the symplectic form in these coordinates is*

$$\Omega = dq^i \wedge dp_i. \tag{3.3}$$

3.2 Poisson Manifolds

Poisson manifolds will generalise Symplectic geometry and the Poisson bracket for a classical mechanical system; accordingly will provide us with a more generalised understanding of dynamics.

Definition 3.4. A *Poisson manifold* is a manifold M equipped with a Poisson bracket (also called Poisson structure), namely a bilinear map $\{,\}$ on $C^\infty(M)$ such that:

- $(C^\infty(M), \{,\})$ is a Lie algebra.
- For each $f, g, h \in C^\infty(M)$ we have the derivation property in the first factor: $\{fg, h\} = \{f, h\}g + f\{g, h\}$ and similarly with the second factor.

Similarly to the symplectic case, given two Poisson manifolds $(M_1, \{,\}_1), (M_2, \{,\}_2)$, we say that a smooth mapping $f : M_1 \to M_2$ is a *Poisson* map if

$$f^*\{F, G\}_2 = \{F, G\}_1, \tag{3.4}$$

for any $F, G \in C^\infty(M)$.

We can express the Poisson bracket in local coordinates $(x_1, ..., x_n)$ as

$$\{F, G\} = \sum_{i,j=1}^{n} x^{ij} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j}, \tag{3.5}$$

where $x^{ij} := \{x_i, x_j\}$ is called the *Poisson matrix* in these coordinates.

Consider a Poisson manifold M and the map $G \mapsto \{G, H\}$ for $H, G \in C^\infty(M)$. By the derivation properties of the Poisson bracket on M it is clear this map is a derivation on $C^\infty(M)$. It is a well-known result of differential geometry that such derivations are in one to one correspondence with vector fields on M. Thus for $H \in C^\infty(M)$ we can define a unique vector field

$$X_H(G) := \{G, H\}, \tag{3.6}$$

called the *Hamiltonian vector field* associated with H.

Example 3.5 (Poisson Structure from Symplectic)**.** If (M, Ω) be a symplectic manifold then it is also a Poisson manifold when equipped with the bracket

$$\{F, G\}(x) := \Omega(X_F[x], X_G[x]), \tag{3.7}$$

where $F, G : M \to \mathbb{R}$ are smooth functions and $x \in M$. Indeed the bilinearity follows from the bilinearity of two-forms. Moreover we have

$$X_{FG} = FX_G + GX_F, \tag{3.8}$$

giving us the derivation property:

$$\{FG, H\} = X_H[FG] = FX_H[G] + GX_H[F] = F\{G, H\} + G\{F, H\}. \tag{3.9}$$

□

In this essay we will analyse the **Lie-Poisson Structure**. Let $\mathfrak{g}$ be a Lie algebra; then we can define a Poisson structure on its dual $\mathfrak{g}^*$ as follows. Let $e_1, ..., e_n$ be a basis for $\mathfrak{g}$ and recall that we can write the commutator relations as

$$[e_i, e_j] = f^k{}_{ij} e_k, \quad f^k{}_{ij} \in \mathbb{R}. \tag{3.10}$$

Let $e^1, ..., e^n$ be the basis of $\mathfrak{g}^*$ dual to $e_1, ..., e_n$, defined as $e^i := e_i^*$, where for $X = \alpha_i e^i \in \mathfrak{g}$ we have $e_i^*(X) = \langle X, e_i \rangle := \alpha_i$. Throughout let $\mu = x_i e^i$, we claim that, for $F, G \in C^\infty(\mathfrak{g}^*)$, the bilinear map

$$\{F, G\}(\mu) = x^{ij} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j}, \tag{3.11}$$

is a Poisson structure on $C^\infty(\mathfrak{g}^*)$, where $x^{ij} = [e_i, e_j]^*$ is the Poisson matrix. Unless confusion arises we will usually drop the μ dependence in $\{F, G\}(\mu)$. The bilinearity and derivation property of the Poisson bracket arises from the linearity and derivation properties of partial derivatives. The Jacobi identity and derivation property follow after observing $\{e_k^*, e_l^*\} = x^{ij} \partial_i e_k^* \partial_j e_l^* = [e_k, e_l]^*$ and by the bilinearity of the bracket this implies $\{\xi^*, \gamma^*\} = [\xi, \gamma]^*$ for $\gamma, \xi \in \mathfrak{g}$. From here it is obvious that

$$\{\{x_i, x_j\}, x_k\} = [[e_i, e_j], e_k]^*.$$

Thus the Jacobi identity follows immediately from that of the Lie bracket.

To find a coordinate-free expression of (3.11), given $\mu \in \mathfrak{g}^*$ and $F \in C^\infty(\mathfrak{g}^*)$; the push forward $F_* : T_\mu(\mathfrak{g}^*) \to \mathbb{R}$ can be regarded as an element of $\mathfrak{g}$. Indeed $\mathfrak{g}^*$ as a vector space is naturally isomorphic to its tangent space $T_\mu(\mathfrak{g}^*)$, thus the push forward can be considered as an element of $(\mathfrak{g}^*)^*$, which itself is naturally isomorphic to $\mathfrak{g}$. Hence we identify the push forward at $\mu \in \mathfrak{g}^*$ with an element $d_\mu F \in \mathfrak{g}$. We can write this in terms of the basis as $d_\mu F = \sum_{i=1}^n \frac{\partial F}{\partial x_i}(\mu) e_i$.

Finally, we state our definition:

Definition 3.6. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The Lie-Poisson bracket (or Lie-Poisson structure) is the Poisson bracket on $C^\infty(\mathfrak{g}^*)$:

$$\{F, G\}(\mu) := \Big\langle \mu, [d_\mu F, d_\mu G] \Big\rangle, \quad F, G \in C^\infty(\mathfrak{g}^*), \mu \in \mathfrak{g}^* \tag{3.12}$$

equipping $\mathfrak{g}^*$ with a Poisson manifold structure. Here $d_\mu F, d_\mu G$ are elements of $\mathfrak{g}$ as just explained.

In the coordinate expression (3.11) we can find x^{ij} explicitly:

$$x^{ij} = \{x_i, x_j\} = [e_i, e_j]^*,$$
$$[e_i, e_j]^* = (f^k{}_{ij} e_k)^* = f^k{}_{ij} e_k^* = f^k{}_{ij} x_k.$$

Thus Equation (3.11) reduces to:

$$\{F, G\} = \sum_{i,j=1}^n f^k{}_{ij} x_k \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j}. \tag{3.13}$$

Example 3.7. When $\mathfrak{g}$ is one of the Bianchi Lie algebras from Equations (2.24) we have that the only nonzero structure constants are:

$$\begin{gathered} f^2{}_{12} = a, \quad f^3{}_{12} = n_3, \\ f^1{}_{23} = n_1, \\ f^2{}_{31} = n_2, \quad f^3{}_{31} = -a. \end{gathered} \tag{3.14}$$

For this case Equation (3.11) becomes:

$$\begin{gathered} \{F, G\} = x_1(f^1{}_{23}(\partial_2 F \partial_3 G - \partial_3 F \partial_2 G)) \\ + x_2(f^2{}_{12}(\partial_1 F \partial_2 G - \partial_2 F \partial_1 G) + f^2{}_{31}(\partial_3 F \partial_1 G - \partial_1 F \partial_3 G)) \\ + x_3(f^3{}_{12}(\partial_1 F \partial_2 G - \partial_2 F \partial_1 G) + f^3{}_{31}(\partial_3 F \partial_1 G - \partial_1 F \partial_3 G)), \end{gathered} \tag{3.15}$$

and substituting (3.14) gives us:

$$\begin{aligned}\{F,G\} = x_1(n_1(\partial_2 F\partial_3 G - \partial_3 F\partial_2 G))\\ +x_2(a(\partial_1 F\partial_2 G - \partial_2 F\partial_1 G) + n_2(\partial_3 F\partial_1 G - \partial_1 F\partial_3 G))\\ +x_3(n_3(\partial_1 F\partial_2 G - \partial_2 F\partial_1 G) - a(\partial_3 F\partial_1 G - \partial_1 F\partial_3 G)),\end{aligned} \tag{3.16}$$

for any $F, G \in C^\infty(\mathfrak{g}^*)$. □

3.3 Poisson Dynamics

As we alluded to before, Poisson geometry will generalise the idea of Hamiltonian dynamics for a classical mechanical system. We now aim to further this by analysing the equations of motion in terms of the Poisson bracket and consider invariants of motion.

Before looking at dynamics we show one important property of Hamiltonian vector fields.

Lemma 3.8. Let M be a Poisson manifold and $H, G \in C^\infty(M)$. We have:

$$[X_H, X_G] = -X_{\{H,G\}}. \tag{3.17}$$

Proof.

$$\begin{aligned}[X_H, X_G][F] &= X_H[X_G[F]] - X_G[X_H[F]]\\ &= \{\{F,G\},H\} - \{\{F,H\},G\}\\ &= -\{F,\{G,H\}\} \quad (Jacobi)\\ &= -X_{\{H,G\}}[F].\end{aligned} \tag{3.18}$$

□

We now look at the main theorem about equations of motion on a Poisson manifold.

Theorem 3.9. *Let ϕ_t be the flow on a Poisson manifold M of a Hamiltonian vector field X_H, where $H : M \to \mathbb{R}$ be a smooth function.*

1. *For $F \in C^\infty(U)$, where $U \subset M$ is open, we have*

   $$\frac{d}{dt}(F \circ \phi_t) = \{F, H\} \circ \phi_t = \{F \circ \phi_t, H\}. \tag{3.19}$$

   Or in other words, $\dot{F} = \{F, H\}$ for all $F \in C^\infty(U)$ iff ϕ_t is the flow of X_H.

2. *If ϕ_t is the flow of X_H then $H \circ \phi_t = H$.*

Proof. Let $x \in M$.

1. By the chain rule the left hand side becomes:

   $$\frac{d}{dt}F(\phi_t(x)) = dF\big|_{\phi_t(x)}\frac{d}{dt}\phi_t(x).$$

   The right hand side is:

   $$\{F,H\}\big|_{\phi_t(x)} = dF\big|_{\phi_t(x)}X_H[\phi_t(x)]$$

   and it follows from the Hahn Banach theorem [12, Theorem 1.1.2, pp. 4-7] that these are equal in an open set $U \subset M$ for any $F \in C^\infty(M)$ iff $X_H[\phi_t(x)] = \frac{d}{dt}\phi_t(x)$ i.e ϕ_t is the flow of X_H with initial point x. Conversely if ϕ_t is a flow so that

   $$X_H[\phi_t(x)] = (\phi_t)_*(X_H[x]),$$

using the chain rule again we see:

$$\begin{aligned}\frac{d}{dt}F(\phi_t(x)) &= dF\big|_{\phi_t(x)} X_H[\phi_t(x)] \\ &= dF\big|_{\phi_t(x)} (\phi_t)_*(X_H[x]) \\ &= d((F \circ \phi_t)(x) X_H[x] \\ &= \{F \circ \phi_t, H\}(x).\end{aligned} \tag{3.20}$$

2. Follows from 1. using $H = F$.

$\square$

We want to compare the local dynamical equations of due to both Symplectic and Poisson geometry. In the Symplectic case, we have local coordinates (by Darboux's theorem) $x := (q^1, ..., q^n, p_1, ..., p_n)$ and the Hamiltonian equations of motion will read

$$\vec{\dot{x}} = J\nabla H, \tag{3.21}$$

where J:=

$$\begin{bmatrix} 0 & I_n \\ -\mathbb{I}_n & 0 \end{bmatrix}. \tag{3.22}$$

We compare this to the local Poisson manifold equations of motion. Let $(x_1, ..., x_n)$ be local coordinates, where n can be either even or odd (as opposed to the symplectic case). If we let Λ be the Poisson matrix in these coordinates, the dynamical equations $\dot{x}_i = \{H, x_i\}$ become

$$\vec{\dot{x}} = \Lambda\nabla H. \tag{3.23}$$

Thus we can see the similarities between the two formulations. However, we stress that the Poisson structure allows for dynamics on a space of arbitrary dimension.

Definition 3.10. Let $C \in C^\infty(U)$ for some open subset, $U \subset M$, of a Poisson manifold M. We say that C is a *Casimir function* of the Poisson structure if for all $H \in C^\infty(M)$ we have

$$\{H, C\} = 0. \tag{3.24}$$

An important Corollary of Theorem 3.9 is that if $G, H \in C^\infty(M)$, then G is constant along the integral curves of X_H iff $\{G, H\} = 0$. The Casimir functions are those which have trivial dynamics, that is C will be constant along the flow of all Hamiltonian vector fields. We note that functions G which are constant along integral curves of X_H will be the central focus of Chapter 4.

We finish by showing that for any choice in Hamiltonian, the dynamics of the system will live on a level set of the Casimir function C. Indeed, by the skew-symmetry of the Poisson bracket, we have $\{C, C\} = 0$ and so C is constant along the integral curves of X_C. Let $f, g \in C^\infty(M)$, using the Jacobi equation we see

$$\{C, \{f, g\}\} = \{f, \{C, g\}\} - \{g, \{H, f\}\} = 0, \tag{3.25}$$

where the terms on the right-hand side are zero by definition of a Casimir function; and so $\{C, \{f, g\}\} = 0$ and the dynamics are restricted to level sets of C.

"Thus for any choice of a Hamiltonian, the resulting dynamics will be constrained to live on a set on which L is constant."

3.4 Equations of Motion for $\mathfrak{g}^*$

We want to now consider what the equations of motion and Hamiltonian vector fields for a Hamiltonian $H \in C^\infty(\mathfrak{g}^*)$ look like in terms of the **Lie-Poisson bracket**. Recall the definition of the Lie-Poisson bracket in Equation (3.12) and let $F \in C^\infty(\mathfrak{g}^*)$. We have

$$\begin{aligned} \{F, H\}(\mu) &= \langle \mu, [d_\mu F, d_\mu H] \rangle \\ &= \langle \mu, -ad_{d_\mu H} d_\mu F \rangle \\ &= -\langle ad^*_{d_\mu H}\mu, d_\mu F \rangle. \end{aligned} \tag{3.26}$$

Moreover by the chain rule we have

$$\frac{dF}{dt} = dF(\mu)\frac{d\mu}{dt} = \langle \frac{d\mu}{dt}, d_\mu F \rangle \tag{3.27}$$

and non-degeneracy of the pairing implies that the equations of motion are

$$\frac{d\mu}{dt} = -ad^*_{d_\mu H}\mu. \tag{3.28}$$

Thus we have that the Hamiltonian vector fields of the Lie-Poisson structure are of the form

$$X_H[\mu] = -ad^*_{d_\mu H}\mu, \tag{3.29}$$

for a Hamiltonian H.

Using Equation (3.13) we can write a coordinate expression for the equations of motion:

$$\frac{dF}{dt} = \sum_{i,j=1}^{n} f^k{}_{ij} x_k \frac{\partial F}{\partial x_i}\frac{\partial H}{\partial x_j}. \tag{3.30}$$

The problem is too general to solve for any sort of dynamics other than the trivial dynamics in this essay.

So we will turn our attention to finding Casimir functions of the Lie-Poisson structure. A Casimir function of the Lie Poisson structure will satisfy:

$$X_C[F] = \sum_{i,j=1}^{n} f^k{}_{ij} x_k \frac{\partial F}{\partial x_i}\frac{\partial C}{\partial x_j} = 0, \quad \forall F \in C^\infty(M). \tag{3.31}$$

In particular for $F = x_l$, $l = 1, ..., n$ we must still have $\{x_l, C\} = 0$. This gives us

$$X_C[x_l] = \sum_{i,j=1}^{n} f^k{}_{ij} x_k \delta^l_i \frac{\partial C}{\partial x_j} = 0, \tag{3.32}$$

$$\implies X_C[x_l] = \sum_{j=1}^{n} f^k{}_{lj} x_k \delta^l_i \frac{\partial C}{\partial x_j} = 0, \quad l = 1, ..., n. \tag{3.33}$$

This will give us a system of n partial differential equations which we hope to solve for a Casimir function C on some open set $U \subset M$. We now look at the specific case of the Bianchi Lie algebras.

Example 3.11. For the Bianchi Lie algebras, using Equation (3.16) we deduce that if C is a Casimir function we must have:

$$\begin{aligned} (ax_2 + n_3x_3)\frac{\partial C}{\partial x_2} + (ax_3 - n_2x_2)\frac{\partial C}{\partial x_3} &= 0, \\ -(ax_2 + n_3x_3)\frac{\partial C}{\partial x_1} + n_1x_1\frac{\partial C}{\partial x_3} &= 0, \\ -(ax_3 - n_2x_2)\frac{\partial C}{\partial x_1} - n_1x_1\frac{\partial C}{\partial x_2} &= 0. \end{aligned} \tag{3.34}$$

Identifying $\mathfrak{g}^* \equiv \mathbb{R}^3$ with basis (x_2, x_2, x_3) I found in [8] open sets and Casimir functions for each Bianchi Lie group as outlined in Table 3.1.

Table 3.1: Casimir/Coadjoint Invariant Functions of the Bianchi Groups.

Type	Invariant	Open Set	0-D Orbits
I	0	$\mathbb{R}^3$	Everywhere
II	x_1	$\mathbb{R}^3 - \{x_1 = 0\}$	$x_1 = 0$
VII_0	$x_1^2 + x_2^2$	$\mathbb{R}^3 - \{x_3 \text{ axis}\}$	x_3 axis
VI_0	$x_1^2 - x_2^2$	$\mathbb{R}^3 - \{x_3 \text{ axis}\}$	x_3 axis
IX	$x_1^2 + x_2^2 + x_3^2$	$\mathbb{R}^3 - \{(0,0,0)\}$	The origin
VIII	$x_1^2 + x_2^2 - x_3^2$	$\mathbb{R}^3 - \{(0,0,0)\}$	The origin
V	x_2/x_3 x_3/x_2	$\mathbb{R}^3 - \{x_3 = 0\}$ $\mathbb{R}^3 - \{x_2 = 0\}$	x_1 axis
IV	$x_3 \exp(-\frac{x_2}{x_3})$	$\mathbb{R}^3 - \{x_3 = 0\}$	x_1 axis
VII_a	$(x_2^2 + x_3^2)\exp(-2a(\arctan(\frac{x_2}{x_3})))$	Cylindrical Coordinates: $\mathbb{R}^3 - \{(x_1, 0, 0)\}$	x_1 axis
III	$x_2 + x_3$	$\mathbb{R}^* - \{x_2 = x_3\}$	$x_2 = x_3$
$\mathrm{VI}_{a>1}$	$(x_2 - x_3)(x_2 + x_3)^{\frac{1+a}{1-a}}$	$\mathbb{R}^3 - \{x_2 + x_3 = 0\}$	x_1 axis
$\mathrm{VI}_{a<1}$	$(x_2 - x_3)(x_2 + x_3)^{\frac{1+a}{1-a}}$	$\mathbb{R}^3 - \{x_2 = x_3 = 0\}$	x_1 axis

For Bianchi $\mathfrak{g}^*$, it was completely necessary to use Poisson geometry to describe the mechanics since the configuration space (which is isomorphic to $\mathbb{R}^3$) is three-dimensional and thus symplectic geometry cannot be applied.

In the next section, we will see that Poisson manifolds decompose into symplectic manifolds called symplectic leaves. For the Lie-Poisson bracket the symplectic leaves are coadjoint orbits. Moreover, I showed in [8] that under certain conditions the level sets of the Casimir functions on an open subset of $\mathfrak{g}^*$ are the symplectic leaves themselves. The above table gives a partially complete classification for the coadjoint orbits of each Bianchi group.

While dynamics on our manifold cannot be described by Symplectic geometry, it does foliate into symplectic manifolds which can be described by level sets of the Casimir functions which are invariant under all Hamiltonian vector fields. In fact, our Poisson structure restricted to a leaf is exactly the symplectic form; a proof of this fact can be found in [4]. By our reasoning at the end of the previous section, the dynamics are restricted to the level sets of the Casimir functions, which are symplectic leaves and thus symplectic manifolds.

3.5 Symplectic Foliation

We have mentioned how Poisson manifolds will decompose into symplectic manifolds called 'symplectic leaves' and we next want to give a brief overview of the Theorems used to show this and apply them to the Lie-Poisson structure.

At the end of the previous section, we discussed how the dynamics on the Poisson manifold will be restricted to the level sets of the Casimir functions. We will see that under certain conditions the level sets of Casimir functions are exactly the symplectic leaves, and so the Poisson manifold will decompose into level sets of its Casimir functions.

To motivate this we state without proof the following theorems on the symplectic foliation of a Poisson manifold, the proofs of which can be found in [4, Section 5.3].

Theorem 3.12. *Let M be a Poisson manifold. Then M is the disjoint union of immersed submanifolds* *whose tangent spaces are spanned by its Hamiltonian vector fields. These submanifolds are symplectic manifolds called symplectic leaves. We call the decomposition the symplectic foliation of M.*

Theorem 3.13. *A function on a Poisson manifold M is a Casimir function if and only if F is constant on each symplectic leaf.*

These two theorems tell us that given a Poisson manifold M, it decomposes into the union of symplectic manifolds where the Poisson structure restricted to a symplectic leaf is the symplectic form. In fact, Theorem 3.13 goes further, which is explained in the following theorem, the proof of which can also be found in [5, Ch. 3].

Theorem 3.14. *Let M be a Poisson manifold of dimension n and let U be a nonempty open subset. Let $F_1, ..., F_r \in C^{\infty}(U)$ be such that:*

1. *The rank of the Poisson matrix is constant of U and equal to $n - r$.*
2. *The functions $F_1, ..., F_r$ are Casimir functions of the restriction of the Poisson structure to U.*
3. *For each point ξ of U the differentials $(dF_i)_{\xi}$ are independent.*

Then the symplectic leaves on U are the level sets of the map $(F_1, ..., F_r) : U \to \mathbb{R}^r$.

Symplectic foliation thus provides a beautiful connection between the two types of manifolds we have discussed in this essay. Now we wish to show that for the Lie-Poisson bracket the symplectic leaves are exactly the coadjoint orbits and the Casimir functions are the coadjoint invariant functions.

Let G be a connected Lie group (hence why we defined the Bianchi groups as being connected) whose Lie algebra is $\mathfrak{g}$. We show that the set of infinitesimal generators of the coadjoint action coincides with the set of Hamiltonian vector fields on $\mathfrak{g}^*$, so that we can apply Theorem 3.12. In Equation (3.29) we showed that the Hamiltonian vector fields were

$$X_H[\mu] = -ad^*_{d_\mu H}\mu, \tag{3.35}$$

for $\mu \in \mathfrak{g}^*$ and a Hamiltonian H. Now for $\xi \in \mathfrak{g}$ we have $d_\mu \xi^* = \xi^*$ where $\xi^* := \langle \mu, \xi \rangle$. Thus it follows that the set of Hamiltonian vector fields at $\mu \in \mathfrak{g}^*$ is

$$\{ad^*_{d_\mu F}\mu : F \in C^{\infty}(\mathfrak{g}^*)\} = \{ad^*_{\xi}\mu : \xi \in \mathfrak{g}\}, \tag{3.36}$$

which gives us our result after realising that G being connected means the coadjoint orbits are connected and so the symplectic leaves are coadjoint orbits by Theorem 3.12.
Moreover, it follows from Theorem 3.13 that the Casimir functions of the Lie-Poisson bracket are those that are constant on symplectic leaves, i.e. Ad^* invariant functions. In [8] I showed that for the Bianchi Lie groups; we could apply Theorem 3.14to show that the symplectic leaves were level sets of the Casimir functions we found.

Chapter 4

Momentum Maps and Noether's Theorem

We now aim to use the theory of Poisson geometry as developed in the previous chapter to provide a geometric understanding of Noether's theorem. A usual first course in the Lagrangian and Hamiltonian formulation of Classical Mechanics will introduce aspiring physicists to the notion of how symmetries give rise to conserved quantities via Noether's theorem. First examples are usually symmetry of time translation, which implies conservation of energy; symmetry under rotation gives conservation of angular momentum; translational symmetry implies conservation of momentum; a reference for these derivations can be found in [6]. Our geometric generalisation of conserved quantities arising from symmetries will be realised through the momentum map.

4.1 Momentum Maps

We begin by recalling that if Φ_g is a left action of a Lie group G on a manifold M then for each $g \in G$ the map $\Phi_g : M \to M$ is a smooth map. Furthermore, if M is a Poisson manifold and if for each $g \in G$ we have that Φ_g is a Poisson map, i.e.

$$\Phi_g^*\{F, G\} = \{\Phi_g^* F, \Phi_g^* G\}, \tag{4.1}$$

for any $F, G \in C^\infty(M)$: then we say that the action is a *Poisson action*. Let $\xi \in \mathfrak{g}$; then differentiating (4.1) in the direction of ξ we get:

$$\frac{d}{dt}(\exp(t\xi)\{F, G\})\big|_{t=0} = \{\frac{d}{dt}(\exp(t\xi)F)\big|_{t=0}, G\} + \{F, \frac{d}{dt}(\exp(t\xi)G)\big|_{t=0}\}. \tag{4.2}$$

Recalling from Equation (2.8) the definition of the infinitesimal action we have:

$$\bar{\xi}\{F, G\} = \{\bar{\xi}F, G\} + \{F, \bar{\xi}G\}; \tag{4.3}$$

in this case, we say that the action is an *infinitesimal Poisson automorphism.*

However we are interested in a slightly stronger condition: namely, that there exists a global Hamiltonian $J(\xi) \in C^\infty(M)$ for the infinitesimal action $\bar{\xi}$, i.e:

$$X_{J(\xi)} = \bar{\xi}. \tag{4.4}$$

We will say that a Lie algebra *acts canonically* on a Poisson manifold if (4.4) is satisfied. We are now ready to define a momentum map.

Definition 4.1. Let $\mathfrak{g}$ be a Lie algebra which acts canonically on a Poisson manifold M. Suppose that there is a linear map $J : \mathfrak{g} \to C^\infty(M)$ such that J satisfies (4.4) for all $\xi \in \mathfrak{g}$. The map

$$\mathbf{J} : M \to \mathfrak{g}^*$$

$$\langle \mathbf{J}(x), \xi \rangle := J(\xi)(x), \tag{4.5}$$

for $\xi \in \mathfrak{g}$ and $x \in M$ is called a *momentum map* of the action.

As we alluded to before; if we have symmetries arising from the action of our Lie group we hope to be able to mirror Noether's theorem by finding such a momentum map that will correspond to a conserved quantity of the symmetry.

Returning to the Lie-Poisson structure, we want a map $J : \mathfrak{g} \to C^\infty(\mathfrak{g}^*)$ such that J satisfies (4.4) for all $\xi \in \mathfrak{g}$. We first show that the coadjoint action is a Poisson action. Indeed given $F, G \in C^\infty(\mathfrak{g}^*)$ and letting $\alpha = Ad^*_{g^{-1}}$ we have

$$\begin{aligned} \{F, G\}(\alpha\mu) &= \langle \alpha\mu, [d_{\alpha\mu}F, d_{\alpha\mu}G] \rangle \\ &= \langle \alpha\mu, [Ad_g d_\mu(F \circ \alpha), Ad_g d_\mu(G \circ \alpha)] \rangle \\ &= \langle \mu, [d_\mu(F \circ \alpha), d_\mu(G \circ \alpha)] \rangle \\ &= \{F \circ \alpha, G \circ \alpha\}(\mu); \end{aligned} \tag{4.6}$$

where in the second line we used the identity:

$$d_\alpha F = Ad_g d_\mu(F \circ \alpha). \tag{4.7}$$

Now for the coadjoint action the infinitesimal acition is

$$\bar{\xi} = \frac{d}{dt}(\exp(t\xi)\mu)\big|_{t=0} = -ad^*_\xi \mu, \tag{4.8}$$

for $\mu \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$. Thus (4.4) in this case becomes:

$$ad^*_{d_\mu J(\xi)}\mu = ad^*_\xi \mu. \tag{4.9}$$

Hence the momentum map is the identity on $\mathfrak{g}^*$, which we see after realising

$$J(\xi)(\mu) = \langle \mu, \xi \rangle. \tag{4.10}$$

While in some sense this result is trivial, it will allow us to later classify the allowable Hamiltonians for the Lie-Poisson bracket and show that the dynamics must be generated by Casimir functions.

4.2 Noether's Theorem

As promised, we are now going to provide a geometric interpretation of Noether's theorem for Poisson manifolds. Let $\mathfrak{g}$ act canonically on a Poisson manifold M and assume there exists a momentum map $\mathbf{J} : M \to \mathfrak{g}^*$. Let $H \in C^\infty(M)$ be a $\mathfrak{g}-$ invariant Hamiltonian, that is

$$\bar{\xi}(H) = 0, \quad \forall \xi \in \mathfrak{g}, \tag{4.11}$$

which tells us that $\{J(\xi), H\} = 0$. Thus for any $\xi \in \mathfrak{g}$ it follows that $J(\xi)$ is conserved along the flow of X_H, the Hamiltonian vector field associated with H. Thus $\mathbf{J}$ is conserved. We have proved the following Theorem:

Theorem 4.2 (Noether's Theorem). *Let $\mathfrak{g}$ act canonically on the Poisson manifold M and assume there exists a momentum map $\boldsymbol{J} : M \to \mathfrak{g}^*$. If $H \in C^\infty(M)$ is $\mathfrak{g}-$invariant for all $\xi \in \mathfrak{g}$ then*

$$\boldsymbol{J} \circ \phi_t = \boldsymbol{J}, \tag{4.12}$$

where ϕ_t is the flow of X_H. I.e. $\boldsymbol{J}$ is a constant of motion. If the Lie algebra action comes from a Poisson action Φ_g of G then the invariance hypothesis of H is automatically satisfied.

As we hoped, we have a version of Noether's theorem which tells us that if an action preserves the Poisson structure on a Poisson manifold, then if there exists a momentum map it is a constant of motion. From Classical Mechanics courses, we know that translation invariance should imply that linear momentum is conserved; so we hope that our version of Noether's theorem will describe the same result.
Indeed, let us consider an n particle system and let $\mathbb{R}^{3n}$ act on $T^*\mathbb{R}^{3n} \cong \mathbb{R}^{6n}$ via the action

$$\dot{\vec{x}}(\vec{q}_i, \vec{p^i}) := (\vec{q}_i + \vec{x}, \vec{p^i}). \tag{4.13}$$

For Noether's theorem to be applied we need this to be a canonical action, i.e. Equation (4.4) must be satisfied. First we note that if $\xi \in \mathfrak{g}$ then differentiating the action tells us our infinitesimal action is

$$\bar{\xi}(\vec{q}_i, \vec{p^i}) = (\xi, ..., \xi, 0, ..., 0); \tag{4.14}$$

and so applying Equation (4.4) we find

$$X_{J(\xi)}(\vec{q}_i, \vec{p^i}) = (\frac{\partial J(\xi)}{\partial \vec{p^i}}, -\frac{\partial J(\xi)}{\partial \vec{q}_i}) = (\xi, ..., \xi, 0, ..., 0). \tag{4.15}$$

Thus, it follows that

$$J(\xi)(\vec{q}_i, \vec{p^i}) = (\sum_{i=1}^{n} \vec{p^i})\dot{\xi}, \tag{4.16}$$

and so the momentum map is exactly the linear momentum

$$\mathbf{J}(\vec{q}_i, \vec{p^i}) = (\sum_{i=1}^{n} \vec{p^i}), \tag{4.17}$$

which will be a constant of motion along any $\mathfrak{g}-$invariant Hamiltonian, as we expect.

4.3 Noether's Theorem on $\mathfrak{g}^*$

We will now apply Noether's theorem to the Lie-Poisson structure of the dual Lie algebras $\mathfrak{g}^*$, to find the possible Hamiltonians on $\mathfrak{g}^*$. In some sense the resulting dynamics will be trivial, however, it will allow us to express the tools we have developed thus far in this essay. We will then be able to classify the possible Hamiltonians in the Bianchi case and show that they are the Casimir functions we found before — recalling from Section 3.4 that the dynamics are restricted to level sets of these Casimir functions.

From Equation (4.10) we recall that $\mathbf{J} = Id_{\mathfrak{g}^*}$ is a momentum map for the Lie-Poisson bracket on $\mathfrak{g}^*$. Moreover the derivation in Equations (4.6) showed us that the coadjoint action of G was Poisson. It follows that for any $\mathfrak{g}-$invariant Hamiltonian $H \in C^\infty(\mathfrak{g}^*)$ we must have, by Equation (4.12), that

$$Id_{\mathfrak{g}^*} \circ \phi_t = Id_{\mathfrak{g}^*}, \tag{4.18}$$

where ϕ_t is the flow of X_H. In other words the flow of the Hamiltonian vector field must be the identity.

Equation (4.18) tells us that

$$\frac{d}{dt}\phi_t(x) = X_H(\phi_t(x)) = 0, \tag{4.19}$$

and recalling from Equation 3.31 the form of the Hamiltonian vector fields of the Lie-Poisson bracket in our usual basis for $\mathfrak{g}^*$:

$$X_H[\phi_t(x_1, ..., x_n)] = \sum_{i,j=1}^{n} f^k{}_{ij} x_k \frac{\partial(x_1, ..., x_n)}{\partial x_i} \frac{\partial H}{\partial x_j} = \sum_{i,j=1}^{n} f^k{}_{ij} x_k \frac{\partial H}{\partial x_j} = 0; \tag{4.20}$$

and so we must have

$$\sum_{i,j=1}^{n} f^k{}_{ij} x_k \frac{\partial H}{\partial x_j} = 0. \tag{4.21}$$

Example 4.3. Specializing to the Bianchi Lie groups, we stated in Equation (3.14) the structure constants of the Bianchi Lie algebras; from here it follows that a Hamiltonian on a Bianchi $\mathfrak{g}^*$, H_B, must satisfy

$$\frac{\partial H_B}{\partial x_1}(-ax_2 - n_3x_3 + n_2x_2 - ax_3) + \frac{\partial H_B}{\partial x_2}(ax_2 + n_3x_3 - n_1x_1) + \frac{\partial H_B}{\partial x_3}(n_1x_1 + ax_3 - n_2x_2) = 0. \quad (4.22)$$

It is easy to check that the Casimir functions we defined in Table 3.1 solve this. □

Returning to the general case of the Lie-Poisson bracket take our coadjoint invariant functions (Casimir functions) and analyse which ones solve (4.21). If we recall from Section 2.3, Casimir functions generate trivial dynamics on $\mathfrak{g}^*$, that is they are constant along the flow of all Hamiltonian vector fields; in fact, we will show that the only dynamics on $\mathfrak{g}^*$ are trivial.

To classify the dynamics on $\mathfrak{g}^*$ and show that the only allowable Hamiltonians applicable to Noether's theorem are the Casimir functions, we recall from Section 3.4 that the Ad^* invariant functions are the Casimir functions and so that to apply Noether's theorem we must have that the Hamiltonian is a Casimir function. Theorem 3.13 tells us for the Lie-Poisson bracket the Hamiltonians which can be applied to Noether's Theorem are exactly the Casimir functions, provided the Lie group G is connected. Thus the dynamics on $\mathfrak{g}^*$ must be trivial.

Example 4.4. For the Bianchi Lie groups, which we defined as being simply connected and connected, we were able to provide a classification of the Casimir functions on open subsets of $\mathfrak{g}^*$, thus we have been able to entirely classify the trivial dynamics of the Lie-Poisson structure of the Bianchi Lie-groups and so we have classified the dynamics of the reduced phase space T^*G/G for a three-dimensional space which admits a continuous group of motions. Thus we have classified Hamiltonian dynamics on a three-dimensional configuration space, a task that Poisson geometry was necessary to do.

□

Chapter 5

Reduction

Our final chapter is a technical one to show that the canonical Poisson bracket on the cotangent space T^*G is related to the Lie-Poisson bracket we defined on $\mathfrak{g}^*$. Some of the technical aspects of the theory we develop will be based on bundle theory and the proofs shall be omitted but may be found in [2]. Our proof of the Lie-Poisson Reduction theorem will use the fact that we have derived the Lie-Poisson bracket on $\mathfrak{g}^*$ already; a more constructive proof without assuming this can be found in [4].

5.1 Lie-Poisson Reduction

We now aim to prove the following theorem, which states that the reduced phase space of a Lie group T^*G/G is a Poisson manifold diffeomorphic to the Lie algebra $\mathfrak{g} = T_eG$. Throughout the section, the subscript on the bracket indicates the space in which the bracket is defined.

We first introduce some notation, namely for $\alpha_g \in T_g^*G$ define $O(\alpha_g)$ to be the set

$$O(\alpha_g) := \{\beta \in T^*G : \beta = T^*R_{g^{-1}}(\alpha), \text{ for some } g \in G\}. \tag{5.1}$$

Theorem 5.1 (Lie-Poisson Reduction Theorem)**.** *Suppose $\alpha_g \in T^*G$, the map*

$$\begin{aligned} f : T^*G/G &\to \mathfrak{g}^* \\ O(\alpha_g) &\mapsto T_e^*R_g(\alpha), \end{aligned} \tag{5.2}$$

is both a diffeomorphism and a Poisson map.

The proof that $f : T^*G/G \to \mathfrak{g}^*$ is a diffeomorphism requires bundle theory so we will provide only a brief overview of the proof. We can see this by considering the map

$$\begin{aligned} \lambda : T_g^*G &\to G \times \mathfrak{g}^*, \\ \lambda(\alpha_g) &:= (g, T_e^*R_g(\alpha_g)), \end{aligned} \tag{5.3}$$

which transforms the cotangent lift of the right action on G into the action of G on $G \times \mathfrak{g}^*$ given by

$$g(h,\xi) \equiv \Phi_g(h,\xi) := (gh, \xi), \tag{5.4}$$

and so we have the following diffeomorphisms:

$$T^*G/G \cong (G \times \mathfrak{g}^*)/G \cong \mathfrak{g}^*. \tag{5.5}$$

Thus we have the desired diffeomorphism $T^*G/G \cong \mathfrak{g}^*$.

Now the map $f : T^*G/G \to \mathfrak{g}^*$ provides us with the commutative triangle:

$$\begin{tikzcd} & T^*G/G \arrow[dr, "f"] & \\ T^*G \arrow[ur, "\pi"] \arrow[rr, "\mathbf{J}_L"] & & \mathfrak{g}^* \end{tikzcd}$$

where π is the canonical projection and $\mathbf{J}_L := T_e^* L_g$ is the momentum map of the cotangent lift of right translation on G; so that $J_L = f \circ \pi$. We note that because $R_g : G \to G$ is a diffeomorphism then it can be shown that its cotangent lift is a symplectic map.

We claim that if Φ_g is a Poisson action (cf. Equation (4.1)) of G on a Poisson manifold M then, assuming M/G is well defined and $\pi : M \to M/G$ is a smooth submersion, there exists a unique Poisson structure on M/G such that π is a Poisson map. Indeed, let $\alpha : M/G \to \mathbb{R}$ be a smooth real valued function and consider its pull back $\bar{\alpha} := \pi^*\alpha$ under π, then $\bar{\alpha}([x]) = \pi^*\alpha([x])$ is well defined as it is constant on $G-$orbits. Thus for $\beta, \gamma : M/G \to \mathbb{R}$ we have

$$\{\beta, \gamma\}_{M/G} \circ \pi = \{\beta \circ \pi, \beta \circ \pi\}_M \equiv \{\bar{\beta}, \bar{\gamma}\}_M, \tag{5.6}$$

because π is a Poisson map. Moreover the surjectivity of π tells us that $\{\beta, \gamma\}$ is determined uniquely. Now recalling that Φ_g is a Poisson action and the pull backs of β and γ under π are constant on $G-$orbits we have

$$\begin{aligned} \{\bar{\beta}, \bar{\gamma}\}(\Phi_g(x)) &= (\{\bar{\beta}, \bar{\gamma}\} \circ \Phi_g(x) \\ &= \{\bar{\beta} \circ \Phi_g, \bar{\gamma} \ \circ \Phi_g\}(x) \quad \text{(Poisson Action)} \\ &= \{\bar{\beta}, \bar{\gamma}\}(x), \qquad\qquad \text{(Constant on } G-\text{orbits)} \end{aligned} \tag{5.7}$$

an so $\{\bar{\beta}, \bar{\gamma}\}$ is also constant on $G-$orbits; therefore the bracket is well-defined. Moreover, the derivation properties, bilinearity and Jacobi identity all follow from the bracket on M.

Using this claim it follows that there is a unique Poisson structure on T^*G/G such that π is a Poisson map. The map $\mathbf{J}_L = T_e^* R_g$ is a Poisson map because of equivariance (show); it follows the map f is Poisson, meaning

$$(\{F, G\}_{\mathfrak{g}^*} \circ f)(x) = \{F \circ f, H \circ f\}_{T^*G/G}(x), \tag{5.8}$$

for $F, G \in C^\infty(\mathfrak{g}^*)$ and $x \in T^*G/G$.

To finish the proof, we note that because π is surjective, there exists an $\alpha_g \in T^*G$ for each $x \in T^*G/G$ such that $x = \pi(\alpha_g)$. Moreover recall that $\mathbf{J}_L = f \circ \pi$ is a Poisson map. It follows that

$$\begin{aligned} (\{F, H\}_{\mathfrak{g}^*} \circ f)(x) &= \{F \circ f, H \circ f\}_{\mathfrak{g}^*}(f \circ \pi)(\alpha_g) \\ &= \{F, H\}_{\mathfrak{g}^*} \circ \mathbf{J}_L(\alpha_g) \\ &= \{F \circ \mathbf{J}_L, H \circ \mathbf{J}_L\}_{T^*G}(\alpha_g) \\ &= \{F \circ f, H \circ f\}_{T^*G/G}(\pi(\alpha_g)) \\ &= \{F \circ f, H \circ f\}_{T^*G/G}(x). \end{aligned} \tag{5.9}$$

Chapter 6

Conclusion

6.1 Concluding Remarks

Throughout this essay, we aimed to show the power of Poisson geometry by defining dynamics on a manifold of arbitrary dimension and readily applying it to the case of the Lie-Poisson structure. We found that the dynamics of Poisson geometry were constrained to live on the level sets of the Casimir functions and proved a geometric version of Noether's theorem. Our development of the theory allowed us to understand symmetries of Hamiltonian systems and we applied this to the case of the Bianchi $\mathfrak{g}^*$, a three-dimensional Poisson manifold and showed that the Hamiltonian dynamics were trivial. Moreover, we related the symplectic foliation of $\mathfrak{g}^*$ to the coadjoint orbits of the corresponding Bianchi group G. We concluded the essay with a proof showing that the reduced phase space T^*G/G was a Poisson manifold diffeomorphic to $\mathfrak{g}^*$.

As a final glimpse into the richness of this subject, Section 6.2 will very briefly discuss its relation to quantum theory.

6.2 Glimpsping Further Developments: Geometric Quantization and the Orbit Method

Quantization refers to the process of taking a classical mechanical system and translating it into a quantum analogue by mapping classical observables to quantum operators. When considering a classical system whose phase space is described by a symplectic manifold (M, Ω), geometric quantization aims to quantize M by constructing a map sending observables $f \in C^\infty(M)$ to operators on a Hilbert space $\mathcal{H}$:

$$Q : C^\infty(M) \to Op(\mathcal{H}). \tag{6.1}$$

Dirac proposed certain conditions which such a map should obey; perhaps unsurprisingly, these are known as the *Dirac quantization conditions:*

1. The map Q must be linear on $C^\infty(M)$.
2. The map must relate commutators to Poisson brackets in the following sense:
$$[Q(f), Q(g)] = -i\hbar Q(\{f, g\}). \tag{6.2}$$
3. Constant observables must get mapped to constant operators:
$$Q(f) = f\mathbb{I}, \tag{6.3}$$
for f constant.
4. Completeness: if $\{f_i\}$ form a complete set of observables for the classical system then $\{Q(f_i)\}$ form a complete set of operators for the quantum system.

The exact formulation of geometric quantization is described in [10]. However, the theory of geometric quantization involves understanding line bundles, this would provide content enough for another Part III essay in itself.

In summary: geometric quantization amounts to a two-step process known as pre-quantization and polarization. A pre-quantization on a symplectic manifold (M, Ω) is a complex line bundle B over M together with a connection ∇ and a Hilbert structure H on the line bundle. Here the quantization map is defined as

$$Q(f) := -i\nabla_{X_f} + f. \tag{6.4}$$

Pre-quantization does a good job at mapping observables to operators. However, we must reduce this to a smaller space so that Condition 4. of the Dirac conditions is met. The process of polarization allows us to do exactly this. We stress that this is an extremely brief overview and invite the reader to explore the theory of geometric quantization in [10].

In essence, quantization is concerned with finding unitary representations of the structure of the classical phase space. The Orbit Method, described extensively in [11], concerns itself with finding unitary representations of Lie groups. If we have a Lie group G and know its coadjoint orbits, finding unitary representations of G amounts to performing geometric quantization on quantizable orbits. Indeed, we know from our work in the previous chapters that coadjoint orbits are symplectic manifolds; so quantization can be well defined on them. The orbit method identifies unitary representations of a Lie group G with the action of G on the state space of the geometric quantization.

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